Improving the signal-to-noise ratio in lattice gauge theories

Kurt Langfeld, Hugo Reinhardt, and Oliver Tennert
Institut für Theoretische Physik, Universität Tübingen, D-72076 Tübingen, Germany
(Received 28 April 1997)

Renormalization of composite fields is employed to suppress the statistical noise in lattice gauge calculations. We propose a new action which differs from the standard Wilson action by ''irrelevant'' operators, but suppresses the fluctuations of the plaquettes. We numerically study the Creutz ratios and find a scaling window. The SU(2) mass gap is estimated. We prove that the contributions of the ''irrelevant'' operators to the screening mass decrease towards the continuum limit. The results obtained from the action with noise suppression are compared with those of the standard Wilson action. [S0556-2821(97)01723-2]

PACS number(s): 11.10.Gh, 11.15.Ha, 12.38.Gc

I. INTRODUCTION

The present lattice calculations provide the only rigorous approach to low-energy Yang-Mills theories. After the scaling window had been discovered by Creutz in his pioneering work [1], lattice simulations provided the first information on the ratio of the low-lying glueball masses and string tension for the SU(2) [2] and SU(3) [3,4] gauge groups.

Unfortunately, the limited capacity of computers put severe constraints on the accuracy of ''lattice measurements.'' First, the finite number of lattice links correspond to a finite physical volume. Nowadays, a physical volume of $(1.6\text{ fm})^4$ is available for reasonable values of the lattice spacing (see, e.g., [4]). Secondly, the finite number of independent configurations which are employed to calculate the expectation value of the desired operator implies that the ''lattice measurement'' is contaminated with statistical noise.

Weisz and Symanzik have shown that the situation corresponding to the finite size problem can be significantly improved by using an improved action [5]. In the numerical simulation, an effective action is used which already contains corrections from perturbative radiation. In recent years, much work has been devoted to the development of such improved lattice actions, which are often referred to as ''perfect'' lattice actions [6].

In this paper, we will focus on the noise problem. In order to outline the conceptual nature of the noise problem, we briefly review the arguments presented in [7]. Glue-ball (screening) masses m_g are extracted from correlation functions: i.e.,

$$C(t):=\langle \Phi(t)\Phi(0)\rangle \approx \text{const}\times e^{-m_g t}, \tag{1}$$

where the brackets indicate an average over independent lattice configurations. The statistical error of the desired quantity is measured by the standard deviation [7]

$$\langle \Phi(t)\Phi(0)\Phi(t)\Phi(0)\rangle - C^2(t)\approx \langle \Phi^2(0)\rangle. \tag{2}$$

From perturbation theory, one knows that composite operators acquire new divergences implying that the statistical error of the correlation function $C(t)$, i.e., $\sqrt{\langle \Phi^2(0)\rangle}$, diverges in the continuum limit $a\to 0$. The disastrous and fundamental problem therefore is that the signal-to-noise ratio vanishes in the continuum limit.

In the recent past, two concepts have been established to be important in order to improve the signal-to-noise ratio. First, the choice of a nonlocal operator $\Phi(x)$ in Eq. (1) (which nevertheless carries the quantum numbers of the state under investigation) might result in a composite operator $\Phi^2(x)$ which is free of ultraviolet divergences (*smearing* [8]). Secondly, information from the links of a former update step is used to enhance the signal (*fuzzing* [9]). Hybrid algorithms, which combine ''smearing'' and ''fuzzing,'' as well as an estimate of their impact on the signal-to-noise ratio can be found in [7].

In this paper, we propose a new method to improve the signal-to-noise ratio. The method is inspired from the renormalization procedure for composite operators in continuum quantum field theory. The basic idea is to add to the Wilson action an additional term which vanishes faster than the Wilson action in the continuum limit, i.e., we add an ''irrelevant operator,'' but which suppresses the statistical noise. We study the efficiency of our method by calculating the SU(2) mass gap employing the ''old'' idea of plaquette-plaquette correlations.

The paper is organized as follows. In the next section, we briefly review the renormalization of composite operators in continuum quantum field theory. We then discuss the modifications of the Wilson action by ''irrelevant'' composite fields which yield the suppression of the noise. In the third section, we present our numerical results. Conclusions are left to the final section.

II. RENORMALIZATION OF COMPOSITE OPERATORS

A. In continuum quantum field theory

For illustration purposes, we here consider the continuum quantum field theory of a field $\phi(x)$ which is described by the Euclidean partition function

$$\mathcal{Z}[\eta](g)=\int \mathcal{D}\phi(x)\exp\left\{-S[\phi](g)+\int d^4x\,\eta(x)\phi(x)\right\}, \tag{3}$$

where a regularization is understood in order to make Eq. (3) well defined. For simplicity, we assume that the Euclidean action S contains only one parameter g (e.g., the coupling strength). The external source η linearly couples to the field

$\phi(x)$ implying that functional derivatives of $Z[\eta](g)$ with respect to η yield connected Green's functions

$$\langle T\phi(x_1)\cdots\phi(x_N)\rangle. \tag{4}$$

These Green's functions are generically divergent in four space-time dimensions, if the regulator is removed. Renormalized Green's functions are obtained from the generating functional

$$\mathcal{Z}_R[\eta_R](g_R):=\mathcal{Z}[Z_\eta\eta_R](Z_g g_R) \tag{5}$$

by performing the functional derivative with respect to η_R. Z_η and Z_g are the so-called renormalization constants, and η_R is the renormalized source which accounts for field renormalization, and g_R is the renormalized parameter. In order to guarantee that Eq. (5) yields finite Green's functions, we have tacitly assumed that the field theory (3) is multiplicative renormalizable [10], i.e., all divergences can be absorbed in the renormalization constants $Z_{\eta/g}$.

The crucial observation is that, if we allow for composite field insertions, i.e., if we are interested in the limit $x_1\to x_2$ in the renormalized Green's function, new divergences arise in the partition function $Z_R[\eta_R](g_R)$. In order to renormalize these insertions, we generalize

$$\begin{aligned}\mathcal{Z}_R[\eta_R](g_R)\to\mathcal{Z}_R[\eta_R,j_R](g_R)\\ =\int\mathcal{D}\phi(x)\exp\Big\{-S[\phi](Z_r g)\\ +\int d^4x[Z_\eta\eta_R(x)\phi(x)\\ +Z_j j_R(x)\phi^2(x)]\Big\}.\end{aligned} \tag{6}$$

The additional divergences due to the composite field $\phi^2(x)$ can be absorbed in the renormalization constant Z_j.

The dependence of the renormalization constants on the regulator is of course not known *a priori*. Perturbation theory usually provides a systematic way to extract this dependence [10].

In the context of numerical lattice gauge calculations, the question arises of how one should choose the bare source $j(x)$ as function of the lattice spacing in order to renormalize the composite field insertions and therefore to reduce the statistical noise in the continuum limit. In the next subsection, we will suggest a choice for the source term.

B. In lattice gauge calculations

The partition function of SU(2) lattice Yang-Mills theory is defined as a functional integral over the link variables $U_\mu(x)$: i.e.,

$$\mathcal{Z}=\int\mathcal{D}U_\mu\exp\{-S\}, \tag{7}$$

where the standard Wilson action is given by

$$S_W=\sum_{\{x\}\mu\nu}\beta[1-P_{\mu\nu}(x)]. \tag{8}$$

$P_{\mu\nu}$ is the plaquette, which is built from four link variables: i.e.,

$$P_{\mu\nu}(x)=\frac{1}{2}\mathrm{tr}\{U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x)\}. \tag{9}$$

The functional integral (7) is defined on a lattice with lattice spacing a, which serves as the ultraviolet regulator. In the continuum limit $(a\to 0)$, the plaquette is

$$P_{\mu\nu}(x)=1-\frac{a^4}{4}F^a_{\mu\nu}F^a_{\mu\nu}+O(a^6), \tag{10}$$

where $F^a_{\mu\nu}$ is the usual field strength tensor. In this paper, we will confine ourselves to the plaquette-plaquette correlation function $\langle P_{\mu\nu}(x)P_{\alpha\beta}(0)\rangle$ in order to extract the mass gap of the SU(2) lattice theory. From Eq. (2), it is clear that this correlation function is plagued by a statistical noise which diverges in the continuum limit. From the discussions in the last section, it is now evident that one must add a term $\Sigma_{\{x\}\mu\nu}j(x)P^2_{\mu\nu}(x)$ with a suitable choice of $j(x)$ to the action (8) in order to avoid this divergence. We here propose to perform the numerical simulation using the action

$$S=\sum_{\{x\}\mu\nu}\beta[1-P_{\mu\nu}(x)]+\sum_{\{x\}\mu\nu}j[P_{\mu\nu}(x)-\mathcal{A}]^2, \tag{11}$$

where j and $\mathcal{A}$ are constants. In fact, we will choose $\mathcal{A}$ to be the average value of the plaquette $P_{\mu\nu}$, which will be a function of β and j.

Let us study the naive continuum limit of the action S (11). Using Eq. (10), a direct calculation yields (up to a constant)

$$S=[\beta-2j(1-\mathcal{A})]\frac{a^4}{4}F^a_{\mu\nu}F^a_{\mu\nu}+O(a^6). \tag{12}$$

This implies that the action S cannot be distinguished in the naive continuum limit from Wilson's action (8) with an effective parameter $\beta_{\mathrm{eff}}=\beta-2j(1-\mathcal{A})$. In the quantum continuum limit $(\beta\to\infty)$, the average plaquette and effective inverse temperature β_{eff} are approximately given by [1]

$$\mathcal{A}=1-\frac{3}{4\beta},\quad \beta_{\mathrm{eff}}=\beta-\frac{3j}{2\beta}. \tag{13}$$

If j increases less than linearly with β, the results of the quantum theory using S should agree with those which are obtained by employing the Wilson action.

On the other hand, the term in Eq. (11) proportional to j further constrains the plaquette to its average value $\mathcal{A}$ and therefore suppresses statistical fluctuations around the average value of the plaquette. The key point is that the action S (11) differs from the Wilson action by ''irrelevant'' operators which are chosen to suppress the statistical noise of the plaquette. The price one has to pay is that the average plaquette value must already be known at the beginning of the numerical simulation.

From the last subsection, it is clear that the signal-to-noise ratio stays finite in the continuum limit, if j is appropriately chosen. Let us outline, how this choice must be done in practice.

For this purpose, we briefly recall the renormalization of lattice Yang-Mills theory. In the case of the standard Wilson

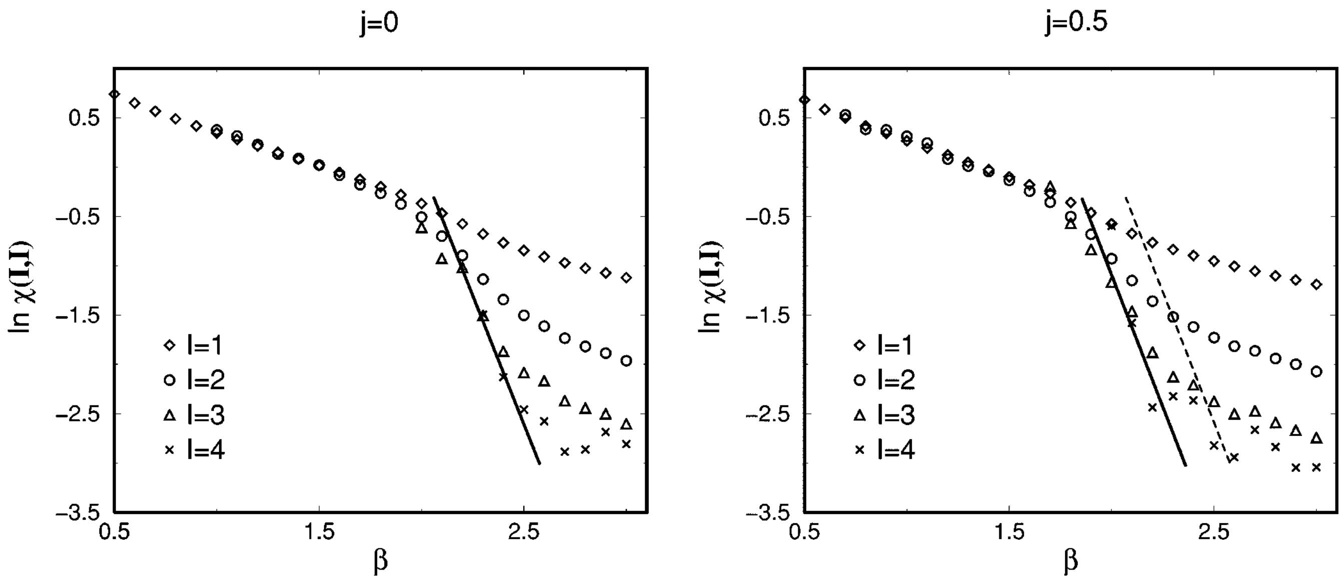

FIG. 1. The Creutz ratios employing the standard Wilson action (left) and using the action S (11) with $j=0.5$ (right). The dashed line in the right-hand picture indicates the perturbative scaling behavior of the case $j=0$.

action, the task is to determine the scale dependence of the coupling constant, which is equivalent to extract the β dependence of the lattice spacing, i.e., $a(\beta)$. This is usually done be calculating κa^2 (κ is the string tension) as a function of β. Choosing a renormalization point $\kappa=440$ MeV, one obtains the ''running'' of the coupling strength from the function $a(\beta)$.

In the case of the action (11) with the noise suppression term, a second renormalization constant, i.e., j, enters the considerations, which must be fixed by an additional renormalization condition. A natural choice is that the signal-to-noise ratio remains nonvanishing, when the continuum limit is approached. Choosing a value for the string tension and a value for the latter ratio determines the regulator (i.e., the lattice spacing) dependence of the parameters β and j. The procedure is possible, since we are dealing with a renormalizable theory.

Numerical simulations deal with finite lattices off the continuum limit implying that the signal-to-noise ratio is nonzero even without noise renormalization. For current standard lattice sizes ($10^4, \ldots, 20^4$ lattice points) and for a SU(2) gauge group, a scaling window $\beta=2, \ldots, 3$ is of interest. For β values smaller than 2, the cutoff, i.e., π/a, becomes too small, whereas for $\beta>3$ finite size effects spoil the lattice measurements. The best that one can do is to suppress the noise for β values within the scaling window. The generic procedure is to choose the tuple (κ,j) consistent with the string tension, and take the value j as large as possible. Note that the ''irrelevant'' terms in the action make a significant contribution to the observables, if j is chosen too large. We will find that $j\in[0,1]$ is a reasonable choice for the SU(2) gauge group and for β in the scaling window. We will show that for $j=1$ a reduction of the noise by a factor of 10 is possible.

III. NUMERICAL RESULTS

A. Creutz ratios

We perform our numerical simulations of the quantum theory employing the action (11) on a 10^4 lattice using the heat bath algorithm by Creutz [1]. Our purpose is to demonstrate the mechanism of noise suppression proposed in the previous sections, rather than to provide new precision measurements. In the latter case, one should resort to ''improved'' actions [5] as well as a larger number of lattice points.

The first task is to calculate the average plaquette $\mathcal{A}$ self-consistently for given values for β and j. We apply the following procedure (we leave it to the reader to develop his own method): before the lattice has reached its thermodynamical equilibrium, we use the lattice average from the previous heat bath step for $\mathcal{A}$. When equilibrium is reached, we calculate the average plaquette taking into account all heat bath steps. In a particular step, we assign the actual value of this average plaquette to $\mathcal{A}$. We then perform a large number of heat bath steps to obtain an accurate value of $\mathcal{A}$, which subsequently enters the numerical calculations of correlation functions, where a smaller number of heat bath steps is sufficient.

The crucial question is whether the scaling limit is reached for finite values of j. In order to answer this question, we calculate the Creutz ratios [1] as a function of β. The left picture of Fig. 1 shows the case $j=0$. These are the Creutz ratios which one obtains using the standard Wilson action. These results are compared with those for $j=0.5$ (in the right picture of Fig. 1). The lines indicate the scaling behavior which has been calculated with the help of the perturbative renormalization group β function. The crucial observation is that the model with nonvanishing source j also approaches the scaling behavior (shown by the lines in Fig. 1) which is predicted by the perturbative renormalization group.

B. Noise suppression

Let us study the efficiency with which the action in Eq. (11) suppresses the statistical error of the average value of the plaquette. For this purpose we consider the probability

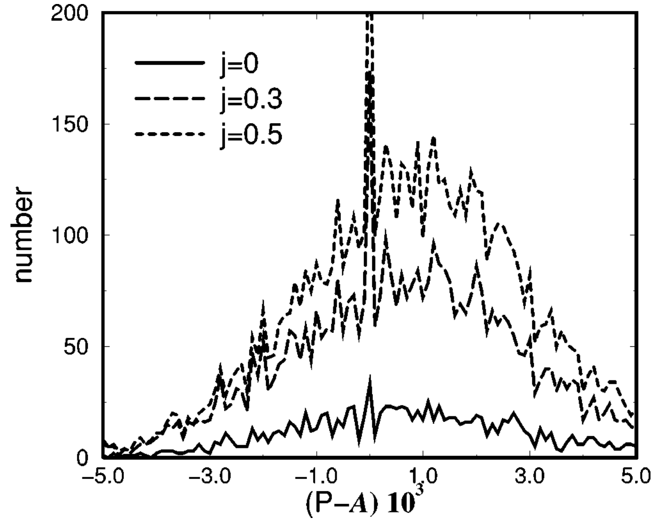

FIG. 2. The distribution of the plaquette around its average value $\mathcal{A}$ for several values of the noise suppression factor j.

distribution of finding a particular value of the plaquette $P_{\mu\nu}$ on a 10^4 lattice in the interval

$$[\mathcal{A}-5\times10^{-3},\mathcal{A}+5\times10^{-3}]. \tag{14}$$

From the numerical point of view, we proceed as follows. We divide the above interval in bins of width 10^{-4}, and calculate the lattice average of the plaquette in a particular heat bath step. We then count the number of average values which correspond to a certain bin. We evaluated 1140 heat bath steps. The numerical result is shown in Fig. 2 for $\beta=2.1$ and $j=0.5$. One clearly observes that the data points are strongly grouped around the corresponding average value $\mathcal{A}$ for large values of the noise suppression factor j. In addition, one observes a sharp peak at $P_{\mu\nu}=\mathcal{A}$. Whether this peak is an artifact due to corrections to the action of order a^8 or whether the peak is necessary to reproduce the standard Yang-Mills action (12) in the scaling limit $\beta\to\infty$, is not clear to us.

In order to quantify the suppression of the noise of the plaquette, using a 4^4 lattice in the numerical calculation is convenient, since fluctuations are large due to finite size effects and since the numerical simulation is fast. We used $\beta=2.2$. The results are summarized in Table I.

n is the number of heat bath steps which were employed to calculate the average plaquette. For each value of n, five

TABLE I. The relative error ϵ of the plaquette value for several values of the noise suppresion factor j.

n	j	ϵ $[10^{-3}]$	n	j	ϵ $[10^{-3}]$
100	0	5.9	300	0	3.9
100	0.5	1.5	300	0.5	0.81
100	1	0.84	300	1	0.21
200	0	4.5	400	0	2.0
200	0.5	0.71	400	0.5	0.46
200	1	0.59	400	1	0.23

independent runs were performed to estimate the relative error ϵ of the plaquette value, i.e., $\epsilon:=\delta\mathcal{A}/\mathcal{A}$. One observes that ϵ decreases with increasing n. This is simply due to the improved statistics. In addition, a reduction of the noise by a factor of 10 is feasible at $j=1$ compared with the case without noise suppression $(j=0)$.

C. The SU(2) mass gap

In this subsection, we will numerically estimate the SU(2) mass gap from the plaquette-plaquette correlation function in order to demonstrate how the action (11) works in practice. The purpose of this subsection is twofold. First, we want to show that the mass gap obtained here is in agreement with the high precision measurements [2] which employ high statistics and an improvement of the signal-to-noise ratio using the fuzzing technique. Secondly, we will compare the results at several values of β in order to estimate the contribution of the ''irrelevant'' operators off the continuum limit. We are aware of the problem that the overlap of the plaquette with glue-ball wave function is small [11]. For high precision measurements, one should therefore employ nonlocal operators. Performing the noise suppression for the case of these operators, however, might be numerically costly. For these first investigations, we therefore confine ourselves to the study of correlations of the plaquette.

Furthermore, we use the source method [12] to calculate the correlation function of the plaquettes. For this purpose, we estimate the average plaquette at the origin from

$$W[\eta]=\frac{\int\mathcal{D}U_\mu\Sigma_{\mu\nu}P_{\mu\nu}(0)\exp\{-S+\Sigma_{\{x\}\mu\nu}\eta_xP_{\mu\nu}(x)\}}{\int\mathcal{D}U_\mu\exp\{-S+\Sigma_{\{x\}\mu\nu}\eta_xP_{\mu\nu}(x)\}}, \tag{15}$$

where S is the action (11). Let $P(x)$ denote $\Sigma_{\mu\nu}P_{\mu\nu}(x)$. It is straightforward to verify that the functional derivative of $W[\eta]$ with respect to the source $\eta(x)$ yields the desired correlation function: i.e.,

$$C(t)=\frac{\delta W[\eta]}{\delta\eta(x)}\bigg|_{\eta=0}=\langle P(x)P(0)\rangle-\langle P(x)\rangle\langle P(0)\rangle. \tag{16}$$

In practice, we are interested in the correlation of the plaquette in time direction implying that one chooses $\eta(x)=\eta(t)$. In fact, one simulates two statistical ensembles. One ensemble is generated with the inverse temperature, the other is obtained by setting the inverse temperature of the time slice $t=0$ to $\beta+\eta$ leaving the remaining β values unchanged [12]. In both ensembles, the average plaquette is obtained as a function of time. The correlation function (16) is obtained by approximating the functional derivative in Eq. (16) by the difference of the expectation values of the plaquette of the two ensembles.

We use 1140 heat bath steps to extract the average plaquette in both ensembles. A typical result for the correlation function as a function of time is shown in Fig. 3, where the noise suppression factor is set to $j=0.5$. The inverse

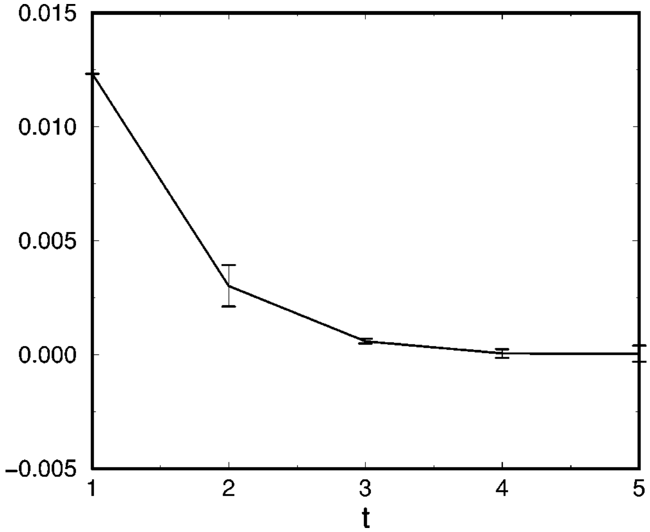

FIG. 3. The correlation function (16) as a function of time t in units of lattice spacings for $\beta=1.9$ and $j=0.5$.

temperature is $\beta=2.1$ guaranteeing that the systems are in the scaling region (see left picture of Fig. 1). One clearly observes an exponential decay of the correlation, where the slope of $\ln C(t)$ provides the SU(2) mass gap in units of the lattice spacing.

It is interesting to compare the value of the mass gap, obtained with and without noise suppression, for β values corresponding to the middle and the onset of the scaling window, respectively. In the latter case, the contribution of ''irrelevant operators'' to the mass gap should be more pronounced compared with the former case. The numerical results are summarized in Table II.

The error bars indicate the uncertainty due to statistical fluctuations. They are extracted from the fit of the correlation function $C(t)$ to the function const$\times \exp\{-mt\}$. The string tension κ sets the scale. We use $\kappa=440$ MeV. L is the extension of our lattice in each direction. One observes that the discrepancy between the values $m/\sqrt{\kappa}$ with and without noise suppression decreases, if the ensemble turns towards the continuum limit, i.e., κa^2 decreases. This shows that the influence of the ''irrelevant'' operators, which distinguishes our action (16) from the standard Wilson action diminishes. Note, however, that at physical lattice sizes where the influence of the ''irrelevant'' operators is small, finite size effects might play a role. This would imply that a large number of lattice points (here we use 10^4 lattice points) is necessary for high precision measurements of the SU(2) mass gap. The numerical result for the mass gap in the scaling region is in agreement with the results of [2].

IV. CONCLUSIONS

We have shown that constraining the plaquette to its average value can be understood as composite field renormalization. The parameter j, which enters the action, acts as a renormalization constant absorbing the divergences arising from the composite nature of fields. These divergences are responsible for the small signal-to-noise ratio in correlation functions of plaquettes. The action (11) with noise suppression differs from the standard Wilson action up to a shift in β only by ''irrelevant operators''; i.e., both actions coincide in the naive continuum limit.

TABLE II. The screening mass m of the plaquette-plaquette correlation for several values of (β,j).

β	j	κa^2	L	ma	$m/\sqrt{\kappa}$
2.1	0	0.577	3.5 fm	1.40 ± 0.14	1.85 ± 0.2
1.9	0.5	0.577	3.5 fm	1.56 ± 0.23	2.06 ± 0.3
2.3	0	0.21	2.1 fm	1.36 ± 0.03	2.97 ± 0.06
2.1	0.5	0.21	2.1 fm	1.46 ± 0.06	3.25 ± 0.13
2.4	0	0.126	1.6 fm	1.43 ± 0.08	4.04 ± 0.3
2.2	0.5	0.115	1.5 fm	1.41 ± 0.07	4.14 ± 0.2

Adopting a conceptual point of view, we have outlined two renormalization conditions which yield the scale dependence of the parameters β and j. Since Yang-Mills theories are renormalizable, the adjustment of the parameters guarantees a nonzero signal-to-noise ratio in the continuum limit.

In practice, numerical simulations use finite lattices and finite values of β implying that the signal-to-noise ratio is finite, too. In this case, the conceptual results can be used to suppress the noise for the finite range of β values under considerations. We have numerically studied the new action (11) on coarse grained lattices consisting of 4^4 and 10^4 lattice points. The Creutz ratios from the numerical data with and without noise suppression show a scaling window in both cases. It turns out that choosing a constant value for j for β lying in the scaling window ($\beta\in[2,3]$ for our present case) is reasonable and suppresses the noise up to a factor of 10. Other choices of the β dependence of j are possible and perhaps more convenient depending on the type of correlation function which is under consideration.

The SU(2) mass gap has been estimated from the plaquette-plaquette correlation function. We have found that the contributions from the ''irrelevant'' operators to the screening mass decrease with increasing values of β. The goal of the noise suppression in this case has mainly been the reduction of the statistical fluctuations of the ''background,'' on top of which the signal exists.

For high precision measurements of glue-ball masses, one should use correlation functions of operators which have a larger overlap with the glue-ball wave function than the plaquettes. In addition, ''perfect'' actions will help to extrapolate to the continuum limit. A generalization of the noise suppression introduced in the present paper to either case seems feasible. In the case of the ''perfect'' actions, one has to ensure that the noise suppression term does not spoil the correct ultraviolet behavior exploited by the ''perfect'' action techique.

ACKNOWLEDGMENT

This work was supported in part by DFG under contract Re 856/1–3.

[1] M. Creutz, Phys. Rev. Lett. **45**, 313 (1980); Phys. Rev. D **21**, 2308 (1980).

[2] B. Carpenter, C. Michael, and M. J. Teper, Phys. Lett. B **198**, 511 (1987); C. Michael and M. J. Teper, Nucl. Phys. **B305**, 453 (1988).

[3] B. A. Berg, A. H. Billoire, and C. Vohwinkel, Phys. Rev. Lett. **57**, 400 (1986); M. Falcioni, M. L. Paciello, G. Parisi, and B. Taglienti, Nucl. Phys. **B251**, 624 (1985); G. Schierholz, in *Lattice '88*, Proceedings of the International Symposium, Batavia, Illinois, edited by A. S. Kronfeld and P. B. Mackenzie [Nucl. Phys. B (Proc. Suppl.) **9**, 244 (1989)].

[4] SESAM Collaboration, G. S. Bali *et al.*, in *Lattice '96*, Proceedings of the International Symposium, St. Louis, Missouri, edited by C. Bernard *et al.*, [Nucl. Phys. B (Proc. Suppl.) **53**, 239 (1997)].

[5] P. Weisz, Nucl. Phys. **B212**, 1 (1983); K. Symanzik, *ibid.* **B226**, 187 (1983); S. Belforte, G. Curci, P. Menotti, and G. P. Paffuti, Phys. Lett. **131B**, 423 (1983); B. Berg, A. Billoire, S. Meyer, and C. Panagiotakopoulos, *ibid.* **133B**, 359 (1983); R. Gupta and A. Patel, Phys. Rev. Lett. **53**, 531 (1984); S. Itoh, Y. Iwasaki, and T. Yoshie, Nucl. Phys. **B250**, 312 (1985).

[6] P. Hasenfratz and F. Niedermayer, Nucl. Phys. **B414**, 785 (1994); W. Bietenholz and U. J. Wiese, *ibid.* **B464**, 319 (1996).

[7] F. Brandstäter, A. S. Kronfeld, and G. Schierholz, Nucl. Phys. **B345**, 709 (1990).

[8] Ape Collaboration, M. Albanese *et al.*, Phys. Lett. B **192**, 163 (1987); **205**, 535 (1988).

[9] M. Teper, Phys. Lett. B **183**, 345 (1986).

[10] See, e.g., Ta-Pei Cheng and Ling-Fong Li, *Gauge Theory of Elementary Particle Physics* (Oxford University Press, New York, 1984); F. J. Yndurain, *Quantum Chromodynamics* (Springer Verlag, Berlin, 1983).

[11] P. de Forcrand, Z. Phys. C **16**, 87 (1986).

[12] M. Falcioni, E. Marinari, M. L. Paciello, G. Parisi, B. Taglienti, and Z. Yi-Cheng, Nucl. Phys. **B215** [**FS7**], 265 (1983); M. Falcioni, M. L. Paciello, G. Parisi, and B. Taglienti, *ibid.* **B251** [**FS13**], 624 (1985).