Asymptotic limits and the structure of the pion form factor

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We use dispersive techniques to address the behavior of the pion form factor as $Q^2 \to \infty$ and $Q^2 \to 0$. We perform the matching with the constraints of perturbative QCD and chiral perturbation theory in the high-energy and low-energy limits, leading to four sum rules. We present a version of the dispersive input which is consistent with the data and with all theoretical constraints. The results indicate that the asymptotic perturbative QCD limit is approached relatively slowly, and give a model-independent determination of low-energy chiral parameters. [S0556-2821(97)02123-1]

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We are fortunate to have the rigorous techniques of perturbative QCD [1] and chiral perturbation theory [2] describing the high- and low-energy domains of the strong interactions, respectively. The only comparatively rigorous techniques that apply to the intermediate energy region are lattice simulations [3] and dispersion relations [4]. Dispersive techniques are increasingly being combined with the other theoretical methods in order to provide as much control as possible throughout all energy regions. The simplest cases are the two point functions of vector and axial-vector currents [5], which are associated with the Weinberg sum rules [6] and other related sum rules. The next simpliest example is the three point function of the pion electromagnetic form factor. The purpose of this paper is to discuss the dispersive treatment of the pion form factor. We apply chiral constraints at low energy and incorporate the behavior of perturbative QCD at the highest energies. This leads to four sum rules, two of which are reasonably obvious and two which are new. In addition, the dispersive treatment allows us to address the question of how fast the form factor approaches the asymptotic QCD behavior [7].

Dispersion relations are sometimes used in setting up a framework for experimental analyses which incorporates the constraints of analyticity. They can be used to check the consistency of real and imaginary parts of a data analysis. Our use here differs from these procedures because we are addressing a different question. Our prime interest is in exploring the matching of the theoretical analyses of different energy regions. For example, within chiral perturbation theory the most interesting dynamical information appears in the real parts of amplitudes and form factors. The chiral coefficients which also occur in the real parts are determined in principle by the matching of the low-energy theory to the full theory of QCD. Dispersion relations provide a semi-phenomenological method to accomplish this matching, relating nontrivial dynamical information in QCD to the parameters of the low-energy expansion. Likewise dispersion relations contain information on the transition to the high-energy domain of perturbative QCD. Our methods are designed to explore some of these connections.

The form factor is defined by

$$\langle \pi^+ | J_\mu^{em} | \pi^+ \rangle = f_\pi(q^2)(p+p')_\mu \tag{1}$$

with $q_\mu = (p'-p)_\mu$ In its twice subtracted form, the dispersion relation for the pion form factor reads

$$f_\pi(q^2) = 1 + Kq^2 + \frac{q^4}{\pi}\int_{4m_\pi^2}^\infty \frac{ds}{s^2}\,\frac{\mathrm{Im} f_\pi(s)}{s-q^2-i\epsilon}. \tag{2}$$

Our results are independent of the number of subtractions, but this form is most useful in presenting our techniques. We have imposed the normalization constraint $f_\pi(0)=1$, and the constant K is a subtraction constant to be determined below.

At the high-energy end, perturbative QCD tells us that the asymptotic behavior of the pion form factor [7], with $Q^2 = -q^2$, is

$$f_\pi(Q^2) = 16\pi \frac{\alpha_s(Q^2) F_\pi^2}{Q^2} = \frac{64\pi^2}{9}\,\frac{F_\pi^2}{Q^2 \ln(Q^2/\Lambda^2)} \tag{3}$$

with $F_\pi = 93$ MeV.

The fact that this decreases faster than $1/Q^2$ implies three sum rules when combined with Eq. (2). The fact that there is no term proportional to Q^2 as $Q^2 \to \infty$ implies a sum rule for the subtraction constant

$$K = \frac{1}{\pi}\int_{4m_\pi^2}^\infty \frac{ds}{s^2}\mathrm{Im} f_\pi(s). \tag{4}$$

Correspondingly, there is no constant term as $Q^2 \to \infty$, which requires a sum rule that can be found by Taylor expanding the denominator at large Q^2, yielding

$$1 = \frac{1}{\pi}\int_{4m_\pi^2}^\infty \frac{ds}{s}\mathrm{Im} f_\pi(s). \tag{5}$$

Finally, the lack of a $1/Q^2$ term in the asymptotic region implies that

$$0 = \frac{1}{\pi}\int_{4m_\pi^2}^\infty ds \quad \mathrm{Im} f_\pi(s). \tag{6}$$

These sum rules are contingent on the convergence of the integrals. This is especially relevant for the last one, but we will see that the integral is just barely convergent.

At the low-energy end, the pion form factor has been calculated to two loops in chiral perturbation theory. The result is

$$f_\pi(q^2)=1+\frac{2\overline{L_9}}{F_\pi^2}q^2+c_V q^4+O(q^6), \tag{7}$$

with

$$\overline{L_9}=L_9^{(r)}(\mu)-\frac{1}{192\pi^2}\left(\ln\frac{m_\pi^2}{\mu^2}+1\right)+\frac{\overline{f_1}m_\pi^2}{16\pi^2F_\pi^2}, \tag{8}$$

$$c_V=\frac{1}{16\pi^2F_\pi^2}\left\{\frac{1}{60m_\pi^2}+\frac{\overline{f_2}}{16\pi^2F_\pi^2}\right\}.$$

In this expression, the parameters $L_9^{(r)}(\mu)$ and c_V, $\overline{f_1}$, $\overline{f_2}$ are renormalized parameters from the E^4 and E^6 chiral Lagrangians, respectively. $L_9^{(r)}(\mu)$ can in principle be measured in other reactions, although it is most common to extract it from this form factor. One obtains a dispersion sum rule for $L_9^{(r)}(\mu)$ by expanding the chiral result around $q^2=0$ to find that $L_9^{(r)}(\mu)$ is related to the subtraction constant K defined above. The precise relation is

$$K=\frac{2\overline{L_9}}{F_\pi^2}=\frac{2L_9^{(r)}(\mu)}{F_\pi^2}-\frac{1}{96\pi^2F_\pi^2}\left(\ln\frac{m_\pi^2}{\mu^2}+1\right). \tag{9}$$

Here and in what follows we drop reference to the chiral constant $\overline{f_1}$ since its effect is so small due to the factor of m_π^2 multiplying it in Eq. (8). Note that relation for $\overline{L_9}$ is independent of the arbitrary scale μ, as the dependence of $L_9^{(r)}(\mu)$ on μ is compensated by the explicit μ behavior displayed above. This exercise can be repeated to find the term at order Q^4, both in the dispersion relation and in the chiral expansion. The result is

$$c_V=\frac{1}{16\pi^2F_\pi^2}\left\{\frac{1}{60m_\pi^2}+\frac{\overline{f_2}}{16\pi^2F_\pi^2}\right\}=\frac{1}{\pi}\int_{4m_\pi^2}^\infty\frac{ds}{s^3}\mathrm{Im}f_\pi(s). \tag{10}$$

The derivation above used a twice subtracted relation; however, this was not a required feature. The same results can be obtained from dispersion relations with differing numbers of subtractions. It is an interesting feature, typical of dispersion relations, that the same sum rule can follow from the constraints on the high-energy end in some derivations yet emerge in the low-energy limit in another. This occurs because dispersion relations tie together the high-energy and low-energy behaviors. As an example, let us briefly describe how these sum rules would be derived using an unsubtracted dispersion relation:

$$f_\pi(q^2)=\frac{1}{\pi}\int_{4m_\pi^2}^\infty ds\,\frac{\mathrm{Im}\ f_\pi(s)}{s-q^2-i\epsilon}. \tag{11}$$

In this case, the sum rules of Eqs. (5), (9), and (10) all follow from Taylor expanding around $q^2=0$, while Eq. (6) follows from the $q^2\to\infty$ limit, whereas we had previously used the high-energy limit to identify the sum rule of Eq. (5). Dispersion relations connect the high- and low-energy limits by providing constraints on the whole analytic function.

We now turn to the construction of a representation of $\mathrm{Im}f_\pi(s)$ which is consistent with both the data and with theoretical constraints. The easiest step is at low energy, where chiral symmetry requires the structure [8]

$$\mathrm{Im}f_\pi(s)=\frac{s(1-4m_\pi^2/s)^{3/2}}{96\pi^2F_\pi^2}\,\theta(s-4m_\pi^2)+O(s^2). \tag{12}$$

In the intermediate-energy region, we have data on both the real and imaginary parts of $f_\pi(s)$. There is nothing surprising here, the physics is just that of the ρ resonance. We take $\mathrm{Im}f_\pi(s)$ from a fit to the data in Ref. [9]. Matching with the low-energy limit is simple, as the resulting function is easily adjusted to approach Eq. (12) as $s\to0$.

For the high-energy end, we need to choose an asymptotic form for $\mathrm{Im}f_\pi(s)$ which yields Eq. (3) when inserted in a dispersion relation. To see that this is the appropriate procedure, we consider dividing the dispersive integral into two pieces, with the transition part $\bar{s}$ being large enough that above $s=\bar{s}$ we are in asymptotic high-energy behavior for $\mathrm{Im}f_\pi(s)$:

$$f_\pi(q^2)=\frac{1}{\pi}\int_{4m_\pi^2}^{\bar{s}}ds\frac{\mathrm{Im}f_\pi(s)}{s+Q^2}+\frac{1}{\pi}\int_{\bar{s}}^\infty ds\,\frac{\mathrm{Im}f_\pi(s)}{s+Q^2}. \tag{13}$$

In the first integral the integrand is finite and the range is finite so that the result is analytic in $1/Q^2$ around $Q^2\to\infty$. As a consequence, the logarithm in the QCD form for the asymptotic limit cannot be reproduced from the first integral, and must come from the $s\to\infty$ behavior of $\mathrm{Im}f_\pi(s)$ in the second integral. The form of the imaginary part which guarantees the proper asymptotic limit is

$$\mathrm{Im}f_\pi(s)=-\frac{64\pi^3}{9}\,\frac{F_\pi^2}{s\ \ln^2(s/\Lambda^2)}. \tag{14}$$

This result is of course expected, as it it the imaginary part which follows from an analytic continuation of Eq. (3). This form also allows the high-energy sum rule, Eq. (6), to converge.

A final step is the matching between the intermediate- and high-energy forms of $\mathrm{Im}f_\pi(s)$. The only model dependence in our procedure comes at this point, as there can be matching procedures which differ in some details. However, the combination of data and sum rules turn out to be highly constraining. We require that the matching be such that we satisfy the high-energy sum rule Eq. (6) and the normalization integral Eq. (5). For these to be satisfied, the negative values of $\mathrm{Im}f_\pi(s)$ obtained from the asymptotic form at large s must extend to fairly low energies in order to be able to cancel the known positive contribution effects of the ρ. This is a powerful constraint. There is certainly some ambiguity in the precise form in the matching region, but we have found a relatively simple solution. This is depicted in Fig. 1, showing a smooth matching slightly above 1 GeV.

If we use this form for $\mathrm{Im}f_\pi(s)$ in a dispersion relation, we clearly have no predictive power in the intermediate-

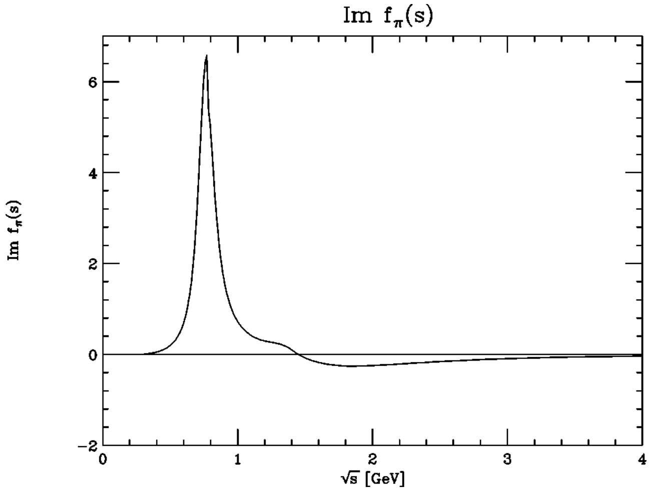

FIG. 1. Fit to the imaginary part of the pion form factor satisfying the consistency constraints described in the text.

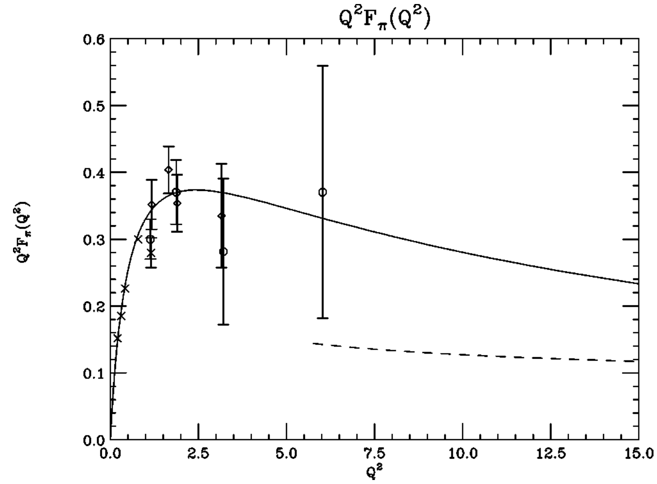

FIG. 2. The real part of the pion form factor at large Q^2. The dashed line indicates the asymptotic prediction of perturbative QCD with $\Lambda = 0.3$ GeV.

energy region where our method is data driven. The predictions come from the approaches to the asymptotic regions. Within a dispersive framework the transitions to the low-energy and high-energy limits are both determined largely by the numerically important intermediate energy region. Our results are presented graphically in Fig. 2. On the low-energy side the structure of the real part of the form factor is governed by the low-energy constraints of Eqs. (9),(10). These are predicted by the dispersion relations to have the form

$$L_9^{(r)}(\mu) = \frac{1}{192\pi^2}\left(\ln\frac{m_\pi^2}{\mu^2}+1\right) + \frac{F_\pi^2}{2\pi}\int_{4m_\pi^2}^{\infty}\frac{ds}{s^2}\mathrm{Im}f_\pi(s) = 0.0074, \tag{15}$$

$$c_V = 4.1\ \mathrm{GeV}^{-4},$$

$$\overline{f_2} = 6.6$$

using $\mu = m_\eta$. The result for $L_9^{(r)}$ agrees with the standard result, derived from the real part of the form factor. This is just a consistency condition for the dispersion relation. Of greater conceptional interest is the way that the dispersion method embodies the underlying physics of vector meson dominance (VMD), and the way that it resolves the issue of the scale dependence of the chiral coefficients in VMD [10]. Vector dominance is motivated by a narrow width approximation to the dispersion integral:

$$\mathrm{Im}f_\pi(s) = \pi m_\rho^2\delta(s - m_\rho^2). \tag{16}$$

Ecker, Pich, and de Rafael argued that VMD determines the chiral coefficients at the scale $\mu^2 = m_\rho^2$. The dispersive approach provides a different answer — VMD determines not simply the chiral coefficient $L_9^{(r)}(\mu)$ but rather the scale independent combination of the coefficient plus a specific combination of chiral logs, i.e., $\overline{L_9}$ in Eq. (9).

On the high-energy side, we see from Fig. 2 that the asymptotic QCD limit is approached rather slowly. In a dispersive framework this is due to the large contributions of the soft physics region, most notably the ρ resonance. The residual effects of the soft physics continues to be larger than the somewhat small perturbative contribution out to reasonably high energies. This result is consistent with quark model calculations [11], but is far less model dependent.

The techniques of dispersion relations provide a partial bridge between the low-energy techniques of chiral perturbation theory and the high-energy techniques of QCD. The simplest exploration of these methods involve two point functions. The present work involves a three point function and hence is a step towards the consideration of yet more difficult matrix elements such as the nonleptonic amplitudes responsible for electromagnetic mass difference [12] or weak decays.

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