

Precision corrections to dispersive bounds on form factors

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We present precision corrections to dispersion relation bounds on form factors in bottom hadron semileptonic decays and analyze their effects on parametrizations derived from these bounds. We incorporate QCD two-loop and nonperturbative corrections to the two-point correlator, consider form factors whose contribution to decay rates is suppressed by lepton mass, and implement more realistic estimates of truncation errors associated with the parametrizations. We include higher resonances in the hadronic sum that, together with heavy quark symmetry relations near zero recoil, further tighten the sum rule bounds. Utilizing all these improvements, we show that each of the six form factors in $\bar{B} \rightarrow D \ell \bar{\nu}$ and $\bar{B} \rightarrow D^* \ell \bar{\nu}$ can be described with 3% or smaller precision using only the overall normalization and one unknown parameter. A similar one-coefficient parametrization of one of the $\Lambda_b \rightarrow \Lambda_c \ell \bar{\nu}$ form factors, together with heavy quark symmetry relations valid to order $1/m^2$, describes the differential baryon decay rate in terms of one unknown parameter and the phenomenologically interesting quantity $\bar{\Lambda}_\Lambda \approx M_{\Lambda_b} - m_b$. We discuss the validity of slope-curvature relations derived by Caprini and Neubert, and present weaker, corrected relations. Finally, we present sample fits of current experimental $\bar{B} \rightarrow D^* \ell \bar{\nu}$ and $\bar{B} \rightarrow D \ell \bar{\nu}$ data to the improved one-parameter expansion. [S0556-2821(97)01823-7]

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I. INTRODUCTION

The decays of bottom hadrons provide fertile ground for explorations of both weak and strong interactions. The empirical smallness of the Cabibbo-Kobayashi-Maskawa (CKM) elements $|V_{cb}|$ and $|V_{ub}|$ implies that b -hadron decays are relatively long lived, and thus more easily analyzed, while the heaviness of the b quark means that the heavy quark effective theory (HQET) [1,2] provides a reliable expansion for describing the approximate decoupling of the b quark from the rest of the hadron, leading to a substantial simplification of the strong-interaction dynamics.

Semileptonic decays of b hadrons are particularly tractable from the theoretical point of view, since the leptonic current may be trivially separated from the hadronic current. Furthermore, focusing upon exclusive decays permits one to avoid questions regarding the validity of quark-hadron duality near kinematic end point regions. On the other hand, exclusive processes must be described in terms of a number of nonperturbative form factors that encapsulate the physics of the hadronization process.

HQET has provided a substantial leap forward in the exclusive analysis, demonstrating that heavy-to-heavy quark transitions at zero recoil are accompanied by a complete overlap of initial- and final-state hadron wave functions [1,3]. Consequently, each form factor possesses a well-defined normalization at this kinematic point. For the case of

$\bar{B}^{(*)} \rightarrow D^{(*)}$ transitions, for example, each of the 20 form factors either vanishes or is proportional to a single universal form factor ξ , which represents this wave function overlap as a function of momentum transfer. Sum rule constraints restrict the slope and magnitude of these form factors [4–6], but otherwise shapes of the form factors are *a priori* undetermined functions of momentum transfer.

However, some recent work [7–18] has demonstrated that one may obtain rather nontrivial and model-independent constraints on the shape of such form factors in the context of dispersion relations. Using basic field theoretic properties such as causality and crossing symmetry, one finds that the shape of a given form factor is determined to high accuracy over its entire kinematic range by its value at only a small number of points [7–10]. Indeed, for the case of $\bar{B} \rightarrow D^{(*)} \ell \bar{\nu}$, the HQET normalization plus only one additional point determines most of the form factors to 3–7% accuracy [7]. Similar analyses provide interesting and useful constraints on $\bar{B} \rightarrow \pi \ell \bar{\nu}$ [7,10–13], $\bar{B} \rightarrow \rho \ell \bar{\nu}$ [14], and $\Lambda_b \rightarrow \Lambda_c \ell \bar{\nu}$ [7].

At the core of this analysis lies a dispersion relation in momentum space that relates the integral of an inclusive production rate to a two-point function evaluated far from physical resonances, where perturbative QCD is quantitatively reliable. The production rate is a sum of positive-definite exclusive rates; it follows that all contributions to the hadronic side serve to saturate more and more closely the bound from the perturbative side. Clearly, two improvements to this procedure include a better calculation of the QCD side (which includes both multiloop effects and nonperturbative vacuum condensates), and the inclusion of more states on the

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hadronic side. The former may be extracted from existing literature, while the latter may be achieved using heavy quark symmetry relations. The inclusion of higher states has previously been investigated for elastic $B \rightarrow B$ scattering [17] and for $\bar{B} \rightarrow \pi \ell \bar{\nu}$ decays [11], but not for $b \rightarrow c$ transitions. For mesonic and baryonic $b \rightarrow c$ transitions, the inclusion of higher states improves considerably the precision of the dispersion-relation constraints, and represents the primary contribution of this work. In addition, we include in this analysis nonzero lepton masses, anticipating future measurements of processes such as $\bar{B} \rightarrow D^{(*)} \tau \bar{\nu}$ or $\bar{B} \rightarrow \pi \tau \bar{\nu}$.

This paper is organized as follows. In Sec. II we define the form factors for the physically-observed semileptonic decays and present expressions for differential decay widths including lepton masses. Section III presents a brief review of the dispersive method for obtaining bounds on the form factors. Ingredients of the form factor parametrizations, including explicit formulas for the QCD results at one and two loops as well as leading nonperturbative effects, appear in Sec. IV, followed by a tabulation of the form factor weighting functions, which are central to the dispersive method. In Sec. V we discuss the inclusion of additional, previously ignored hadronic states into the dispersion relations, and show how heavy quark effective theory may be used to include their effects and thereby tighten form factor constraints. Section VI defines and presents the truncation errors that measure the quality of our form factor parametrizations. In Sec. VII we discuss slope-curvature relations derived by Caprini and Neubert [16], point out an invalidating assumption, and examine the form of the corrected relations. In Sec. VIII we present the results of the current analysis in fits to the latest experimental data, and in Sec. IX we conclude.

II. FORM FACTORS

We begin by defining form factors for the semileptonic decays of interest in terms of hadronic transition matrix elements. From the field theory point of view, it is most convenient to define form factors as coefficients of independent Lorentz structures appearing in the matrix element. However, the combinations of form factors most easily obtained from data are those appearing in a sum of squares in the differential rates, namely, the helicity amplitudes. As seen below, the helicity amplitudes are particular linear combinations of the original form factors, and thus simply form a different basis for the description of the matrix elements. It is these helicity amplitudes we wish to constrain.

The notation used throughout the paper is as follows. The generic $b \rightarrow c$ semileptonic decay is denoted by $H_b \rightarrow H_c \ell \bar{\nu}$, where the hadron H_b has mass M and momentum p , the daughter hadron H_c has mass m and momentum p' , and the charged lepton ℓ has mass m_ℓ . The momentum transfer $t = (p - p')^2$ is the invariant mass squared of the lepton pair (or virtual W). The polar vector and axial vector flavor-changing currents are denoted by $V^\mu = \bar{c} \gamma^\mu b$ and $A^\mu = \bar{c} \gamma^\mu \gamma_5 b$. Finally, it is convenient to define the kinematic invariants

$$t_\pm = (M \pm m)^2,$$

$$k = M \sqrt{\frac{\mathbf{p}^2}{t}} \equiv \sqrt{\frac{(t_+ - t)(t_- - t)}{4t}}, \quad (2.1)$$

where \mathbf{p} is the three-momentum of H_c in the rest frame of H_b . Note that the identity of the decaying quark does not enter into the expressions below except through hadron masses, so the same expressions apply to such decays as $\bar{B} \rightarrow \pi \ell \bar{\nu}$, $D \rightarrow \bar{K}^* \ell \bar{\nu}$, and so on; only the spins of the hadrons are relevant.

The inclusion of charged lepton masses brings into the differential widths a new set of helicity amplitudes, and thus a new laboratory for studying the strong interaction. The contributions of such terms is suppressed by a factor of m_ℓ^2 , and arises through a virtual W with the quantum numbers of a scalar, i.e., a timelike polarization. Angular momentum conservation forbids the decay of such a state to a right-handed antineutrino and a left-handed *massless* charged lepton, so such decays are necessarily accompanied by a helicity-suppression factor m_ℓ^2 . Although the detection of lepton mass-dependent effects is presently beyond the means of current experiments, the gradual accumulation of statistics may eventually make such effects discernible in decays such as $\bar{B} \rightarrow D^{(*)} \tau \bar{\nu}$ or $\bar{B} \rightarrow \pi \tau \bar{\nu}$.

Finally, it should be noted that the differential widths $d\Gamma/dt$ presented below have already been integrated in lepton energy, or equivalently over angles of final-state particles. This reflects only the current thrust of experiment and does not indicate a limitation of the dispersive method described below. Indeed, were the statistics available, it would be interesting to consider the double differential decay distribution $d^2\Gamma/dE_\ell dt$, for then one could probe the parity-violating interference terms between vector and axial vector weak currents. In terms of the QCD side of the dispersion relations described below, one would also need to compute $V-A$ correlators, in addition to $V-V$ and $A-A$.

A. $\bar{B} \rightarrow D \ell \bar{\nu}$

The hadronic matrix element governing the rate of $\bar{B} \rightarrow D \ell \bar{\nu}$ may be described by form factors

$$\langle D(p') | V^\mu | \bar{B}(p) \rangle = f_+(p+p')^\mu + f_-(p-p')^\mu, \quad (2.2)$$

that enter the differential rate as

$$\begin{aligned} \frac{d\Gamma}{dt} = & \frac{G_F^2 |V_{cb}|^2}{192\pi^3 M^3} \frac{k}{t^{5/2}} (t - m_\ell^2)^2 [4k^2 t (2t + m_\ell^2) |f_+|^2 \\ & + 3m_\ell^2 |f_0|^2], \end{aligned} \quad (2.3)$$

where

$$f_0(t) \equiv (M^2 - m^2) f_+(t) + t f_-(t). \quad (2.4)$$

B. $\bar{B} \rightarrow D^* \ell \bar{\nu}$

The matrix elements for $\bar{B} \rightarrow D^* \ell \bar{\nu}$ depend on four form factors,

$$\begin{aligned}\langle D^*(p', \epsilon) | V^\mu | \bar{B}(p) \rangle &= ig \epsilon^{\mu\alpha\beta\gamma} \epsilon_\alpha^* p'_\beta p_\gamma, \\ \langle D^*(p', \epsilon) | A^\mu | \bar{B}(p) \rangle &= f \epsilon^{*\mu} + (\epsilon^* \cdot p) [a_+(p+p')^\mu \\ &\quad + a_-(p-p')^\mu],\end{aligned}\quad (2.5)$$

that enter the differential rate in the combinations

$$\begin{aligned}\frac{d\Gamma}{dt} &= \frac{G_F^2 |V_{cb}|^2}{192\pi^3 M^3} \frac{k}{t^{5/2}} (t-m_\rho^2)^2 \\ &\quad \times \{ (2t+m_\rho^2) [2t|f|^2 + |\mathcal{F}_1|^2 + 2k^2 t^2 |g|^2] \\ &\quad + 3m_\rho^2 k^2 t |\mathcal{F}_2|^2 \},\end{aligned}\quad (2.6)$$

where

$$\begin{aligned}\mathcal{F}_1(t) &\equiv \frac{1}{m} \left[2k^2 t a_+(t) - \frac{1}{2} (t-M^2+m^2) f(t) \right], \\ \mathcal{F}_2(t) &\equiv \frac{1}{m} [f(t) + (M^2-m^2) a_+(t) + t a_-(t)].\end{aligned}\quad (2.7)$$

C. $\Lambda_b \rightarrow \Lambda_c \ell \bar{\nu}$

This decay can be described by six form factors, defined by

$$\begin{aligned}\langle \Lambda_c(p') | V^\mu | \Lambda_b(p) \rangle &= \bar{u}_c(p') [F_1 \gamma^\mu + F_2 v^\mu + F_3 v'^\mu] u_b(p), \\ \langle \Lambda_c(p') | A^\mu | \Lambda_b(p) \rangle &= \bar{u}_c(p') [G_1 \gamma^\mu + G_2 v^\mu + G_3 v'^\mu] \gamma_5 u_b(p),\end{aligned}\quad (2.8)$$

with $v = p/M_{\Lambda_b}$ and $v' = p'/M_{\Lambda_c}$. This gives

$$\begin{aligned}\frac{d\Gamma}{dt} &= \frac{G_F^2 |V_{cb}|^2}{192\pi^3 M^3} \frac{k}{t^{5/2}} (t-m_\rho^2)^2 \\ &\quad \times \{ (t-t)(2t+m_\rho^2) [2t|F_1|^2 + |H_V|^2] \\ &\quad + 3m_\rho^2 (t_+ - t) |F_0|^2 + (t_+ - t)(2t+m_\rho^2) [2t|G_1|^2 \\ &\quad + |H_A|^2] + 3m_\rho^2 (t_- - t) |G_0|^2 \},\end{aligned}\quad (2.9)$$

where

$$\begin{aligned}H_V(t) &= (M+m)F_1 + \frac{1}{2}(t_+ - t) \left(\frac{F_2}{M} + \frac{F_3}{m} \right), \\ H_A(t) &= (M-m)G_1 - \frac{1}{2}(t_- - t) \left(\frac{G_2}{M} + \frac{G_3}{m} \right), \\ F_0(t) &= (M-m)F_1 + \frac{1}{2M}(t+M^2-m^2)F_2 \\ &\quad - \frac{1}{2m}(t-M^2+m^2)F_3,\end{aligned}$$

$$\begin{aligned}G_0(t) &= (M+m)G_1 - \frac{1}{2M}(t+M^2-m^2)G_2 \\ &\quad + \frac{1}{2m}(t-M^2+m^2)G_3.\end{aligned}\quad (2.10)$$

III. REVIEW OF THE DISPERSIVE APPROACH

Constraints on a generic $H_b \rightarrow H_c \ell \bar{\nu}$ form factor $F(t)$ are obtained by noting that the amplitude for production of $H_b \bar{H}_c$ from a virtual W boson is determined by the analytic continuation of $F(t)$ from the semileptonic region of momentum-transfer $m_\rho^2 \leq t \leq t_-$ to the pair-production region $t_+ \leq t$. The idea of the dispersion relation is to constrain $F(t)$ in the pair-production region using perturbative QCD, then use analyticity to translate that constraint into one valid in the semileptonic region [19]. A detailed derivation can be found in [7,8,11]; here we merely outline the essential elements.

In QCD, the two-point function¹ of a flavor-changing current $J = V, A$, or $V-A$,

$$\begin{aligned}\Pi_J^{\mu\nu}(q) &= \frac{1}{q^2} (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi_J^T(q^2) + \frac{q^\mu q^\nu}{q^2} \Pi_J^L(q^2) \\ &\equiv i \int d^4x e^{iqx} \langle 0 | T J^\mu(x) J^{\nu\dagger}(0) | 0 \rangle,\end{aligned}\quad (3.1)$$

is rendered finite by making one or two subtractions, leading to the dispersion relations

$$\begin{aligned}\chi_J^L(q^2) &\equiv \frac{\partial \Pi_J^L}{\partial q^2} = \frac{1}{\pi} \int_0^\infty dt \frac{\text{Im} \Pi_J^L(t)}{(t-q^2)^2}, \\ \chi_J^T(q^2) &\equiv \frac{1}{2} \frac{\partial^2 \Pi_J^T}{\partial (q^2)^2} = \frac{1}{\pi} \int_0^\infty dt \frac{\text{Im} \Pi_J^T(t)}{(t-q^2)^3}.\end{aligned}\quad (3.2)$$

The functions $\chi(q^2)$ may be computed reliably in perturbative QCD for values of q^2 far from the kinematic region where the current J can create resonances: specifically, $(m_b + m_c) \Lambda_{\text{QCD}} \ll (m_b + m_c)^2 - q^2$. For $b \rightarrow c$ or $b \rightarrow u$, $q^2 = 0$ satisfies this condition.

Inserting a complete set of states X into the two-point function relates the Π_J to the production rate of hadrons from a virtual W ,

$$\text{Im} \Pi_J^{T,L} = \frac{1}{2} \sum_X (2\pi)^4 \delta^4(q - p_X) |\langle 0 | J | X \rangle|^2, \quad (3.3)$$

where the sum is over all hadronic states X with the same quantum numbers as the current J , weighted by phase space. Then, from the dispersion relations (3.2), the perturbatively evaluated $\chi(q^2)$ is equal to the integrated production rate of

¹This definition differs slightly from that used in [7–11], and serves to separate $\Pi^{\mu\nu}$ into manifestly spin-0 and spin-1 pieces. Then the functions χ^T, χ^L defined in Eq. (3.2) coincide with χ, χ^L defined in the previous works.

$W^* \rightarrow X$, weighted with a smooth function of momentum transfer squared t . Since the sum is semipositive definite, one may restrict attention to a subset of hadronic states to obtain a strict inequality. In the case of interest, we focus on X being two-particle states of the form $H_b \bar{H}_c$. This places an upper bound on the form factor $F(t)$ in the pair-production region that takes the form

$$\frac{1}{\pi \chi^T(q^2)} \int_{t_+}^{\infty} dt \frac{W(t) |F(t)|^2}{(t-q^2)^3} \leq 1, \quad (3.4)$$

from the $\text{Im } \Pi^T$ dispersion relations in Eq. (3.2). Here $W(t)$ is a computable function of t that depends on the particular form factor under consideration. A similar result holds for Π^L .

Using analyticity to turn Eq. (3.4) into a constraint in the semileptonic region requires that the integrand is analytic below the pair-production threshold $t < t_+$. To do this, we introduce a function

$$z(t; t_s) = \frac{t_s - t}{(\sqrt{t_+ - t} + \sqrt{t_+ - t_s})^2} \quad (3.5)$$

that is real for $t_s < t_+$, zero at $t = t_s$, and a pure phase for $t \geq t_+$. All the poles in the integrand of Eq. (3.4) can be removed by multiplying by various powers of $z(t; t_s)$, provided the positions t_s of the subthreshold poles in $F(t)$ are known. Each pole has a distinct value of t_s , and the product $z(t; t_{s_1}) z(t; t_{s_2}) \cdots$ serves to remove all of them. Such poles arise as the contribution of B_c resonances to the form factor $F(t)$, as well as singularities in the kinematic part of the integrand. After determining these positions phenomenologically, the upper bound on $F(t)$ becomes

$$\frac{1}{\pi} \int_{t_+}^{\infty} dt \left| \frac{dz(t; t_0)}{dt} \right| |\phi(t; t_0) P(t) F(t)|^2 \leq 1, \quad (3.6)$$

where the weight function $\phi(t; t_0)$ (known as an *outer function* in complex analysis) is given by

$$\phi(t; t_0) = \tilde{P}(t) \left[\frac{W(t)}{|dz(t; t_0)/dt| \chi^T(q^2) (t - q^2)^3} \right]^{1/2}. \quad (3.7)$$

The factor $\tilde{P}(t)$ is a product of $z(t; t_s)$'s and $\sqrt{z(t; t_s)}$'s, with t_s chosen to remove the subthreshold singularities and cuts in the kinematic part of the integrand, while the *Blaschke factor* $P(t)$ is a product of $z(t; t_p)$'s with t_p chosen to be the positions of subthreshold poles in $F(t)$. The functions $\phi(t; t_0)$ and $\tilde{P}(t)$ also depend on q^2 , which we leave implicit for notational simplicity, while t_0 is a free parameter to be discussed in Sec. IV.

The quantity $\phi(t; t_0) P(t) F(t)$ may be expanded in a set of orthonormal functions that are proportional to powers of $z(t; t_0)$ [see Eq. (4.21) below]. The function $z(t; t_0)$ has a physical interpretation as a natural scale for the variation of $F(t)$ in the semileptonic region [11], and will play an important role throughout this paper. We exhibit all relevant expressions in terms of both variables z and t in Sec. IV. The

result of the expansion in $z(t; t_0)$ is an expression for $F(t)$ valid even in the semileptonic region,

$$F(t) = \frac{1}{P(t) \phi(t; t_0)} \sum_{n=0}^{\infty} a_n z(t; t_0)^n, \quad (3.8)$$

where, as a result of Eq. (3.6), the coefficients a_n are unknown constants obeying

$$\sum_{n=0}^{\infty} a_n^2 \leq 1. \quad (3.9)$$

For the $b \rightarrow c$ processes that are the main subject of this paper, $z(t; t_0)$ is no larger than 0.07 for any physical momentum transfer $m_c^2 \leq t \leq t_-$, and can be made substantially smaller by a judicious choice of t_0 , so the expansion can be truncated after the first two or three terms.

IV. PARAMETRIZATION INGREDIENTS

Generating a parametrization such as Eq. (3.8) for a particular form factor requires three ingredients: One needs the perturbative evaluation of χ derived from the two-point function for a current J , including the Wilson coefficients of phenomenologically determined condensates. In addition, the functional form of the weighting function ϕ must be computed. Finally, the masses of subthreshold resonances with the same quantum numbers as J must be extracted from experiment or potential models. The function ϕ depends on the form factor under consideration, while χ and P depend only on the current J .

A. QCD evaluation of χ

In the previous section we observed that it suffices to take $q^2 = 0$ in the computation of $\chi_J^{T,L}$ for currents containing a heavy quark. This is convenient since then the perturbative expressions become particularly simple. Corrections to the perturbative result may be included by expressing the two-point function as an operator product expansion (OPE) and including the leading nonperturbative vacuum condensates such as $\langle G^2 \rangle$ and $\langle \bar{q}q \rangle$; the total χ is the sum of the perturbative and condensate terms:

$$\chi = \chi_{\text{pert}} + \chi_{\text{cond}}. \quad (4.1)$$

The full perturbative expressions to two loops may be obtained through a lengthy but straightforward² manipulation of results existing in the literature [20,22]. At $q^2 = 0$, χ_{pert} is only a function of the ratio of quark pole masses $u = m_c/m_b$, and for a vector current $J = V$ is given by

²We have corrected a number of typographical errors in the perturbative results of Ref. [20] and the Wilson Coefficient results of [21] to ensure compliance with various consistency conditions. We also made several comparisons between references to verify their agreement, once these corrections are made.

$$\begin{aligned}
m_b^2 \chi_{\text{pert}}^T(u) &= \frac{1}{32\pi^2(1-u^2)^5} [(1-u^2)(3+4u-21u^2+40u^3-21u^4+4u^5+3u^6)+12u^3(2-3u+2u^2)\ln(u^2)] \\
&+ \frac{\alpha_s}{576\pi^3(1-u^2)^6} [(1-u^2)^2(75+360u-1031u^2+1776u^3-1031u^4+360u^5+75u^6)+4u(1-u^2)(18-99u \\
&+732u^2-1010u^3+732u^4-99u^5+18u^6)\ln(u^2)+4u^3(108-324u+648u^2-456u^3+132u^4+59u^5-12u^6 \\
&-9u^7)\ln^2(u^2)+8(1-u^2)^3(9+12u-32u^2+12u^3+9u^4)\text{Li}_2(1-u^2)], \tag{4.2}
\end{aligned}$$

$$\begin{aligned}
\chi_{\text{pert}}^L(u) &= \frac{1}{8\pi^2(1-u^2)^3} [(1-u^2)(1+u+u^2)(1-4u+u^2)-6u^3\ln(u^2)] + \frac{\alpha_s}{48\pi^3(1-u^2)^4} [(1-u^2)^2(1-36u-22u^2-36u^3 \\
&+u^4)-2u(1-u^2)(9+4u+66u^2+4u^3+9u^4)\ln(u^2)-4u^3(9+18u^2-2u^3-3u^4+u^5)\ln^2(u^2)+8(1-u^2)^3 \\
&\times(1-3u+u^2)\text{Li}_2(1-u^2)], \tag{4.3}
\end{aligned}$$

where the dilogarithm is defined by

$$\text{Li}_2(z) \equiv - \int_0^z dz' \frac{\ln(1-z')}{z'}. \tag{4.4}$$

Expressions for an axial current $J=A$ are obtained from Eqs. (4.2), (4.3) by replacing $u \rightarrow -u$.

It has been pointed out [23] that nonanalytic quark mass dependence, such as in the perturbative results presented above, indicates the inclusion of some infrared effects into the Wilson coefficients (WC's), in conflict with the usual interpretation of the operator product expansion (OPE) as a separation into short- and long-distance effects. A formal analysis reshuffles the WC's in front of each nonperturbative condensate. However, since our analysis requires only the numerical sum of such effects, the total result should be the same in either form.

The leading nonperturbative corrections are supplied by the condensates of dimension less than five, namely the gluon and quark condensates. For a heavy b quark decaying into a quark q of arbitrary mass through a vector current, the contributions from the condensates, derived using Refs. [21,24] and evaluated at $q^2=0$, read

$$\begin{aligned}
\chi_{\text{cond}}^T(u) &= -\langle \bar{q}q \rangle \frac{(2-3u+2u^2)}{2m_b^5(1-u^2)^5} + \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \left\{ \frac{-1}{24m_b^6(1-u^2)^7} [(1-u^2)(2-104u+148u^2 \right. \\
&\left. -270u^3+145u^4-104u^5+5u^6-2u^7)-12u\ln(u^2)(3-5u+17u^2-15u^3+17u^4-5u^5+3u^6)] \right\}, \tag{4.5}
\end{aligned}$$

$$\begin{aligned}
\chi_{\text{cond}}^L(u) &= +\langle \bar{q}q \rangle \frac{1}{m_b^3(1-u^2)^3} + \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \left\{ \frac{1}{12m_b^4(1-u^2)^5} [(1-u^2)(1-21u+10u^2-20u^3 \right. \\
&\left. +u^4-u^5)-3u\ln(u^2)(3-2u+8u^2-2u^3+3u^4)] \right\}. \tag{4.6}
\end{aligned}$$

These expressions superficially appear to diverge in the limit $u \rightarrow 1$. However, in this limit the $\langle m_q \bar{q}q \rangle$ condensate obeys the relation [25]

$$\langle m_q \bar{q}q \rangle = -\frac{1}{12} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle + \mathcal{O}\left(\frac{1}{m_q^2}, \alpha_s\right), \tag{4.7}$$

and Eqs. (4.5), (4.6) simplify to

$$\begin{aligned}
\chi_{\text{cond}}^T(u) &= \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \frac{1}{24(1-u^2)^7 m_b^6 u} \{ (1-u^2)(2-5u+104u^2-145u^3+268u^4-145u^5 \\
&+104u^6-5u^7+2u^8)+12u^2\ln(u^2)(3-5u+17u^2-15u^3+17u^4-5u^5+3u^6) \}, \tag{4.8}
\end{aligned}$$

$$\chi_{\text{cond}}^L(u) = - \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \frac{1}{12(1-u^2)^5 m_b^4 u} \{ (1-u^2)(1-u+20u^2-10u^3+20u^4-u^5+u^6) + 3u^2 \ln(u^2) \} \\ \times (3-2u+8u^2-2u^3+3u^4), \quad (4.9)$$

The expressions (4.8), (4.9) are applicable to $b \rightarrow c$ decays, while Eqs. (4.5), (4.6) are best suited to $b \rightarrow u$ or $c \rightarrow s$ transitions. Expressions for an axial current are obtained by replacing $u \rightarrow -u$. The limiting values for $u \rightarrow 0$ are

$$m_b^2 \chi^T(0) = \frac{3}{32\pi^2} + \frac{\alpha_s}{192\pi^3} (25+4\pi^2) - \frac{1}{m_b^3} \langle \bar{q}q \rangle \\ - \frac{1}{12m_b^4} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle, \\ \chi^L(0) = \frac{1}{8\pi^2} + \frac{\alpha_s}{144\pi^3} (3+4\pi^2) + \frac{1}{m_b^3} \langle \bar{q}q \rangle \\ + \frac{1}{12m_b^4} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle, \quad (4.10)$$

while for $u \rightarrow 1$,

$$m_b^2 \chi^T(+1) = \frac{1}{20\pi^2} + \frac{41\alpha_s}{162\pi^3} - \frac{1}{210m_b^4} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle, \\ m_b^2 \chi^T(-1) = \frac{1}{40\pi^2} + \frac{689\alpha_s}{6480\pi^3} - \frac{1}{140m_b^4} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle, \\ \chi^L(+1) = 0, \\ \chi^L(-1) = \frac{1}{4\pi^2} + \frac{7\alpha_s}{12\pi^3} + \frac{1}{60m_b^4} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle. \quad (4.11)$$

For $b \rightarrow c$, using pole mass values such that $u=0.33$, we have

$$m_b^2 \chi^T(+0.33) = 9.659 \times 10^{-3} \left[1 + 1.42\alpha_s \right. \\ \left. - 4.8 \times 10^{-4} \left(\frac{4.9 \text{ GeV}}{m_b} \right)^4 \left\langle \frac{(\alpha_s G^2/\pi)}{0.02 \text{ GeV}^4} \right\rangle \right], \\ m_b^2 \chi^T(-0.33) = 5.709 \times 10^{-3} \left[1 + 1.32\alpha_s \right. \\ \left. - 6.8 \times 10^{-4} \left(\frac{4.9 \text{ GeV}}{m_b} \right)^4 \left\langle \frac{(\alpha_s G^2/\pi)}{0.02 \text{ GeV}^4} \right\rangle \right],$$

$$\chi^L(+0.33) = 3.713 \times 10^{-3} \left[1 + 1.37\alpha_s \right. \\ \left. - 5.3 \times 10^{-4} \left(\frac{4.9 \text{ GeV}}{m_b} \right)^4 \left\langle \frac{(\alpha_s G^2/\pi)}{0.02 \text{ GeV}^4} \right\rangle \right], \\ \chi^L(-0.33) = 2.162 \times 10^{-3} \left[1 + 0.64\alpha_s \right. \\ \left. + 2.2 \times 10^{-4} \left(\frac{4.9 \text{ GeV}}{m_b} \right)^4 \left\langle \frac{(\alpha_s G^2/\pi)}{0.02 \text{ GeV}^4} \right\rangle \right], \quad (4.12)$$

where the central value $\langle (\alpha_s/\pi) G^2 \rangle = 0.02 \text{ GeV}^4$ is taken from Ref. [26], and the pole mass value $m_b = 4.9 \text{ GeV}$ is from [27]. We also use $\alpha_s(m_b) = 0.22$ [27], and since the coefficient of the gluon condensate is tiny, we ignore it in our numerical analysis for $b \rightarrow c$.

B. Weighting functions ϕ

To obtain the general form of the weighting functions ϕ defined in Eq. (3.6), first observe that the quantities Π^L, Π^T are, respectively, the $\mu = \nu = 0$ and $\mu = \nu = 1, 2$, or 3 components of $\Pi^{\mu\nu}$ evaluated in the center of mass frame, $q^\mu = (\sqrt{t}, \mathbf{0})$. Then the generic expression for the contribution of a particular form factor $F(t)$ to the polarization tensor may be denoted by

$$\text{Im } \Pi^T \geq \frac{n_I}{K\pi} (t-t_+)^{a/2} (t-t_-)^{b/2} t^{-c} |F(t)|^2 \theta(t-t_+), \\ \text{Im } \Pi^L \geq \frac{n_I}{K\pi} (t-t_+)^{a/2} (t-t_-)^{b/2} t^{-(c+1)} |F(t)|^2 \theta(t-t_+), \quad (4.13)$$

where K, a, b , and c are integers determined by the form factor $F(t)$, and n_I is an isospin Clebsch-Gordan factor, which is 2, 3/2, and 1 for $\bar{B} \rightarrow D^{(*)}$, $B \rightarrow \pi$, and $\Lambda_b \rightarrow \Lambda_c$ transitions, respectively. Also, let $\chi = \chi^T$ or χ^L denote the generic QCD function appropriate to the quantum numbers of the form factor.³ The expressions for the weighting functions are readily derived from Eq. (4.13), and are given by

³In terms of previous notation, one finds for the meson form factors considered in Refs. [7,8] $K = 2^p \kappa^{-1} / \pi \chi M^2$, $a = b = p$, $c = s + p - 3$, whereas in Ref. [11] one finds $K = 3 \cdot 2^s$, $a = b = w$, and $c = p$. For the baryon form factors considered in [7], the relation is given by $a = 2p + 1$, $b = 3 - 2p$, $c = s + 1$, and $K = 2 \kappa^{-1} / \pi \chi M^2$ or $K = 2 \kappa^{-1} / \pi \chi^L$, depending upon whether the form factor appears with χ or χ^L .

TABLE I. Factors entering Eq. (4.14) or Eq. (4.23) for the meson form factors F_i in $\bar{B} \rightarrow D^{(*)}$.

F_i	K	χ	a	b	c
f_+	48	$\chi^T(+u)$	3	3	2
f_0	16	$\chi^L(+u)$	1	1	1
f	24	$\chi^T(-u)$	1	1	1
\mathcal{F}_1	48	$\chi^T(-u)$	1	1	2
g	96	$\chi^T(+u)$	3	3	1
\mathcal{F}_2	64	$\chi^L(-u)$	3	3	1

$$\begin{aligned} \phi_i(t; t_0) &= \sqrt{\frac{n_I}{K\pi\chi}} \left(\frac{t_+ - t}{t_+ - t_0} \right)^{1/4} (\sqrt{t_+ - t} + \sqrt{t_+ - t_0}) \\ &\quad \times (t_+ - t)^{a/4} (\sqrt{t_+ - t} + \sqrt{t_+ - t_-})^{b/2} \\ &\quad \times (\sqrt{t_+ - t} + \sqrt{t_+})^{-(c+3)}. \end{aligned} \quad (4.14)$$

The values of the parameters K, a, b, c for each form factor, as well as the relevant χ , are given in Table I for $\bar{B} \rightarrow D^{(*)}$ transitions, and in Table II for $\Lambda_b \rightarrow \Lambda_c$. Although Eq. (4.14) assumes $q^2=0$, it is easy to generalize to arbitrary $Q^2 \equiv -q^2$: Simply evaluate the perturbative functions $\chi(Q^2)$ at the given value, and multiply ϕ by

$$\left(\frac{\sqrt{t_+ - t} + \sqrt{t_+}}{\sqrt{t_+ - t} + \sqrt{t_+ + Q^2}} \right)^d, \quad (4.15)$$

with $d=3$ if the form factor involves χ^T and $d=2$ if it involves χ^L .

While the momentum-transfer variable $t=(p-p')^2$ is useful for heavy-to-light decays and has an obvious physical meaning, it is often more convenient when dealing with heavy-to-heavy transitions such as $b \rightarrow c$ to use a kinematic variable that helps disentangle long-distance physics from the heavy quark scale. One such variable is

$$w \equiv v \cdot v' = \frac{M^2 + m^2 - t}{2Mm}. \quad (4.16)$$

In the b rest frame, w depends only on the energy transfer to the light degrees of freedom, in units of Λ_{QCD} . It is due to this property that $b \rightarrow c$ form factors are related to each other in the heavy quark limit at equal values of w . In the semileptonic region, the variable $z(t; t_0)$ has the same physical property, so form factors will be related at equal values of z

TABLE II. Factors entering Eq. (4.14) or Eq. (4.23) for the baryon form factors F_i .

F_i	K	χ	a	b	c
F_0	8	$\chi^L(+u)$	3	1	1
F_1	12	$\chi^T(+u)$	1	3	1
H_V	24	$\chi^T(+u)$	1	3	2
G_0	8	$\chi^L(-u)$	1	3	1
G_1	12	$\chi^T(-u)$	3	1	1
H_A	24	$\chi^T(-u)$	3	1	2

as well. We can demonstrate this by expressing z as a function of w without reference to heavy meson masses,

$$z(w; N) = \frac{\sqrt{1+w} - \sqrt{2N}}{\sqrt{1+w} + \sqrt{2N}}, \quad (4.17)$$

where N is a free parameter related to t_0 by

$$N = \frac{t_+ - t_0}{t_+ - t_-}, \quad (4.18)$$

so that $z(w; N)$ vanishes at $w=2N-1$. With t, t_0 related to w, N by Eqs. (4.16) and (4.18), Eq. (4.17) is simply a rewriting of Eq. (3.5), $z(t; t_0) = z(w; N)$. The advantage in using $z(w; N)$ for $b \rightarrow c$ transitions is that its definition is process independent.

For the semileptonic decay $H_b \rightarrow H_c \ell \bar{\nu}$, the limiting values of z are given by $m_\ell^2 \leq t \leq t_-$, or

$$z_{\min} = - \left(\frac{\sqrt{N}-1}{\sqrt{N}+1} \right) \quad (4.19)$$

and

$$z_{\max} = \frac{\sqrt{(1+r)^2 - \delta^2} - 2\sqrt{Nr}}{\sqrt{(1+r)^2 - \delta^2} + 2\sqrt{Nr}}. \quad (4.20)$$

where $r = m/M$ and $\delta = m_\ell/M$.

The dispersion relation (3.6), written now entirely in terms of z , reads

$$\frac{1}{2\pi i} \int_C \frac{dz}{z} |\phi(z)P(z)F(z)|^2 \leq 1, \quad (4.21)$$

where C is the unit circle in the complex z plane, the Blaschke factor for a pole at $z_p \equiv z(t_p; t_0)$ (which is real for subthreshold resonance masses) is

$$z(t; t_p) = \frac{z - z_p}{1 - z z_p}, \quad (4.22)$$

for $z = z(t; t_0)$ and any t_0 , with $P(z)$ being the product of all such factors, and the weighting functions (4.14) are given by

$$\begin{aligned} \phi(z; N) &= M^{(1/2)(a+b)-(c+2)} \\ &\quad \times \sqrt{\frac{n_I}{K\pi\chi}} 2^{(1/2)(a+b)+2} N^{a/4+1/2} r^{(1/4)(a+b)+1/2} \\ &\quad \times (1+z)^{(1/2)(a+1)} (1-z)^{c+(1/2)(3-a-b)} \\ &\quad \times [(\sqrt{N}-1)z + (\sqrt{N}+1)]^{b/2} \\ &\quad \times [(1+r)(1-z) + 2\sqrt{Nr}(1+z)]^{-(c+3)}. \end{aligned} \quad (4.23)$$

Finally, evaluation of $\phi(z; N)$ at nonzero values of $Q^2 = -q^2$ is accomplished by multiplying Eq. (4.23) by

$$\left(\frac{(1+r)(1-z) + 2\sqrt{Nr}(1+z)}{\sqrt{(1+r)^2 + Q^2/M^2(1-z) + 2\sqrt{Nr}(1+z)}} \right)^d, \quad (4.24)$$

where d is defined as in Eq. (4.15).

TABLE III. Calculated B_c pole masses used in this work.

Type	Masses (GeV)
Vector	6.337, 6.899, 7.012, 7.280
	7.350, 7.594, 7.646, 7.872, 7.913
Axial vector	6.730, 6.736, 7.135, 7.142
	7.470, 7.470, 7.757, 7.757
Scalar	6.700, 7.108, 7.470, 7.757
Pseudoscalar	6.264, 6.856, 7.244, 7.562, 7.844

C. Subthreshold resonances and $P(t)$

The Blaschke factor $P(t)$ for a form factor describing $H_b \rightarrow H_c \ell \bar{\nu}$ depends on the masses of B_c resonances below the $H_b \bar{H}_c$ pair-production threshold. The Blaschke factors are simply products of $z(t; t_s)$ with t_s evaluated at the invariant mass squared $t_s \rightarrow t_p$ of each such resonance with the same spin-parity as the current J ,

$$P(t) = \prod_p z(t; t_p). \quad (4.25)$$

For $b \rightarrow c$ transition from factors, the masses of the relevant B_c -type resonances can be accurately estimated from potential models [28,29]. We compile in Table III the masses computed in Ref. [29].

For heavy-to-light form factors there is no formal limit in which the light quark becomes nonrelativistic, and potential model calculations are less reliable. This is not a problem for $B \rightarrow \pi$, where the only subthreshold resonance, the B^* , is experimentally observed, but for decays to other light states such as $B \rightarrow \rho$, the presence and masses of additional subthreshold resonances must be taken from models. Once these uncertainties are accounted for, simple parametrizations should be reliable. For example, the model of [30] indicates that the form factor f for $B \rightarrow \rho$ has only one narrow subthreshold pole. Indeed, this pole appears to have been observed (mixed with others) by ALEPH, DELPHI, and OPAL (see [27] for analysis and references on the “ $B_j^*(5732)$ ”). Even accounting for significant uncertainties in its mass, this leads to an accurate parametrization using the overall normalization and two unknown coefficients. It is important to estimate the uncertainties from model-dependent poles on a case by case basis [14].

This applies as well to subthreshold branch cuts due to multiparticle states and anomalous thresholds. A model-dependent analysis [7,8] suggests these are negligible for $b \rightarrow c$ transitions. Qualitatively, this result comes about because cuts are a much less severe form of nonanalytic behavior than poles. Whether cuts continue to be unimportant for $B \rightarrow \rho$ transitions requires a more detailed analysis.

V. ADDITIONAL STATES

The effects of higher states in the dispersion relation depend on the flavor of the $b \rightarrow q$ current under consideration. Henceforth, we specialize to $b \rightarrow c$ transitions, for which HQET is most useful.

A. Contributions to the dispersion relation

We have observed that contributions to the original dispersion relation [see Eq. (3.3)] are semipositive definite, so each additional state coupling to the vacuum through the current J serves to further saturate the bound supplied by the QCD parton-level calculation. The inclusion of only a single two-particle $\bar{B}\bar{D}$ or $\bar{B}\bar{D}^*$ state in obtaining these bounds is relatively weak, since such exclusive states account for only a small portion of the inclusive total. In general, the hadronic side includes also B_c resonances, a continuum of states such as $\bar{B}\bar{D}\pi\pi$, and so on. While it is desirable to include as many of these states as possible, it is not clear how to include them in a model-independent fashion; the chief exceptions are two-particle states related to one another via heavy quark spin symmetry, namely, the four states $\bar{B}^{(*)}\bar{D}^{(*)}$. While form factors for transitions such as $\bar{B}^{(*)} \rightarrow D^{(*)}$ are not physically accessible through semileptonic decays, their normalization is nonetheless known via HQET, allowing an additional strengthening of the dispersive bounds.

The contributions (4.13) of these states to the dispersion relations are of the form

$$\text{Im } \Pi(t) \geq \sum_i \kappa_i(t) |F_i(t)|^2, \quad (5.1)$$

where the sum is over all helicity amplitudes F_i arising from pair production of $\bar{B}^{(*)}\bar{D}^{(*)}$. The weight functions ϕ are readily obtained from the kinematic prefactors $\kappa_i(t)$, while the relation between the helicity amplitudes and the original form factors may be obtained by choosing definite polarizations of the \bar{B}^* , \bar{D}^* , and the virtual W .

For $\bar{B}^* \rightarrow D$ transitions, the form factors are defined by

$$\begin{aligned} \langle D(p') | V^\mu | \bar{B}^*(p, \epsilon) \rangle &= i \hat{g} \epsilon^{\mu\alpha\beta\gamma} \epsilon_{\alpha p \beta p' \gamma}, \\ \langle D^*(p') | A^\mu | \bar{B}^*(p, \epsilon) \rangle &= \hat{f} \epsilon^\mu + (\epsilon \cdot p') [\hat{a}_+(p' + p)^\mu \\ &\quad + \hat{a}_-(p' - p)^\mu]. \end{aligned} \quad (5.2)$$

The functions ϕ for $\bar{B}^* \rightarrow D$ transitions are identical to those for $\bar{B} \rightarrow D^*$, with the simple replacement $M \leftrightarrow m$. The helicity amplitudes possess factors of M and m , and need not be invariant under this exchange. This is true in particular for \mathcal{F}_1 and \mathcal{F}_2 ; here we find that [compare Eq. (2.7)]

$$\begin{aligned} \hat{\mathcal{F}}_1(t) &\equiv \frac{1}{M} \left[2k^2 t \hat{a}_+(t) - \frac{1}{2} (t - m^2 + M^2) \hat{f}(t) \right], \\ \hat{\mathcal{F}}_2(t) &\equiv \frac{1}{M} [\hat{f}(t) - (M^2 - m^2) \hat{a}_+(t) + t \hat{a}_-(t)]. \end{aligned} \quad (5.3)$$

The transition $\bar{B}^* \rightarrow D^*$ possesses ten independent vector current form factors, which we define as [31]

$$\begin{aligned}
& \frac{1}{\sqrt{Mm}} \langle D^*(\epsilon_D, v') | V^\mu | \bar{B}^*(\epsilon_B, v) \rangle \\
&= f_4(\epsilon_B \cdot \epsilon_D^*) v'^\mu + f_5(\epsilon_B \cdot \epsilon_D^*) v^\mu + f_6(v' \cdot \epsilon_B) \\
&\quad \times (v \cdot \epsilon_D^*) v^\mu + f_7(v' \cdot \epsilon_B)(v \cdot \epsilon_D^*) v'^\mu + f_8(v \cdot \epsilon_D^*) \epsilon_B^\mu \\
&\quad + f_9(v' \cdot \epsilon_B) \epsilon_D^{*\mu}, \\
& \frac{1}{\sqrt{Mm}} \langle D^*(\epsilon_D, v') | A^\mu | \bar{B}^*(\epsilon_B, v) \rangle \\
&= +if_{10} \epsilon_{\alpha\beta\gamma}^\mu \epsilon_B^\alpha \epsilon_D^{*\beta} v'^\gamma + if_{11} \epsilon_{\alpha\beta\gamma}^\mu \epsilon_B^\alpha \epsilon_D^{*\beta} v^\gamma + if_{12} \\
&\quad \times [(v' \cdot \epsilon_B) \epsilon_{\alpha\beta\gamma}^\mu \epsilon_D^{*\alpha} v'^\beta v^\gamma + \epsilon_{\alpha\beta\gamma\delta} \epsilon_B^\alpha \epsilon_D^{*\beta} v'^\gamma v^\delta v'^\mu] \\
&\quad + if_{13} [(v \cdot \epsilon_D) \epsilon_{\alpha\beta\gamma}^\mu \epsilon_B^{*\alpha} v'^\beta v^\gamma \\
&\quad - \epsilon_{\alpha\beta\gamma\delta} \epsilon_B^\alpha \epsilon_D^{*\beta} v'^\gamma v^\delta v'^\mu]. \tag{5.4}
\end{aligned}$$

The combinations of these form factors appearing as helicity amplitudes may be denoted

$$\begin{aligned}
V_{++} &= \frac{f_9}{\sqrt{Mm}}, \\
V_{+0} &= \frac{f_8}{\sqrt{Mm}}, \\
V_{0+} &= \frac{1}{\sqrt{2Mm}} (Mf_4 + mf_5), \\
V_{00} &= \frac{1}{4(Mm)^{5/2}} \{Mm(t - M^2 - m^2)(Mf_4 + mf_5) \\
&\quad - 2k^2 t(mf_6 + Mf_7) + Mm[m(t \\
&\quad + M^2 - m^2)f_8 + M(t - M^2 + m^2)f_9]\}, \\
S_{00} &= \frac{1}{(2Mm)^{5/2}} \{Mm(t - M^2 - m^2)[M(t - M^2 + m^2)f_4 \\
&\quad - m(t + M^2 - m^2)f_5] + 2k^2 t[m(t + M^2 - m^2)f_6 \\
&\quad - M(t - M^2 + m^2)f_7] - 4k^2 tMm(mf_8 - Mf_9)\}, \\
S_{0+} &= \frac{1}{\sqrt{4Mm}} [M(t - M^2 + m^2)f_4 - m(t + M^2 - m^2)f_5], \\
A_{++} &= \frac{1}{2(Mm)^{3/2}} \{Mm[(t - M^2 - m^2)f_{10} - 2Mmf_{11}] \\
&\quad - 2k^2 tf_{12}\}, \\
A_{0+} &= \frac{1}{2(Mm)^{3/2}} \{M^2 m(t - M^2 + m^2)f_{10} - Mm^2(t + M^2 \\
&\quad - m^2)f_{11} + 2k^2 t(Mf_{12} - mf_{13})\},
\end{aligned}$$

TABLE IV. Factors entering Eq. (4.14) or Eq. (4.23) for the meson form factors F_i in $\bar{B}^* \rightarrow D^*$.

F_i	K	χ	a	b	c
V_{++}	96	$\chi^T(+u)$	3	3	1
V_{+0}	96	$\chi^T(+u)$	3	3	1
V_{0+}	48	$\chi^T(+u)$	3	3	2
V_{00}	48	$\chi^T(+u)$	3	3	2
S_{00}	8	$\chi^L(+u)$	1	1	1
S_{0+}	8	$\chi^L(+u)$	1	1	1
A_{++}	24	$\chi^T(-u)$	1	1	1
A_{0+}	24	$\chi^T(-u)$	1	1	2
A_{+0}	24	$\chi^T(-u)$	1	1	1
P_{0+}	32	$\chi^L(-u)$	3	3	1

$$\begin{aligned}
A_{+0} &= \frac{1}{2(Mm)^{3/2}} \{Mm[2Mmf_{10} - (t - M^2 - m^2)f_{11}] \\
&\quad + 2k^2 tf_{13}\},
\end{aligned}$$

$$\begin{aligned}
P_{0+} &= \frac{1}{2(Mm)^{3/2}} \{2Mm(Mf_{10} + mf_{11}) + [M(t - M^2 + m^2)f_{12} \\
&\quad + m(t + M^2 - m^2)f_{13}]\}. \tag{5.5}
\end{aligned}$$

The labels V, S, A, P reflect the spin parity (vector, scalar, axial vector, or pseudoscalar) of the virtual W , while the subscripts denote the helicities of the W^* and D^* in the decay of the \bar{B}^* .

Only amplitudes of a fixed spin-parity enter each dispersion relation. For example, the four V helicity amplitudes enter the vector current dispersion relation for Π^T . Above all the appropriate pair-production thresholds, the contribution to $\text{Im } \Pi_V^T$ from $\bar{B}\bar{D}$, $\bar{B}^*\bar{D}$, $\bar{B}\bar{D}^*$, $\bar{B}^*\bar{D}^*$, and $\Lambda_b \bar{\Lambda}_c$ states is

$$\begin{aligned}
\text{Im } \Pi_V^T &\geq \frac{k^3}{6\pi\sqrt{t}} [2(|f_+|^2 + |V_{0+}|^2 + |V_{00}|^2) \\
&\quad + t(|g|^2 + |\hat{g}|^2 + |V_{++}|^2 + |V_{+0}|^2)] \\
&\quad + \frac{k(t-t_-)}{12\pi t^{3/2}} [2t|F_1|^2 + |H_V|^2], \tag{5.6}
\end{aligned}$$

where we have included an isospin factor $n_I=2$ for the mesons, and k is defined in Eq. (2.1). Using Eq. (4.13), it is straightforward to compute the weighting functions ϕ for all the $\bar{B}^*\bar{D}^*$ form factors. For each such F_i , one obtains a parametrization of the form Eq. (3.8), with the unknown expansion coefficients denoted by b_{in} ,

$$F_i(z) = \frac{1}{P_i(z)} \sum_{n=0}^{\infty} b_{in} z^n. \tag{5.7}$$

The ϕ parameters [see Eqs. (4.14) or (4.23)] a, b, c, K and the relevant χ for $\bar{B}^* \rightarrow D^*$ are given in Table IV.

Substituting this expansion into the dispersion relation (4.21) gives

$$\sum_{i=0}^H \sum_{n=0}^{\infty} b_{in}^2 \leq 1. \quad (5.8)$$

Included in the sum are all helicity amplitudes $i=0, \dots, H$ for processes with the right quantum numbers to couple to the current J . It is clear that the constraint on a particular helicity amplitude F_i can be strengthened if it is possible to relate the various b_{in} , i.e., if one can relate the form factors. This is accomplished with the help of heavy quark symmetry.

B. Form factors in the heavy quark limit

In general, form factors are not related by heavy quark symmetry throughout the pair production region [32], but may be related in the semileptonic region. This fact has been exploited to improve constraints on the $B \rightarrow B$ elastic form factor [17] and the dominant $\bar{B} \rightarrow \pi \ell \bar{\nu}$ form factor [11]. The situation in the present case is conceptually analogous, although algebraically more cumbersome.

In the semileptonic region, the 20 form factors of $\bar{B}^{(*)} \rightarrow D^{(*)}$ reduce to only one in the heavy quark limit, the universal Isgur-Wise function $\xi(w)$, with $\xi(1)=1$. Likewise, the six baryon form factors in $\Lambda_b \rightarrow \Lambda_c$ reduce to another universal function [33], which we may denote $\zeta(w)$, with $\zeta(1)=1$. Recalling that $r=m/M$, the relation of the helicity amplitudes to ξ and ζ in the heavy quark limit are given by

$$\begin{aligned} f_+ &= \frac{1}{2} \mathcal{F}_2 = -\frac{1}{2} \hat{\mathcal{F}}_2 = -\frac{1}{\sqrt{2}} V_{0+} = V_{00} = \frac{1}{2} P_{0+} = \frac{(1+r)}{2\sqrt{r}} \xi, \\ f_0 &= \mathcal{F}_1 = \hat{\mathcal{F}}_1 = -\sqrt{2} S_{00} = S_{0+} = -A_{0+} \\ &= M^2 \sqrt{r} (1-r)(1+w) \xi, \\ g &= -\hat{g} = V_{++} = V_{+0} = \frac{1}{M\sqrt{r}} \xi, \\ f &= -\hat{f} = -A_{++} = A_{+0} = M\sqrt{r}(1+w) \xi, \end{aligned} \quad (5.9)$$

for mesons, and by

$$\begin{aligned} F_0 &= H_A = M(1-r) \zeta, \\ F_1 &= G_1 = \zeta, \\ H_V &= G_0 = M(1+r) \zeta \end{aligned} \quad (5.10)$$

for baryons. The only helicity amplitudes in this list protected by Luke's theorem [3,34] from $1/M$ corrections at $w=1$ are f, \hat{f} , and G_1 .

In the strict heavy quark limit, our dispersion relation constraints become useless because the form factors in Eqs. (5.9) and (5.10) develop an essential singularity due to an infinite number of poles just below threshold [35]. The description Eq. (3.8) then contains no information, because the Blaschke factor $P(t)$ goes to zero in the semileptonic region.

For finite masses our parametrizations are well behaved, and heavy quark relations are valid up to $1/m$ corrections. In

this case, some of the form factors in Eqs. (5.9) or (5.10) will have more constraining parametrizations than others because their Blaschke factors, which reflect the number and positions of subthreshold resonances, will be larger. If one views the universality of the Isgur-Wise function as arising from the dominance of the essential singularity in each of the form factors, one might expect $1/m$ corrections to be larger for form factors with fewer subthreshold poles. Of course, it is always possible that the residues of the various poles could conspire to keep $1/m$ corrections small, since Blaschke factors alone permit [36] any residues consistent with the dispersion relation bounds.

We use the heavy quark relations (5.9) and (5.10) to tighten our bounds on parametrization coefficients. When we use these bounds to quote smaller errors on our parametrizations, we allow for substantial deviations due to $1/m$ effects, thereby minimizing errors induced by assuming full heavy quark symmetry. However, when we use these bounds to test heavy quark symmetry by constraining the slope or curvature of the Isgur-Wise function, one should bear in mind the possibility that heavy quark violations could be larger in form factors with very few subthreshold poles (such as f_0, S_{00} , or S_{0+}) than in those that are typically measured experimentally.

C. Bounding parameterization coefficients

We now use the heavy quark relations of the higher resonance helicity amplitudes to improve the constraints. For concreteness, consider the form factor $g(w)$. Near zero recoil, $w=1$, heavy quark symmetry relates it to six other form factors appearing in the dispersion relation,

$$\begin{aligned} g &= -\hat{g} = V_{++} = V_{+0} = \frac{2}{M+m} f_+ = \frac{2}{M+m} V_{00} \\ &= -\frac{\sqrt{2}}{M+m} V_{0+}. \end{aligned} \quad (5.11)$$

While exact heavy quark spin symmetry implies that the functional dependence of these form factors is the same for all values of w corresponding to semileptonic decay, for physical masses we demand only that the normalization and first and second derivatives of these form factors are roughly equal at $w=1$. This suffices to provide lower bounds, assuming full heavy quark symmetry, on the contributions of the form factors in Eq. (5.11) to the sum in Eq. (5.8). Once computed, it is straightforward to weaken these lower bounds by including factors indicating the violation of heavy quark symmetry.

Since each form factor F_i in Eq. (5.11) has a parametrization of the form (5.7), the coefficients a_n in the expansion of g can be related to the coefficients b_{in} in the expansion of F_i by

$$\sum_n a_n z^n \approx \frac{1}{C_i(z)} \sum_n b_{in} z^n, \quad (5.12)$$

where \approx means only the normalization and first two derivatives are equal at $w=1$. The functions

$$C_i = \frac{\Xi_g P_i \phi_i}{\Xi_i P_g \phi_g} \quad (5.13)$$

are given by ratios of Blaschke factors, weighting functions, and symmetry factors $\Xi_i(z)$ chosen so that

$$\xi(z) = \Xi_i(z) F_i(z) \quad (5.14)$$

in the heavy quark limit. The kinematic factors $\Xi_i(w)$ appear in Eqs. (5.9) and (5.10). By choosing the same value of N for each form factor, we ensure that the kinematic variable $z(w; N)$ is process independent, most of the z dependence in $\phi_g(z)/\phi_i(z)$ cancels out, and the $C_i(z)$ become quite simple.

Numerically, the values of N that optimize our constraints correspond to $N = 1 + 2\epsilon$ with $\epsilon \approx 0.05$, and ignoring terms of order ϵ^2 in Eq. (5.12) is a good approximation for parameter values a_n that saturate their bounds. Evaluating (5.12) and its first and second derivatives at $w = 1$ ($z \approx -\epsilon/2$), we find, for each i ,

$$\begin{aligned} b_0 &= C a_0, \\ b_1 &= C' a_0 + C a_1, \\ b_2 &= \frac{1}{2} C'' a_0 + C' a_1 + C a_2 + \frac{3}{2} \epsilon (b_3 - b_3^{\text{HQS}}), \end{aligned} \quad (5.15)$$

where C , $C' = dC/dz$ and $C'' = d^2C/dz^2$ are evaluated at $z = 0$ (not $w = 1$), and b_3^{HQS} is the value b_3 would have if the third derivative of Eq. (5.12) yielded a valid relation. Departures from the heavy quark symmetry limit tend to increase as we take higher derivatives of Eq. (5.12), so one might expect substantial corrections to $b_3 = b_3^{\text{HQS}}$. However, corrections to this relation are multiplied by ϵ , so we may justifiably ignore the factor $3\epsilon(b_3 - b_3^{\text{HQS}})/2$ in b_2 .

Substituting Eq. (5.15) into

$$\sum_{n=0}^{\infty} a_n^2 + \sum_{i=1}^H \sum_{n=0}^{\infty} b_{in}^2 \leq 1 \quad (5.16)$$

gives more stringent bounds on the coefficients a_n . For the form factor g , they yield $-0.37 \leq a_1 \leq 0.40$ and $-0.49 \leq a_2 \leq 0.47$, compared to $|a_1|, |a_2| \leq 1$ from Eq. (3.9). Bounds for the other form factor coefficients can be obtained in a straightforward fashion by singling out a different set of b_{in} in Eq. (5.12).

The bounds on a_1 are useful as tests of heavy quark symmetry. For example, parametrizing f_0 and constraining $-0.62 \leq a_1 \leq 0.58$ restricts the slope of the Isgur-Wise function, up to $1/m_c$ effects, to $-0.3 \leq \rho^2 \equiv -(d\xi/dw)|_{w=1} \leq 1.8$. Bounds on a_2 are useful for decreasing truncation errors of our one-coefficient parametrizations, as described in the next section. In this case, however, we do not want to rely on exact heavy quark symmetry, because we wish to be as conservative as possible with how much the symmetry improves the truncation error, so we allow for explicit violations to the infinite mass limit. Such violations are potentially largest for the bounds on a_2 , because they depend on relations involving second derivatives. Allowing for the total contribution from the higher-spin states to be as low as 60% of their infinite-mass value gives bounds on a_2 of $-0.58 \leq a_2 \leq 0.57$

TABLE V. One-coefficient parameterizations using optimized N . Bounds on a_1 ignore heavy quark violation. Bounds on a_2 allow for violation as described in Sec. V. The truncation error is relative to the normalization of $\Xi_F F$ at $w = 1$.

F	N_{optimal}	Combined bounds from $\bar{B}^{(*)} \bar{D}^{(*)}$		$T(\Xi_F F)$
f_+	1.108	$-0.23 \leq a_1 \leq 0.20$	$-0.55 \leq a_2 \leq 0.58$	2.6%
f_0	1.109	$-0.62 \leq a_1 \leq 0.58$	$-0.78 \leq a_2 \leq 0.85$	0.7%
f	1.093	$-0.37 \leq a_1 \leq 0.39$	$-0.58 \leq a_2 \leq 0.56$	1.8%
\mathcal{F}_1	1.093	$-0.06 \leq a_1 \leq 0.06$	$-0.11 \leq a_2 \leq 0.10$	2.1%
g	1.093	$-0.37 \leq a_1 \leq 0.40$	$-0.58 \leq a_2 \leq 0.57$	1.2%
\mathcal{F}_2	1.093	$-0.41 \leq a_1 \leq 0.45$	$-0.59 \leq a_2 \leq 0.57$	0.6%
F_0	1.081	$-1 \leq a_1 \leq 1$	$-1 \leq a_2 \leq 1$	6.3%
G_0	1.080	$-1 \leq a_1 \leq 1$	$-1 \leq a_2 \leq 1$	9.4%

for g , which leads to a corresponding decrease in the error induced by truncating the expansion (3.8) by 40% [see Eq. (6.4) below]. In summary, bounds on a_1 are obtained using full, unbroken heavy quark symmetry, and may be used to test the accuracy of this symmetry. The bounds on a_1 do not enter the construction of our parametrizations. The derivation of the bounds on a_2 allows for substantial violations of heavy quark symmetry. These bounds enter into the truncation errors we quote on our parametrizations.

Since the baryonic and mesonic form factors are not related by heavy quark symmetry, Eq. (5.16) only applies in the baryonic case to the pairs F_1, H_V and G_1, H_A . Of these, only H_V and H_A receive substantial improvements to their truncation errors, which are proportional to a_3 [see Eq. (6.4)]. While $1/m_c$ corrections to relations involving a_3 and b_3 could *a priori* be large, they are known in terms of one constant $\bar{\Lambda}_\Lambda$ for the baryons [37], and are not particularly large. For example, the relation between H_V and F_1 is independent of w , to $O(1/m_c^2)$. Allowing for the contributions to the dispersion relation from F_1 and G_1 to be as small as 50% of their heavy quark symmetry values gives $|a_3|_{\text{max}} = 0.57$ for H_V and $|a_3|_{\text{max}} = 0.29$ for H_A .

The bounds on a_1 ignoring heavy quark violation, and the bounds on a_2 allowing for deviations from the heavy quark limit as described above, are given in Tables V and VI.

VI. TRUNCATION ERRORS

To fit data with our parametrizations, we must truncate the series (3.8) after a finite number Q of unknown coeffi-

TABLE VI. Two-coefficient parametrizations using optimized N . Bounds on a_1 ignore heavy quark violation. Bounds on a_2 allow for violation as described in Sec. V. The truncation error is relative to the normalization of $\Xi_F F$ at $w = 1$.

F	N_{optimal}	Combined bounds from $\bar{B}^{(*)} \bar{D}^{(*)}$		$T(\Xi_F F)$
F_1	1.104	$-0.90 \leq a_1 \leq 0.90$	$-0.97 \leq a_2 \leq 0.97$	7.0%
H_V	1.104	$-0.31 \leq a_1 \leq 0.31$	$-0.53 \leq a_2 \leq 0.53$	12%
G_1	1.104	$-0.98 \leq a_1 \leq 0.98$	$-0.99 \leq a_2 \leq 0.99$	9.1%
H_A	1.104	$-0.15 \leq a_1 \leq 0.15$	$-0.26 \leq a_2 \leq 0.26$	18%
F_0	1.104	$-1 \leq a_1 \leq 1$	$-1 \leq a_2 \leq 1$	0.35%
G_0	1.104	$-1 \leq a_1 \leq 1$	$-1 \leq a_2 \leq 1$	0.53%

icients a_n . This introduces a truncation error which can be minimized by choosing an appropriate value of N [7] (or equivalently, t_0). Rather than use a_0 as a free parameter as in [8], we solve for a_0 so that the form factor is automatically normalized at $w=1$ to $F(1)$. If $N=1$, then $z(1;N)=0$, and this parametrization coincides with that in [8]. For $N \geq 1$, the parametrization (3.8), including the solution for a_0 , may be reexpressed as

$$\Xi_F(w)F(w) = \frac{\Xi_F(w)}{P(w)\phi(w;N)} \left\{ P(1)\phi(1;N)F(1) + \sum_{n=1}^Q a_n [z^n(w;N) - z^n(1;N)] \right\}, \quad (6.1)$$

where the product $\Xi_F F$ is normalized to coincide with the Isgur-Wise function $\xi(w)$ [or $\zeta(w)$ for baryons] in the heavy quark limit. The full form factor is of course given by the $Q \rightarrow \infty$ sum, while approximations F^{fit} are obtained by truncating at finite Q ; then the fit coefficients a_1, \dots, a_Q can be chosen so that the difference between the parametrization F^{fit} and the actual form factor F is given by

$$\begin{aligned} \Xi_F(w)(F(w) - F^{\text{fit}}(w)) &= \frac{\Xi_F(w)}{P(w)\phi(w)} \sum_{n=Q+1}^{\infty} a_n (z^n - z_{\min}^n) \\ &\approx \frac{\Xi_F(w)}{P(w)\phi(w)} a_{Q+1} (z^{Q+1} - z_{\min}^{Q+1}), \end{aligned} \quad (6.2)$$

where we have ignored numerically unimportant higher order terms. Using the Schwarz inequality, the boundedness condition $\sum_n a_n^2 \leq 1$, and the geometric series sum, these higher order terms can be shown to be smaller than

$$\begin{aligned} &\left| \frac{\Xi_F(w)}{P(w)\phi(w)} \sum_{n=Q+2}^{\infty} a_n (z^n - z_{\min}^n) \right| \\ &\leq \left| \frac{\Xi_F(w)}{P(w)\phi(w)} \right| \sqrt{\sum_{n=Q+2}^{\infty} |a_n|^2 \sum_{n=Q+2}^{\infty} (z^n - z_{\min}^n)^2} \\ &\leq \left| \frac{\Xi_F(w)}{P(w)\phi(w)} (z^{Q+2} - z_{\min}^{Q+2}) \right| + O(z^{Q+4}), \end{aligned} \quad (6.3)$$

which, since $|z| \leq 0.04$ for the cases at hand, can be justifiably ignored. Thus, to good accuracy, the maximal possible truncation error $T(\Xi_F F)$ for any physically allowed z is

$$T(\Xi_F F) = \max \left| \frac{\Xi_F(w)(z^{Q+1} - z_{\min}^{Q+1})}{P(w)\phi(w)} \right| |a_{Q+1}|_{\max}, \quad (6.4)$$

where Q is the highest power of z used in the parametrization fit and $|a_{Q+1}|_{\max}$ is the maximal allowed value of $|a_{Q+1}|$. The inclusion of higher states leads to tighter bounds on $|a_{Q+1}|_{\max}$ and thus smaller truncation errors.

The truncation error vanishes at $w=1$, where the normalization is fixed. For optimal values of N , $z_{\min} \approx -z_{\max}$, so for one-coefficient parametrizations the truncation error drops sharply for some w near w_{\max} . This leads to significantly

smaller truncation errors than in previous works [7,8]. For two-coefficient parameterizations, the factor $(z^3 - z_{\min}^3)$ adds constructively near w_{\max} ; for a given bound on $|a_3|$, this leads to a larger truncation error than in previous work [7]. This is unavoidable as long as only a_1 and a_2 are fit parameters, i.e., as long as a_0 is chosen to enforce the normalization at zero recoil, so that the truncation error vanishes at $z=z_{\min}$ rather than $z=0$.

Our truncation errors for one- and two-coefficient parametrizations of the various form factors in $\bar{B} \rightarrow D \ell \bar{\nu}$, $\bar{B} \rightarrow D^* \ell \bar{\nu}$, and $\Lambda_b \rightarrow \Lambda_c \ell \bar{\nu}$ are shown in Tables V and VI. The value of the free parameter N has been optimized for each form factor and number of fit parameters to produce the smallest truncation errors. The bounds on a_1 come from Eq. (5.16) and the heavy quark symmetry relation Eq. (5.15), and may be used as tests of heavy quark symmetry. The bounds on a_2 enter our truncation errors, so allowance for heavy quark symmetry violation has been made as described in the previous section. The truncation error is expressed as a percentage of $\Xi_F(1)F(1)$, which equals unity in the heavy quark limit. The truncation errors in Table V are typically less than half those of previous parametrizations, while the errors in Table VI are either better or worse by nearly a factor of 2, depending on the form factor [7] (we have corrected an oversight in this reference, which used incorrect Blaschke factors for F_0 and G_0).

Note the especially small size of the truncation error for the form factors f_0 and \mathcal{F}_2 ; if we consider all form factors related by heavy quark symmetry, then the champion in this respect is S_{0+} , with a truncation error of only 0.56%. However, as we discuss in the next section, actually fitting data to such form factors introduces much larger $1/m$ uncertainties.

For baryon form factors, the large number of sub-threshold poles typically ensures that at least two parameters are required. Even with two parameters and the spin-symmetry improvements, the truncation errors are significant. Using three parameters reduces the truncation errors to negligible levels.

A way to circumvent the relatively large truncation errors on most of the baryon form factors is revealed by an interesting feature of Table V: The $\Lambda_b \rightarrow \Lambda_c \ell \bar{\nu}$ form factor F_0 is well-described (to about 6%) by a one-coefficient parametrization. The contribution from F_0 to the decay rate is suppressed by the lepton mass, so it is difficult to observe. However, F_0 is related by heavy quark symmetry to all the other baryon form factors [see Eq. (5.10)]. Unlike in meson decays, the relation including $1/m_c$ effects is known in terms of one constant [37] $\bar{\Lambda}_\Lambda \approx M_{\Lambda_b} - m_b$. Thus, the differential decay rate can be described in terms of two constants, a_1 and $\bar{\Lambda}_\Lambda$. Using the value of $|V_{cb}|$ obtained from $\bar{B} \rightarrow D^{(*)} \ell \bar{\nu}$ and known hadron masses should then allow a determination of $\bar{\Lambda} \approx M_B - m_b$. This quantity is important because it enters both exclusive and inclusive semileptonic decay distributions as a $1/m_c$ correction.

In addition to errors incurred by truncating the expansion (6.1), there are a number of uncertainties arising from various approximations we have made. We enumerated in [8] a list of uncertainties which must be estimated for a reliable determination of the quality of the dispersive bounds, and

found that we could allow for their effects by increasing truncation errors by 40%. Since then, we have greatly reduced or eliminated many of these uncertainties. In [7] we saw that the inclusion of the additional parameter N (or t_0) permits a dramatic reduction of the truncation error, and the more realistic definition of the truncation error used here reduces the uncertainties of one-parameter fits even more. The simultaneous inclusion of all $\bar{B}^{(*)}\bar{D}^{(*)}$ states on the hadronic side of the dispersion relation serves to help saturate the bound from the partonic side, and the explicit inclusion of two-loop perturbative and leading nonperturbative effects eliminates them as a source of uncertainty. The uncertainty in B_c pole positions is most significant for poles near threshold, and leads to larger truncation errors only if the pole masses have been overestimated. As pointed out in [7], branch cuts from multiparticle states below threshold can be ignored if they violate isospin. The only uncertainty from [8] that has not been reduced is due to the choice of pole quark masses in the perturbative calculation. Combining the remaining uncertainties as in [8], we find that their effects may be allowed for by increasing $B \rightarrow D, D^*$ truncation errors by 20% (e.g., the conservative truncation error for f would be 2.2%) and $\Lambda_b \rightarrow \Lambda_c$ truncation errors by 30%. In nearly all cases, this small increase makes no practical difference.

VII. SLOPE AND CURVATURE RELATIONS

A set of interesting relations between the slopes and curvatures of $\bar{B} \rightarrow D \ell \bar{\nu}$ and $\bar{B} \rightarrow D^* \ell \bar{\nu}$ form factors has been derived by Caprini and Neubert [16]. Here we examine these relations in the context of the parametrization formalism, point out and circumvent an invalidating assumption, and discuss the utility of the new, valid, relations.

We have seen that each of the form factors F for $\bar{B}^{(*)} \rightarrow D^{(*)} \ell \bar{\nu}$ can be expanded in a series:

$$F(z) = \frac{1}{P(z)\phi(z)} \sum_{n=0}^{\infty} a_n z^n, \quad (7.1)$$

where, in order to compare with [16], we have set $t_0 = t_-$ ($N=1$). From Eq. (3.9), the coefficients a_n obey

$$a_0^2 + a_1^2 + a_2^2 \leq 1. \quad (7.2)$$

Expressing a_0, a_1 , and a_2 in terms of z derivatives of $P(z)\phi(z)F(z)$ at $z=0$, Eq. (7.2) gives

$$\begin{aligned} & [P(z)\phi(z)F(z)]^2 \Big|_{z=0} + \left[\frac{d}{dz} (P(z)\phi(z)F(z)) \right]^2 \Big|_{z=0} \\ & + \frac{1}{4} \left[\frac{d^2}{dz^2} (P(z)\phi(z)F(z)) \right]^2 \Big|_{z=0} \leq 1. \end{aligned} \quad (7.3)$$

For $t_0 = t_-$, $z=0$ corresponds to $w=1$. Then Eq. (7.3) constrains the slope $-\rho^2$ and curvature c defined by

$$F(w) = F(1) - \rho^2(w-1) + c(w-1)^2 + O[(w-1)^3], \quad (7.4)$$

to lie within an ellipse.

Equation (7.3) is the starting point of reference [16], with one critical simplification: They consider the form factor $\tilde{f}_0 \equiv f_0 / [(M_B - M_D) \sqrt{M_B M_D} (1+w)]$, and argue that it does not receive contributions from scalar B_c mesons that would generate poles in f_0 . This allows them to set $P(z) = 1$, leading to a nearly linear relation between ρ^2 and c ,

$$c \approx 0.72\rho^2 - 0.09. \quad (7.5)$$

Their argument is based on the assumption that the scalar B_c mesons are broad resonances because they can decay into two-particle intermediate states such as $B_c(0^-) + \eta$. Unfortunately, the very potential models they cite refute this idea. For example, Ref. [29] has the mass of the scalar 2^3P_0 state, 6.700 GeV, below the two-particle threshold for an η plus the $1^1S_0 B_c$ ground state, $6.264 + 0.548 = 6.812$ GeV. The scalar 2^3P_0 is thus essentially stable with respect to hadronic transitions, since transitions involving one pion are suppressed by isospin, two pions by parity (or phase space, in decays to a B_c^*), and three pions by phase space. All the references [28] that calculate the relevant scalar masses and widths agree that there are two narrow, scalar B_c resonances below the $\bar{B}\bar{D}$ threshold. It is also worth noting that the lowest-lying charmonium scalar state χ_{c0} is narrow.

While subthreshold branch cuts from states containing at least a b and a c quark may be legitimately ignored [7,8], it is well known that poles play an essential role in the shape of the form factor [11,35]. For this reason, the slope-convexity relations derived in [16] are invalid.

New relations can be derived by simply including the two scalar B_c states in $P(t)$. We use masses from [29], which agree with other potential model determinations to better than 1%. It is algebraically straightforward to input physical masses and expand our parametrization in powers of $(w-1)$,

$$\begin{aligned} \tilde{f}_0(w) = & \tilde{f}_0(1) + [+1.72a_1 - 0.77\tilde{f}_0(1)](w-1) \\ & + [-1.74a_1 + 0.21a_2 + 0.55\tilde{f}_0(1)](w-1)^2 \\ & + [+1.41a_1 - 0.27a_2 - 0.38\tilde{f}_0(1)](w-1)^3 \\ & + [-1.03a_1 + 0.25a_2 + 0.25\tilde{f}_0(1)](w-1)^4 + \dots, \end{aligned} \quad (7.6)$$

and solve for the coefficient c of $(w-1)^2$ in terms of the coefficient $-\rho^2$ of $(w-1)$. We find

$$c = 1.02\rho^2 + 0.21a_2 - 0.23\tilde{f}_0(1). \quad (7.7)$$

This is not a very interesting relation because the unknown coefficient a_2 , which can be as large as ± 1 ($\approx \pm 0.6$ if we include the contribution of higher states and ignore heavy quark symmetry violation), significantly affects the slope-convexity relationship. Had we ignored the Blaschke factor $P(t)$, the coefficient of a_2 would have been 0.07.

For $N=1$, our usual truncation analysis shows that a_2 can contribute at most 4% to $\tilde{f}_0(w)$. This may seem surprising, given its obvious importance in Eqs. (7.6) and (7.7). The 4% value arises from a cancellation of the a_2 dependence among the various $(w-1)^n$ coefficients; note the alternating signs of the a_n coefficients. The cancellation is not accidental, but

reflects the naturalness of an expansion in $z(w;N)$ rather than $(w-1)$. This effect is highlighted by the observation that the $(w-1)^3$ term in (7.6) can be as large as 40% at $w_{\max}=1.6$, indicating that \tilde{f}_0 must be expanded to rather high order in $(w-1)$ if percent-level accuracy is desired.

The expansion (7.6) can, alternatively, be used to test heavy quark symmetry by placing a restriction on the slope $-\rho^2$. If we include the contribution from the spin-related states $\bar{B}^{(*)}\bar{D}^{(*)}$, a_1 is restricted to $-0.61 \leq a_1 \leq 0.59$, leading to

$$-0.26 \leq \rho^2 \leq 1.84. \quad (7.8)$$

The same relation can be derived using the form factors S_{0+} or S_{00} . These bounds are somewhat weaker than those derived from Bjorken [4] and Voloshin [5] inequalities,⁴ which restrict $0.22 \leq \rho^2 \leq 1.15$ once $O(\alpha_s)$ corrections have been included [38].

VIII. EXPERIMENTAL FITS

While constraints obtained from unobserved form factors such as f_0 may serve as tests of heavy quark symmetry, they are not well suited for fitting to data. The reason is that once the truncation error on a form factor is sufficiently small (a few percent), heavy quark symmetry violating effects of order 20–30% become the main concern. For $\bar{B} \rightarrow D \ell \bar{\nu}$, using the parametrization of f_+ avoids any dependence on heavy quark symmetry. For $\bar{B} \rightarrow D^* \ell \bar{\nu}$, using the form factor f minimizes the dependence on heavy quark symmetry. This is because on the one hand, the ratio

$$\frac{f}{M_B \sqrt{M_B M_{D^*}} (1+w)g} = \frac{\hat{C}_1^5}{\hat{C}_1} \left[1 + \frac{\bar{\Lambda}}{2m_c} (w-2) \right] + O\left(\frac{1}{m_b}, \frac{1}{m_c} \right). \quad (8.1)$$

is given in terms of a single [3], roughly determined constant $\bar{\Lambda} \approx 300\text{--}600$ MeV and known [39] perturbative functions $C_1^5/C_1 = 1 + O(\alpha_s)$, while on the other hand the ratio $a_+/g = -1/2$ is determined using only spin symmetry, which is expected to hold more precisely than full flavor-spin symmetry [40]. We use f rather than g because it is protected from $1/m$ corrections [3] at zero recoil.

The purpose of the following fits is not to extract the best value of $|V_{cb}|$, since only the experimental groups themselves can correctly account for efficiencies, resolutions, smearing effects, etc. Rather, since we expect the approximate results from using the QCD-derived parametrization to survive these experimental corrections, these fits may be used to motivate a more thorough analysis.

⁴The physics leading to these results is quite different: The Bjorken and Voloshin inequalities use perturbative QCD to bound exclusive form factors directly in the semileptonic region, while the dispersion constraints use both perturbative QCD in the pair-production region and the phenomenological mass spectrum in the unphysical region $t_- < t < t_+$.

Form factors for $\Lambda_b \rightarrow \Lambda_c \ell \bar{\nu}$ may be extracted in the near future at CDF [41] or LEP [42]. Current data for $\bar{B} \rightarrow D^{(*)} \ell \bar{\nu}$ decay spectra are available from CLEO [43], ALEPH [44], OPAL [45], and DELPHI [46]; older data exists from ARGUS [47]. Some progress has been made towards measuring individual form factors in $\bar{B} \rightarrow D^* \ell \bar{\nu}$ [48]. When this is finally accomplished, the parametrizations for individual form factors can be applied without recourse to heavy quark symmetry, except for the uncertainties in the value of $F(1)$.

In the meantime, one must rely on the heavy quark symmetry prediction that the $\bar{B} \rightarrow D^* \ell \bar{\nu}$ differential rate is proportional to a function $\mathcal{F}(w)$ that is normalized to $1 + O(1/m^2)$ at zero recoil and is proportional to the Isgur-Wise function $\xi(x) = \Xi_f(w)f(w)$ up to $1/m_c$ corrections. We may then use the one-coefficient, QCD-derived parametrization of f obtained from Eqs. (5.9) and (6.1) to extract from data the values of $|V_{cb}|\mathcal{F}(1)$ and $a_1/\mathcal{F}(1)$:

$$\frac{\mathcal{F}(w)}{\mathcal{F}(1)} = \frac{1}{(1+w)P_f(w)\phi_f(w;N)} \left\{ 2P_f(1)\phi_f(1;N) + \frac{a_1}{\mathcal{F}(1)} \frac{[z(w;N) - z(1;N)]}{\sqrt{M_B M_{D^*}}} \right\}, \quad (8.2)$$

where $N=1.093$, $z(w;N)$ is defined in Eq. (4.17), $\phi_f(w;N)$ is given by Eq. (4.23) and Table I, and $P_f(w)$ is determined from Eqs. (4.25), (4.16), and the first four vector masses in Table III.

The procedure is precisely as detailed in [8], except that we now have only one fit coefficient instead of two [7]. Moreover, the improvements described above reduce our one-parameter truncation errors to no more than 3%. We fit to the experiments whose differential distributions are easily available. A χ^2 per degree of freedom (N_{DF}) fit using our QCD dispersion relation bounds (QCD fit) to CLEO data [43] gives

$$\text{QCD fit} \quad 10^3 \times |V_{cb}|\mathcal{F}(1) = 36.9_{-2.1}^{+2.0},$$

$$\frac{a_1}{\mathcal{F}(1)} = 0.000_{-0.019}^{+0.022}, \quad (8.3)$$

$$\text{Linear fit} \quad 10^3 \times |V_{cb}|\mathcal{F}(1) = 35.1_{-1.9}^{+1.9}, \quad (8.4)$$

with $\chi_{\min}^2/N_{\text{DF}}=0.67$, while a fit to ALEPH data [44] gives

$$\text{QCD fit} \quad 10^3 \times |V_{cb}|\mathcal{F}(1) = 31.9_{-2.4}^{+2.4},$$

$$\frac{a_1}{\mathcal{F}(1)} = 0.096_{-0.034}^{+0.040}, \quad (8.5)$$

$$\text{Linear fit} \quad 10^3 \times |V_{cb}|\mathcal{F}(1) = 31.9_{-1.8}^{+1.8}, \quad (8.6)$$

with $\chi_{\min}^2/N_{\text{DF}}=0.74$. We have also included the values quoted by the experimental groups using a linear fit for comparison, and only statistical errors at one standard deviation are listed.

For $\bar{B} \rightarrow D \ell \bar{\nu}$, we can fit the parametrization of f_+ ,

$$\frac{f_+(w)}{f_+(1)} = \frac{1}{P_{f_+}(w)\phi_{f_+}(w;N)} \left\{ P_{f_+}(1)\phi_{f_+}(1;N) + \frac{a_1}{f_+(1)} [z(w;N) - z(1;N)] \right\}, \quad (8.7)$$

directly to data, without the need to invoke heavy quark symmetry. Here $N=1.108$ and P_{f_+} depends on the first three vector masses in Table III.

In this case, a fit to ALEPH data [44] yields

$$\text{QCD Fit} \quad 10^3 \times |V_{cb}| \mathcal{F}_D(1) = 29.2_{-7.3}^{+7.3},$$

$$\frac{a_1}{\mathcal{F}_D(1)} = 0.095_{-0.066}^{+0.110}, \quad (8.8)$$

$$\text{Linear fit} \quad 10^3 \times |V_{cb}| \mathcal{F}_D(1) = 27.8_{-6.8}^{+6.8}, \quad (8.9)$$

with⁵ $\chi_{\min}^2/N_{\text{DF}}=1.94$, and now $\mathcal{F}_D(1) = \Xi_{f_+}(1)f_+(1) = 2\sqrt{r}f_+(1)/(1+r)$.

For CLEO $\bar{B} \rightarrow D^* \ell \bar{\nu}$ data, which exhibits discernible curvature, our central values of $|V_{cb}| \mathcal{F}(1)$ lie at the upper 1 σ boundary of the linear-fit result, while for ALEPH data, which is extremely flat, the central values are nearly the same. The statistical errors are larger because the QCD parametrization allows for curvature.

The ALEPH $\bar{B} \rightarrow D \ell \bar{\nu}$ data presents an interesting area in which to test our results: The shape of their data will have to change as the statistical errors are reduced, if it is to be consistent with QCD.

IX. CONCLUSIONS

Form factors can be reliably bounded in the pair-production region of momentum space by perturbative QCD calculations. Analyticity, crossing symmetry, and dispersion relations may then be used to translate these bounds into constraints in the phenomenologically interesting semileptonic region.

While these constraints typically imply rather weak bounds on the slopes of form factors [10,17,18,36], quite stringent bounds can be obtained if the form factor at two or more points is known. The constraints actually imply an infinite number of increasingly stringent bounds, depending on the number of points at which the form factor is known [10,12]. All of these bounds are automatically obeyed if the form factor is parametrized as in Eq. (3.8), even if the parametrization is truncated after a few terms.

In this paper, we have eliminated some of the uncertainty involved in the derivation of these parametrizations by including two-loop perturbative corrections to the partonic side of the dispersion relation. We have also presented parametrizations for form factors whose contribution to semileptonic decay rates is suppressed by the lepton mass.

We examined a relation between the slope and curvature of the Isgur-Wise function derived by Caprini and Neubert

[16], and pointed out the presence of subthreshold singularities that invalidate their analysis. Once these singularities are correctly accounted for, the slope-convexity relations become rather weak. Bounds on the slope of the Isgur-Wise function made by ignoring finite mass corrections are also fairly weak. We point out that even with strong constraints on the slope and curvature, higher-order terms in a $(w-1)$ expansion of the Isgur-Wise function can be quite large. The parametrization in Eq. (3.8) does not suffer from this limitation.

For $b \rightarrow c$ transitions, we reduced the *truncation errors* that describe the accuracy of such parametrizations. This was accomplished in part by using a parametrization that is automatically normalized at zero recoil to a quantity $\mathcal{F}(1)$, which must be supplied by some other method such as heavy quark symmetry. More importantly, we included the contributions of higher states in the hadronic side of the dispersion relation. These states, the $\bar{B}^* \bar{D}$ and $\bar{B}^* \bar{D}^*$ pairs, are related to $\bar{B} \rightarrow D \ell \bar{\nu}$ and $\bar{B} \rightarrow D^* \ell \bar{\nu}$ form factors in the semileptonic region by heavy quark symmetry. Even in the presence of substantial heavy quark symmetry violation, these relations place lower bounds on the contribution from the higher states to the dispersion relation that lead to tighter upper bounds on the magnitudes of the unknown parametrization coefficients. This in turn reduces the truncation errors on the parametrizations of the various form factors.

For most of the $\bar{B} \rightarrow D, D^*$ form factors, the inclusion of higher states reduces truncation errors by roughly 40%. In one case, \mathcal{F}_1 , the truncation error is reduced by as much as a factor of 10. This hefty improvement arises because \mathcal{F}_1 contributes very little to the dispersion relation, so it is far from saturating the perturbative bound until its spin-symmetry partners are included as well. After including all $\bar{B}^{(*)} \bar{D}^{(*)}$ states, we find that each of the six form factors governing $\bar{B} \rightarrow D \ell \bar{\nu}$ and $\bar{B} \rightarrow D^* \ell \bar{\nu}$ is described to better than 3% accuracy using only one unknown parameter. This should be a considerable aid in experimentally disentangling the various form factors in differential decay distributions.

For $\Lambda_b \rightarrow \Lambda_c \ell \bar{\nu}$ decays, there are no spin-symmetry partners to help saturate the dispersion relation bound. However, the presence of more than one helicity amplitude in the same bound achieves the same effect. This is most dramatic in the case of H_A , whose truncation error is reduced by a factor of 4. Of greater interest is the observation that the form factor F_0 can be described at the 6% level using only one unknown coefficient. Because the $1/m_c$ corrections to heavy quark symmetry relations among the baryonic form factors are given in terms of one additional parameter $\bar{\Lambda}_\Lambda$ the entire decay distribution can be described using only two unknown constants. This should allow a relatively clean extraction of the phenomenologically interesting quantity $\bar{\Lambda}_\Lambda$.

Finally, we used a parametrization of the form factor f and heavy quark relations to extract $|V_{cb}|$ from $\bar{B} \rightarrow D^* \ell \bar{\nu}$ data. This choice of form factor minimizes the dependence on heavy quark symmetry. We expect the qualitative features of this extraction to persist even after the effects of experimental resolution, smearing, etc., are properly incorporated. Our analysis suggests that the implicit error associated with the choice of parametrization is comparable

⁵The larger χ^2/N_{DF} is due to their binned data point at $w=1.55$, which suggests a peculiar upturn of the form factor near w_{\max} .

to the statistical errors normally quoted. Similar statements apply to $\bar{B} \rightarrow D/\bar{\nu}$, except that in this case no reliance on heavy quark symmetry is necessary, since we can parametrize the form factor f_+ directly.

Further improvements may be possible by including additional higher states in the dispersion bound, or perhaps by weighing the dispersion integral differently. One could also readily incorporate approximate SU(3) symmetry by using an effective $n_I = 2.5$ isospin factor [16] in $\bar{B} \rightarrow D, D^*$ decays, which would decrease truncation errors by an additional 10%. While such improvements would be welcome for the baryonic form factors, their utility for $\bar{B} \rightarrow D, D^*$ form factors is not clear. This is because, once truncation errors are at the few-percent level, the overwhelming source of uncertainty comes from heavy quark symmetry violations, which are expected to be of order 30%. Such uncertainties highlight the importance of extracting individual form factors, which can be parametrized using one coefficient without recourse to heavy quark symmetry.

Note added. The essential role of the scalar B_c poles in the slope-convexity relations of Caprini and Neubert has been pointed out independently by L. Lellouch (private communication). A corrected slope-convexity relation and other topics related to those in this work are in preparation by these authors [49]. Additional criticism of the neglect of the scalar poles appears in Ref. [50].

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