Manifestly gauge covariant treatment of lattice chiral fermions. II

Kiyoshi Okuyama* and Hiroshi Suzuki†
Department of Physics, Ibaraki University, Mito 310, Japan
(Received 22 July 1997)

We propose a formulation of chiral fermions on a lattice, on the basis of a lattice extension of the covariant regularization scheme in continuum field theory. The species doublers do not emerge. The real part of the effective action is just one-half of that of Dirac-Wilson fermion and is always gauge invariant even with a finite lattice spacing. The gauge invariance of the imaginary part, on the other hand, sets a severe constraint that is a lattice analog of the gauge anomaly-free condition. For real gauge representations, the imaginary part identically vanishes and the gauge invariance becomes exact. [S0556-2821(97)04723-1]

PACS number(s): 11.15.Ha, 11.30.Rd

Inspired by the covariant regularization scheme [1–3] in the continuum field theory, one of us recently proposed a manifestly gauge covariant treatment of chiral fermions on a lattice [4]. However, the proposal heavily relied on the notion in perturbation theory and its validity was demonstrated only in the continuum limit. Many important issues, such as the integrability (see below), were also not clarified there. In this article, we remedy these points and try to set up a truly nonperturbative framework with the same strategy.

The basic idea of [4] is the following: At present, it seems impossible to construct a lattice action of chiral fermions that *explicitly* distinguishes gauge anomaly-free representations from anomalous ones. This implies that we cannot expect a sensible manifestly gauge invariant lattice formulation because it will not reproduce in the continuum limit the gauge anomaly for the anomalous cases. If one nevertheless forces the manifest gauge invariance, the species doublers [5,6], which cancel the gauge anomaly, will emerge; thus we have to break the gauge symmetry at a certain stage. With these observations, a formulation that preserves the gauge symmetry as much as possible in *both* the anomalous and nonanomalous cases seems desirable.

The covariant regularization [2,3] is such a regularization scheme in the continuum theory. The scheme does not spoil *all* the gauge invariance even in anomalous cases; instead it sacrifices Bose symmetry among gauge vertices in a fermion one-loop diagram. In this scheme, one starts with a regularized gauge current operator (the covariant gauge current)

$$\begin{aligned}\langle J^{\mu b}(x)\rangle &= \langle \overline{\psi}(x) T^b \gamma^\mu P_R \psi(x)\rangle \\ &= -\lim_{y\to x} \mathrm{tr} T^b \gamma^\mu P_R G(x,y) \\ &\equiv -\lim_{y\to x} \mathrm{tr} T^b \gamma^\mu P_R f(\not{D}^2/\Lambda^2)\,\frac{-1}{i\not{D}}\,\delta(x-y), \quad (1)\end{aligned}$$

where $P_R \equiv (1+\gamma_5)/2$ is the chirality projection operator and $\not{D} \equiv \gamma^\mu(\partial_\mu + iA_\mu^b T^b)$ is the covariant derivative; note that *Dirac* propagator is used. In Eq. (1), Λ is the cutoff parameter and the regulating factor $f(t)$ satisfies $f(0)=1$ and $f(\infty)=f'(\infty)=f''(\infty)=\cdots=0$. The definition immediately follows the gauge *covariance* of the current operator, namely, under the gauge transformation on the background gauge field $A_\mu(x) \to -iV(x)\partial_\mu V^\dagger(x) + V(x)A_\mu(x)V^\dagger(x)$, the gauge current transforms gauge covariantly:

$$\langle J^{\mu b}(x)\rangle \to -\lim_{y\to x} \mathrm{tr}[V^\dagger(x) T^b V(x)]\gamma^\mu P_R f(\not{D}^2/\Lambda^2)\,\frac{-1}{i\not{D}} \times \delta(x-y). \quad (2)$$

In other words, the gauge invariance at external gauge vertices of a fermion one-loop diagram *except* that of $J^{\mu b}(x)$ is preserved in the scheme. Because of this Bose asymmetric treatment of gauge vertices, the gauge invariance can be "maximally" preserved even in anomalous cases. As a consequence, the gauge anomaly has the covariant form.

Once the gauge current operator is defined in this way, the effective action $\Gamma[A]$ might be obtained from the relation

$$\langle J^{\mu b}(x)\rangle = -\frac{\delta\Gamma[A]}{\delta A_\mu^b(x)}. \quad (3)$$

However, such a functional $\Gamma[A]$ exists only if the covariant gauge anomaly vanishes. The simplest way to see this is to note the covariant anomaly does not satisfy the Wess-Zumino consistency condition [7], which is a consequence of the integrability (3). The integrability or the Bose symmetry, however, is restored when we can further impose the gauge invariance on the $J^{\mu b}(x)$ vertex, i.e., anomaly-free cases. In fact, for anomaly-free cases, one can write down a formula of $\Gamma[A]$ [3],

$$\Gamma[A] = -\int_0^1 dg \int d^4x\, A_\mu^b(x)\langle J^{\mu b}(x)\rangle_g, \quad (4)$$

where the gauge current on the right-hand side is the covariant current (1) and the subscript g means it is evaluated by a covariant derivative with a coupling constant g, $\not{D}_g \equiv \gamma^\mu(\partial_\mu + igA_\mu^b T^b)$. When the gauge anomaly is absent, one can prove [3] that the integrable current (3) coincides with the covariant one (1) (in the infinite cutoff limit $\Lambda\to\infty$). In

*Electronic address: okuyama@mito.ipc.ibaraki.ac.jp
†Electronic address: hsuzuki@mito.ipc.ibaraki.ac.jp

this scheme, therefore, anomalous cases are distinguished by the nonintegrability without explicitly spoiling all the gauge invariance.

The covariant current (1) is not in general integrable, i.e., not a functional derivative of something. This means that in particular it cannot be written as a functional derivative of the functional integral of a certain action [8]. However, one may directly work with the fermion propagator and the gauge current operator as in Eq. (1). This is also true in the lattice theory; the crucial point of our approach is to ''forget'' about the action [4].

Let us now translate the above strategy of covariant regularization into the lattice language as much as possible. Of course, there is a wide freedom to do so, partially corresponding to the freedom of regulating factor $f(t)$. However, the details of the extension should not be important and we first require the following: (i) The expression reduces to the continuum analog in the naive (or classical) continuum limit, (ii) the lattice propagator has no doubler's pole; and (iii) the lattice fermion propagator transforms gauge covariantly, namely, under the gauge transformation on the link variable $U_\mu(x)\rightarrow V(x)U_\mu(x)V^\dagger(x+a^\mu)$, the propagator transforms as $G(x,y)\rightarrow V(x)G(x,y)V^\dagger(y)$.

For definiteness and simplicity, we will use the Wilson propagator [9] in this article:

$$G(x,y)\equiv\frac{-1}{i\not{D}(x)+R(x)}\,\delta(x,y)=\delta(x,y)\,\frac{1}{i\overleftarrow{\not{D}}(y)+\overleftarrow{R}(y)},\tag{5}$$

where the delta function on the lattice is defined by $\delta(x,y)\equiv\delta_{x,y}/a^4$; $\not{D}(x)$ is the lattice covariant derivative and $R(x)$ is the Wilson term:

$$\not{D}(x)\equiv\sum_\mu\gamma^\mu\frac{1}{2a}[U_\mu(x)e^{a\partial_\mu}-e^{-a\partial_\mu}U_\mu^\dagger(x)],$$
$$R(x)\equiv\frac{r}{2a}\sum_\mu[U_\mu(x)e^{a\partial_\mu}+e^{-a\partial_\mu}U_\mu^\dagger(x)-2]\tag{6}$$

and

$$\overleftarrow{\not{D}}(x)\equiv-\sum_\mu\gamma^\mu\frac{1}{2a}[U_\mu(x)e^{-a\overleftarrow{\partial}_\mu}-e^{a\overleftarrow{\partial}_\mu}U_\mu^\dagger(x)],$$
$$\overleftarrow{R}(x)\equiv-\frac{r}{2a}\sum_\mu[U_\mu(x)e^{-a\overleftarrow{\partial}_\mu}+e^{a\overleftarrow{\partial}_\mu}U_\mu^\dagger(x)-2].\tag{7}$$

In the above expressions, a is the lattice spacing and $\exp(\pm a\partial_\mu)$ is the translation operator in the μ direction by a unit lattice spacing. The equality of the two expressions (5) follows from two equivalent forms of the Wilson action:

$$S[\psi,\overline{\psi},U]=a^4\sum_x\overline{\psi}(x)[i\not{D}(x)+R(x)]\psi(x)$$
$$=-a^4\sum_x\overline{\psi}(x)[i\overleftarrow{\not{D}}(x)+\overleftarrow{R}(x)]\psi(x).\tag{8}$$

In contrast with the continuum Dirac propagator in Eq. (1), the Wilson term mixes the right-handed and left-handed chiralities.[1] However, we do not think that this is so problematical because the physical particle picture emerges only in the continuum limit and in the continuum limit we expect this chirality mixing due to the Wilson term to vanish. Note that the Wilson propagator nevertheless has the required gauge covariance property.

As the lattice analog of the covariant gauge current, therefore, we shall study

$$\Delta[U,\delta U]\equiv-a^4\sum_x\operatorname{tr}[i\,\delta\not{D}(x)P_R+\tfrac{1}{2}\delta R(x)]G(x,y)|_{y=x}$$
$$=a^4\sum_x\operatorname{tr}G(y,x)[i\,\delta\overleftarrow{\not{D}}(x)P_R+\tfrac{1}{2}\delta\overleftarrow{R}(x)]|_{y=x},\tag{9}$$

where δU represents an infinitesimal variation of the link variable and its conjugate is defined by $\delta U_\mu^\dagger(x)=-U_\mu^\dagger(x)\,\delta U_\mu(x)U_\mu^\dagger(x)$. The second expression follows from the definitions (6) and (7) and the fact that we can freely shift the ''integration variable'' x. As the analog of Eq. (3), we identify it with the variation of the effective action

$$\Delta[U,\delta U]=\delta\Gamma[U].\tag{10}$$

The defining relations (9) and (10) are suggested by the *naive* relation $\exp\Gamma[U]=\int\mathcal{D}\psi\,\mathcal{D}\overline{\psi}\,\exp[a^4\Sigma_x\overline{\psi}(x)i\not{D}(x)P_R\psi(x)]$. [The variation of the Wilson term $\delta R(x)$ in Eq. (9) will be necessary for the integrability.] The integrability (10) is of course not a trivial statement and will be investigated below.

We first note the manifest gauge covariance of $\Delta[U,\delta U]$:

$$\Delta[V(x)U_\mu(x)V^\dagger(x+a^\mu),\delta U_\mu(x)]$$
$$=\Delta[U_\mu(x),V^\dagger(x)\,\delta U_\mu(x)V(x+a^\mu)].\tag{11}$$

That is, $\Delta[U,\delta U]$ behaves gauge covariantly under the gauge transformation on the background U. This is an analogous relation to Eq. (2).

Next, we separate the ''would-be variation'' $\Delta[U,\delta U]$ into the real and imaginary parts. We note that the following relations hold for an arbitrary matrix $m(x)$:

$$\not{D}(x)^*m(x)=-[m(x)^T\overleftarrow{\not{D}}(x)]^T,$$
$$R(x)^*m(x)=-[m(x)^T\overleftarrow{R}(x)]^T,\tag{12}$$

where $T^{b*}=T^{bT}$, $\gamma^{\mu*}=-\gamma^{\mu T}$, and $\gamma_5^*=\gamma_5^T$ have been used. Using these relations, we find

$$G(x,y)^*=\gamma_5^TG(y,x)^T\gamma_5^T.\tag{13}$$

From Eqs. (12) and (13), the complex conjugate of $\Delta[U,\delta U]$ is given by

[1]One may even avoid this chiral symmetry breaking by making use of more ingenious propagator in [10]. See [4].

$$\Delta[U,\delta U]^* = a^4\sum_x \operatorname{tr} G(y,x)[i\,\delta\overleftarrow{\not{D}}(x)P_L + \tfrac{1}{2}\delta\overleftarrow{R}(x)]|_{y=x}$$

$$= -a^4\sum_x \operatorname{tr}[i\,\delta\not{D}(x)P_L + \tfrac{1}{2}\delta R(x)]G(x,y)|_{y=x}. \tag{14}$$

Then a comparison with Eq. (9) shows that the real and imaginary parts are respectively given by

$$\mathrm{Re}\Delta[U,\delta U] = -\tfrac{1}{2}a^4\sum_x \operatorname{tr}[i\,\delta\not{D}(x) + \delta R(x)]G(x,y)|_{y=x} \tag{15}$$

and

$$i\ \mathrm{Im}\Delta[U,\delta U] = -\tfrac{1}{2}a^4\sum_x \operatorname{tr} i\,\delta\not{D}(x)\gamma_5 G(x,y)|_{y=x}$$

$$= \tfrac{1}{2}a^4\sum_x \operatorname{tr} G(y,x)i\,\delta\overleftarrow{\not{D}}(x)\gamma_5|_{y=x}. \tag{16}$$

Now, for the *real* part of $\Delta[U,\delta U]$ [Eq. (15)], we immediately see the integrability and the gauge *invariance*. By the gauge invariance, we mean that the would-be variation of the effective action $\Delta[U,\delta U]$ vanishes along the direction of the gauge degrees of freedom. That is,

$$\mathrm{Re}\Delta[U,\delta_\lambda U]=0 \quad \text{for}$$

$$\delta_\lambda U_\mu(x) \equiv -i\lambda(x)U_\mu(x) + iU_\mu(x)\lambda(x+a^\mu), \tag{17}$$

where $\lambda(x)=\lambda^b(x)T^b$. One can easily verify this relation by using above definitions. This gauge invariance property of the real part is almost trivial in our construction because Re $\Delta[U,\delta_\lambda U]$ is simply one-half of that of the Dirac-Wilson fermion:

$$\mathrm{Re}\Delta[U,\delta U] = \delta\Gamma_1[U],$$

$$\Gamma_1[U] \equiv \tfrac{1}{2}\ \ln\det[i\not{D}(x)+R(x)]. \tag{18}$$

Note that the last expression is well defined and not a formal one with the lattice regularization. Therefore, for the real part, we arrived at a quite simple picture: The real part of $\Delta[U,\delta U]$ can always be regarded as a variation of the effective action $\Gamma_1[U]$, which is just one-half of the effective action of the Dirac-Wilson fermion. In other words, the chiral determinant obtained by ‘‘integrating’’ Re $\Delta[U,\delta U]$ gives rise to the square root of the Dirac-Wilson determinant. Although the gauge invariance of the real part of the effective action is almost trivial in this way, this is very interesting because the gauge invariance of the real part is one of main achievements of recent research [11–14]. In our approach, the origin of this property of the real part may be traced to the basic idea of covariant regularization, i.e., maximal gauge invariance. We note that our treatment of the real part turned out to be almost identical to that of [14].

The gauge invariance of the imaginary part, on the other hand, is difficult. A short calculation shows that

$$i\ \mathrm{Im}\Delta[U,\delta_\lambda U] = a^4\sum_x \lambda^b(x)\mathcal{A}^b(x), \tag{19}$$

where $\mathcal{A}^b(x)$ is given by

$$\mathcal{A}^b(x) \equiv -\tfrac{1}{2}\ \operatorname{tr}[G(y,x)\overleftarrow{\not{D}}(x)\gamma_5 T^b - T^b\gamma_5\not{D}(x)G(x,y)]|_{y=x}. \tag{20}$$

In fact, this is a lattice analog of the gauge anomaly: By considering the *axial* rotations $\psi(x)\to\exp[i\theta^b(x)T^b\gamma_5]\psi(x)$ and $\bar\psi(x)\to\bar\psi(x)\ \exp[i\theta^b(x)T^b\gamma_5]$ in the Wilson action (8), we can compute $\mathcal{A}^b(x)$ in the continuum limit [5,15] and find the covariant gauge anomaly

$$\lim_{a\to 0}\mathcal{A}^b(x) = \frac{i}{32\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}\ \operatorname{tr}\ T^b F_{\mu\nu}F_{\rho\sigma}. \tag{21}$$

Therefore, if the gauge representation is anomaly-free, the imaginary part of $\Delta[U,\delta U]$ vanishes along the gauge variation *in the continuum limit* and the effective action becomes gauge invariant; this is the expected property. However, this is not sufficient for the gauge invariance with a *finite* lattice spacing. It is clear that $\mathcal{A}^b(x)=0$ with a finite lattice spacing is a much stronger condition than the anomaly-free condition in the continuum theory. We can furthermore show that the integrability of the imaginary part also requires $\mathcal{A}^b(x)=0$ (see the Appendix), thus the integrability does not hold unless $\mathcal{A}^b(x)=0$.

Therefore, we again face the usual difficulty of lattice chiral gauge theory that the gauge mode decouples only in the continuum limit, even in anomaly-free cases. Although the natural lattice extension of the covariant regularization provides a simple picture for a treatment of the real part of the effective action, it does not solve the main difficulty of anomaly-free *complex* representations in the lattice chiral gauge theory. For the general discussion on the imaginary part of the effective action of lattice chiral fermion, see [16]. Equation (21) suggests that the difficulty of our approach might be avoided only by invoking the double-limit procedure in [14].

However, at least for *real* gauge representations, we can show that the above problems of the gauge invariance and the integrability do not occur at all. This is because the imaginary part of $\Delta[U,\delta U]$ [Eq. (16)] identically vanishes for real representations. The demonstration is straightforward: For a real representation T^b, there exists a unitary matrix u that maps T^b into the conjugate representation $uT^bu^\dagger=-T^{b*}=-T^{bT}$. We then insert $u^\dagger C^{-1}Cu=1$ into the first line of Eq. (16). (C is the charge conjugation matrix $C\gamma^\mu C^{-1}=-\gamma^{\mu T}$ and thus $C\gamma_5C^{-1}=\gamma_5^T$.) Then, by noting

$$Cu\,\delta\not{D}(x)u^\dagger C^{-1}m(x) = -[m(x)^T\delta\overleftarrow{\not{D}}(x)]^T,$$

$$CuG(x,y)u^\dagger C^{-1} = G(y,x)^T, \tag{22}$$

we find

$$i\ \mathrm{Im}\ \Delta[U,\delta U] = -\tfrac{1}{2}a^4\sum_x \operatorname{tr} G(y,x)i\,\delta\overleftarrow{\not{D}}(x)\gamma_5|_{y=x}. \tag{23}$$

A comparison with Eq. (16) shows that the imaginary part of $\Delta[U,\delta U]$ identically vanishes; $\Delta[U,\delta U]$ is purely real.

Therefore, the treatment of real representations is simple: The variation of the effective action is given by Eq. (15), which is nothing but one-half of that of the Dirac-Wilson fermion. We note that, although this seems almost trivial, the square root of the Dirac determinant in general cannot be expressed as a functional integral of a local action. In particular, it seems impossible to construct a gauge invariant Wilson action for an odd number of chiral fermions in a pseudoreal representation. The expression of the variation of effective action (15) furthermore seems congenial to the Metropolis algorithm, in which the *difference* of the effective action between two gauge field configurations is the basic building block. Thus we propose the use of Eq. (15). We have also established the reality of the variation that is required in the Metropolis algorithm. Of course, since Eq. (15) represents only an infinitesimal change of the effective action, presumably one has to divide a finite variation associated with the update of a link variable into sufficiently many pieces.

Concerning the actual numerical application, we have to investigate also the necessity of the fine-tuning. Although usually the Wilson fermion requires the fine-tuning to restore the chiral symmetry [5], we do not see the necessity in our formula (15): The configuration of the link variable is kept fixed when computing the variation $\Delta[U,\delta U]$ and the original Wilson propagator (5) as it stands is used. Therefore, for us, it seems that the ''back reaction'' of the gauge field dynamics does not modify the above properties.

The overlap formulation [11,12] also possesses desired properties such that the real part of the effective action is gauge invariant and there is no need for the fine-tuning. However, the overlap has the remarkable property [11] that a relation of nontrivial topological gauge field configurations and the fermionic zero mode is explicit. In our approach, an investigation on such a ''global property'' has to be postponed to a future work.

Finally, we comment on the relation to the continuum theory. By parametrizing the link variable as $U_\mu(x) = \exp[iaA_\mu^b(x)T^b]$, the gauge current is defined by

$$\begin{aligned}\langle J^{\mu b}(x)\rangle &\equiv -\frac{\Delta[U,\delta U]}{a^4\,\delta A_\mu^b(x)}\\ &= -\mathrm{tr}\int_0^1 d\beta\, e^{\beta iaA_\mu(x)}T^b e^{-\beta iaA_\mu(x)}\\ &\quad\times\frac{1}{2}\left[\left(\gamma^\mu P_R-\frac{ir}{2}\right)U_\mu(x)G(x+a^\mu,x)\right.\\ &\quad\left.+\left(\gamma^\mu P_R+\frac{ir}{2}\right)G(x,x+a^\mu)U_\mu^\dagger(x)\right].\end{aligned} \tag{24}$$

The fermion one-loop vertex functions are defined accordingly:

$$\langle J^{\mu b}(x)\rangle \equiv \sum_{n=1}^{\infty}\frac{1}{n!}\prod_{j=1}^{n}\left[a^4\sum_{x_j,\mu_j,b_j}A_{\mu_j}^{b_j}(x_j)\int_{-\pi/a}^{\pi/a}\frac{d^4p_j}{(2\pi)^4}\,e^{ip_j(x-x_j)}e^{-iap_{j_{\mu_j}}/2}\right]\Gamma^{\mu\mu_1\cdots\mu_n bb_1\cdots b_n}(p_1,p_2,\ldots,p_n). \tag{25}$$

When a new lattice formulation is proposed, it is important to examine the continuum limit in the perturbative treatment. However, in our formulation, the real part of the gauge current (24) is just one-half of that of the conventional Wilson fermion. Therefore, for the real part, Ward identities associated with the gauge symmetry [5], which are linear relations among vertex functions, trivially hold. Also all the perturbative calculations for the vertex functions of the Wilson fermion can be used by simply dividing by 2. For example, we may use the result of [17] for the vacuum polarization tensor (because of γ_5, the imaginary part does not contribute to this function) to yield

$$\begin{aligned}\lim_{a\to 0}\Gamma^{\mu\nu bc}(p) &= -\frac{1}{24\pi^2}\,\mathrm{tr}T^bT^c(p^\mu p^\nu - g^{\mu\nu}p^2)\\ &\quad\times\left[\ln\frac{4\pi}{-a^2p^2}-\gamma+\frac{5}{3}-12\pi^2L(r)\right],\end{aligned} \tag{26}$$

where the function $L(\lambda)$ is given by Eq. (3.25) of [17].

For the imaginary part of the gauge current (24), our construction (16) is quite faithful to the idea of covariant regularization. For example, using the gauge covariance (11), we can derive Ward identities associated with the gauge invariance at external vertices [4]:

$$p_\nu \lim_{a\to 0}\Gamma^{\mu\nu bc}(p)=0,$$

$$\begin{aligned}&p_\nu\lim_{a\to 0}\Gamma^{\mu\nu\rho bcd}(p,q)+if^{bce}\lim_{a\to 0}\Gamma^{\mu\rho ed}(q)\\ &-if^{cde}\lim_{a\to 0}\Gamma^{\mu\rho be}(p+q)=0,\end{aligned} \tag{27}$$

and so on. Equation (21), on the other hand, shows that we have the covariant gauge anomaly, which completely vanishes for anomaly-free cases without any gauge noninvariant counterterms. Therefore, assuming that the Lorentz covariance is restored, we can expect that the continuum limit of our formulation reproduces all the results of the covariant regularization in the continuum theory.

We thank K. Haga for collaboration in the early stage. We are grateful to Professor K. Fujikawa for discussions and to Professor H. Banerjee and Professor P. Mitra for answering our question on [3]. We are also grateful to Professor S. V. Zenkin for helpful information. The work of H.S. was supported in part by the Ministry of Education Grant-in-Aid for Scientific Researh, Grant Nos. 09740187, 09226203, and 08640348.

APPENDIX

In this appendix we present a relation between the integrability of the imaginary part of $\Delta[U,\delta U]$ and the ''anomaly-free condition'' $\mathcal{A}^b(x)=0$. First we define a quantity

$$K^\mu(x)_{ij}\equiv\frac{i\ \mathrm{Im}\Delta[U,\delta U]}{a^4\delta U_\mu(x)_{ji}}. \tag{A1}$$

We perform the infinitesimal gauge transformation $\delta_\lambda U$ in Eq. (17) on the both sides of this equation. On the left-hand side, the gauge transformation may be generated by a differential operator:

$$G^b(y)\equiv\sum_\nu\left\{-i[T^bU_\nu(y)]_{lk}\frac{\delta}{\delta U_\nu(y)_{lk}}+i[U_\nu(y-a^\nu)T^b]_{lk}\frac{\delta}{\delta U_\nu(y-a^\nu)_{lk}}\right\}. \tag{A2}$$

It is easy to see that $\Sigma_x\lambda^b(x)G^b(x)$ generates the infinitesimal gauge transformation. Then we can cast the gauge variation on the left-hand side into the form

$$\begin{aligned}G^b(y)K^\mu(x)_{ij}=&\frac{\delta}{\delta U_\mu(x)_{ji}}\mathcal{A}^b(y)\\&+i[K^\mu(x)T^b]_{ij}\delta_{x,y}\\&-i[T^bK^\mu(x)]_{ij}\delta_{x+a^\mu,y}\\&+\sum_\nu\{-i[T^bU_\nu(y)]_{lk}\mathcal{R}^{\mu\nu}_{ij,kl}(x,y)\\&+i[U_\nu(y-a^\nu)T^b]_{lk}\mathcal{R}^{\mu\nu}_{ij,kl}(x,y-a^\nu)\}.\end{aligned} \tag{A3}$$

In deriving this identity, we first interchanged the places of ij and kl. This produced a ''functional rotation'' of K:

$$\mathcal{R}^{\mu\nu}_{ij,kl}(x,y)\equiv\frac{\delta K^\mu(x)_{ij}}{\delta U_\nu(y)_{lk}}-\frac{\delta K^\nu(y)_{kl}}{\delta U_\mu(x)_{ji}}. \tag{A4}$$

We then changed the order of the derivative and U. This produced the commutator term in the second line of Eq. (A3).

Now the right-hand side of Eq. (A1) transforms gauge covariantly under the infinitesimal gauge transformation. This can be written as

$$G^b(y)\ \frac{i\ \mathrm{Im}\Delta[U,\delta U]}{a^4\delta U_\mu(x)_{ji}}=i[K^\mu(x)T^b]_{ij}\delta_{x,y}-i[T^bK^\mu(x)]_{ij}\delta_{x+a^\mu,y}. \tag{A5}$$

Therefore, from Eqs. (A3) and (A5) we find

$$\frac{\delta}{\delta U_\mu(x)_{ji}}\mathcal{A}^b(y)=\sum_\nu\{i[T^bU_\nu(y)]_{lk}\mathcal{R}^{\mu\nu}_{ij,kl}(x,y)-i[U_\nu(y-a^\nu)T^b]_{lk}\mathcal{R}^{\mu\nu}_{ij,kl}(x,y-a^\nu)\}. \tag{A6}$$

The right-hand side of this equation can be regarded as the covariant divergence of the functional rotation $\mathcal{R}$. We can interpret this identity from two different viewpoints. First, if the lattice gauge anomaly $\mathcal{A}^b(x)$ vanishes, then the covariant divergence of the functional rotation $\mathcal{R}$ vanishes. A relation similar to Eq. (A6) exists in the continuum theory and when the gauge anomaly is absent, it can be used to show the functional rotation of the covariant gauge current vanishes. This fact was used to show the integrability of the covariant current in anomaly-free cases [3]. In our present lattice case, unfortunately, we could not prove that the corresponding statement that the covariant conservation, (A6) equals 0, implies the vanishing of $\mathcal{R}$. *If* the functional rotation (A4) itself is zero, then Poincaré's lemma may be used to show the (local) integrability of the imaginary part:

$$i\ \mathrm{Im}\Delta[U,\delta U]=\delta\Gamma_2[U]. \tag{A7}$$

On the contrary, if we assume the integrability (A7), we have $\mathcal{R}=0$ and Eq. (A6) shows $\mathcal{A}^b(x)$ is independent of U. However, we can directly compute $\mathcal{A}^b(x)$ for $U=1$ with a finite lattice spacing and find $\mathcal{A}^b(x)=0$ for $U=1$. Consequently, the integrability requires the lattice anomaly-free condition $\mathcal{A}^b(x)=0$.

[1] J. Schwinger, Phys. Rev. **82**, 664 (1951).

[2] K. Fujikawa, Phys. Rev. D **29**, 285 (1984); Nucl. Phys. **B428**, 169 (1994).

[3] H. Banerjee, R. Banerjee, and P. Mitra, Z. Phys. C **32**, 445 (1986).

[4] H. Suzuki, Phys. Rev. D **55**, 2994 (1997).

[5] L. H. Karsten and J. Smit, Nucl. Phys. **B183**, 103 (1981).

[6] H. B. Nielsen and M. Ninomiya, Nucl. Phys. **B185**, 20 (1981).

[7] J. Wess and B. Zumino, Phys. Lett. **37B**, 95 (1971).

[8] See, however, K. Okuyama and H. Suzuki, Phys. Lett. B **382**, 117 (1996), and references therein.

[9] K. G. Wilson, in *New Phenomena in Subnuclear Physics*, edited by A. Zichichi (Plenum, New York, 1977).

[10] C. Rebbi, Phys. Lett. B **186**, 200 (1987).

[11] R. Narayanan and H. Neuberger, Nucl. Phys. **B412**, 574 (1994); **B443**, 305 (1995); **B477**, 521 (1996); Phys. Lett. B **358**, 303 (1995); **380**, 291 (1996); R. Narayanan and P. Vranas, hep-lat/9702005.

[12] S. Aoki and R. B. Levien, Phys. Rev. D **51**, 3790 (1995); S. Randjbar-Daemi and J. Strathdee, Phys. Lett. B **348**, 543 (1995); Phys. Rev. D **51**, 6617 (1995); Nucl. Phys. **B443**, 386 (1995); **B461**, 305 (1996); T. Kawano and Y. Kikukawa, hep-lat/9501032; M. F. L. Golterman and Y. Shamir, Phys. Lett. B **353**, 84 (1995); **359**, 422(E) (1995); C. D. Fosco, Int. J. Mod. Phys. A **11**, 3987 (1996); Y. Kikukawa and S. Miyazaki, Prog. Theor. Phys. **96**, 1189 (1996); Y. Kikukawa,

hep-lat/9705024; N. Maru and J. Nishimura, hep-th/9705152; A. Yamada, hep-lat/9705040.

[13] S. V. Zenkin, Phys. Lett. B **395**, 283 (1997).

[14] G. T. Bodwin and E. V. Kovács, in *Lattice'92*, Proceedings of the International Symposium, Amsterdam, the Netherlands, edited by J. Smit and P. van Baal [Nucl. Phys. B (Proc. Suppl.) **30**, 617 (1993)]; P. Hernándaz and R. Sundrum, Nucl. Phys. **B455**, 287 (1995); **B472**, 334 (1996); G. T. Bodwin, Phys. Rev. D **54**, 6497 (1996).

[15] E. Seiler and I. Stamatescu, Phys. Rev. D **25**, 2177 (1982).

[16] S. D. Pietra, V. D. Pietra, and L. Alvarez-Gaume, Commun. Math. Phys. **109**, 691 (1987); M. Gockeler and G. Schierholz, in *Lattice '92* (Ref. [14]), p. 609, and references therein.

[17] H. Kawai, R. Nakayama, and K. Seo, Nucl. Phys. **B189**, 40 (1981).