# How $\pi^0 \to \gamma\gamma$ changes with temperature

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At zero temperature, in the chiral limit the amplitude for $\pi^0$ to decay into two photons is directly related to the coefficient of the axial anomaly. At any nonzero temperature, this direct relationship is lost: while the coefficient of the axial anomaly is independent of temperature, in a thermal bath the anomalous Ward identities do not uniquely constrain the amplitude for $\pi^0 \to \gamma\gamma$. Explicit calculation shows that to lowest order about zero temperature this amplitude decreases. [S0556-2821(97)00423-2]

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## I. INTRODUCTION

In field theory, currents which are conserved classically may not be quantum mechanically [1]. For example, in massless QED the conservation of the axial vector current is violated by the axial anomaly. The Adler-Bardeen theorem states that with the proper regularization scheme, the coefficient of the anomaly, as computed at one loop order, is exact to all orders in perturbation theory. Moreover, the axial anomaly does not change if the fermions propagate in either a thermal bath or a Fermi sea [2–4].

In vacuum, one of the most striking manifestations of the axial anomaly is the decay of a neutral pion into two photons, as the amplitude is directly proportional to the coefficient of the axial anomaly in QED [1,5]. A natural supposition is then that because the axial anomaly does not change with temperature, neither does the amplitude for $\pi^0 \to \gamma\gamma$ [6,7].

In this paper we show that the story is more involved. We compute with a gauged nonlinear sigma model [8] which properly incorporates all anomalies by inclusion of the Wess-Zumino-Witten (WZW) term [9–13]. The effective Lagrangian for $\pi^0 \to \gamma\gamma$ is

$$\mathcal{L}_{\pi\gamma\gamma} = \left( \frac{e^2 N_c}{48\pi^2} \right) \frac{1}{f_\pi} \pi^0 F_{\alpha\beta} \tilde{F}^{\alpha\beta}, \tag{1.1}$$

where $f_\pi \sim 93$ MeV is the pion decay constant, $N_c = 3$ is the number of colors, etc.

In Sec. II we start by computing the effects of pion loops on the amplitude of Eq. (1.1), using the WZW action to one loop order in vacuum [14]. The form of the WZW action is constrained by topology [10], so after the dust of calculation settles, in vacuum the result is trivial: the only effect of the pion loops is to change a bare pion decay constant into a renormalized $f_\pi$.

We then extend the calculations to soft, cool pions at low temperature [15]. In this paper we work exclusively with two flavors in the chiral limit. The restriction to two flavors is done for ease of calculation, and is otherwise inessential. The chiral limit, $m_\pi = 0$, is assumed because then the pion decay amplitude is directly related to the axial anomaly; for calculations at $m_\pi \neq 0$ at nonzero temperature; see [7,16]. We believe that our results are relevant for $m_\pi \neq 0$ (as in vacuum), but more detailed analysis is required to establish this.

At nonzero temperature, calculations in a background field formalism [15] show that to $\sim T^2/f_\pi^2$, the zero temperature pion decay constant is replaced by a temperature-dependent form [17,18]:

$$f_\pi(T) = \left( 1 - \frac{1}{12} \frac{T^2}{f_\pi^2} \right) f_\pi . \tag{1.2}$$

Thus a second guess for the change of $\mathcal{L}_{\pi\gamma\gamma}$ with temperature would be that the zero temperature $f_\pi$ is replaced by $f_\pi(T)$. Since $f_\pi(T)$ decreases to $\sim T^2/f_\pi^2$, if true the amplitude, $\sim 1/f_\pi(T)$, would increase to this order.

In Sec. III we evaluate precisely the same diagrams as at zero temperature to $\sim T^2/f_\pi^2$. The result, Eq. (3.10), is the sum of two terms: one has exactly the form of Eq. (1.1), with $f_\pi$ replaced by $f_\pi(T)$, but there is also a second term, special to nonzero temperature. This type of term was derived recently in nonlinear sigma models in the absence of gauge fields [15]: it is nonlocal, analogous to the hard thermal loops of hot gauge theories [19]. For $\pi^0 \to \gamma\gamma$, to order $\sim T^2/f_\pi^2$ we find that the sum of these two terms is such that instead of increasing, like $\sim 1/f_\pi(T)$, the amplitude decreases, like $\sim f_\pi(T)$, Eq. (3.9).

In Sec. IV we give a general analysis of the relationship between the chiral Ward identities and the amplitude for $\pi^0 \to \gamma\gamma$. As is standard [5], we use the anomalous Ward identity to relate a three point function of currents to the amplitude for $\pi^0 \to \gamma\gamma$. At zero temperature, this relationship is precise because of the Sutherland-Veltman theorem [20] (in a slight abuse of terminology). The proof of the Sutherland-Veltman theorem depends crucially upon Lorentz invariance. A heat bath, however, provides a preferred rest frame; extending an analysis of Itoyama and Mueller [3], we show that consequently, the Sutherland-Veltman theorem does not apply at any nonzero temperature. This is why $\pi^0 \to \gamma\gamma$ changes with temperature, even though the anomaly does not: besides the contributions from $\pi^0 \to \gamma\gamma$, because there is no Sutherland-Veltman theorem at nonzero temperature, there are other terms which enter to ensure that the Adler-Bardeen theorem is satisfied.

In Sec. V we demonstrate these general arguments by computing the correlator between one axial and two vector currents in the nonlinear sigma model to one loop order, $\sim T^2/f_\pi^2$. At nonzero temperature, new tensor structures arise

 

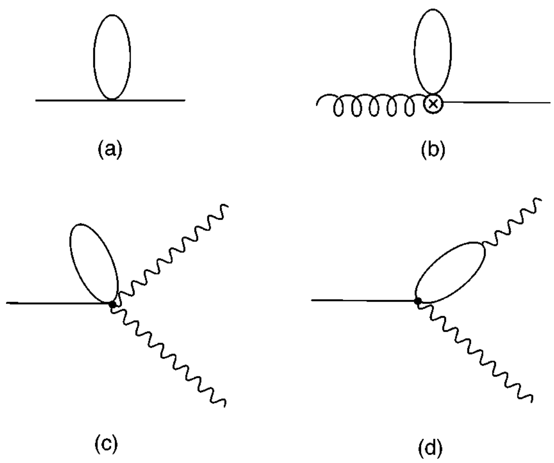


FIG. 1. One loop corrections to the $\pi\gamma\gamma$ vertex.

in this correlator; these structures are why the Sutherland-Veltman theorem is inapplicable at $T\neq 0$. Nevertheless, when all terms are added together, we find that the Adler-Bardeen theorem remains valid to one loop order. This is a useful and nontrivial check of our result for $\pi^0\to\gamma\gamma$.

Technical details are relegated to several appendixes. The WZW action is discussed in Appendix A. Various formulas for hard thermal loops are collected in Appendix B. We use the imaginary time formalism at nonzero temperature in this paper, but show in Appendix C how the same results follow in the real time formalism. Lastly, in Appendix D we compute another anomalous amplitude, that for $\gamma\to\pi\pi\pi$ [22], at low temperature.

While the principal concern of our work is thermal field theories, we hope that some of our dicussion, especially that in Sec. IV, might be of more general interest. Perhaps understanding why $\pi^0\to\gamma\gamma$ is not tied to the axial anomaly at nonzero temperature helps us better understand this relation at zero temperature.

## II. $\pi\to\gamma\gamma$ IN VACUUM

We start by computing the effects of pion loops on the amplitude for $\pi\to\gamma\gamma$ in vacuum. At one loop order, the relevant diagrams are Figs. 1(a)–1(d): Fig. 1(a) gives the pion field renormalization constant, $Z_\pi$; Fig. 1(b), the renormalized pion decay constant $f_\pi$, while corrections to the amplitude itself are given by Figs. 1(c) and 1(d).

For the ''tadpole'' type diagrams of Figs. 1(a), 1(b), and 1(c) we use a trick. To compute Fig. 1(a) we expand the full Lagrangian, Eq. (A10), to quartic order in the pion field:

$$\mathcal{L}=\frac{1}{2}(\partial_\alpha\vec{\pi})^2+\frac{1}{6f_b^2}[(\vec{\pi}\cdot\partial_\alpha\vec{\pi})^2-\vec{\pi}^2(\partial_\alpha\vec{\pi})^2]+\cdots. \tag{2.1}$$

In all expressions from the appendix, we need to use the bare pion decay constant, $f_b$, instead of the renormalized quantity $f_\pi$, as one loop effects change $f_b$ into $f_\pi$. For the quartic terms, contracting two out of the four pion fields in all possible ways gives

$$\langle\mathcal{L}\rangle\simeq\frac{1}{2}\left(1-\frac{2}{3}\frac{\mathcal{I}_0}{f_b^2}\right)(\partial_\alpha\vec{\pi})^2\equiv\frac{1}{2}(\partial_\alpha\vec{\pi}_r)^2. \tag{2.2}$$

where

$$\mathcal{I}_0=\langle\pi^2\rangle=\int\frac{d^4K}{(2\pi)^4}\frac{1}{K^2}. \tag{2.3}$$

While this integral is quadratically divergent, we ignore regularization, since its actual value is irrelevant for our purposes. In Eq. (2.2) $\pi_r=\pi/\sqrt{Z_\pi}$ is the renormalized pion field, and so

$$Z_\pi=1+\frac{2}{3}\frac{\mathcal{I}_0}{f_b^2}. \tag{2.4}$$

For the pion decay constant, instead of Fig. 1(b) we expand the axial vector current $J^a_{5,\alpha}$ of Eq. (A11) to cubic order in the pion field:

$$J^a_{5,\alpha}=f_b\partial_\alpha\pi^a-\frac{2}{3f_b}(\vec{\pi}^2\partial_\alpha\pi^a-\pi^a\vec{\pi}\cdot\partial_\alpha\vec{\pi})+\cdots. \tag{2.5}$$

Contracting all pairs of pion fields,

$$\langle J^a_{5,\alpha}\rangle=\left(1-\frac{4}{3}\frac{\mathcal{I}_0}{f_b^2}\right)f_b\partial_\alpha\pi^a\equiv f_\pi\partial_\alpha\pi^a_r, \tag{2.6}$$

so that

$$f_\pi=\left(1-\frac{\mathcal{I}_0}{f_b^2}\right)f_b. \tag{2.7}$$

As is typical of nonlinear sigma models, unphysical, off-shell quantities such as $Z_\pi$, Eq. (2.4), depend upon the parametrization of the coset space, while physical expressions, such as that for $f_\pi$ in Eq. (2.7), do not.

Turning to the amplitude for $\pi^0\to\gamma\gamma$, from Eq. (1.1) it is

$$\mathcal{M}=g^b_{\pi\gamma\gamma}\varepsilon_{\alpha\beta\gamma\delta}\epsilon_1^\alpha\epsilon_2^\beta P_1^\gamma P_2^\delta; \tag{2.8}$$

$P_1$ and $P_2$, and $\epsilon_1$ and $\epsilon_2$ are the momenta and the polarization vectors of the two photons, both of which lie on the mass shell, $P_1^2=P_2^2=P_1\cdot\epsilon_1=P_2\cdot\epsilon_2=0$. At tree level, the bare coupling $g^b_{\pi\gamma\gamma}$ satisfies

$$f_b g^b_{\pi\gamma\gamma}=\frac{e^2N_c}{12\pi^2}. \tag{2.9}$$

The right-hand side of Eq. (2.9) is precisely the coefficient of the axial anomaly in QED [1].

To evaluate Fig. 1(c) we expand the anomalous current for the coupling to two photons, Eqs. (A7), (A11), and (A14), to cubic order in the pion field:

$$\mathcal{L}_{\pi\gamma\gamma} = \left(\frac{e^2 N_c}{48\pi^2}\right)\frac{1}{f_b}\varepsilon_{\alpha\beta\gamma\delta}F_{\alpha\beta}A_\gamma\left[\left(1-\frac{2}{3}\frac{\vec{\pi}^2}{f_b^2}\right)\partial_\delta \pi^3 + \frac{2}{3f_b^2}\vec{\pi}\cdot\partial_\delta\vec{\pi}\pi^3\right]+\cdots. \tag{2.10}$$

Contracting two pion fields,

$$\langle \mathcal{L}_{\pi\gamma\gamma}\rangle \simeq \left(1-\frac{4}{3}\frac{\mathcal{I}_0}{f_b^2}\right)\left(\frac{e^2 N_c}{48\pi^2}\right)\varepsilon_{\alpha\beta\gamma\delta}F_{\alpha\beta}A_\gamma \partial_\delta \pi^3. \tag{2.11}$$

The two distinct diagrams of Fig. 1(d) require more effort:

$$\mathcal{M}^d = 8\left(\frac{e^2 N_c}{48\pi^2}\right)\frac{1}{f_b^3}\varepsilon_{\alpha\beta\gamma\delta}P_1^\beta P_2^\delta \epsilon_1^\alpha \Gamma_{\gamma\sigma}(P_2)\epsilon_2^\sigma + (P_1,\epsilon_1 \rightleftharpoons P_2,\epsilon_2), \tag{2.12}$$

where

$$\Gamma^{\alpha\beta}(P) = \int \frac{d^4K}{(2\pi)^4}\frac{K^\alpha K^\beta}{K^2(K-P)^2} \equiv \frac{\mathcal{I}_0}{2}[\delta^{\alpha\beta} - \Pi^{\alpha\beta}(P)]. \tag{2.13}$$

This peculiar separation of terms in $\Gamma^{\alpha\beta}(P)$ is done in anticipation of the results at nonzero temperature. In vacuum, for $P^2=0$, $\Pi^{\alpha\beta}(P)\sim P^\alpha P^\beta$, and so because of the Levi-Civita symbol, $\Pi^{\alpha\beta}(P)$ does not contribute to $\mathcal{M}^d$. Hence Eq. (2.12) reduces to a form proportional to the original term in Eq. (2.8). Putting everything together, the renormalized coupling $g_{\pi\gamma\gamma}$ equals

$$g_{\pi\gamma\gamma} = \left[1+\left(\frac{1}{3}-\frac{4}{3}+2\right)\frac{\mathcal{I}_0}{f_b^2}\right]g^b_{\pi\gamma\gamma} = \left(1+\frac{\mathcal{I}_0}{f_b^2}\right)g^b_{\pi\gamma\gamma}. \tag{2.14}$$

The 1/3 comes from a factor of $\sqrt{Z_\pi}$ for the renormalized pion field, Fig. 1(a) and Eq. (2.4), the $-4/3$ from Fig. 1(c), Eq. (2.11), and the 2 from Fig. 1(d), Eqs. (2.12) and (2.13). Hence at one loop order,

$$f_\pi g_{\pi\gamma\gamma} = \frac{e^2 N_c}{12\pi^2}. \tag{2.15}$$

Comparing Eqs. (2.9) and (2.15), we see that the anomaly is not renormalized to one loop order [1]: separate divergences in $f_\pi$ and $g_{\pi\gamma\gamma}$ cancel in the product [14].

### III. $\pi \rightarrow \gamma\gamma$ AT LOW TEMPERATURE

We now compute the decay for a cool pion, at a temperature $T \ll f_\pi$ [15]. The diagrams are identical, the only difference is that we need to compute at $T \neq 0$. For the tadpole diagrams of Figs. 1(a), 1(b), and 1(c), the integral is the analogy of Eq. (2.3):

$$\langle \pi^2 \rangle = T\sum_{n=-\infty}^{+\infty}\int \frac{d^3k}{(2\pi)^3}\frac{1}{K^2}. \tag{3.1}$$

We use the imaginary time formalism, $K^2 = k_0^2 + \vec{k}^2$, $k=|\vec{k}|$, and $k^0 = 2\pi n T$ for integral $n$; after doing the sum over $n$, the Bose-Einstein statistical distribution function, $n(\omega) = 1/[\exp(\omega/T)-1]$, appears:

$$\langle \pi^2 \rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2k}[1+2n(k)] \equiv \mathcal{I}_0 + \mathcal{I}_T = \mathcal{I}_0 + \frac{T^2}{12}; \tag{3.2}$$

$\mathcal{I}_0$ is the value of the integral at zero temperature. Henceforth we drop the $T=0$ part of any integrals, assuming that they turn bare into renormalized quantities, such as $f_b$ into $f_\pi$, throughout.

The calculation of temperature-dependent corrections to the pion decay constant proceeds as in the previous section. Ignoring ultraviolet renormalization, in Eq. (2.7) we replace $\mathcal{I}_0$ by $\mathcal{I}_T$, and $f_b$ by $f_\pi$, to obtain

$$f_\pi(T) = \left(1-\frac{\mathcal{I}_T}{f_\pi^2}\right)f_\pi = \left(1-\frac{1}{12}\frac{T^2}{f_\pi^2}\right)f_\pi. \tag{3.3}$$

which was quoted in the introduction, Eq. (1.2).

Thus if anything unusual happens at nonzero temperature, it can only be from the diagram of Fig. 1(d). Unlike the tadpole diagrams, this diagram has nontrivial momentum dependence, and so we must be more precise in specifying the external momenta. To compute scattering in a thermal bath, we continue the euclidean momenta $p^0$ to a minkowski energy $\omega$ by $p^0 = -i\omega + 0^+$. Following [15] we further assume that each momentum is not only cool but soft, taking both $|\omega|, p \ll T \ll f_\pi$.

For scattering between soft, cool pions, in [15] we showed that the leading temperature corrections are directly analogous to the hard thermal loops of hot gauge theories [19]. We used the background field method, but only in the absence of external gauge fields. While the perturbative calculations which follow are thus less elegant, they illustrate the physics more directly. From the perspective of [15], there is nothing special about $\pi^0 \rightarrow \gamma\gamma$; the connection to the axial anomaly will be clarified later.

We introduce $\delta\Gamma^{\alpha\beta}(P)$,

$$\delta\Gamma^{\alpha\beta}(P) = \frac{T^2}{24}[\delta^{\alpha\beta} - \delta\Pi^{\alpha\beta}(P)], \tag{3.4}$$

with

$$\frac{T^2}{24}\delta\Pi^{\alpha\beta}(P) \approx T\sum_{n=-\infty}^{+\infty}\int \frac{d^3k}{(2\pi)^3}\left\{\frac{\delta^{\alpha\beta}}{2}\frac{1}{K^2} - \frac{K^\alpha K^\beta}{K^2(K-P)^2}\right\}. \tag{3.5}$$

The $\approx$ sign denotes that only the hard thermal loops in the integral are retained, which we denote by $\delta\Gamma^{\alpha\beta}(P)$ and $\delta\Pi^{\alpha\beta}(P)$. The hard thermal loops are the terms $\sim T^2$, and are given explicitly in Appendix B.

Up to an overall constant, $\delta\Pi^{\alpha\beta}$ is the same hard thermal loop as appears in the polarization tensor for a photon in thermal equilibrium. For a thermal photon, the screening of time-dependent electric and magnetic fields implies that the mass shell is at $P^2 \sim e^2 T^2$. For the sake of simplicity we assume that the photons do not thermalize; then the only photons which propagate are transverse modes on the light cone, $P_1^2 = P_2^2 = 0$. The polarization vectors for these modes are purely spatial vectors which satisfy $P \cdot \epsilon = 0$.

From Eq. (2.12), the contribution to the amplitude from Fig. 1(d), $\mathcal{M}^d$, involves $\Gamma^{\alpha\beta}(P)\epsilon^\beta$ for one of the two photons on their mass shell. Using Eq. (B6) of Appendix B,

$$\delta\Pi^{\alpha\beta}(P)\epsilon^\beta|_{P^2=0} = \epsilon^\alpha, \tag{3.6}$$

where $\epsilon^\beta$ is the polarization vector for the photon with momentum $P$. Only the first term on the right-hand side of Eq. (B7) contributes, as terms in $\delta\Pi^{\alpha\beta}(P)$ which are $\sim P^\beta$ or $n^\beta$ drop out after contraction with $\epsilon^\beta$. From the definition of $\delta\Gamma^{\alpha\beta}$, Eq. (3.4),

$$\delta\Gamma^{\alpha\beta}(P)\epsilon^\beta|_{P^2=0} = 0. \tag{3.7}$$

Consequently, while at zero temperature Fig. 1(d) contributes to the amplitude for $\pi^0 \to \gamma\gamma$, to leading order at nonzero temperature its contribution vanishes identically, Eq. (3.7).

Knowing that Fig. 1(d) does not contribute, it is then easy to read off the one loop corrections to the coupling $g_{\pi\gamma\gamma}$ to $\sim T^2/f_\pi^2$, $g_{\pi\gamma\gamma}(T)$. As in Eq. (3.3), we start with Eq. (2.14), and then replace $\mathcal{I}_0$ by $\mathcal{I}_T$, and $f_b$ by $f_\pi$. We keep the 1/3 from Fig. 1(a), the $-4/3$ from Fig. 1(c), but replace the $+2$ from Fig. 1(d) by 0, to obtain

$$g_{\pi\gamma\gamma}(T) = \left[1 + \left(\frac{1}{3} - \frac{4}{3} + 0\right)\frac{\mathcal{I}_T}{f_\pi^2}\right] g_{\pi\gamma\gamma} = \left(1 - \frac{\mathcal{I}_T}{f_\pi^2}\right) g_{\pi\gamma\gamma}. \tag{3.8}$$

Notice that while Eq. (2.7) is precisely analogous to Eq. (3.3), because of the difference in Fig. 1(d), Eq. (2.14) is not analogous to Eq. (3.8) — there is a difference in sign. Consequently,

$$g_{\pi\gamma\gamma}(T) = \left(1 - \frac{1}{12}\frac{T^2}{f_\pi^2}\right) g_{\pi\gamma\gamma}. \tag{3.9}$$

As discussed in the Introduction, naively one might guess that Eq. (2.15) generalizes to nonzero temperature just by replacing $f_\pi$ and $g_{\pi\gamma\gamma}$ with $f_\pi(T)$ and $g_{\pi\gamma\gamma}(T)$, respectively. This is wrong: to $\sim T^2/f_\pi^2$, instead of $g_{\pi\gamma\gamma}(T) \sim 1/f_\pi(T)$, as would be guessed from Eq. (2.15), instead $g_{\pi\gamma\gamma}(T) \sim f_\pi(T)$, Eq. (3.9). We do not know why, to leading order about low temperature, $g_{\pi\gamma\gamma}(T)$ decreases in exactly the same manner as $f_\pi(T)$.

Our result in Eq. (3.8) differs from that found by Dobado, Alvarez-Estrada, and Gomez [7]. These authors consider the same model, but find $g_{\pi\gamma\gamma}(T) = g_{\pi\gamma\gamma}$ to $\sim T^2/f_\pi^2$. Our results agree except for Fig. 1(d), which we believe was treated incorrectly [23].

Before continuing, following [15] we construct the effective Lagrangian for $\pi^0 \to \gamma\gamma$ to $\sim T^2/f_\pi^2$. In $\delta\Gamma^{\alpha\beta}(P)$ of Eq. (3.4), the term $\sim \delta^{\alpha\beta}$ is easy to include. At zero temperature, Eq. (2.13), this term is the only part of Fig. 1(d) which contributes, $+2$ in Eq. (2.14), and turns $1/f_b$ in $g_{\pi\gamma\gamma}$ into $1/f_\pi$ in $g^r_{\pi\gamma\gamma}$. Thus to $\sim T^2/f_\pi^2$, the effect of Figs. 1(a), 1(c), and the term $\sim \delta^{\alpha\beta}$ in $\delta\Gamma^{\alpha\beta}(P)$ of Fig. 1(d) is just to change $1/f_\pi$ into $1/f_\pi(T)$ in the original Lagrangian, $\mathcal{L}_{\pi\gamma\gamma}$ of Eq. (1.1).

Including the term $\delta\Pi^{\alpha\beta}(P)$ is less trivial. Because of Eq. (B1), it must be constructed out of transverse quantities. Using Eq. (B8), we find that to $\sim T^2/f_\pi^2$, the effective Lagrangian for $\pi^0 \to \gamma\gamma$ is

$$\mathcal{L}_{\pi^0\gamma\gamma}(T) = \left(\frac{e^2 N_c}{48\pi^2}\right)\frac{1}{f_\pi(T)}\pi^0 F_{\alpha\beta}\tilde{F}^{\alpha\beta} - \frac{T^2}{12 f_\pi^2}\left(\frac{e^2 N_c}{48\pi^2}\right)\int \frac{d\Omega_{\hat{k}}}{4\pi} H_{\gamma\alpha}\frac{\hat{K}^\alpha \hat{K}^\beta}{-(\partial\cdot\hat{K})^2} F_{\gamma\beta}. \tag{3.10}$$

In this expression $\tilde{F}^{\alpha\beta} = \epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}/2$,

$$H_{\alpha\beta} = \partial_\alpha H_\beta - \partial_\alpha H_\beta, \tag{3.11}$$

and

$$H_\alpha = \frac{1}{f_\pi}\varepsilon_{\alpha\beta\gamma\delta}F_{\beta\gamma}\partial_\delta \pi^0. \tag{3.12}$$

The vector $\hat{K} = (i, \hat{k})$ and the integration over the angle $\hat{k}$ are discussed following Eq. (B2).

The nonlocal term in Eq. (3.10) is specific to finite temperature. At zero temperature, there is no other term besides Eq. (1.1) that contributes to $\pi^0 \to \gamma\gamma$ for photons on the mass-shell. In the terminology of the nonlinear sigma model [8], the operator of Eq. (1.1) is $O(P^4)$, while operators at next to leading order are $O(P^6)$. These operators, however, are either proportional to $P_1^2$ or $P_2^2$, and so vanish on the photon(s) mass shell, or $m_\pi^2$, and so vanish in the chiral limit. Thus in the vacuum, the only possible change in Eq. (1.1) is the transmutation of $f_b$ into $f_\pi$, Sec. II. At nonzero temperature, however, there are new nonlocal terms which arise, Eq. (3.10). Because they are nonlocal, these new terms are also $O(P^4)$, and so as important as Eq. (1.1) [15]. This is the technical reason why the amplitude for $\pi^0 \to \gamma\gamma$ depends nontrivially upon temperature.

## IV. $\pi^0 \to \gamma\gamma$ AND THE AXIAL ANOMALY

In the previous section we found that the amplitude for $\pi^0 \to \gamma\gamma$ diminishes to leading order in an expansion about low temperature, Eq. (3.9). The question we address in this section is why is this amplitude tied to the coefficient of the axial anomaly at zero temperature, Eq. (2.15), but not at nonzero temperature?

We work in the chiral limit to leading order about zero temperature, $\sim T^2/f_\pi^2$, because then we can make certain technical assumptions which simplify the discussion. The general case is considered at the end of this section.

Define the vector current $J_\alpha$ and the axial vector current in the isospin-3 direction, $J^3_{5,\gamma}$. The vector current is conserved,

$$\partial^\alpha J_\alpha = 0, \tag{4.1}$$

while the axial current is anomalous,

$$\partial^\alpha J^3_{5,\alpha} = -\frac{e^2 N_c}{48\pi^2} F_{\alpha\beta}\widetilde{F}^{\alpha\beta}. \tag{4.2}$$

By the Adler-Bardeen theorem, the coefficient of the right-hand side is exact to one loop order [1], and is independent of temperature and density [2–4].

One quantity we can compute is the (thermal) three point Green's function between two vector, and one axial vector, current:

$$\mathcal{T}_{\alpha\beta\gamma}(P_1,P_2;T) = -ie^2 \int d^4X_1 d^4X_2 e^{i(P_1\cdot X_1 + P_2\cdot X_2)} \times \frac{\mathrm{Tr}[e^{-H/T} J_\alpha(X_1) J_\beta(X_2) J^3_{5,\gamma}(0)]}{\mathrm{Tr}(e^{-H/T})}, \tag{4.3}$$

where $H$ is the Hamiltonian. Then $\mathcal{T}_{\alpha\beta\gamma}$ satisfies current conservation,

$$P_1^\alpha \mathcal{T}_{\alpha\beta\gamma} = P_2^\beta \mathcal{T}_{\alpha\beta\gamma} = 0, \tag{4.4}$$

and the anomalous Ward identity,

$$Q^\gamma \mathcal{T}_{\alpha\beta\gamma} = -\frac{e^2 N_c}{12\pi^2} \varepsilon_{\alpha\beta\gamma\delta} P_1^\gamma P_2^\delta, \tag{4.5}$$

$Q = P_1 + P_2$.

To relate the anomalous Ward identity to the amplitude for pion decay we follow Shore and Veneziano [5]. At low temperature the pion couples to the axial current as

$$\langle 0 | J^a_{5,\alpha} | \pi^b(Q) \rangle = i Q_\alpha f_\pi \delta^{ab}. \tag{4.6}$$

(This is not valid to $\sim T^4/f_\pi^4$; then the relation is more complicated, [18].)

To obtain the amplitude for $\pi^0 \to \gamma\gamma$, we introduce $Q^2$ times the matrix element between two QED currents and a pion:

$$\mathcal{T}_{\alpha\beta} = e^2 Q^2 \int d^4X_1 d^4X_2 e^{i(P_1\cdot X_1 + P_2\cdot X_2)} \times \frac{\mathrm{Tr}[e^{-H/T} J_\alpha(X_1) J_\beta(X_2) \pi(0)]}{\mathrm{Tr}(e^{-H/T})}. \tag{4.7}$$

This is related to the pion decay amplitude, Eq. (2.8), as

$$\mathcal{M} = \lim_{Q^2\to 0} \epsilon_1^\alpha \epsilon_2^\beta \mathcal{T}_{\alpha\beta}. \tag{4.8}$$

Subtracting the one pion pole term from Eq. (4.3) gives $\hat{\mathcal{T}}_{\alpha\beta\gamma}$, which by construction is one pion irreducible:

$$\hat{\mathcal{T}}_{\alpha\beta\gamma} = \mathcal{T}_{\alpha\beta\gamma} + f_\pi Q_\gamma \frac{1}{Q^2} \mathcal{T}_{\alpha\beta}. \tag{4.9}$$

Again, by current conservation

$$P_1^\alpha \hat{\mathcal{T}}_{\alpha\beta\gamma} = P_2^\beta \hat{\mathcal{T}}_{\alpha\beta\gamma} = 0, \tag{4.10}$$

while the anomalous Ward identity, Eq. (4.5), becomes

$$Q^\gamma \hat{\mathcal{T}}_{\alpha\beta\gamma} = f_\pi \mathcal{T}_{\alpha\beta} - \frac{e^2 N_c}{12\pi^2} \varepsilon_{\alpha\beta\gamma\delta} P_1^\gamma P_2^\delta. \tag{4.11}$$

The trick is now to try to deduce general relations using just Eqs. (4.10), (4.11), and Bose symmetry between the two photons, $P_1, \alpha \rightleftharpoons P_2, \beta$.

We first discuss zero temperature, where we can invoke Euclidean invariance. The most general pseudotensor $\hat{\mathcal{T}}_{\alpha\beta\gamma}$ which satisfies all of our conditions involves three terms:

$$\hat{\mathcal{T}}_{\alpha\beta\gamma} = T_1 \varepsilon_{\alpha\beta\gamma\delta}(P_1^\delta - P_2^\delta) + T_2(\varepsilon_{\alpha\gamma\delta\kappa} P_2^\beta - \varepsilon_{\beta\gamma\delta\kappa} P_1^\alpha) P_1^\delta P_2^\kappa + T_3(\varepsilon_{\alpha\gamma\delta\kappa} P_1^\beta - \varepsilon_{\beta\gamma\delta\kappa} P_2^\alpha) P_1^\delta P_2^\kappa. \tag{4.12}$$

Current conservation, Eq. (4.10), gives

$$T_1 + P_1^2 T_2 + P_1 \cdot P_2 T_3 = 0, \tag{4.13}$$

while from the anomalous Ward identity, Eq. (4.11),

$$-2T_1 = f_\pi g_{\pi\gamma\gamma} - \frac{e^2 N_c}{12\pi^2}. \tag{4.14}$$

Combining these two relations we obtain

$$2P_1^2 T_2 + 2P_1 \cdot P_2 T_3 = f_\pi g_{\pi\gamma\gamma} - \frac{e^2 N_c}{12\pi^2}. \tag{4.15}$$

Putting the photons on their mass shell $P_1^2 = P_2^2$, the left-hand side in Eq. (4.15) reduces to $Q^2 T_3$. This is zero on the pion mass shell, $Q^2 \to 0$, since by definition $\hat{\mathcal{T}}$ is one pion irreducible, and so cannot have a pole $\sim 1/Q^2$. Hence the left-hand side of Eq. (4.15) vanishes, and we obtain the desired relation between $g_{\pi\gamma\gamma}$ and the coefficient of the axial anomaly,

$$0 = f_\pi g_{\pi\gamma\gamma} - \frac{e^2 N_c}{12\pi^2}, \tag{4.16}$$

which is Eq. (2.13).

This analysis, and especially the tensor decomposition of Eq. (4.12), is identical to the derivation of the Sutherland-Veltman theorem [20]. Historically, this theorem predated the anomaly, and was used to conclude that $g_{\pi\gamma\gamma} = 0$. By adding the axial anomaly through the anomalous Ward identity of Eq. (4.11), however, we obtain $g_{\pi\gamma\gamma} \sim e^2 N_c / f_\pi$, Eq. (4.16). From a modern perspective, then, it is precisely the Sutherland-Veltman theorem which relates the axial anomaly to the amplitude for $\pi^0 \to \gamma\gamma$, and tells us at zero temperature the left-hand side of Eq. (4.16) vanishes.

This is no longer true at nonzero temperature. Following Itoyama and Mueller [3], we write the most general tensor decomposition of $\hat{\mathcal{T}}_{\alpha\beta\gamma}$. In a thermal bath, however, Euclidean symmetry is lost, and the rest frame of the thermal bath, which we take as $n_\mu = (1, \vec{0})$, enters. Some of the possible tensors include

$$\hat{\mathcal{T}}_{\alpha\beta\gamma}=T_1\varepsilon_{\alpha\beta\gamma\delta}(P_1^\delta-P_2^\delta)+T_2(\varepsilon_{\alpha\gamma\delta\kappa}P_2^\beta-\varepsilon_{\beta\gamma\delta\kappa}P_1^\alpha)P_1^\delta P_2^\kappa$$
$$+T_3(\varepsilon_{\alpha\gamma\delta\kappa}P_1^\beta-\varepsilon_{\beta\gamma\delta\kappa}P_2^\alpha)P_1^\delta P_2^\kappa$$
$$+T_4 n\cdot Q\varepsilon_{\alpha\beta\delta\kappa}P_1^\delta P_2^\kappa n^\gamma+T_5(n\cdot P_2\varepsilon_{\alpha\gamma\delta\kappa}n^\beta$$
$$-n\cdot P_1\varepsilon_{\beta\gamma\delta\kappa}n^\alpha)P_1^\delta P_2^\kappa+\cdots. \quad (4.17)$$

We have only included the terms in $\hat{\mathcal{T}}$ which contribute to the Ward identities of Eq. (4.10) and (4.11). Current conservation gives

$$T_1+P_1^2T_2+P_1\cdot P_2T_3+(n\cdot P_1)^2T_5=0, \quad (4.18)$$

while the anomalous Ward identity fixes

$$-2T_1+(n\cdot Q)^2T_4=f_\pi(T)g_{\pi\gamma\gamma}(T)-\frac{e^2N_c}{12\pi^2}, \quad (4.19)$$

from which follows

$$2P_1^2T_2+2P_1\cdot P_2T_3+(n\cdot Q)^2T_4+2(n\cdot P_1)^2T_5$$
$$=f_\pi(T)g_{\pi\gamma\gamma}(T)-\frac{e^2N_c}{12\pi^2}. \quad (4.20)$$

Putting all fields on their mass shell, $P_1^2=P_2^2=0$, $Q^2\to0$, and assuming as before that $T_3$ has no pole $\sim1/Q^2$, we obtain

$$(n\cdot Q)^2T_4+2(n\cdot P_1)^2T_5=f_\pi(T)g_{\pi\gamma\gamma}(T)-\frac{e^2N_c}{12\pi^2}. \quad (4.21)$$

Since the terms on the left-hand side involve only $n\cdot Q$ and $n\cdot P_1$, there is no reason for them to vanish even if $P_1^2=P_2^2=Q^2=0$; thus the direct connection between $g_{\pi\gamma\gamma}(T)$, $f_\pi(T)$, and the anomaly is lost. We stress that, as always, the Adler-Bardeen theorem remains valid, and gives Eq. (4.20). It is only the Sutherland-Veltman theorem which no longer applies at nonzero temperature.

The above analysis only applies to leading order in low temperature, $\sim T^2/f_\pi^2$. This is because beyond leading order, the pion mass shell is no longer at $Q^2=0$ [18]; also, as photons thermalize, their mass shell moves off the light cone. This is incorporated by using Eq. (4.20) instead of Eq. (4.21). For instance, we can understand how the anomalous Ward identity is satisfied in a chirally symmetric phase. From explicit calculation in a constituent quark model [21], $\pi^0\to\gamma\gamma$ vanishes once chiral symmetry is restored. This does not conflict with the anomalous Ward identity since even if $g_{\pi\gamma\gamma}(T)=0$, there are other terms which can ensure that Eq. (4.20) is satisfied. For photons which do not thermalize, so $P_1^2=P_2^2=0$, even assuming that at the chiral critical point that the pion mass shell is $Q^2=0$, the tensors $T_4$ and/or $T_5$ will in general be nonzero.

Our analysis agrees with the results of Contreras and Loewe [16], who computed the triangle diagram at $T\neq0$ with massive fermions. Their result, Eq. (1.2), is directly related to $\hat{\mathcal{T}}_{\alpha\beta\gamma}$. There are two terms: the first is regular, temperature dependent, and gives the amplitude for $\pi^0\to\gamma\gamma$, while the second is the anomaly, and is independent of temperature.

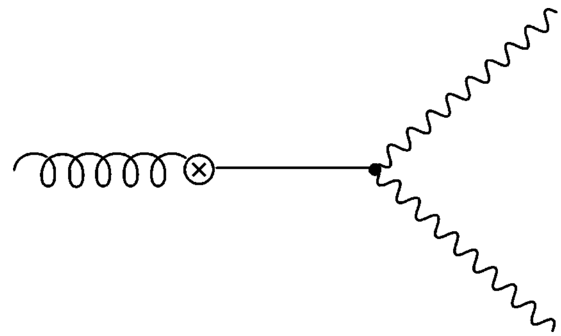

FIG. 2. One pion reducible contribution to $\mathcal{T}_{\alpha\beta\delta}$ at tree level.

At first sight it might appear peculiar that the Sutherland-Veltman theorem applies at zero temperature, but fails at any nonzero temperature. Even at zero temperature, however, the Sutherland-Veltman theorem only applies in the chiral limit when both photons are on their mass shell. Without all of these conditions, the left-hand side of Eq. (4.13) does not vanish, and $g_{\pi\gamma\gamma}$ is not given by Eq. (4.14). An example of this occurs when one (or both) of the photons are off the mass shell [24], even if $Q^2\to0$. In particular, in the limit of large $P^2$, it is known that $g_{\pi\gamma\gamma}\sim e^2f_\pi/P^2$ [24]. Thus on the right-hand side of Eq. (4.13), we can neglect $f_\pi g_{\pi\gamma\gamma}\sim1/P^2$ relative to the anomaly term, $-e^2N_c/(12\pi^2)$, and Eq. (4.13) tells us that a combination of $T_2$ and $T_3$ are given entirely by the anomaly, $\sim e^2N_c/P^2$.

## V. THE ADLER-BARDEEN THEOREM AT LOW TEMPERATURE

In this section we calculate the correlator $\hat{\mathcal{T}}_{\alpha\beta\gamma}$ to $\sim T^2/f_\pi^2$ in the nonlinear sigma model. This allows us to check explicitly the temperature dependence of $f_\pi(T)$ and $g_{\pi\gamma\gamma}(T)$ found previously in Sec. III. It is also illuminating to see exactly which amplitudes enter at one loop order in the nonlinear sigma model, and how they conspire to satisfy the Adler-Bardeen theorem.

At tree level in the nonlinear sigma model, there is no direct coupling between two vector and one axial vector currents, so $T_1=T_2=T_3=0$. Thus the anomalous Ward identity, Eq. (4.5), is satisfied entirely by the one pion reducible term, Eq. (4.11), with $\hat{\mathcal{T}}_{\alpha\beta\delta}=0$. This is illustrated in Fig. 2.

Contributions to $\hat{\mathcal{T}}_{\alpha\beta\gamma}$ are generated at one loop order. In order to compute these, it is necessary to compute corrections to the axial current. We have computed these in two ways. The most direct is to follow the original method of Wess and Zumino [9]. Besides the photon field $A_\alpha$, which couples to the electromagnetic current $J_\alpha$, we also introduce an external field $A_{5,\alpha}^3$ which couples to the axial current $J_{5,\alpha}^3$. One then differentiates the generating functional of Wess and Zumino with respect to the external fields $A_\alpha$ and $A_{5,\alpha}^3$. At one loop order, the terms required are $\sim\pi^+\pi^-A_{5,\alpha}^3A_\beta$ and $\sim\pi^+\pi^-A_{5,\alpha}^3A_\beta A_\gamma$; $\pi^\pm=(\pi^1\pm i\pi^2)/\sqrt{2}$ are the charged pion fields. After lengthy calculation, we find

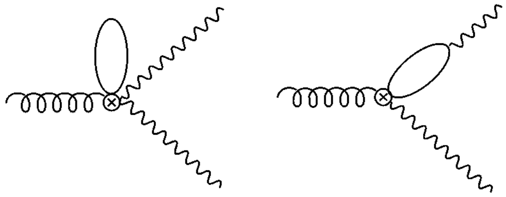


FIG. 3. One pion irreducible contributions to $\mathcal{T}_{\alpha\beta\delta}$ at one loop order.

$$\Delta\mathcal{L} = \left(\frac{eN_c}{48\pi^2 f_\pi^2}\right)\epsilon_{\alpha\beta\gamma\delta}F_{\alpha\beta}A^3_{5,\gamma}[i(\pi^-\partial_\delta\pi^+ - \pi^+\partial_\delta\pi^-) - 2e\pi^+\pi^-A_\delta]. \quad (5.1)$$

To leading order in $e$, this $\mathcal{L}$ is invariant under an electromagnetic gauge transformation:

$$\pi^\pm(x) \to \exp^{\pm ie\theta(x)}\pi^\pm(x),$$

$$A_\alpha(x) \to A_\alpha(x) - \partial_\alpha\theta(x). \quad (5.2)$$

The terms in $\mathcal{L}$ can be viewed as corrections to the axial current. In Appendix A we compute these corrections using the Noether construction of the axial current. This is somewhat delicate, since it is necessary to start with a Lagrangian density (and not merely a Lagrangian) which is manifestly gauge invariant. The result, Eq. (A17), agrees with Eq. (5.1).

Through the diagrams of Fig. 3, the couplings in $\Delta\mathcal{L}$ generate the one pion irreducible terms in $\hat{\mathcal{T}}_{\alpha\beta\gamma}$.

We begin with the results at zero temperature. Given the couplings in $\Delta\mathcal{L}$, the only tensor structure which arises from Fig. 3 is $\sim\epsilon_{\alpha\beta\gamma\delta}(P_1-P_2)^\delta$. Comparing with the tensors in Eq. (4.12), then, at one loop order automatically $T_2=T_3=0$. For $T_1$, we find

$$T_1 = (2-2)\left(\frac{\mathcal{I}_0}{f_b^2}\right)\frac{e^2N_c}{24\pi^2} = 0. \quad (5.3)$$

The contribution $\sim +2$ is from Fig. 3(a), that $\sim -2$ from the two diagrams of Figure 3(b). Because all of the $T_i$'s vanish, current conservation (4.13) and the anomaly equation (4.14), are satisfied rather trivially. That the latter vanishes is equivalent to the Sutherland-Veltman theorem, Eq. (4.16). This explains our results in Sec. II, where we found that at zero temperature, the renormalization of $f_\pi$ and $g_{\pi\gamma\gamma}$ exactly compensate each other.

At nonzero temperature we can evaluate $T_1 \ldots T_5$ using the results of Sec. III. Fig. 3(a) is a tadpole diagram, and so as at zero temperature, just contributes to $T_1$. Thus the only possibility for a new tensor structure is from the hard thermal loop in Fig. 3(b). For one ordering of momenta, Fig. 3(b) gives

$$-i\frac{e^2N_c}{12\pi^2 f_\pi^2}\epsilon_{\gamma\delta\beta\kappa}P_2^\delta\,\delta\Gamma^{\kappa\alpha}(P_1), \quad (5.4)$$

where $\delta\Gamma^{\kappa\alpha}$ is given in Eq. (3.4). Putting the photon momentum $P_1$ on its mass shell, we require the result for $\delta\Pi^{\kappa\alpha}(P_1)$ in Eq. (B6). We drop all terms in $\delta\Pi^{\kappa\alpha}(P_1)\sim P_1^\alpha$, Eq. (B6), since they vanish upon contraction with the photon polarization tensor. The term $\sim\delta^{\kappa\alpha}$ in Eq. (B6) cancels against the same term in Eq. (3.4); thus at nonzero temperature, there is no contribution to $T_1$ from Fig. 3(b), and

$$T_1 = (2+0)\left(\frac{\mathcal{I}_T}{f_\pi^2}\right)\frac{e^2N_c}{24\pi^2}. \quad (5.5)$$

Comparing to Eq. (5.3), again Fig. 3(a) gives the term $\sim +2$, and Fig. 3(b) the term $\sim 0$. That Fig. 3(b) does not contribute is just like the same result for $g_{\pi\gamma\gamma}(T)$, Fig. 1(d) and Eq. (3.8).

Figure 3(b) will contribute, however, through the term $\sim n^\alpha P_1^\kappa/(n\cdot P_1)$ in $\delta\Pi^{\kappa\alpha}(P_1)$ in Eq. (B6). Comparing with the tensor decomposition of Eq. (4.17), we find

$$T_5 = -\frac{1}{(n\cdot P_1)^2}\left(\frac{\mathcal{I}_T}{f_\pi^2}\right)\frac{e^2N_c}{12\pi^2}. \quad (5.6)$$

Since these are the only diagrams at one loop order,

$$T_2 = T_3 = T_4 = 0. \quad (5.7)$$

Consequently,

$$-2T_1 = 2(n\cdot P_1)^2T_5 = f_\pi(T)g_{\pi\gamma\gamma}(T) - \frac{e^2N_c}{12\pi^2}. \quad (5.8)$$

Thus we can see that our results satisfy current conservation, Eq. (4.18) and the anomaly equation of (4.21). Last but not least, the latter only holds given $f_\pi(T)$ and $g_{\pi\gamma\gamma}(T)$ in Eqs. (1.1) and (3.8), and so provides a nontrivial check of these calculations.

Notice that while there is a factor of $1/(n\cdot P_1)^2$ in $T_5$, it is essentially kinematic in origin, as envisioned by Itoyama and Mueller [3]. One factor of $n\cdot P_1$ arises from the definition of $T_5$, Eq. (4.17), while the other can be seen to arise from a directional singularity, $\sim p^i/p$, in the integral of Eq. (B6). Further, notice that there is a (logarithmic) collinear divergence in the integral of Eq. (B6). This singularity drops out of the full amplitude after contracting with the polarization tensor of the photon.

## VI. CONCLUSIONS

In this paper we have concentrated exclusively on the anomalous decay of $\pi^0\to\gamma\gamma$. We have done so because it is the most familiar anomalous decay, and because the connection to the axial anomaly is especially close. For the collisions of heavy ions at ultrarelativistic energies, though, this is a purely academic point, since any high temperature plasma flies apart long before a $\pi^0$ has a chance to decay electromagnetically.

Our basic point is much more general, however. While the axial anomaly for fermions is independent of temperature, anomalous decays of mesonic fields change with temperature. Processes of obvious interest for hadronic systems include $\eta'$ into two gluons and the decays of the $\omega$ meson [21]. Electroweak processes include couplings of the axion. Whether the changes in these couplings are physically relevant can only be determined after detailed calculation; what is certain is that they do change. In a cosmological context, the mechanisms of electroweak baryogenesis involve the effective coupling of a pseudoscalar field to the Pontryagin density for the $SU(2)_L$ gauge field. In vacuum, this coupling is directly related to the anomalous divergence of the baryon current. It was argued in [25] that this coupling must be suppressed above the electroweak phase transition; presumably this can be understood from our analysis. Lastly, we note that the 't Hooft anomaly matching conditions [26] strongly constrain the appearance of massless bound states in confining theories. Due to the failure of the Sutherland-Veltman theorem, Sec. IV, these conditions are clearly much less restrictive at nonzero temperature.

If nothing else, perhaps this gives us a greater appreciation of the wonder of the fermion axial anomaly. While every other anomalous decays changes in complicated and detailed ways, that alone remains inviolate, always.

## ACKNOWLEDGMENTS

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## APPENDIX A: WZW ACTION

In this appendix we review the Wess-Zumino-Witten (WZW) model coupled to an external photon field. Along with establishing notation, this also enables us to discuss a novel form of the WZW model, mentioned recently [12,13], and to comment about the two currents in the two-flavor version of the WZW model.

For $n_f$ flavors, the model is constructed from a $n_f \times n_f$ unitary matrix $g$, $g^\dagger g = 1$. In the absence of gauge fields, the action is the sum of two terms, $\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_{\rm WZW}$, with $\mathcal{S}_0 = \int d^4x \mathcal{L}_0$ the usual action for a nonlinear sigma model:

$$\mathcal{L}_0 = \frac{f_\pi^2}{4}\,{\rm tr}(\partial_\alpha g^\dagger \partial_\alpha g). \tag{A1}$$

The generators of $SU(n_f)$ are the matrices $\lambda^a$, normalized as ${\rm tr}(\lambda^a \lambda^b) = 2\delta^{ab}$. This Lagrangian is invariant under global $SU(n_f)_{\ell} \times SU(n_f)_r$ unitary transformations, $g(x) \to \Omega_{\ell}\, g(x) \Omega_r^\dagger$. For example, under axial transformations, for which $\Omega_{\ell} = \Omega_r^\dagger$, the corresponding conserved current is

$$\mathcal{J}_{5,\alpha} = R_\alpha + L_\alpha = g^\dagger \partial_\alpha g + (\partial_\alpha g) g^\dagger. \tag{A2}$$

The second piece of the action is the Wess-Zumino-Witten term,

$$\mathcal{S}_{\rm WZW} = -i\,\frac{N_c}{240\pi^2}\int d^5x\, \varepsilon_{\alpha\beta\gamma\delta\sigma}\; tr(R_\alpha R_\beta R_\gamma R_\delta R_\sigma), \tag{A3}$$

where the integral is over a five-dimensional region whose boundary is four-dimensional spacetime.

We wish to couple $g$ to a photon field $A_\alpha(x)$ in a gauge invariant manner. For three flavors the charge matrix is $Q = \mathbf{1}/6 + \lambda_3/2$, and so we introduce the covariant derivative, $D_\alpha = \partial_\alpha + ieA_\alpha[Q,\cdot]$. By construction, both $g$ and $D_\alpha g$ transform covariantly under local U(1) gauge transformations, $A_\alpha \to A_\alpha + \partial_\alpha \theta(x)$, $\Theta(x) = \exp[-ieQ\theta(x)]$, $g \to \Theta g \Theta^\dagger$, $D_\alpha g \to \Theta(D_\alpha g)\Theta^\dagger$.

As usual, to make $\mathcal{L}_0$ gauge invariant we simply replace the ordinary by the covariant derivative, $\tilde{\mathcal{L}}_0 = f_\pi^2 tr|D_\alpha g|^2/4$. We can also do this for $\mathcal{S}_{\rm WZW}$ by replacing $R_\alpha$ with the gauge covariant $\tilde{R}_\alpha = g^\dagger D_\alpha g$. Using $\tilde{R}_\alpha$ in $\mathcal{S}_{\rm WZW}$ gives something which is manifestly gauge invariant, but incomplete, since it can be shown that the ensuing equations of motion depend on the fifth dimension. In five dimensions, however, there are several other gauge invariant terms which can be added, involving powers of the (Abelian, gauge invariant) field strength, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. The correct action then follows uniquely by requiring that the equations of motion are independent of the fifth dimension [12,13],

$$\begin{aligned}\tilde{\mathcal{S}}_{\rm WZW} = -i\,\frac{N_c}{240\pi^2}\int d^5x\,\varepsilon_{\alpha\beta\gamma\delta\sigma}\Big\{ & {\rm tr}(\tilde{R}_\alpha \tilde{R}_\beta \tilde{R}_\gamma \tilde{R}_\delta \tilde{R}_\sigma) \\ & + 5F_{\alpha\beta}\,{\rm tr}[Q(\tilde{L}_\gamma \tilde{L}_\delta \tilde{L}_\sigma + \tilde{R}_\gamma \tilde{R}_\delta \tilde{R}_\sigma)] \\ & - 10F_{\alpha\beta}F_{\gamma\delta} tr\Big[Q^2(\tilde{L}_\sigma + \tilde{R}_\sigma) + \frac{1}{2}Qg^\dagger Q D_\sigma g \\ & - \frac{1}{2}QgQD_\sigma g^\dagger\Big]\Big\}.\end{aligned} \tag{A4}$$

This expression is manifestly gauge invariant but not obviously independent of the fifth dimension [27]. This is in contrast to the usual action, in which the terms which couple to the gauge field are manifestly four dimensional, but not evidently gauge invariant: $\tilde{\mathcal{S}}_{\rm WZW} = \mathcal{S}_{\rm WZW} + \int d^4x \mathcal{L}^A_{\rm WZW}$, where

$$\mathcal{L}^A_{\rm WZW} = -eN_c A_\alpha \mathcal{J}_\alpha + ie^2 N_c \varepsilon_{\alpha\beta\gamma\delta}\partial_\alpha A_\beta A_\gamma \mathcal{K}_\delta, \tag{A5}$$

where

$$\mathcal{J}_\alpha = \frac{1}{48\pi^2}\varepsilon_{\alpha\beta\gamma\delta}\;{\rm tr}\,\{Q(L_\beta L_\gamma L_\delta + R_\beta R_\gamma R_\delta)\}, \tag{A6}$$

and

$$\mathcal{K}_\alpha = \frac{1}{24\pi^2}\,{\rm tr}\Big\{Q^2(L_\alpha + R_\alpha) + \frac{1}{2}(QgQg^\dagger L_\alpha + Qg^\dagger QgR_\alpha)\Big\}. \tag{A7}$$

The Lagrangian (A5) is the form given by Witten [10]: it is gauge invariant up to boundary terms.

Henceforth we follow Brihaye, Pak, and Rossi [11] and restrict ourselves to two flavors. Introducing the pion field $\pi^a$,

$$g = \exp\left(i\,\frac{\pi^a \lambda^a}{f_\pi}\right), \tag{A8}$$

then for SU(2)

$$g = \cos\phi + i\,\frac{\pi^a\lambda^a}{\sqrt{\vec{\pi}^2}}\sin\phi, \quad \phi = \frac{\sqrt{\vec{\pi}^2}}{f_\pi}, \tag{A9}$$

$\vec{\pi}^2=\pi^a\pi^a$. The original Lagrangian $\mathcal{L}_0$ becomes

$$\mathcal{L}_0=\frac{1}{2}\frac{\sin^2\phi}{\phi^2}(\partial_\alpha\vec{\pi})^2+\frac{1}{2f_\pi^2}\frac{(\phi^2-\sin^2\phi)}{\phi^4}(\vec{\pi}\cdot\partial_\alpha\vec{\pi})^2. \tag{A10}$$

The axial current of Eq. (A2) is

$$\mathcal{J}^a_{5,\alpha}=4i\left(\frac{\sin\phi\,\cos\phi}{\phi}\partial_\alpha\pi^a+\frac{(\phi-\sin\phi\,\cos\phi)}{f_\pi^2\phi^3}\vec{\pi}\cdot\partial_\alpha\vec{\pi}\pi^a\right). \tag{A11}$$

In the WZW term, $S_{\rm WZW}$ vanishes, while using the identity,

$$\varepsilon^{abc}\varepsilon_{\alpha\beta\gamma\delta}\left(\vec{\pi}\cdot\partial_\beta\vec{\pi}\pi^a-\frac{\vec{\pi}^2}{3}\partial_\beta\pi^a\right)\partial_\gamma\pi^b\partial_\delta\pi^c=0, \tag{A12}$$

we find that

$$\mathcal{J}_\alpha=\frac{1}{72\pi^2f_\pi^3}\frac{\sin^2\phi}{\phi^2}\varepsilon^{abc}\varepsilon_{\alpha\beta\gamma\delta}\partial_\beta\pi^a\partial_\gamma\pi^b\partial_\delta\pi^c. \tag{A13}$$

The charge $Q=\mathbf{1}/6+\lambda_3$; in going from Eq. (A6) to Eq. (A13), it turns out that the contribution from the piece $\sim\lambda_3$ drops out. This demonstrates that $\mathcal{J}_\alpha$ is directly proportional to the baryon current [10]. Further, we find that $\mathcal{K}_\alpha$ is proportional to the axial current in the isospin-3 direction, Eq. (A11),

$$\mathcal{K}_\alpha=\frac{1}{96\pi^2f_\pi}\mathcal{J}^3_{5,\alpha}. \tag{A14}$$

That for two flavors $\mathcal{J}_\alpha$ and $\mathcal{K}_\alpha$ are proportional to the baryon and axial isospin currents does not seem to have been recognized previously. Notice that $\mathcal{K}_\alpha$ is only proportional to the axial current for the original Lagrangian, $\mathcal{L}_0$; as is discussed in Sec V, the complete axial current includes contributions from the WZW term. We do not know if $\mathcal{J}_\alpha$ and $\mathcal{K}_\alpha$ are equal to the analogous currents for three or more flavors.

To conclude, we discuss how to compute the axial current $J^3_{5,\mu}$ of Sec. V using the Noether construction. The general form for the gauged axial current is rather involved. Since we compute perturbatively, however, we just compute the current in the same way. Under an infinitesimal axial rotation, $\Omega_{\ell}=\Omega_r^\dagger=exp(i\omega)$, the pion field transforms nonlinearly,

$$\pi^a\to\pi^a+f_\pi\omega^a+\frac{1}{3f_\pi}(-\vec{\pi}^2\omega^a+\vec{\pi}\cdot\vec{\omega}\pi^a)+\cdots, \tag{A15}$$

to $\sim\omega$ and $\sim\pi^3$. After performing such a transformation, the requisite current is then the coefficient of $\partial_\alpha\omega^3$.

In the present case, however, while the Lagrangian is (of course) gauge invariant, the Lagrangian density need not be. In particular, the Lagrangian density of Eq. (A5) is not gauge invariant; under a local gauge transformation, it transforms by a total derivative. To the order at which we compute, we add the following term to the Lagrangian density:

$$-\frac{e^2N_c}{24\pi^2}\epsilon_{\alpha\beta\gamma\delta}\partial_\alpha(\pi^3A_\beta\partial_\gamma A_\delta)-\frac{ieN_c}{24\pi^2}\epsilon_{\alpha\beta\gamma\delta}\partial_\alpha[A_\beta(\pi^+\partial_\gamma\pi^- -\pi^-\partial_\gamma\pi^+)\partial_\delta\pi^3]. \tag{A16}$$

For example, the first piece contributes a Chern-Simons term to the current. With the addition of Eq. (A16), the Lagrangian density is manifestly gauge invariant, and the axial current is

$$J^3_{5,\alpha}\approx f_\pi\partial_\alpha\pi^3+\frac{2}{3f_\pi}(\vec{\pi}\cdot\partial_\alpha\vec{\pi}\pi^3-\vec{\pi}^2\partial_\alpha\pi^3)+\frac{ieN_c}{24\pi^2f_\pi^2}\varepsilon_{\alpha\beta\gamma\delta}\partial_\beta A_\gamma(\pi^+\partial_\delta\pi^- -\pi^-\partial_\delta\pi^+)-\frac{e^2N_c}{12\pi^2}\left(\frac{\pi^+\pi^-}{f_\pi^2}\right)\varepsilon_{\alpha\beta\gamma\delta}A_\beta\partial_\gamma A_\delta+\cdots \tag{A17}$$

When $A_\alpha=0$, $J^3_{5,\alpha}$ reduces to the axial vector current in the absence of electromagnetism, $\mathcal{J}^3_{5,\alpha}$, as can be verified by expanding Eq. (A11). The terms linear and quadratic in $A_\alpha$ agree with the calculation from the Wess-Zumino consistency condition, Eq. (5.1).

This ambiguity in the construction of the axial current is familiar. Instead of the gauge invariant, anomalous current used in Sec. IV, by subtracting off a Chern-Simons term, we can choose to work instead with a current which is conserved but not gauge invariant.

## APPENDIX B: HARD THERMAL LOOPS

We collect some results of hard thermal loops, with minor differences in notation from previous work [19]. The simplest hard thermal loop is the integral of Eq. (3.1), $\mathcal{I}_T=T^2/12$. After that, there is $\delta\Pi^{\alpha\beta}(P)$ of Eqs. (3.4) and (3.5). By definition the hard thermal loop includes only the terms $\sim T^2$ in the integral, in the limit of soft external momentum $P\ll K\sim T$. In this approximation, $\Pi^{\alpha\beta}(P)$ is tranverse:

$$P^\alpha\delta\Pi^{\alpha\beta}(P)=0. \tag{B1}$$

One way of writing $\delta\Pi^{\alpha\beta}(P)$ is

$$\delta\Pi^{\alpha\beta}(P)=2n^\alpha n^\beta+2\int\frac{d\Omega}{4\pi}\omega\frac{\hat{K}^\alpha\hat{K}^\beta}{P\cdot\hat{K}}, \tag{B2}$$

where $n_\alpha=(1,\vec{0})$ and $\hat{K}=(i,\hat{k})$; $\hat{k}$ is a three vector of unit norm, $\hat{k}^2=1$, so that $\hat{K}^\alpha$ is null, $\hat{K}^2=0$, and $P\cdot\hat{K}=ip_0+\vec{p}\cdot\hat{k}=\omega+p\cos\theta$. It is a dummy variable, in that one integrates over all directions of $\hat{k}$, as $\int d\Omega_{\hat{k}}/(4\pi)$. In component form,

$$\delta\Pi^{00}(P)=-2Q_1(z),$$

$$\delta\Pi^{0i}(P)=-2izQ_1(z)\hat{p}^i,$$

$$\delta\Pi^{ij}(P)=2z^2Q_1(z)\hat{p}^i\hat{p}^j-\frac{2}{5}\left(Q_3(z)-Q_1(z)-\frac{5}{3}\right)\times(\delta_{ij}-\hat{p}^i\hat{p}^j), \quad \text{(B3)}$$

where $z=\omega/p$, and the $Q_i(z)$ are Legendre functions of the second kind,

$$Q_1(z)=\frac{z}{2}\ln\left(\frac{z+1}{z-1}\right)-1, \quad \text{(B4)}$$

$$Q_3(z)=\frac{z(5z^2-3)}{4}\ln\left(\frac{z+1}{z-1}\right)-\frac{5}{2}z^2+\frac{2}{3}. \quad \text{(B5)}$$

For $\pi^0\rightarrow\gamma\gamma$, we need the value of $\delta\Pi^{\alpha\beta}(P)$ near the light cone, $\omega\sim p^+$,

$$\delta\Pi^{\alpha\beta}\sim\delta^{\alpha\beta}+\frac{1}{n\cdot P}(n^\alpha P^\beta+P^\alpha n^\beta)-\left[\ln\left(\frac{2p}{\omega-p}\right)-1\right]\frac{P^\alpha P^\beta}{p^2}, \quad \text{(B6)}$$

where $n^\alpha=(1,\vec{0})$.

An equivalent but useful expression for Eq. (B2) is

$$\Pi^{\alpha\beta}(P)=\int\frac{d\Omega}{4\pi}\left[\delta^{\alpha\beta}+\frac{\hat{K}^\alpha\hat{K}^\beta}{(\hat{K}\cdot P)^2}P^2-\frac{P^\alpha\hat{K}^\beta+\hat{K}^\alpha P^\beta}{(\widehat{K\cdot P})}\right]; \quad \text{(B7)}$$

from which follows

$$T\sum\int\frac{d^3p}{(2\pi)^3}X_\alpha\delta\Pi^{\alpha\beta}(P)Y_\beta=\int d^4x\int\frac{d\Omega_{\hat{k}}}{4\pi}(\partial_\mu X_\alpha-\partial_\alpha X_\mu)\frac{\hat{K}^\alpha\hat{K}^\beta}{-(\partial\cdot\hat{K})^2}\times(\partial_\mu Y_\beta-\partial_\beta Y_\mu). \quad \text{(B8)}$$

The Abelian field strengths of the vector fields $X^\alpha$ and $Y^\beta$ enter because $\delta\Pi^{\alpha\beta}$ is transverse, Eq. (B1).

## APPENDIX C: $\pi\rightarrow\gamma\gamma$ IN THE REAL TIME FORMALISM

We found in Sec. III that the contribution from Fig. 1(d) vanishes to order $T^2/f_\pi^2$. In this appendix we check this result using the real time approach. In this instance, the real time method is simple and not problematic. It also shows that the diagram has no temperature-dependent terms whatsoever: the entire diagram is equal, identically, to its value at zero temperature.

After dropping irrelevant terms proportional to $P_1^\alpha$, Fig. 1(d) is proportional to the integral

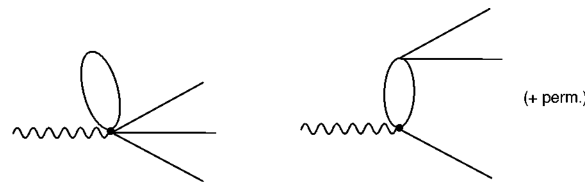


FIG. 4. One loop corrections to the $\pi\pi\pi\gamma$ vertex.

$$\epsilon_{\beta\gamma\delta\kappa}P_2^\kappa\int\frac{d^4K}{(2\pi)^3}(2K-P_1)^\alpha K^\gamma(K-P_1)^\delta\times\left(\frac{n(\omega_k)}{(K-P_1)^2}\delta(K^2)+\frac{n(\omega_{k-p_1})}{K^2}\delta[(K-P_1)^2]\right)$$
$$=-2\epsilon_{\beta\gamma\delta\kappa}P_1^\delta P_2^\kappa\int\frac{d^4K}{(2\pi)^3}\frac{K^\alpha K^\gamma}{K\cdot P_1}\delta(K^2)n(\omega_k), \quad \text{(C1)}$$

$\omega_k=k$, $\omega_{k-p_1}=|\vec{k}-\vec{p}_1|$. The physical amplitude is obtained by contraction with the polarization vectors for the two photons. If $\vec{p}_1$ lies in the $z$ direction, say, then $\alpha$ must lie along the $x$ or $y$ directions. The angular $\phi$ integration will then vanish unless $\gamma=\alpha$. Thus the integral reduces to

$$\int\frac{d^4K}{(2\pi)^3}\frac{\omega_k^2\sin^2\theta}{K\cdot P_1}\frac{n(\omega_k)}{\omega_k}[\delta(k_0-\omega_K)+\delta(k_0+\omega_K)]$$
$$=\int\frac{d^3k}{(2\pi)^3}\frac{2\cos\theta}{p_1^0}n(\omega_k). \quad \text{(C2)}$$

This integral vanishes after integration over $\theta$; note that this physical amplitude is free of any collinear divergences. This confirms our results obtained with the imaginary time formalism.

## APPENDIX D: $\gamma\rightarrow\pi\pi\pi$ AT LOW TEMPERATURE

In this section we compute the one loop corrections to the amplitude $\gamma\rightarrow\pi\pi\pi$ for soft, cool pions. This provides another, less trivial, example of hard thermal loops. Corrections to the five dimensional Wess-Zumino-Witten term, Eq. (A3), have been computed in [15] using a background field method; it would be interesting using this method to compute the one loop corrections to the gauged WZW model, Eq. (A5).

To one-loop order, the corrections to $\gamma\rightarrow\pi\pi\pi$ are those of Figs. 4(a) and 4(b).

Figure 4(a) is a tadpole diagram, which is most easily computing by expanding Eq. (A13) in the pion field,

$$\mathcal{L}_{\gamma\pi\pi\pi}\simeq-\frac{eN_c}{72\pi^2f_b^3}\left(1-\frac{\vec{\pi}^2}{3f_\pi^2}\right)A_\alpha\varepsilon^{abc}\varepsilon_{\alpha\beta\gamma\delta}\partial_\beta\pi^a\partial_\gamma\pi^b\partial_\delta\pi^c, \quad \text{(D1)}$$

and then compute as in Eq. (2.2).

At tree level, the amplitude for $\pi\pi\pi\gamma$ scattering is

$$\mathcal{M}=\kappa_b\varepsilon^{abc}\ \varepsilon_{\alpha\beta\gamma\delta}\epsilon^{\alpha}P^a_{\beta}P^b_{\gamma}P^c_{\delta}, \tag{D2}$$

where $P^a$ is the momentum of the pion with isospin $a$, etc., and

$$f_b^3\kappa_b=\frac{ieN_c}{72\pi^2}. \tag{D3}$$

From Eq. (A10), the amplitude for the scattering between four pions is

$$\mathcal{A}=-\frac{1}{f_b^2}\bigg[\delta^{ab}\delta^{cd}P^2_{ab}+\delta^{ac}\delta^{bd}P^2_{ac}+\delta^{ad}\delta^{bc}P^2_{ad} -\frac{\Sigma P_i^2}{3}(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc})\bigg], \tag{D4}$$

where $P_{ab}=P_a+P_b$, etc.

The contribution of Fig. 4(b) is

$$\mathcal{M}^b=\frac{3i}{f_b^2}\frac{eN_c}{72\pi^2f_b^3}\varepsilon_{\alpha\beta\gamma\delta}\epsilon_{\alpha}P^c_{\delta}(P^a_{\gamma}+P^b_{\gamma})\Gamma^{\beta\kappa}(P_{ab})(P^a_{\kappa}-P^b_{\kappa}) + \text{permutations}, \tag{D5}$$

where $\Gamma^{\alpha\beta}(P_{ab})$ is the integral of Eq. (2.13).

In the vacuum, $\Gamma^{\alpha\beta}(P)=\delta^{\alpha\beta}\mathcal{I}_0/2$, up to terms $\sim P^{\alpha}P^{\beta}$ which drop out of the amplitude. To $O(P^4)$,

$$f_b^3\kappa_b=\left(1+(1-1+3)\frac{\mathcal{I}_0}{f^2}\right)\frac{ieN_c}{72\pi^2}. \tag{D6}$$

The 1 comes from $Z_{\pi}^{3/2}$, Fig. 1(a) and Eq. (2.4), the $-1$ from Fig. 4(a) and Eq. (D1), the $+3$ from Fig. 4(b) and Eq. (D5). Using Eq. (2.7), then, to one loop order in vacuum (D3) renormalizes with no change in form,

$$f_{\pi}^3\kappa=\frac{ieN_c}{72\pi^2}, \tag{D7}$$

analogous to Eqs. (2.9) and (2.15).

At low temperature, $\Gamma^{\alpha\beta}$ is replaced by $\delta\Gamma^{\alpha\beta}$ of Eqs. (3.4) and (3.5). Using Eq. (B8), we find that the effective Lagrangian for $\gamma\to\pi\pi\pi$ is similar to that for $\pi^0\to\gamma\gamma$, Eq. (3.10). One term is as at zero temperature, with $f_{\pi}$ replaced by $f_{\pi}(T)$, while the second is a hard thermal loop:

$$\mathcal{L}_{\pi\pi\pi\gamma}(T)=-\frac{eN_c}{72\pi^2f_{\pi}(T)^3}\quad A_{\alpha}\varepsilon^{abc}\varepsilon_{\alpha\beta\gamma\delta}\partial_{\beta}\pi^a\partial_{\gamma}\pi^b\partial_{\delta}\pi^c -\frac{T^2}{12f_{\pi}^2}\frac{eN_c}{48\pi^2f_{\pi}^3}\varepsilon^{abc}\int\frac{d\Omega_{\hat{k}}}{4\pi}H^a_{\gamma\alpha}\frac{\hat{K}^{\alpha}\hat{K}^{\beta}}{-(\partial\cdot\hat{K})^2}J^{bc}_{\gamma\beta}, \tag{D8}$$

where

$$H^a_{\alpha,\beta}=\partial_{\alpha}(\varepsilon_{\beta\gamma\delta\kappa}F_{\gamma\delta}\partial_{\kappa}\pi^a)-(\alpha\leftrightarrow\beta), \tag{D9}$$

$$J^{bc}_{\alpha\beta}=\partial_{\alpha}\pi^b\partial_{\beta}\pi^c-\partial_{\beta}\pi^b\partial_{\alpha}\pi^c.$$


[1] S. L. Adler, Phys. Rev. **177**, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento A **60**, 147 (1969); S. L. Adler and W. A. Bardeen, Phys. Rev. **182**, 1517 (1969); B. Zumino, Y. S. Wu, and A. Zee, Nucl. Phys. **B239**, 477 (1984); M. A. Shifman, Phys. Rep. **209**, 341 (1991).

[2] L. Dolan and R. Jackiw, Phys. Rev. D **9**, 3320 (1974); M. Reuter and W. Dittrich, *ibid.* **32**, 513 (1985); F. Ruiz Ruiz and R. Alvarez-Estrada, Phys. Lett. B **180**, 153 (1986); A. Das and A. Karev, Phys. Rev. D **36**, 623 (1987); R. Baier and E. Pilon, Z. Phys. C **52**, 339 (1991).

[3] H. Itoyama and A. H. Mueller, Nucl. Phys. **B218**, 349 (1983).

[4] Y.-L. Liu and G.-J. Ni, Phys. Rev. D **38**, 3840 (1988); X.-G. He and G. C. Joshi, *ibid.* **41**, 3796 (1990); R.-T. Wang, J. Phys. A **23**, 5555 (1990); A. A. Rawlinson, D. Jackson, and R. J. Crewther, Z. Phys. C **56**, 679 (1992); A. V. Smilga, Phys. Rev. D **45**, 1378 (1992); A. Gomez Nicola and R. F. Alvarez-Estrada, Int. J. Mod. Phys. A **9**, 1423 (1994).

[5] R. Jackiw, *Current Algebra and Anomalies* (Princeton University Press, Princeton, NJ, 1985); C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980); G. M. Shore and G. Veneziano, Nucl. Phys. **B381**, 3 (1992).

[6] A. Dobado, R. F. Alvarez-Estrada, and A. Gomez Nicola, Phys. Lett. B **319**, 238 (1993).

[7] A. Dobado, R. F. Alvarez-Estrada, and A. Gomez Nicola, Phys. Lett. B **324**, 345 (1994).

[8] S. Weinberg, Physica A **96**, 327 (1979); J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) **158**, 142 (1984); J. F. Donoghue, E. Golowich, and B. R. Holstein, *Dynamics of the Standard Model* (Cambridge University Press, New York, 1992); G. Ecker, Prog. Part. Nucl. Phys. **35**, 1 (1995).

[9] J. Wess and B. Zumino, Phys. Lett. **37B**, 95 (1971).

[10] E. Witten, Nucl. Phys. **B223**, 422 (1983).

[11] O. Kaymakcalan and J. Schechter, Phys. Rev. D **31**, 1109 (1985); Y. Brihaye, N. Pak, and P. Rossi, Nucl. Phys. **B254**, 71 (1985); U. G. Meissner, Phys. Rep. **161**, 213 (1988).

[12] C. M. Hull and B. Spence, Mod. Phys. Lett. A **6**, 969 (1991).

[13] M. Creutz and M. Tytgat, Phys. Rev. Lett. **76**, 4671 (1996); M. Creutz and M. Tytgat, in *Lattice '96*, Proceedings of the International Symposium, St. Louis, Missouri, edited by C. Bernard *et al.* [Nucl. Phys. (Proc. Suppl.) **53**, 651 (1997)], hep-lat/9608007.

[14] B. R. Holstein, J. F. Donoghue, and Y. C. R. Lin, Phys. Rev. Lett. **55**, 2766 (1985); J. F. Donoghue and D. Wyler, Nucl. Phys. **B316**, 289 (1989); J. Bijnens, A. Bramon, and F. Cornet, Z. Phys. C **46**, 599 (1990); R. Akhoury and A. Alfakih, Ann. Phys. (N.Y.) **210**, 81 (1991); J. Bijnens, Nucl. Phys. **B367**, 709 (1991); Int. J. Mod. Phys. A **8**, 3045 (1993), and references therein.

[15] R. D. Pisarski and M. Tytgat, Phys. Rev. Lett. **78**, 3622 (1997).

[16] C. Contreras and M. Loewe, Z. Phys. C **40**, 253 (1988); T. Hashimoto, K. Hirose, T. Kanki, and O. Miyamura, Phys. Rev. D **37**, 3331 (1988); B. Pin-Zhen and J. Rafelski, Mod. Phys.

Lett. A **7**, 2493 (1992); **7**, 3439(E) (1992); A. Gomez Nicola and R. F. Alvarez-Estrada, Z. Phys. C **60**, 711 (1993); S. P. Klevansky, Nucl. Phys. **A575**, 605 (1994); D. Blaschke, M. Jaminon, Yu. L. Kalinovsky, P. Petrow, S. Schmidt, and B. Van den Bossche, *ibid.* **A592**, 561 (1995).

[17] P. Binetruy and M. K. Gaillard, Phys. Rev. D **32**, 931 (1985); J. Gasser and H. Leutwyler, Phys. Lett. B **184**, 83 (1987); M. Dey, V. L. Eletsky, and B. L. Ioffe, *ibid.* **252**, 620 (1990); A. Bochkarev and J. Kapusta, Phys. Rev. D **54**, 4066 (1996).

[18] R. D. Pisarski and M. Tytgat, Phys. Rev. D **54**, 2989 (1996).

[19] R. D. Pisarski, Phys. Rev. Lett. **63**, 1129 (1989); E. Braaten and R. D. Pisarski, Nucl. Phys. **B337**, 569 (1990); **B339**, 310 (1990); Phys. Rev. D **45**, R1827 (1992); J. Frenkel and J. C. Taylor, Nucl. Phys. **B334**, 199 (1990); **B374**, 156 (1992); J. C. Taylor and S. M. H. Wong, *ibid.* **B346**, 115 (1990); J. P. Blaizot and E. Iancu, *ibid.* **B390**, 589 (1993); **B417**, 608 (1994); **B434**, 662 (1995); R. Efraty and V. P. Nair, Phys. Rev. D **47**, 5601 (1993); R. Jackiw and V. P. Nair, *ibid.* **48**, 4991 (1993); R. Jackiw, Q. Liu, and C. Lucchesi, *ibid.* **49**, 6787 (1994); P. F. Kelly, Q. Liu, C. Lecchesi, and C. Manuel, *ibid.* **50**, 4209 (1994).

[20] D. G. Sutherland, Nucl. Phys. **B2**, 433 (1967); M. Veltman, Proc. R. Soc. London **A301**, 107 (1967).

[21] R. D. Pisarski, in *From Thermal Field Theory to Neural Networks: A Day to Remember Tanguy Altherr*, edited by P. Aurenche, P. Sorba, and G. Veneziano (World Scientific, Singapore, 1996); Phys. Rev. Lett. **76**, 3084 (1996); R. Baier, M. Dirks, and O. Kober, Phys. Rev. D **54**, 2222 (1996).

[22] L. Ametller, J. Bijnens, A. Bramon, and F. Cornet, Phys. Lett. B **303**, 140 (1993); J. Bijnens, A. Bramon, and F. Cornet, Z. Phys. C **46**, 599 (1990).

[23] $\Gamma^{\alpha\beta}(P)$, Eq. (2.13), equals $I^{\alpha\beta}(K)$, Eq. (5) of Ref. [7]. The terms $\sim T^2$ in $\Gamma^{\alpha\beta}(P)$ define $\delta\Gamma^{\alpha\beta}(P)$, Eq. (3.4). From Fig. 1(b) of Ref. [7], which is our Fig. 1(d), in Eq. (12) they obtain $I^{ij}(K) \sim +\delta^{ij}T^2/12+\cdots$. This does not agree with Eq. (3.7), for which the coefficient of the term $\sim\delta^{ij}$ vanishes.

[24] S. J. Brodsky and G. P. Lepage, Phys. Lett. **87B**, 359 (1979); Phys. Rev. D **22**, 2157 (1980); **24**, 1808 (1981); L. Arnellos, W. J. Marciano, and Z. Parsa, Nucl. Phys. **B196**, 371 (1982); A. Anselm, A. Johansen, E. Leader, and L. Lukaszuk, hep-ph/9603444.

[25] M. Dine and S. Thomas, Phys. Lett. B **328**, 73 (1994).

[26] G. 't Hooft, in *Recent Developments in Gauge Theories*, edited by G. 't Hooft *et al.* (Plenum Press, New York, 1980).

[27] This method also allows one to gauge an arbitary subgroup $H$ of $SU(n_f)\times SU(n_f)$, whether or not $H$ is anomaly-free. If the $H$ anomalies do not cancel, however, there is a doubling of the effective number of degrees of freedom, similar to the doubling which occurs for fermions on a lattice [13].