O(N) quantum fields in curved spacetime

S. A. Ramsey* and B. L. Hu†
Department of Physics, University of Maryland, College Park, Maryland 20742-4111
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For the O(N) field theory with $\lambda\Phi^4$ self-coupling, we construct the two-particle-irreducible (2PI) closed-time-path (CTP) effective action in a general curved spacetime. From this we derive a set of coupled equations for the mean field and its variance. They are useful for studying the nonperturbative, nonequilibrium dynamics of a quantum field when full back reactions of the quantum field on the curved spacetime, as well as the fluctuations on the mean field, are required. Applications to phase transitions in the early Universe such as at the Planck scale or in the reheating phase of chaotic inflation are under investigation. [S0556-2821(97)05814-1]

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I. INTRODUCTION

One major direction of research on quantum field theory in curved spacetime [1–3] since the 1980s has been the application of interacting quantum fields to the consideration of symmetry breaking and phase transitions in the early universe, from the Planck to the grand unified energy scales [4–12]. In a series of work, Hu, O'Connor, Shen, Sinha, and Stylianopoulos [13–21] systematically investigated the effect of spacetime curvature, dynamics, and finite temperature in causing a symmetry restoration of interacting quantum fields in curved spacetime. In general one wants to see how quantum fluctuations φ around a mean field $\hat{\phi}$ change as a function of these parameters. For this purpose, the two-particle-irreducible (2PI) effective action was constructed for an N-component scalar O(N) model with quartic interaction [14,10,19]. Hu and O'Connor [19] found that the spectrum of the small-fluctuation operator contains interesting information concerning how infrared behavior of the system depends on the geometry and topology. The equation for $\hat{\phi}$ containing contributions from the variance of the fluctuation field $\langle\varphi^2\rangle$ depicts how the mean field evolves in time. This program explored two of the three essential elements of an investigation of a phase transition [16], the geometry and topology and the field theory and infrared behavior aspects, but not the nonequilibrium statistical-mechanical aspect.

For this and other reasons, Calzetta and Hu [22] started exploring the closed-time-path (CTP) or Schwinger-Keldysh formalism [23–26], which is formulated with an ''in-in'' boundary condition. Because the CTP effective action produces a real and causal equation of motion [27,28], it is well suited for particle production back-reaction problems [29–31]. Use of the CTP formalism in conjunction with the 2PI effective action [32] and the Wigner function [33] enabled Calzetta and Hu to construct a quantum kinetic field theory (in flat spacetime), deriving the Boltzmann field equation from first principles [34]. The necessary ingredients were then in place for an analysis of nonequilibrium phase transitions [35]. In recent years these tools (CTP, 2PI) have indeed been applied to the problems of heavy-ion collisions, pair production in strong electric fields [36], disoriented chiral condensates [37,38], and reheating in inflationary cosmology [39]. However, none of these recent works has included curved spacetime effects in a self-consistent manner, where the spacetime governs the evolution of a quantum field and is, in turn, governed by the quantum field dynamics. This is especially important for Planck scale processes involving quantum fluctuations with back reaction, such as particle creation [40], galaxy formation [41], preheating, and thermalization in chaotic inflation [42,43].

With this paper we return to the problems begun by Calzetta, Hu, and O'Connor a decade ago. We wish to derive the coupled equations for the evolution of the mean field and its variance for the O(N) model in curved spacetime, which should provide a solid and versatile platform for studies of phase transitions in the early universe. The first order of business is to construct the CTP-2PI effective action in a general curved spacetime. The evolution equations are derived from it. We must also deal with the divergences arising in it. From the vantage point of the correlation hierarchy (and the associated master effective action) as applied to a nonequilibrium quantum field [44], there is *a priori* no reason why one should stop at the 2PI effective action. Indeed, the 2PI effective action corresponds to a further approximation from the two-loop truncation of the master effective action constructed from the full Schwinger-Dyson hierarchy [45,44]. For problems where the mean field and the two-point function give an adequate description (which is not the case near the critical point, where one has to be careful), the CTP-2PI effective action is sufficient. In particular, the 2PI effective action contains the commonly used large-N, time-dependent Hartree-Fock, and one-loop approximations.

The O(N) model has been usefully applied to a great variety of problems in field theory and statistical mechanics [46]. The O(N) field theory has the advantage that it affords use of the $1/N$ expansion [32,36], which yields nonperturbative evolution equations in the regime of strong mean field (it yields local, coupled dynamical equations for the mean field and the mode functions of the fluctuation field). Recently it has been applied to problems of nonequilibrium phase transitions [47–49]. In the preheating problem studied in the

*Electronic address: sramsey@physics.umd.edu
†Electronic address: hu@umdhep.umd.edu

following paper [50], we shall see that this is particularly important for chaotic inflation scenarios [51], in which the inflaton mean-field amplitude can be on the order of the Planck mass at the end of the slow-roll period [52,53]. The $1/N$ expansion has many attractive features, as it is known to preserve the Ward identities of the O(N) model [54] and to yield a covariantly conserved energy-momentum tensor [55]. Furthermore, in the limit of large N, the quantum effective action for the matter fields can be interpreted as a leading-order term in the expansion of the full (matter plus gravity) quantum effective action [55].

Mazzitelli and Paz [72] have studied the $\lambda\Phi^4$ and O(N) field theories in a general curved spacetime in the Gaussian and large-N approximations, respectively. Their approach differs from ours in that it is based on a Gaussian factorization which does not permit systematic improvement either in the loop expansion or in the $1/N$ approximation. In contrast, our approach is based on a closed-time-path formulation of the correlation dynamics, and the evolution equation we obtain for the two-point function contains a two-loop radiative damping contribution which is not present in the large-N approximation. At leading order in the large-N approximation, our results agree with theirs, so that their renormalization counterterms can be directly applied to the mean field and gap equations derived here.

This paper is organized as follows. Sections II and III present self-contained summaries of the two essential theoretical methodologies employed in this study, the closed-time-path formalism and the two-particle-irreducible effective action. The adaptation of these tools to the quantum dynamics of a $\lambda\Phi^4$ field theory in curved spacetime is presented in Sec. IV. The O(N) scalar field theory is treated in Sec. V.

Throughout this paper we use units in which $c=1$. Planck's constant $\hbar$ is shown explicitly (i.e., not set equal to 1) except in those sections where noted. In these units, Newton's constant is $G=\hbar M_{\rm P}^{-2}$, where $M_{\rm P}$ is the Planck mass. We work with a four-dimensional spacetime manifold, and follow the sign conventions[1] of Birrell and Davies [2] for the metric tensor $g_{\mu\nu}$, the Riemann curvature tensor $R_{\mu\nu\sigma\rho}$, and the Einstein tensor $G_{\mu\nu}$. We use Greek letters to denote spacetime indices. The beginning Latin letters a,b,c,d,e,f are used as time branch indices (see Sec. II), and the middle Latin letters i,j,k,l,m,n are used as indices in the O(N) space (see Sec. V). The Einstein summation convention over repeated indices is employed. Covariant differentiation is denoted with a nabla ∇_μ or a semicolon.

II. SCHWINGER-KELDYSH FORMALISM

The Schwinger-Keldysh or ''closed-time-path'' (CTP) formalism is a powerful method for deriving real and causal evolution equations for expectation values of quantum operators for nonequilibrium fields, i.e., for quantum systems where the density matrix $\boldsymbol{\rho}$ and the Hamiltonian H do not commute, $[H,\boldsymbol{\rho}]\neq 0$. This can occur, for example, in a field theory quantized on a dynamical background spacetime and also in an interacting field theory with nonequilibrium initial conditions. The methods discussed here are well suited to studying the dynamics of an open quantum system. Excellent reviews of the Schwinger-Keldysh method are Zhou *et al.* [26] as applied to nonequilibrium quantum field theory and Calzetta and Hu [22] as applied to back reaction in semiclassical gravity. In this section we briefly review the Schwinger-Keldysh method in the context of an interacting scalar field theory in Minkowski space, with vacuum boundary conditions.

Consider a scalar field Φ in Minkowski space with a $\lambda\Phi^4$ self-interaction. Studying the semiclassical properties of the theory consists of taking the degrees of freedom to be the classical field $\hat{\phi}$ and fluctuations φ about the classical field configuration. The equation of motion for small oscillations of $\hat{\phi}$ about the stable quantum-corrected equilibrium configuration is obtained via a variational principle from the effective action $\Gamma[\hat{\phi}]$ [58]. In the conventional Schwinger-DeWitt or ''in-out'' approach [1,57], one couples a c-number source J (which is a function on $\mathbb{M}^4$) to the field ϕ and computes the vacuum persistence amplitude in the presence of the source J. This amplitude has a path integral representation

$$Z[J]=\exp\left(\frac{i}{\hbar}W[J]\right)=\int D\phi\exp\left[\frac{i}{\hbar}\left(S^{\rm F}[\phi]+\int d^4x J(x)\phi(x)\right)\right], \tag{2.1}$$

where the functional integral is a sum over classical histories of the ϕ field which are pure negative frequency [i.e., all spatial Fourier modes of ϕ have a time dependence like $\exp(i\omega t),\omega>0$] in the asymptotic past and pure positive frequency [$\sim\exp(-i\omega t)$] in the asymptotic future.[2]

In a general nonequilibrium setting, such as in a curved or dynamical spacetime or when $\hat{\phi}$ is time dependent, the notion of positive frequency in the asymptotic past is in general different from that in the asymptotic future. Hence, the ''in'' vacuum state $|0,\text{in}\rangle$ defined at $x^0=-\infty$ and the ''out'' vacuum state $|0,\text{out}\rangle$ defined at $x^0=\infty$ are not necessarily equivalent. The generating functional $Z[J]$ defined in Eq. (2.1) is then the vacuum persistence amplitude

$$\langle 0,\text{out}|0,\text{in}\rangle_J=\langle 0,\text{out}|T\exp\left(\frac{i}{\hbar}\int d^4x J(x)\Phi_{\rm H}(x)\right)|0,\text{in}\rangle, \tag{2.2}$$

where $\Phi_{\rm H}(x)$ is the Heisenberg field operator for the theory without the source J. This amplitude is in general complex. It follows that the classical field obtained by functional differentiation of $-i\hbar\ln Z[J]$ is the matrix element $\langle 0,\text{out}|\Phi_{\rm H}|0,\text{in}\rangle$ which will in general be complex. In addition, the dependence of $\hat{\phi}_J\equiv\delta W/\delta J$ on J will not, in gen-

[1]In the classification scheme of Misner, Thorne, and Wheeler [56], the sign convention of Birrell and Davies [2] is classified as $(+,+,+)$.

[2]It is noted that these boundary conditions on the functional integral are equivalent (up to an overall normalization) to adding a small imaginary term $-i\epsilon\phi^2$ to the classical action, where $\epsilon>0$.

eral, be causal [28,27]. In curved spacetime, the energy-momentum tensor $\langle T_{\mu\nu}\rangle$ is obtained by functional differentiation of W with respect to $g^{\mu\nu}$, which at one loop yields a complex matrix element of $T_{\mu\nu}(\Phi_{\rm H})$ between the "in" and "out" vacua, where $\Phi_{\rm H}$ is the Heisenberg field operator and $T_{\mu\nu}(\phi)$ is the classical energy-momentum tensor for the field [1].

In the closed-time-path formalism, real and causal dynamics for $\hat{\phi}$ can be obtained, as well as the expectation value of the energy-momentum tensor. Let $x^0=x^0_\star$ be far to the future of any dynamics we wish to study. It is not necessary to assume that $\lambda=0$ or that the Hamiltonian is time independent at $x^0=x^0_\star$. As in the previous "in-out" approach, suppose we wish to compute the quantum-corrected equation governing the classical field $\hat{\phi}$. Let $M=\{(x^0,\vec{x})|-\infty\leqslant x^0\leqslant x^0_\star\}$ be the portion of Minkowski space to the past of time $x^0_\star$. We start by defining a new manifold as a quotient space,

$$\mathcal{M}=(M\times\{+,-\})/\sim, \tag{2.3}$$

where $\sim$ is an equivalence relation defined by the rules

$$\begin{aligned}(x,+)&\sim(x',+)\quad \text{if } x=x',\\ (x,-)&\sim(x',-)\quad \text{if } x=x', \\ (x,+)&\sim(x',-)\quad \text{if } x=x' \text{ and } x^0=x^0_\star.\end{aligned} \tag{2.4}$$

The manifold $\mathcal{M}$ is orientable, provided we reverse the sign of the volume form between the + and − pieces of the manifold. It is then straightforward to generalize the usual effective action construction to the new manifold $\mathcal{M}$. With the volume form on $\mathcal{M}$, we can generalize the classical action $S^{\rm F}$ to $\mathcal{M}$,

$$\mathcal{S}^{\rm F}[\phi_+,\phi_-]=S^{\rm F}[\phi_+]-S^{\rm F}[\phi_-], \tag{2.5}$$

where $S^{\rm F}[\phi]$ is the classical action on M, and ϕ_+ and ϕ_- denote the ϕ field on the + and − branches of $\mathcal{M}$, respectively. The spacetime integrations in the right-hand side of Eq. (2.5) are understood to be over M. In order for $\phi_\pm$ to be a function on $\mathcal{M}$, we must have

$$\phi_+(x)|_{x^0_\star}=\phi_-(x)|_{x^0_\star}. \tag{2.6}$$

The generating functional of vacuum n-point functions (i.e., expectation values in the $|0,\text{in}\rangle$ vacuum) for this theory is then defined by

$$Z[J_+,J_-]=\int_{\rm ctp} D\phi_+ D\phi_- \exp\left[\frac{i}{\hbar}\left(\mathcal{S}^{\rm F}[\phi_+,\phi_-] + \int_M d^4x(J_+\phi_+ - J_-\phi_-)\right)\right], \tag{2.7}$$

where J_+ and J_- are c-number sources on the + and − branches of $\mathcal{M}$, respectively. The designation "ctp" indicates that the functional integrals in Eq. (2.7) are over all field configurations (ϕ_+,ϕ_-) such that (i) $\phi_+=\phi_-$ at the $x^0=x^0_\star$ hypersurface and (ii) ϕ_+ (ϕ_-) consists of pure negative (positive) frequency modes at $x^0=-\infty$. It is not necessary for the normal derivatives of ϕ_+ and ϕ_- to be equal at $x^0=x^0_\star$. Because the theory is free in the asymptotic past, a positive frequency mode[3] is a solution to the spatial-Fourier transformed Euler-Lagrange equation for ϕ whose asymptotic behavior at $x^0=-\infty$ is $\exp(-i\omega x^0)$, for $\omega>0$.

The generating functional for connected diagrams is then defined by

$$W[J_+,J_-]=-i\hbar\,\ln Z[J_+,J_-]. \tag{2.8}$$

Classical fields on both + and − branches are then defined as

$$\hat{\phi}_a(x)_{J_\pm}=c^{ab}\frac{\delta W[J_+,J_-]}{\delta J_b(x)}, \tag{2.9}$$

where a,b are time branch indices with index set $\{+,-\}$. The matrix c^{ab} is defined by $c^{++}=1, c^{--}=-1$, and $c^{+-}=c^{-+}=0$. The functional differentiation in Eq. (2.9) is carried out with variations in δJ_+ and δJ_- which satisfy the constraint that $\delta J_+=\delta J_-$ on the $x^0=x^0_\star$ hypersurface. The $J_\pm$ subscript in Eq. (2.9) indicates the functional dependence on $J_\pm$, which has been shown to be causal [28,27]. In the limit $J_+=J_-\equiv J$, the classical fields on the + and − time branches become equal,

$$(\hat{\phi}_+(x)_{J_\pm})|_{J_+=J_-\equiv J}=(\hat{\phi}_-(x)_{J_\pm})|_{J_+=J_-\equiv J} \equiv\hat{\phi}(x)_J={}_J\langle 0,\text{in}|\Phi_{\rm H}(x)|0,\text{in}\rangle_J, \tag{2.10}$$

where $|0,\text{in}\rangle$ is the state which has evolved from the vacuum at t_0 under the interaction $\Phi_{\rm H}J$, and becomes the expectation value $\langle\Phi_{\rm H}\rangle$ in the limit $J=0$. The effective action is defined via the usual Legendre transform, with c^{ab} now acting as a "metric" on the internal 2×2 CTP field space,

$$\Gamma[\hat{\phi}_+,\hat{\phi}_-]=W[J_+,J_-]-c^{ab}\int_M d^4x J_a(x)\hat{\phi}_b(x), \tag{2.11}$$

where the J subscripts on $\hat{\phi}_\pm$ are suppressed and the functional dependence of $J_\pm$ on $\hat{\phi}$ via inversion of Eq. (2.9) is understood. By direct computation, the inverse of Eq. (2.9) is found to be

$$J_a(x)_{\hat{\phi}_\pm}=-c_{ab}\frac{\delta\Gamma[\hat{\phi}_+,\hat{\phi}_-]}{\delta\hat{\phi}_b(x)}, \tag{2.12}$$

where we have indicated the explicit functional dependence of $J_\pm$ on $\hat{\phi}_\pm$ with a subscript, and c_{ab} is the inverse of the

[3]Here, the choice of vacuum boundary conditions corresponds to adding a small imaginary part $i\epsilon(\phi_+{}^2-\phi_-{}^2)$ to the classical action $\mathcal{S}^{\rm F}$. Alternatively, the boundary conditions correspond to the usual prescription $m^2\to m^2-i\epsilon$ in $S^{\rm F}[\phi]$, but with $S_{\rm ctp}$ now redefined as $\mathcal{S}^{\rm F}[\phi_+,\phi_-]=S^{\rm F}[\phi_+]-S^{\rm F}[\phi_-]^\star$, where $\star$ denotes complex conjugation [22].

matrix c^{ab} defined above. In the limit $\hat{\phi}_+ = \hat{\phi}_- \equiv \hat{\phi}$, this yields the evolution equation for the expectation value ${}_J\langle \Phi_{\rm H} \rangle_J \equiv \hat{\phi}_J$ in the state which has evolved from $|0, \text{in}\rangle$ under the source interaction $J\Phi_{\rm H}$. The evolution equation for $\hat{\phi}$, the vacuum expectation value $\langle 0, \text{in}|\Phi_{\rm H}|0, \text{in}\rangle$, is therefore

$$\left.\frac{\delta\Gamma[\hat{\phi}_+, \hat{\phi}_-]}{\delta \hat{\phi}_+}\right|_{\hat{\phi}_+ = \hat{\phi}_- \equiv \hat{\phi}} = -\left.\frac{\delta\Gamma[\hat{\phi}_+, \hat{\phi}_-]}{\delta \hat{\phi}_-}\right|_{\hat{\phi}_+ = \hat{\phi}_- \equiv \hat{\phi}} = 0. \tag{2.13}$$

Using Eqs. (2.12) and (2.11), an integro-differential equation for Γ can be derived [28], in which the functional differentiations of Γ with respect to $\hat{\phi}_\pm$ are carried out with the constraint that the variations of $\hat{\phi}_\pm$ satisfy $\delta\hat{\phi}_+ = \delta\hat{\phi}_-$ when $x^0 = x^0_\star$. The difference $\phi_a - \hat{\phi}_a$ is naturally interpreted as the fluctuations of a particular history ϕ_a about the ''classical'' field configuration $\hat{\phi}_a$. Let us therefore define the *fluctuation field* $\varphi_a \equiv \phi_a - \hat{\phi}_a$ or, in terms of Heisenberg field operators,

$$\varphi_{\rm H} \equiv \Phi_{\rm H} - \langle \Phi_{\rm H} \rangle = \Phi_{\rm H} - \hat{\phi}, \tag{2.14}$$

where angular brackets around the Heisenberg field operator $\Phi_{\rm H}$ denote an expectation value of $\Phi_{\rm H}$ in the (time-independent) quantum state of the system. Performing the change of variables $\phi_a \to \varphi_a$ in the functional integral, where

$$\varphi_a \equiv \phi_a - \hat{\phi}_a, \tag{2.15}$$

we obtain

$$\Gamma[\hat{\phi}_+, \hat{\phi}_-] = -i\hbar \ln\left\{ \int_{\rm ctp} D\varphi_+ D\varphi_- \times \exp\left[\frac{i}{\hbar}\left(\mathcal{S}^{\rm F}[\hat{\phi}_+ + \varphi_+, \hat{\phi}_- + \varphi_-] - \frac{\delta\Gamma[\hat{\phi}_+, \hat{\phi}_-]}{\delta\hat{\phi}_a}\varphi_a \right)\right]\right\}. \tag{2.16}$$

This functional integro-differential equation has a formal solution [58]

$$\Gamma[\hat{\phi}_+, \hat{\phi}_-] = \mathcal{S}^{\rm F}[\hat{\phi}_+, \hat{\phi}_-] - \frac{i\hbar}{2}\ln\det(\mathcal{A}^{-1}_{ab}) + \Gamma_1[\hat{\phi}_+, \hat{\phi}_-], \tag{2.17}$$

where $\mathcal{A}^{ab}(x, x')$, the second functional derivative of the classical action with respect to the field $\phi_\pm$, is

$$i\mathcal{A}^{ab}(x, x') = \frac{\delta^2 \mathcal{S}^{\rm F}}{\delta\phi_a(x)\phi_b(x')}[\hat{\phi}_+, \hat{\phi}_-]. \tag{2.18}$$

The inverse of $\mathcal{A}^{ab}$ is the one-loop propagator for the fluctuation field ϕ. The functional Γ_1 in Eq. (2.16) is defined as $-i\hbar$ times the sum of all one-particle-irreducible vacuum-to-vacuum graphs with propagator given by $\mathcal{A}^{-1}_{ab}(x, x')$ and vertices given by a ''shifted action'' $\mathcal{S}^{\rm F}_{\rm int}$, defined by

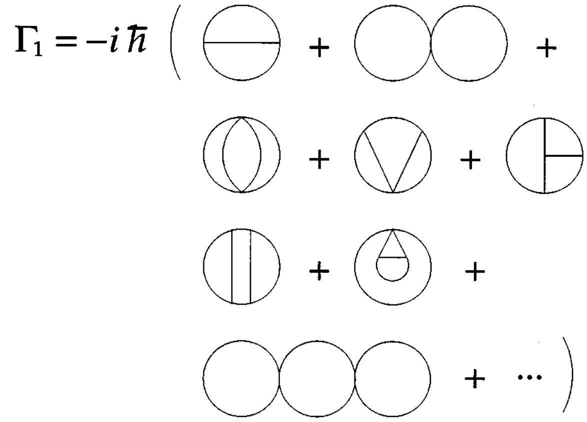

FIG. 1. Diagrammatic expansion for Γ_1. Lines represent the propagator $\mathcal{A}^{-1}_{ab}(x,x')$, and vertices terminating three lines are proportional to $\hat{\phi}$. Each vertex carries spacetime (x) and CTP $(+,-)$ labels.

$$\mathcal{S}^{\rm F}_{\rm int}[\varphi_+, \varphi_-] = \mathcal{S}^{\rm F}[\varphi_+ + \hat{\phi}_+, \varphi_- + \hat{\phi}_-] - \mathcal{S}^{\rm F}[\hat{\phi}_+, \hat{\phi}_-] - \int_M d^4x \left(\frac{\delta\mathcal{S}^{\rm F}}{\delta\phi_a}[\hat{\phi}_\pm]\right)\varphi_a - \frac{1}{2}\int_M d^4x \times \int_M d^4x' \left(\frac{\delta^2\mathcal{S}^{\rm F}}{\delta\phi_a(x)\delta\phi_b(x')}[\hat{\phi}_\pm]\right) \times \varphi_a(x)\varphi_b(x'). \tag{2.19}$$

For simplicity, we do not explicitly indicate the functional dependence of $\mathcal{S}^{\rm F}_{\rm int}$ on $\hat{\phi}_\pm$. Figure 1 shows the diagrammatic expansion for Γ_1. Each vertex carries a spacetime label in M and a time branch label in $\{+,-\}$. The lowest-order contribution is order $\hbar^2$, i.e., at two loops. The propagator $\mathcal{A}^{-1}$ does not depend on $\hbar$. The $\ln\det\mathcal{A}$ term in Eq. (2.17) is the one-loop (order $\hbar$) term in the CTP effective action. The CTP effective action, as a functional of $\hat{\phi}_\pm$, can be computed to any desired order in the loop expansion using Eq. (2.17). In general, this action contains divergences at each order in the loop expansion, which need to be renormalized.

Functionally differentiating $\Gamma[\hat{\phi}_+, \hat{\phi}_-]$ with respect to either $\hat{\phi}_+$ or $\hat{\phi}_-$ and making the identification $\hat{\phi}_+ = \hat{\phi}_- = \hat{\phi}$ [as shown in Eq. (2.13)] yields a dynamical, real, and causal evolution equation for the mean field $\hat{\phi}$. Thus the 1PI effective action $\Gamma[\hat{\phi}_\pm]$ yields *mean-field* dynamics for the theory, which is a lowest-order truncation of the correlation hierarchy [45,44]. However, for a detailed study of nonperturbative growth of quantum fluctuations relevant to nonequilibrium mean-field dynamics (or a symmetry-breaking phase transition), it is also necessary to obtain dynamical equations for the *variance* of $\Phi_{\rm H}$,

$$\langle \Phi^2_{\rm H}\rangle - \langle \Phi_{\rm H}\rangle^2 = \langle \Phi^2_{\rm H}\rangle - \hat{\phi}^2 = \langle \varphi^2_{\rm H}\rangle \equiv \hbar G_{++}(x,x), \tag{2.20}$$

where $\hbar G_{++}(x,x')$ is the time-ordered Green function for the fluctuation field $\varphi_{\rm H}$, $\langle T(\varphi(x)_{\rm H}\varphi(x')_{\rm H})\rangle$. A higher-order truncation of the correlation hierarchy is needed in order to explicitly follow the growth of quantum fluctuations; the 2PI effective action, to which we now turn, serves this purpose.

III. TWO-PARTICLE-IRREDUCIBLE FORMALISM

In a nonperturbative study of nonequilibrium field dynamics in the regime where quantum fluctuations are significant, the 1PI effective action is inadequate because it does not permit a derivation of the evolution equations for the mean field $\langle \Phi_{\rm H}\rangle$ *and* variance $\langle \varphi_{\rm H}^2\rangle$, at a *consistent* order in a nonperturbative expansion scheme. In addition, the initial data for the mean field $\hat{\phi}$ do not contain any information about the quantum state for fluctuations φ around the mean field. The two-particle-irreducible (2PI) effective action method can be used to obtain nonperturbative dynamical equations for both the mean field $\hat{\phi}(x)$ and two-point function $G(x,y)$, which contains the variance, as shown in Eq. (2.20). The 2PI method generalizes the 1PI effective action to an action $\Gamma[\hat{\phi},G]$ which is a functional of possible histories for both $\hat{\phi}$ and G. Alternatively, the 2PI effective action can be viewed as a truncation of the master effective action to second order in the correlation hierarchy [44]. In this section we briefly review how the 2PI method works; more thorough presentations can be found in [32,34].

Unlike the 1PI method where the mean field is fixed to be $\hat{\phi}$, the 2PI method fixes the mean field to be $\hat{\phi}$ and the sum of all self-energy diagrams to be G. This drastically reduces the number of independent diagrams which must be computed in order to obtain $\Gamma[\hat{\phi},G]$ [45]. Coupled dynamical equations for the evolution of $\hat{\phi}$ and G are obtained by separately varying $\Gamma[\hat{\phi},G]$ with respect to G and $\hat{\phi}$. Imposing $\delta\Gamma/\delta\hat{\phi}=0$ yields an evolution equation for the mean field $\hat{\phi}$, and setting $\delta\Gamma/\delta G=0$ yields an evolution equation for G, the ‘‘gap’’ equation. The variance $\langle \varphi_{\rm H}^2\rangle$ is the coincidence limit of the two-point function $\hbar G$, as seen from Eq. (2.20). In a nonequilibrium setting, the closed-time-path method should be used in conjunction with the 2PI formalism in order to obtain real and causal dynamics for $\hat{\phi}$ and G [34,35,45].

Let us apply the 2PI method to a scalar $\lambda\phi^4$ theory in Minkowski space, with vacuum initial conditions. In a direct generalization of Sec. II, both a local source $J_a(x)$ and nonlocal source $K_{ab}(x,x')$ (which are c-number functions on $\mathcal{M}$) are coupled to the field via $\hbar c^{ab}J_a\phi_a$ and $\hbar c^{ab}c^{cd}K_{ac}(x,x')\phi_b(x)\phi_d(x')$ interactions. Following Eq. (2.7), the CTP generating functional is defined as a vacuum persistence amplitude in the presence of the sources J and K, which has the path integral representation

$$Z[J,K]=\int_{\rm ctp} D\phi_+ D\phi_- \exp\Bigg[\frac{i}{\hbar}\Bigg(\mathcal{S}^{\rm F}[\phi_+,\phi_-] + \int_M d^4x c^{ab}J_a\phi_b + \frac{1}{2}\int_M d^4x\int_M d^4x' c^{ab}c^{cd} \times K_{ac}(x,x')\phi_b(x)\phi_d(x')\Bigg)\Bigg]. \tag{3.1}$$

Here, $\mathcal{S}^{\rm F}$ is as defined in Eq. (2.5), and we are using $Z[J,K]$ as a shorthand for $Z[J_+,J_-;K_{++},K_{--},K_{+-},K_{-+}]$. The generating functional for normalized n-point functions (connected diagrams) is defined by

$$W[J,K]=-i\hbar \ln Z[J,K]. \tag{3.2}$$

The ‘‘classical’’ field $\hat{\phi}_a(x)_{JK}$ and two-point function $G_{ab}(x,x')_{JK}$ are then given by

$$\hat{\phi}_a(x)_{JK}=c_{ab}\frac{\delta W[J,K]}{\delta J_b(x)}, \tag{3.3a}$$

$$\hbar G_{ab}(x,x')_{JK}=2c_{ac}c_{bd}\frac{\delta W[J,K]}{\delta K_{cd}(x,x')}-\hat{\phi}_a(x)_{JK}\hat{\phi}_b(x')_{JK}, \tag{3.3b}$$

where we use the subscript JK to indicate that $\hat{\phi}_a$ and G_{ab} are functionals of the sources J and K.

In the limit $K=J=0$, the classical field $\hat{\phi}_a$ satisfies

$$(\hat{\phi}_+)_{J=K=0}=(\hat{\phi}_-)_{J=K=0}=\langle\phi|\Phi_{\rm H}|\phi\rangle\equiv\hat{\phi}, \tag{3.4}$$

i.e., it becomes the expectation value of the Heisenberg field operator $\Phi_{\rm H}$ in the quantum state $|\phi\rangle$ (the mean field). In the same limit, the two-point function G_{ab} is the CTP propagator for the fluctuation field defined by Eq. (2.14). The four components of the CTP propagator are, for $J=K=0$,

$$\hbar G_{++}(x,x')_{|J=K=0}=\langle\phi|T(\varphi_{\rm H}(x)\varphi_{\rm H}(x'))|\phi\rangle, \tag{3.5a}$$

$$\hbar G_{--}(x,x')_{|J=K=0}=\langle\phi|\widetilde{T}(\varphi_{\rm H}(x)\varphi_{\rm H}(x'))|\phi\rangle, \tag{3.5b}$$

$$\hbar G_{+-}(x,x')_{|J=K=0}=\langle\phi|\varphi_{\rm H}(x')\varphi_{\rm H}(x)|\phi\rangle, \tag{3.5c}$$

$$\hbar G_{-+}(x,x')_{|J=K=0}=\langle\phi|\varphi_{\rm H}(x)\varphi_{\rm H}(x')|\phi\rangle, \tag{3.5d}$$

in the Heisenberg picture. In the coincidence limit $x'=x$, all four components above are equivalent to the variance $\langle\varphi_{\rm H}^2\rangle$ defined in Eq. (2.20). Provided we can invert Eqs. (3.3a) and (3.3b) to obtain J and K in terms of $\hat{\phi}$ and G, the 2PI effective action can be defined as the double Legendre transform (in both J and K) of $W[J,K]$

$$\Gamma[\hat{\phi},G]=W[J,K]-\int_M d^4x c^{ab}J_a(x)\hat{\phi}_b(x) -\frac{1}{2}\int_M d^4x\int_M d^4x' c^{ab}c^{cd}K_{ac}(x,x')[\hbar G_{bd}(x,x') +\hat{\phi}_b(x)\hat{\phi}_d(x')]. \tag{3.6}$$

As with $W[J,K]$, we are using $\Gamma[\hat{\phi},G]$ as a shorthand for $\Gamma[\hat{\phi}_+,\hat{\phi}_-;G_{++},G_{--},G_{+-},G_{-+}]$. The JK subscripting of $\hat{\phi}$ and G is suppressed, but the functional dependence of $\hat{\phi}$ and G on J and K through inversion of Eqs. (3.3a) and (3.3b) is understood. By direct functional differentiation of Eq. (3.6), the inverses of Eqs. (3.3a) and (3.3b) are found to be

$$\frac{\delta\Gamma[\hat{\phi},G]}{\delta\hat{\phi}_a(x)} = -c^{ab}J_b(x)_{\hat{\phi}G} - \frac{1}{2}c^{ab}c^{cd}\int_M d^4x'\,(K_{bd}(x,x')_{\hat{\phi}G} + K_{db}(x',x)_{\hat{\phi}G})\hat{\phi}_c(x'), \tag{3.7a}$$

$$\frac{\delta\Gamma[\hat{\phi},G]}{\delta G_{ab}(x,x')} = -\frac{\hbar}{2}c^{ac}c^{bd}K_{cd}(x,x')_{\hat{\phi}G}, \tag{3.7b}$$

where the subscript ‘‘$\hat{\phi}G$’’ indicates that K and J are functionals of $\hat{\phi}$ and G. Once $\Gamma[\hat{\phi},G]$ has been calculated, the evolution equations for $\hat{\phi}$ and G are given by

$$\left.\frac{\delta\Gamma[\hat{\phi},G]}{\delta\hat{\phi}_a(x)}\right|_{\hat{\phi}_+=\hat{\phi}_-\equiv\hat{\phi}} = 0, \tag{3.8a}$$

$$\left.\frac{\delta\Gamma[\hat{\phi},G]}{\delta G_{ab}(x,y)}\right|_{\hat{\phi}_+=\hat{\phi}_-\equiv\hat{\phi}} = 0. \tag{3.8b}$$

Of course, the two equations contained in Eq. (3.8a) are not independent, just as in Eq. (2.13). In addition, only two of equations (3.8b) are independent, one on the diagonal and one off diagonal in the ‘‘internal’’ CTP space. Using both Eqs. (3.6) and (3.1), an equation for $\Gamma[\hat{\phi},G]$ in terms of the sources K and J can be derived,

$$\Gamma[\hat{\phi},G] = -i\hbar\,\ln\left\{\int_{\rm ctp} D\phi_+ D\phi_- \exp\left[\frac{i}{\hbar}\left(\mathcal{S}^{\rm F}[\phi_+,\phi_-] + c^{ab}\int_M d^4x J_a(x)[\phi_b(x)-\hat{\phi}_b(x)] + \frac{1}{2}c^{ac}c^{bd}\int_M d^4x\int_M d^4x' K_{ab}(x,x')[\phi_c(x)\phi_d(x') - \hat{\phi}_c(x)\hat{\phi}_d(x') - \hbar G_{cd}(x,x')]\right)\right]\right\}. \tag{3.9}$$

The sources K and J in the right-hand side of Eq. (3.9) are functionals of $\hat{\phi}$, through Eqs. (3.7a) and (3.7b). Expressing this functional dependence, we obtain a functional integrodifferential equation for Γ,

$$\Gamma[\hat{\phi},G] = \int_M d^4x\int_M d^4x'\,\frac{\delta\Gamma[\hat{\phi},G]}{\delta G_{ba}(x',x)}G_{ab}(x,x') - i\hbar\,\ln\left\{\int_{\rm ctp} D\phi_+ D\phi_- \exp\left[\frac{i}{\hbar}\left(\mathcal{S}^{\rm F}[\phi_+,\phi_-] - \int_M d^4x\frac{\delta\Gamma[\hat{\phi},G]}{\delta\hat{\phi}_a}(\phi_a-\hat{\phi}_a) - \frac{1}{\hbar}\int_M d^4x\int_M d^4x'\,\frac{\delta\Gamma[\hat{\phi},G]}{\delta G_{ba}(x',x)}[\phi_a(x)-\hat{\phi}_a(x)][\phi_b(x')-\hat{\phi}_b(x')]\right)\right]\right\}. \tag{3.10}$$

We have dropped the JK subscripting because the functional derivatives in the equation are only with respect to $\hat{\phi}$ and G. As in Sec. II, a change of variables $D\phi_\pm \to D\varphi_\pm$ is carried out in the functional integral. The resulting equation

$$\Gamma[\hat{\phi},G] = \int_M d^4x\int_M d^4x'\,\frac{\delta\Gamma[\hat{\phi},G]}{\delta G_{ba}(x',x)}G_{ab}(x,x') - i\hbar\,\ln\left\{\int_{\rm ctp} D\varphi_+ D\varphi_- \exp\left[\frac{i}{\hbar}\left(\mathcal{S}^{\rm F}[\varphi_+ + \hat{\phi}_+,\varphi_- + \hat{\phi}_-] - \int_M d^4x\frac{\delta\Gamma[\hat{\phi},G]}{\delta\hat{\phi}_a}\varphi_a - \frac{1}{\hbar}\int_M d^4x\int_M d^4x'\,\frac{\delta\Gamma[\hat{\phi},G]}{\delta G_{ba}(x',x)}\varphi_a(x)\varphi_b(x')\right)\right]\right\} \tag{3.11}$$

has the formal solution [32]

$$\Gamma[\hat{\phi},G] = \mathcal{S}^{\rm F}[\hat{\phi}] - \frac{i\hbar}{2}\ln\det(G_{ab}) + \frac{i\hbar}{2}\int_M d^4x \times\int_M d^4x'\,\mathcal{A}^{ab}(x',x)G_{ab}(x,x') + \Gamma_2[\hat{\phi},G], \tag{3.12}$$

where $\mathcal{A}^{ab}$ is the second functional derivative of the classical action $\mathcal{S}^{\rm F}$, evaluated at $\hat{\phi}_a$. The functional Γ_2 is $-i\hbar$ times the sum of all two-particle-irreducible vacuum-to-vacuum diagrams with lines given by G_{ab} and vertices given by a shifted action $\mathcal{S}^{\rm F}_{\rm int}$. We have, for $\mathcal{A}^{ab}$,

$$i\mathcal{A}^{ab}(x,x') = \frac{\delta^2\mathcal{S}^{\rm F}}{\delta\phi_a(x)\phi_b(x')}[\hat{\phi}] \tag{3.13}$$

and, for $\mathcal{S}^{\rm F}_{\rm int}$,

$$\mathcal{S}^{\rm F}_{\rm int}[\varphi] = \mathcal{S}^{\rm F}[\varphi+\hat{\phi}] - \mathcal{S}^{\rm F}[\hat{\phi}] - \int_M d^4x\left(\frac{\delta\mathcal{S}^{\rm F}}{\delta\phi_a}[\hat{\phi}]\right)\varphi_a - \frac{i}{2}\int_M d^4x\int_M d^4x'\,\mathcal{A}^{ab}(x,x')\varphi_a(x)\varphi_b(x'). \tag{3.14}$$

The shifted action for the $\lambda\phi^4$ scalar field theory is

$$\mathcal{S}^{\rm F}_{\rm int}[\varphi] = S^{\rm F}_{\rm int}[\varphi_+] - S^{\rm F}_{\rm int}[\varphi_-], \tag{3.15}$$

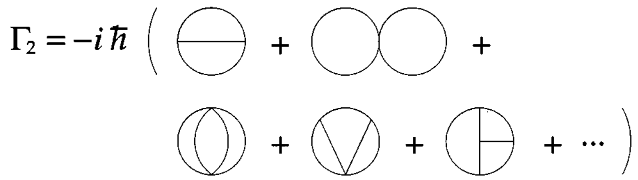

FIG. 2. Diagrammatic expansion for Γ_2. Lines represent the propagator G, and vertices are given by $\mathcal{S}^{\mathrm{F}}_{\mathrm{int}}$. The vertices terminating three lines are proportional to $\hat{\phi}$.

in terms of

$$S^{\mathrm{F}}_{\mathrm{int}}[\varphi] = -\frac{\lambda}{6}\int_M d^4x\left(\frac{1}{4}\varphi^4 + \hat{\phi}\varphi^3\right), \qquad (3.16)$$

where the functional dependence of $\mathcal{S}^{\mathrm{F}}_{\mathrm{int}}$ on $\hat{\phi}_{\pm}$ is not shown explicitly. Two types of vertices appear in Eq. (3.16): a vertex which terminates four lines and a vertex terminating three lines which is proportional to the mean field $\hat{\phi}$. The expansion for Γ_2 in terms of G and $\hat{\phi}$ is depicted graphically up to three-loop order in Fig. 2. Each vertex carries a spacetime label in M and time branch label in $\{+,-\}$. In general, the 2PI effective action contains divergences at each order in the loop expansion. It has been shown formally that if the field theory is renormalizable in the ''in-out'' formulation, then the ''in-in'' equations of motion are renormalizable [34]. In the closed-time-path formalism it is easier to carry out explicit renormalization in the equations of motion, i.e., the mean-field and gap equations, which we will do in Sec. VD.

Various approximations to the full quantum dynamics can be obtained by truncating the diagrammatic expansion for Γ_2. Throwing away Γ_2 in its entirety would yield the one-loop approximation. In Fig. 2, there are two two-loop diagrams, the ''double bubble'' and the ''setting sun.'' Retaining just the double-bubble diagram yields the time-dependent Hartree-Fock approximation [32]. Retaining both diagrams gives a two-loop approximation to the theory.[4] This approximation will yield a non-time-reversal-invariant mean-field equation above threshold, due to the setting sun diagram [44]. The time-reversal noninvariance of the mean-field equation generated by the 2PI effective action is a consequence of the fact that the 2PI effective action really corresponds to a further approximation from the two-loop truncation (in the sense of topology of vacuum graphs) of the *master* effective action [44]. The two-loop truncation of the master effective action is a functional $\Gamma_{l=2}[\hat{\phi},G,C_3]$ which depends on the mean field $\hat{\phi}$, the two-point function G, and the three-point function C_3. The four-point function C_4 also appears, but is not dynamical due to a constraint. The full set of equations,

$$\frac{\delta\Gamma_{l=2}[\hat{\phi},G,C_3]}{\delta\hat{\phi}_a} = 0, \qquad (3.17a)$$

$$\frac{\delta\Gamma_{l=2}[\hat{\phi},G,C_3]}{\delta G_{ab}} = 0, \qquad (3.17b)$$

$$\frac{\delta\Gamma_{l=2}[\hat{\phi},G,C_3]}{\delta (C_3)_{abc}} = 0, \qquad (3.17c)$$

is time-reversal invariant. However, the 2PI effective action is obtained by solving Eq. (3.17c) with *a given choice of causal boundary conditions* and substituting the resulting C_3 into $\Gamma_{l=2}$, to obtain $\Gamma_2[\hat{\phi},G]$. This ''slaving'' of C_3 to $\hat{\phi}$ and G with a particular choice of boundary conditions is what breaks the time-reversal invariance of the theory [44]. In the paper following [50] where we discuss the preheating dynamics, we work with further approximations which discard the setting sun diagram, and thus regain time-reversal-invariant equations.

IV. $\lambda\Phi^4$ FIELD THEORY IN CURVED SPACETIME

In this section the quantum dynamics of a scalar $\lambda\Phi^4$ field theory is formulated in semiclassical gravity, where the matter fields are quantized on a curved classical background spacetime.[5] The two-particle-irreducible effective action is used in conjunction with the CTP formalism to obtain coupled evolution equations for the mean field $\langle\Phi_{\mathrm{H}}\rangle$ and variance $\langle\Phi^2_{\mathrm{H}}\rangle - \langle\Phi_{\mathrm{H}}\rangle^2$ in the $\lambda\Phi^4$ model which are manifestly covariant.

Let us consider a quartically self-interacting scalar field ϕ in a globally hyperbolic, curved background spacetime with metric tensor $g_{\mu\nu}$. The diffeomorphism-invariant classical action for this system is

$$S[\phi,g^{\mu\nu}] = S^{\mathrm{G}}[g^{\mu\nu}] + S^{\mathrm{F}}[\phi,g^{\mu\nu}], \qquad (4.1)$$

where $g^{\mu\nu}$ is the contravariant metric tensor, and S^{G} and S^{F} are the classical actions of the gravity and scalar field sectors of the theory, respectively. For the scalar field action, we have

$$S^{\mathrm{F}}[\phi,g^{\mu\nu}] = -\frac{1}{2}\int d^4x\sqrt{-g}\left[\phi(\Box + m^2 + \xi R)\phi + \frac{\lambda}{12}\phi^4\right], \qquad (4.2)$$

where ξ is the (dimensionless) coupling constant to gravity (necessary in order for the field theory to be renormalizable [59]), $\Box$ is the Laplace-Beltrami operator in terms of the covariant derivative ∇_μ, and R is the scalar curvature. The constant m has units of inverse length, and the ϕ self-coupling λ has units of $1/\hbar$. Following standard procedure in semiclassical gravity [2], we define the semiclassical action for gravity to be

$$S^{\mathrm{G}}[g^{\mu\nu}] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}[R - 2\Lambda + cR^2 + bR^{\alpha\beta}R_{\alpha\beta}$$

[4] A different approximation, the $1/N$ expansion, is used in Sec. V to study the nonequilibrium dynamics of the O(N) field theory.

[5] The semiclassical approximation is consistent with a truncation of the quantum effective action for matter fields and gravity perturbations at one loop [i.e., at order $O(\hbar)$] [29] or [in the case of the O(N) field theory studied here] at leading order in the $1/N$ expansion [55,19].

$$+aR^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}], \tag{4.3}$$

where a, b, and c are constants with dimensions of length squared, $R_{\alpha\beta\gamma\delta}$ is the Riemann tensor, $R_{\alpha\beta}$ is the Ricci tensor, Λ is the ''cosmological constant'' (with units of inverse length-squared), $\sqrt{-g}$ is the square root of the determinant of $g_{\mu\nu}$, and G (with units of length divided by mass) is Newton's constant. As a result of the generalized Gauss-Bonnet theorem [60], the constants a, b, and c are not all independent in four spacetime dimensions; let us, therefore, set $a=0$. Classical Einstein gravity is obtained by setting $b=0$ and $c=0$. Minimal and conformal coupling (for the ϕ field to gravity) correspond to setting $\xi=0$ and $\xi=1/6$, respectively.

The motivation for including the arbitrary coupling ξ and the higher-order curvature terms R^2 and $R^{\alpha\beta}R_{\alpha\beta}$ in the classical action S is that we wish to study the semiclassical dynamics of the theory. In the semiclassical gravity field equation and matter field equations, divergences arise which require a renormalization of b, c, G, Λ, m, ξ, and λ [2]. These quantities are understood to be bare; their observable counterparts are renormalized.

The classical Euler-Lagrange equation for ϕ is obtained by functionally differentiating $S^{\mathrm{F}}[\phi,g^{\mu\nu}]$ with respect to ϕ, and setting $\delta S^{\mathrm{F}}/\delta\phi=0$,

$$\left(\Box+m^2+\xi R+\frac{\lambda}{6}\phi^2\right)\phi=0. \tag{4.4}$$

The Euler-Lagrange equation for the metric $g_{\mu\nu}$ is obtained by functional differentiation of S with respect to $g^{\mu\nu}$ (it is assumed that the variations $\delta\phi$ and $\delta g^{\mu\nu}$ are restricted so that no boundary terms arise),

$$G_{\mu\nu}+\Lambda g_{\mu\nu}+c\ {}^{(1)}H_{\mu\nu}+b\ {}^{(2)}H_{\mu\nu}=-8\pi G T_{\mu\nu}, \tag{4.5}$$

where the tensors $G_{\mu\nu}$, ${}^{(1)}H_{\mu\nu}$, and ${}^{(2)}H_{\mu\nu}$ are defined by [61,62] and $T_{\mu\nu}$ is the classical energy-momentum tensor,

$$T_{\mu\nu}=(1-2\xi)\phi_{;\mu}\phi_{;\nu}+\left(2\xi-\frac{1}{2}\right)g_{\mu\nu}g^{\rho\sigma}\phi_{;\rho}\phi_{;\sigma}-2\xi\phi_{;\mu\nu}\phi$$
$$+2\xi g_{\mu\nu}\phi\Box\phi-\xi G_{\mu\nu}\phi^2+\frac{1}{2}g_{\mu\nu}\left(m^2+\frac{\lambda}{12}\phi^2\right)\phi^2. \tag{4.6}$$

We are interested in the dynamics of expectation values in the semiclassical theory, which in nonequilibrium field theory does *not* follow directly from functional differentiation of the usual Schwinger-DeWitt or ''in-out'' effective action. Instead, the Schwinger-Keldysh formalism (reviewed in Sec. II) should be used. Here we discuss the implementation of the Schwinger-Keldysh method in curved spacetime.

The first step is to generalize the closed-time-path (CTP) manifold $\mathcal{M}$, defined in Eq. (2.3), to curved spacetime. Let $\Sigma_\star$ be a Cauchy hypersurface chosen so that its past domain of dependence [63], $D_-(\Sigma_\star)$, contains all of the dynamics we wish to study. Let us then define the manifold (with boundary)

$$M\equiv D_-(\Sigma_\star). \tag{4.7}$$

The CTP manifold $\mathcal{M}$ is defined following Eq. (2.3) as a quotient space constructed by identification on the hypersurface $\Sigma_\star\subset\partial M$ as in Eq. (2.3) where the equivalence relation is the same as Eq. (2.4) except that the matching of $+$ and $-$ time branches is now done on $\Sigma_\star$. We construct an orientation on $\mathcal{M}$ using the canonical volume form from $M,\boldsymbol{\epsilon}_M$, and define the volume form on $\mathcal{M}$ to be

$$\boldsymbol{\epsilon}_{\mathcal{M}}=\begin{cases}\boldsymbol{\epsilon}_M & \text{on } M\times\{+\},\\ -\boldsymbol{\epsilon}_M & \text{on } M\times\{-\}.\end{cases} \tag{4.8}$$

Finally, we let ϕ and $g^{\mu\nu}$ be independent on the $+$ and $-$ branches of $\mathcal{M}$, provided that $g_+^{\mu\nu}=g_-^{\mu\nu}$ and $\phi_+=\phi_-$ on $\Sigma_\star$. In other words, ϕ and $g^{\mu\nu}$ must be a scalar and a tensor, respectively, on $\mathcal{M}$. In terms of the volume form $\boldsymbol{\epsilon}_M$, we can write a scalar field action on $\mathcal{M}$,

$$\mathcal{S}^{\mathrm{F}}[\phi_\pm,g_\pm^{\mu\nu}]=S^{\mathrm{F}}[\phi_+,g_+^{\mu\nu}]-S^{\mathrm{F}}[\phi_-,g_-^{\mu\nu}], \tag{4.9}$$

where $S^{\mathrm{F}}[\phi]$ is given by Eq. (4.2), and $g_\pm^{\mu\nu}$ is the metric tensor on the $+$ and $-$ branches of $\mathcal{M}$. Using Eq. (4.3) we can similarly define the gravity action $\mathcal{S}^{\mathrm{G}}$ on $\mathcal{M}$,

$$\mathcal{S}^{\mathrm{G}}[g_+^{\mu\nu},g_-^{\mu\nu}]=S^{\mathrm{G}}[g_+^{\mu\nu}]-S^{\mathrm{G}}[g_-^{\mu\nu}], \tag{4.10}$$

where it is understood that only configurations of $g_\pm^{\mu\nu}$ satisfying the constraint $g_+^{\mu\nu}=g_-^{\mu\nu}$ on $\Sigma_\star$ are permitted.

In semiclassical gravity the scalar field theory (with action S^{F}) is quantized on a classical background spacetime, with metric $g_{\mu\nu}$, whose dynamics is determined self-consistently by the semiclassical geometrodynamical field equation. Let us denote the Heisenberg-picture field operator for the canonically quantized ϕ field by Φ_{H}. We wish to compute the quantum effective action Γ for this scalar field theory, using the two-particle-irreducible (2PI) method described in Sec. III. In terms of $\mathcal{S}^{\mathrm{F}}$ (now defined on the curved manifold $\mathcal{M}$), we define the 2PI, CTP generating functional $Z[J,K,g^{\mu\nu}]$ as follows:

$$Z[J,K,g^{\mu\nu}]=\int_{\mathrm{ctp}}D\phi_+D\phi_-\exp\Bigg[\frac{i}{\hbar}\Bigg(\mathcal{S}^{\mathrm{F}}[\phi_\pm,g_\pm^{\mu\nu}]$$
$$+\int_M d^4x\sqrt{-g_c}c^{abc}J_a\phi_b+\frac{1}{2}\int_M d^4x\sqrt{-g_{a'}}$$
$$\times\int_M d^4x'\sqrt{-g'_{c'}}c^{aba'}c^{cdc'}$$
$$\times K_{ac}(x,x')\phi_b(x)\phi_d(x')\Bigg)\Bigg], \tag{4.11}$$

where we have written $Z[J,K,g^{\mu\nu}]$ as a shorthand for $Z[J_\pm,K_{\pm\pm},g_\pm^{\mu\nu}]$. The three-index symbol c^{abc} is defined by

$$c^{abc}=\begin{cases}1 & \text{if } a=b=c=+,\\ -1 & \text{if } a=b=c=-,\\ 0 & \text{otherwise.}\end{cases} \tag{4.12}$$

The boundary conditions on the functional integral define the initial quantum state (assumed here to be pure). In this and a subsequent paper (in which preheating dynamics of inflationary cosmology is studied [50]), we are interested in the case of a quantum state corresponding to a nonzero mean field $\hat{\phi}$, with vacuum fluctuations around the mean field. This entails a definition of the vacuum state for the *fluctuation field*, defined in Eq. (2.14). In curved spacetime in general, there does not exist a unique Poincaré-invariant vacuum state for a quantum field [1,3]. For an asymptotically free field theory, a choice of ''in'' vacuum state corresponds to a choice of a particular orthonormal basis of solutions of the covariant Klein-Gordon equation with which to canonically quantize the field.

From Eq. (4.11), we can derive the two-particle-irreducible (2PI) effective action $\Gamma[\hat{\phi},G,g^{\mu\nu}]$ following the method of Sec. III, with the understanding that Γ now depends functionally on the metric $g_{\pm}^{\mu\nu}$ on the $+$ and $-$ time branches. The functional differentiations should be carried out using a covariant generalization of the Dirac δ function to the manifold M [2]. The functional integrodifferential equation (3.11) for the CTP-2PI effective action can then be generalized to the curved spacetime M in a straightforward fashion, modulo the curved-spacetime ambiguities in the boundary conditions of the functional integral (4.11).

The (bare) semiclassical field equations for the variance, mean field, and metric can then be expressed in terms of variations of $\mathcal{S}^{\mathrm{G}}[g^{\mu\nu}]+\Gamma[\hat{\phi},G,g^{\mu\nu}]$ with respect to $G_{\pm\pm}$, $\phi_{\pm}$, and $g_{\pm}^{\mu\nu}$, respectively, followed by metric and mean-field identifications between the $+$ and $-$ time branches,

$$\left.\frac{\delta(\mathcal{S}^{\mathrm{G}}[g^{\mu\nu}]+\Gamma[\hat{\phi},G,g^{\mu\nu}])}{\delta g_a^{\mu\nu}}\right|_{\hat{\phi}_+=\hat{\phi}_-=\hat{\phi};\;\; g_+^{\mu\nu}=g_-^{\mu\nu}=g^{\mu\nu}}=0, \tag{4.13a}$$

$$\left.\frac{\delta\Gamma[\hat{\phi},G,g^{\mu\nu}]}{\delta\hat{\phi}_a}\right|_{\hat{\phi}_+=\hat{\phi}_-=\hat{\phi};\;\; g_+^{\mu\nu}=g_-^{\mu\nu}=g^{\mu\nu}}=0, \tag{4.13b}$$

$$\left.\frac{\delta\Gamma[\hat{\phi},G,g^{\mu\nu}]}{\delta G_{ab}}\right|_{\hat{\phi}_+=\hat{\phi}_-=\hat{\phi};\;\; g_+^{\mu\nu}=g_-^{\mu\nu}=g^{\mu\nu}}=0. \tag{4.13c}$$

As above, CTP indices are suppressed inside functional arguments. Equations (4.13) constitute the semiclassical approximation to the full quantum dynamics for the system described by the classical action $S^{\mathrm{G}}[g^{\mu\nu}]+S^{\mathrm{F}}[\phi,g^{\mu\nu}]$. The semiclassical field equation (with bare parameters) for $g^{\mu\nu}$ is obtained directly from Eq. (4.13a),

$$G_{\mu\nu}+\Lambda g_{\mu\nu}+c\;{}^{(1)}H_{\mu\nu}+b\;{}^{(2)}H_{\mu\nu}=-8\pi G\langle T_{\mu\nu}\rangle, \tag{4.14}$$

where $\langle T_{\mu\nu}\rangle$ is the (unrenormalized) energy-momentum tensor defined by

$$\langle T_{\mu\nu}\rangle=\frac{2}{\sqrt{-g}}\left.\left(\frac{\delta\Gamma[\hat{\phi},G,g^{\mu\nu}]}{\delta g_+^{\mu\nu}}\right)\right|_{\hat{\phi}_+=\hat{\phi}_-=\hat{\phi};\;\; g_+^{\mu\nu}=g_-^{\mu\nu}=g^{\mu\nu}}. \tag{4.15}$$

Equation (4.14) gives the spacetime dynamics; the dynamics of $\hat{\phi}$ and G are given by the mean-field and gap equations (4.13b) and (4.13c). In Eq. (4.15), the angle brackets denote an expectation value of the energy-momentum tensor (with the Heisenberg field operator Φ_{H} substituted for ϕ in the classical theory) with respect to a quantum state $|\phi\rangle$ defined by the boundary conditions on the functional integral in Eq. (4.11). In four spacetime dimensions (unrenormalized) $\langle T_{\mu\nu}\rangle$ has divergences which can be absorbed by the renormalization of G, Λ, b, and c [2]. This renormalization should be carried out in the field equations rather than in the CTP effective action [28].

The energy-momentum tensor as defined in Eq. (4.15) is obtained by variation of the 2PI effective action Γ, which is a functional of the metric $g_{\pm}^{\mu\nu}$ on both the $+$ and $-$ time branches. From Eq. (4.11), it is possible to derive $\Gamma[\hat{\phi},G,g^{\mu\nu}]$ as an arbitrary functional of $g_+^{\mu\nu}$ and $g_-^{\mu\nu}$. However, in practice it is often easier to work in the simplified case where the metric is fixed to be the same on both the $+$ and $-$ time branches, i.e.,

$$g_+^{\mu\nu}=g_-^{\mu\nu}\equiv g^{\mu\nu}, \tag{4.16}$$

in the computation of $\Gamma[\hat{\phi},G,g^{\mu\nu}]$. Once $\Gamma[\hat{\phi},G,g^{\mu\nu}]$ (or some consistent truncation of the full quantum effective action for $\mathcal{S}^{\mathrm{F}}$) has been computed, it is then straightforward to determine how $\Gamma[\hat{\phi},G,g^{\mu\nu}]$ should be generalized to the case of an arbitrary metric on $\mathcal{M}$, for which $g_+^{\mu\nu}$ and $g_-^{\mu\nu}$ are independent. The bare energy-momentum tensor $\langle T_{\mu\nu}\rangle$ can then be computed using Eq. (4.15). Accordingly, in Sec. V, we fix $g_+^{\mu\nu}=g_-^{\mu\nu}\equiv g^{\mu\nu}$ in the calculation of $\Gamma[\hat{\phi},G,g^{\mu\nu}]$.

The semiclassical Einstein equation is a subcase of the general geometrodynamical field equation (4.14), obtained (after renormalization) by setting the renormalized $b=c=\Lambda=0$ (assuming no cosmological constant) [2]:

$$G_{\mu\nu}=-8\pi G\langle T_{\mu\nu}\rangle. \tag{4.17}$$

Having shown how to derive coupled evolution equations for the mean field, variance, and metric tensor in semiclassical gravity, we now turn our attention to the scalar O(N) model in curved spacetime.

V. O(N) FIELD THEORY IN CURVED SPACETIME

In this section we derive coupled nonperturbative dynamical equations for the mean field $\hat{\phi}$ and variance $\langle\varphi_{\mathrm{H}}^2\rangle$ for the minimally coupled O(N) scalar field theory with quartic self-interaction and unbroken symmetry. The background spacetime dynamics is given by the semiclassical Einstein equation. These equations take into consideration the back reaction of quantum particle production on the mean field, and quantum fields on the dynamical spacetime, self-consistently. In our model the Heisenberg-picture quantum state $|\phi\rangle$ is a coherent state for the field Φ_{H} at the initial time η_0, in which the expectation value $\langle\Phi_{\mathrm{H}}\rangle$ is spatially homogeneous. The coherent state is defined with respect to the adiabatic vacuum constructed via matching of WKB and exact mode functions for the fluctuation field in some asymptotic region of spacetime.

The O(N) field theory has the property that a systematic expansion in powers of $1/N$ yields a nonperturbative reorganization of the diagrammatic hierarchy which preserves the

Ward identities order by order [54]. Unlike perturbation theory in the coupling constant, which is an expansion of the theory around the vacuum configuration, the $1/N$ expansion entails an enhancement of the mean field by $\sqrt{N}$; this corresponds to the opposite limit of strong mean field. (This is precisely the situation which can arise in chaotic inflation at the end of the slow-roll period, where the inflaton mean field amplitude can be as large as $M_{\rm P}/3$ [50].) As discussed in Secs. II and III, the nonequilibrium initial conditions for the mean field as well as the nonperturbative aspect of the dynamics requires use of both closed-time-path and two-particle-irreducible methods. The $1/N$ expansion can be achieved as a further approximation from the two-loop, two-particle-irreducible truncation of the Schwinger-Dyson equations.

Although in this study we assume a pure state, the 2PI formalism is also useful for an open system calculation, in which the mean field is defined as the trace of the product of the reduced density matrix $\boldsymbol{\rho}$ and the Heisenberg field operator $\Phi_{\rm H}$, $\mathrm{Tr}(\boldsymbol{\rho}\Phi_{\rm H})$. When the position-basis matrix element $\langle \phi_1 | \boldsymbol{\rho}(\eta_0) | \phi_2 \rangle$ can be expressed as a Gaussian functional of ϕ_1 and ϕ_2, the nonlocal source K can encompass the initial conditions coming from $\boldsymbol{\rho}(t_0)$ in a natural way [34]. In order to incorporate a density matrix whose initial condition is beyond Gaussian order in the position basis, one can work with a higher-order truncation of the master effective action [44]. The leading-order $1/N$ approximation is equivalent to assuming a Gaussian initial density matrix; therefore, the 2PI effective action is adequate for our purposes.

A. Classical action for the O(N) theory

The O(N) field theory consists of N spinless fields $\vec{\phi} = \{\phi^i\}$, $i = 1, \ldots, N$, with an action which is invariant under the N-dimensional real orthogonal group. The generally covariant classical action for the O(N) theory (with quartic self-interaction) plus gravity is given by

$$S[\phi^i, g^{\mu\nu}] = S^{\rm G}[g^{\mu\nu}] + S^{\rm F}[\phi^i, g^{\mu\nu}], \tag{5.1}$$

where $S^{\rm G}[g^{\mu\nu}]$ is defined in Eq. (4.3) for the spacetime manifold M with metric $g_{\mu\nu}$, and the matter field action $S^{\rm F}[\phi^i, g^{\mu\nu}]$ is given by

$$S^{\rm F}[\phi^i, g_{\mu\nu}] = -\frac{1}{2}\int_M d^4x \sqrt{-g}\left[\vec{\phi}\cdot(\Box + m^2 + \xi R)\vec{\phi} + \frac{\lambda}{4N}(\vec{\phi}\cdot\vec{\phi})^2\right]. \tag{5.2}$$

The O(N) inner product is defined by[6]

$$\vec{\phi}\cdot\vec{\phi} = \phi^i \phi^j \delta_{ij}. \tag{5.3}$$

In Eq. (5.2), λ is a (bare) coupling constant with dimensions of $1/\hbar$, and ξ is the (bare) dimensionless coupling to gravity. The classical Euler-Lagrange equations are obtained by variation of the action S separately with respect to the metric tensor $g_{\mu\nu}$ and the matter fields ϕ^i. In the classical action (5.2), the O(N) symmetry is unbroken. However, the O(N) symmetry can be spontaneously broken, for example, by changing m^2 to $-m^2$ in $S^{\rm F}$. In the symmetry-breaking case with tachyonic mass, the stable equilibrium configuration is found to be

$$\vec{\phi}\cdot\vec{\phi} = \frac{2Nm^2}{\lambda} \equiv v^2, \tag{5.4}$$

which is a constant. If we wish to study the action for oscillations about the symmetry-broken equilibrium configuration, the O(N) invariance of Eq. (5.2) implies that we can choose the minimum to be in any direction; we choose it to be in the first, i.e., $(\phi^1)^2 = v^2$. In terms of the shifted field $\sigma = \phi^1 - v$ and the unshifted fields (the ''pions'') $\pi^i = \phi^i$, $i = 1, \ldots, N-1$, the action becomes

$$\begin{aligned} S^{\rm F}[\sigma, \vec{\pi}, g^{\mu\nu}] = -\frac{1}{2}\int_M d^4x \sqrt{-g}\Bigg[&\sigma(\Box + m^2 + \xi R)\sigma \\ &+ \vec{\pi}\cdot(\Box + m^2 + \xi R)\vec{\pi} + 2(m^2 + \xi R)\sigma^2 \\ &+ 2\sqrt{\frac{\lambda}{2}} M\sigma^3 + 2\sqrt{\frac{\lambda}{2}} M \vec{\pi}\cdot\vec{\pi}\sigma \\ &+ \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}\vec{\pi}\cdot\vec{\pi}\sigma^2 + \frac{\lambda}{4}(\vec{\pi}\cdot\vec{\pi})^2\Bigg]. \end{aligned} \tag{5.5}$$

One can show that the effective mass of each of the ''pions'' $\vec{\pi}$ (defined as the second derivative of the potential) is zero, due to Goldstone's theorem. The theorem holds for the quantum-corrected effective potential as well [64]. In this paper we study the unbroken symmetry case, in order to focus on the parametric amplification of quantum fluctuations; this avoids the additional complications which arise in spontaneous symmetry breaking, e.g., infrared divergences [19,65,35].

B. Quantum generating functional

We aim at deriving the mean-field and gap equations at two-loop order. The 2PI generating functional for the O(N) theory in curved spacetime is defined using the closed-time-path method in terms of c-number sources J^i_a and nonlocal c-number sources K^{ij}_{ab} on the CTP manifold $\mathcal{M}$,

[6]In our index notation, the Latin letters i,j,k,l,m,n are used to designate O(N) indices (with index set $\{1, \ldots, N\}$), while the Latin letters a,b,c,d,e,f are used below to designate CTP indices (with index set $\{+,-\}$).

$$Z[J^i_a,K^{ij}_{ab},g_{\mu\nu}]=\prod_{i,a}\int_{\rm ctp}D\phi^i_a\exp\Bigg[\frac{i}{\hbar}\Bigg(\mathcal{S}^{\rm F}[\phi,g^{\mu\nu}]+\int_M d^4x\sqrt{-g}c^{ab}\vec{J}_a\cdot\vec{\phi}_b+\frac{1}{2}\int_M d^4x\sqrt{-g}\int_M d^4x'\sqrt{-g'}c^{ab}c^{cd}\times K^{ij}_{ac}(x,x')\phi^k_b(x)\phi^l_d(x')\delta_{ik}\delta_{jl}\Bigg)\Bigg],\tag{5.6}$$

where the CTP classical action is defined as in Eq. (4.9), with $\phi^i_\pm$ replacing $\phi_\pm$, and the time branch indices on $g^{\mu\nu}$ suppressed. The sources J^i_a are coupled to the field by the O(N) vector inner product

$$\vec{J}_a\cdot\vec{\phi}_b=J^i_a\phi^j_b\delta_{ij}.\tag{5.7}$$

The time branch labels on the metric tensor are suppressed for simplicity of notation.[7] The designation ‘‘CTP’’ on the functional integral indicates that we sum only over field configurations for ϕ^i_a on $\mathcal{M}$ which satisfy the condition $\phi^i_+=\phi^i_-$ on $\Sigma_\star$, where $\Sigma_\star$ is defined in Sec. IV. In addition, the boundary conditions on the asymptotic past field configurations for $\phi^i_\pm$ in the functional integral correspond to a choice of ‘‘in’’ quantum state $|\phi\rangle$ for the system. The generating functional for normalized expectation values is given by Eq. (3.2), with the additional functional dependence of both W and Z on $g^{\mu\nu}$ understood. As above, we henceforth omit all indices in functional arguments. In terms of this functional, we can define the ‘‘classical’’ field $\hat{\phi}$ and two-point function G by functional differentiation,

$$\hat{\phi}^i_a(x)=\frac{c_{ab}}{\sqrt{-g}}\frac{\delta W}{\delta J^j_b(x)}\delta^{ij},\tag{5.8}$$

$$\hat{\phi}^i_a(x)\hat{\phi}^j_b(x')+\hbar G^{ij}_{ab}(x,x')=2\frac{c_{ac}}{\sqrt{-g}}\frac{c_{bd}}{\sqrt{-g'}}\frac{\delta W}{\delta K^{lm}_{cd}(x,x')}\delta^{ik}\delta^{jl}.\tag{5.9}$$

In the zero-source limit $K^{ij}_{ab}=J^i_a=0$, the classical field $\hat{\phi}^i_a$ satisfies

$$(\hat{\phi}^i_+)_{J=K=0}=(\hat{\phi}^i_-)_{J=K=0}=\langle\phi|\Phi^i_{\rm H}|\phi\rangle\equiv\hat{\phi}^i\tag{5.10}$$

as an expectation value of the Heisenberg field operator $\Phi^i_{\rm H}$ in the quantum state $|\phi\rangle$. The fluctuation field is defined [as in Eq. (2.14)] in terms of the Heisenberg field operator $\Phi_{\rm H}$ and the mean field $\hat{\phi}$ (times the identity operator),

$$\varphi^i_{\rm H}=\Phi^i_{\rm H}-\hat{\phi}^i.\tag{5.11}$$

In the same limit $J=K=0$, the two-point function G^{ij}_{ab} becomes the CTP propagator for the fluctuation field. The four components of the CTP propagator are (for $J^i_a=K^{ij}_{ab}=0$)

$$\hbar G^{ij}_{++}(x,x')_{|J=K=0}=\langle\phi|T(\varphi^i_{\rm H}(x)\varphi^j_{\rm H}(x'))|\phi\rangle,\tag{5.12a}$$

$$\hbar G^{ij}_{--}(x,x')_{|J=K=0}=\langle\phi|\widetilde{T}(\varphi^i_{\rm H}(x)\varphi^j_{\rm H}(x'))|\phi\rangle,\tag{5.12b}$$

$$\hbar G^{ij}_{+-}(x,x')_{J=K=0}=\langle\phi|\varphi^j_{\rm H}(x')\varphi^i_{\rm H}(x)|\phi\rangle,\tag{5.12c}$$

$$\hbar G^{ij}_{-+}(x,x')_{J=K=0}=\langle\phi|\varphi^i_{\rm H}(x)\varphi^j_{\rm H}(x')|\phi\rangle.\tag{5.12d}$$

In the coincidence limit $x'=x$, all four components above are equivalent to the mean-squared fluctuations (variance) about the mean field $\hat{\phi}^i$,

$$\hbar G^{ii}_{++}(x,x)_{|J=K=0}=\langle\phi|(\varphi^i_{\rm H})^2|\phi\rangle=\langle(\varphi^i_{\rm H})^2\rangle.\tag{5.13}$$

Provided that the above equations can be inverted to give J^i_a and K^{ij}_{ab} in terms of $\hat{\phi}^i_a$ and G^{ij}_{ab}, we can define the 2PI effective action as a double Legendre transform of W,

$$\Gamma[\hat{\phi},G,g^{\mu\nu}]=W[J,K,g^{\mu\nu}]-\int_M d^4x\sqrt{-g}c^{ab}J^i_a\hat{\phi}^j_b\delta_{ij}-\frac{1}{2}\int_M d^4x\sqrt{-g}\int_M d^4x'\sqrt{-g'}c^{ab}c^{cd}\times K^{ij}_{ac}(x,x')[\hbar G^{kl}_{bd}(x,x')+\hat{\phi}^k_b(x)\hat{\phi}^l_d(x')]\delta_{ik}\delta_{jl},\tag{5.14}$$

where J^i_a and K^{ij}_{ab} above denote the inverses of Eqs. (5.8) and (5.9). From this equation, it is clear that the inverses of Eqs. (5.8) and (5.9) can be obtained by straightforward functional differentiation of Γ,

$$\frac{1}{\sqrt{-g}}\frac{\delta\Gamma}{\delta\hat{\phi}^i_a(x)}=c^{ab}\delta_{ij}\Bigg(-J^j_b(x)-\frac{1}{2}c^{cd}\int_M d^4x'\sqrt{-g'}\times[K^{jk}_{bd}(x,x')+K^{jk}_{db}(x',x)]\hat{\phi}^l_d\delta_{kl}\Bigg),\tag{5.15}$$

$$\frac{1}{\sqrt{-g}}\frac{\delta\Gamma}{\delta G^{ij}_{ab}(x,x')}\frac{1}{\sqrt{-g'}}=-\frac{\hbar}{2}c^{ac}c^{bd}K^{jk}_{bd}(x',x).\tag{5.16}$$

Performing the usual field shifting involved in the background field approach [58], it can be shown that the 2PI effective action which satisfies Eqs. (5.14), (5.15), and (5.16) can be written

[7]The suppression of CTP indices on the metric tensor does not prevent computation of the expectation value of the stress-energy-momentum tensor. It will be clear how to reinstate the time branch indices after the 2PI effective action has been explicitly computed, e.g., in the large-N approximation.

$$\Gamma[\hat{\phi},G,g^{\mu\nu}]=\mathcal{S}^{\rm F}[\hat{\phi},g^{\mu\nu}]-\frac{i\hbar}{2}\ln\det[(G^{ij}_{ab})^{-1}]+\frac{i\hbar}{2}\int_M d^4x\sqrt{-g}\int_M d^4x'\sqrt{-g'}\mathcal{A}^{ab}_{ij}(x',x)\times G^{ij}_{ab}(x,x')+\Gamma_2[\hat{\phi},G,g^{\mu\nu}], \tag{5.17}$$

where the kernel $\mathcal{A}$ is the second functional derivative of the classical action with respect to the field ϕ,

$$i\mathcal{A}^{ab}_{ij}(x,x')=\frac{1}{\sqrt{-g}}\left(\frac{\delta^2\mathcal{S}^{\rm F}}{\delta\phi^i_a(x)\,\phi^j_b(x')}[\hat{\phi},g^{\mu\nu}]\right)\frac{1}{\sqrt{-g'}}, \tag{5.18}$$

and Γ_2 is a functional to be defined below. Evaluating $\mathcal{A}^{ab}_{ij}$ by differentiation of Eq. (5.2), we find

$$i\mathcal{A}^{ab}_{ij}(x,x')=-\left\{\delta_{ij}c^{ab}[\Box+m^2+\xi R(x)]+\frac{\lambda}{2N}c^{abcd}[[\hat{\phi}^k_c(x)\hat{\phi}^l_d(x)]\delta_{ij}\delta_{kl}+2\hat{\phi}^k_c(x)\hat{\phi}^l_d(x)\delta_{ik}\delta_{jl}]\right\}\delta^4(x-x')\frac{1}{\sqrt{-g'}}, \tag{5.19}$$

where the four-index symbol c^{abcd} is defined in exact analogy with Eq. (4.12). In Eq. (5.17), Γ_2 is $-i\hbar$ times the sum of all two-particle-irreducible vacuum-to-vacuum graphs with propagator G and vertices given by the shifted action $\mathcal{S}^{\rm F}_{\rm int}$, defined by

$$\mathcal{S}^{\rm F}_{\rm int}[\varphi,g^{\mu\nu}]=\mathcal{S}^{\rm F}[\varphi+\hat{\phi},g^{\mu\nu}]-\mathcal{S}^{\rm F}[\hat{\phi},g^{\mu\nu}]-\int_M d^4x\left(\frac{\delta\mathcal{S}^{\rm F}}{\delta\phi^i_a}[\hat{\phi},g^{\mu\nu}]\right)\varphi^i_a-\frac{1}{2}\int_M d^4x\int_M d^4x'\left(\frac{\delta^2\mathcal{S}^{\rm F}}{\delta\phi^i_a(x)\,\phi^j_b(x')}[\hat{\phi},g^{\mu\nu}]\right)\times\varphi^i_a(x)\varphi^j_b(x'). \tag{5.20}$$

The expansion of Γ_2 in terms of G and $\hat{\phi}$ is depicted graphically in Fig. 2. From Eqs. (5.20) and (5.2), $\mathcal{S}^{\rm F}_{\rm int}$ is easily evaluated, and we find

$$\mathcal{S}^{\rm F}_{\rm int}[\varphi,g^{\mu\nu}]=S^{\rm F}_{\rm int}[\varphi_+,g^{\mu\nu}]-S^{\rm F}_{\rm int}[\varphi_-,g^{\mu\nu}], \tag{5.21}$$

in terms of an action $S^{\rm F}_{\rm int}$ on M defined by

$$S^{\rm F}_{\rm int}[\varphi,g^{\mu\nu}]=-\frac{\lambda}{2N}\int_M d^4x\sqrt{-g}\left[\frac{1}{4}(\vec{\varphi}\cdot\vec{\varphi})^2+(\vec{\hat{\phi}}\cdot\vec{\varphi})(\vec{\varphi}\cdot\vec{\varphi})\right]. \tag{5.22}$$

The two types of vertices in Fig. 2 are readily apparent in Eq. (5.22). The first term corresponds to the vertex which terminates four lines; the second term corresponds to the vertex which terminates three lines and is proportional to $\hat{\phi}$.

The action Γ including the full diagrammatic series for Γ_2 gives the full dynamics for $\hat{\phi}$ and G in the O(N) theory. It is of course not feasible to obtain an exact, closed-form expression for Γ_2 in this model. Various approximations to the full 2PI effective action can be obtained by truncating the diagrammatic expansion for Γ_2. Which approximation is most appropriate depends on the physical problem under consideration.

(1) Retaining both the ''setting-sun'' and the ''double-bubble'' diagrams of Fig. 2 corresponds to the two-loop, two-particle-irreducible approximation [44]. This approximation contains two-particle scattering through the setting-sun diagram.

(2) A truncation of Γ_2 retaining only the ''double-bubble'' diagram of Fig. 2 yields equations for $\hat{\phi}$ and G which correspond to the time-dependent Hartree-Fock approximation to the full quantum dynamics [32,36]. This approximation does not preserve Goldstone's theorem, but is energy conserving (in Minkowski space) [36].

(3) Retaining only the $({\rm tr}G^{ij}_{ab})^2$ piece of the double-bubble diagram corresponds to taking the leading order $1/N$ approximation, shown below in Sec. V C.

(4) A much simpler approximation consists of discarding Γ_2 altogether. This yields the one-loop approximation, whose limitations have been extensively documented in the literature [34,45,44,35].

Let us first evaluate the 2PI effective action at two loops [19,41]. This is the most general of the various approximations described above. Here both two-loop diagrams in Fig. 2 are retained. The 2PI effective action is given by Eq. (5.17), and in this approximation, Γ_2 is given by

$$\Gamma_2[\hat{\phi},G,g^{\mu\nu}]=\frac{\lambda\hbar^2}{4N}\left[-\frac{1}{2}c^{abcd}\int_M d^4x\sqrt{-g}[G^{ij}_{ab}(x,x)G^{kl}_{cd}(x,x)+2G^{ik}_{ab}(x,x)G^{jl}_{cd}(x,x)]\delta_{ij}\delta_{kl}+\frac{i\lambda}{N}c^{abcd}c^{a'b'c'd'}\int_M d^4x\sqrt{-g}\int_M d^4x'\sqrt{-g'}\hat{\phi}^i_a(x)\hat{\phi}^{i'}_{a'}(x')[G^{ii'}_{bb'}(x,x')G^{jj'}_{cc'}(x,x')G^{kk'}_{dd'}(x,x')+2G^{ij'}_{bd'}(x,x')G^{jk'}_{cc'}(x,x')G^{ki'}_{db'}(x,x')]\delta_{jk}\delta_{j'k'}\right]. \tag{5.23}$$

Functional differentiation of Γ with respect to $\hat{\phi}$ and G leads to the mean-field and gap equations, respectively. The gap equation obtained at two loops is given by

$$(G^{-1})^{ab}_{ij}(x,x') = \mathcal{A}^{ab}_{ij}(x,x') + \frac{i\lambda\hbar}{2N} c^{abcd}\delta^4(x-x')[\delta^{ij}\delta_{kl}G^{kl}_{cd}(x,x) + 2G^{ij}_{cd}(x,x)]$$
$$+ \frac{\hbar\lambda^2}{2N^2} c^{acde} c^{bc'd'e'} \delta_{kk'}\delta_{ll'}[\hat\phi^i_c(x)\hat\phi^j_{c'}(x')G^{kl}_{dd'}(x,x')G^{k'l'}_{ee'}(x,x') + 2\hat\phi^k_c(x)\hat\phi^l_{c'}(x')G^{k'l'}_{dd'}(x,x')G^{ij}_{ee'}(x,x')$$
$$+ 2\hat\phi^i_c(x)\hat\phi^k_{c'}(x')G^{lj}_{dd'}(x,x')G^{l'k'}_{ee'}(x,x') + 2\hat\phi^k_c(x)\hat\phi^l_{c'}(x')G^{k'j}_{dd'}(x,x')G^{il'}_{ee'}(x,x')$$
$$+ 2\hat\phi^k_c(x)\hat\phi^j_{c'}(x')G^{k'l}_{dd'}(x,x')G^{jl'}_{ee'}(x,x')]. \tag{5.24}$$

The mean-field equation is found to be

$$\left(c^{cb}(\Box + m^2 + \xi R) + c^{abcd}\frac{\lambda}{2N}\hat\phi^i_a\hat\phi^j_d\delta_{ij}\right)\hat\phi^m_b - \frac{i\hbar^2\lambda^2}{4N^2}\int_M d^4x'\sqrt{-g'}\,\Sigma^{cm}(x,x')$$
$$+ \frac{\hbar\lambda c^{abcd}}{2N}\{\delta_{ij}\hat\phi^m_d G^{ij}_{ab}(x,x) + \delta_{jl}\delta^m_i\hat\phi^l_d[G^{ij}_{ab}(x,x) + G^{ji}_{ab}(x,x)]\}, \tag{5.25}$$

in terms of a nonlocal function $\Sigma^{cm}(x,x')$ defined by

$$\Sigma^{em}(x,x') = c^{ebcd}c^{a'b'c'd'}\hat\phi^i_{a'}(x')[G^{mi'}_{bb'}(x,x')G^{jj'}_{cc'}(x,x')G^{kk'}_{dd'}(x,x') + 2G^{mj'}_{bd'}(x,x')G^{jk'}_{cc'}(x,x')G^{ki'}_{b'd}(x,x')$$
$$+ G^{i'm}_{b'b}(x',x)G^{jj'}_{c'c}(x',x)G^{kk'}_{d'd}(x',x) + 2G^{i'j'}_{b'd}(x',x)G^{kk'}_{c'c}(x',x)G^{jm}_{d'b}(x',x)]\delta_{jk}\delta_{j'k'}\delta_{ii'}. \tag{5.26}$$

Taking the limit $\hat\phi^i_+ = \hat\phi^i_- = \hat\phi^i$ in Eqs. (5.24) and (5.25) yields coupled equations for the mean field $\hat\phi^i$ and the CTP propagators G^{ij}_{ab}, on the fixed background spacetime $g^{\mu\nu}$. The equations, as well as the semiclassical Einstein equation obtained from Eq. (4.13a), are real and causal, and correspond to expectation values in the $\hat\phi_+ = \hat\phi_- = \hat\phi$ limit. The $O(\lambda^2)$ parts of the above equations are nonlocal and dissipative. The nonlocal aspect makes numerical solution difficult; the dissipative aspect will be addressed in a future publication [66]. One can regain the perturbative (amplitude) expansion for the CTP effective action at two loops by expanding the one-loop CTP propagators in Eq. (5.25) in a functional power series in $\hat\phi$.

C. Large-N approximation

We now carry out the $1/N$ expansion to obtain local, covariant, nonperturbative mean-field and gap equations for the O(N) field theory in a general curved spacetime. The $1/N$ expansion is a controlled nonperturbative approximation scheme which can be used to study nonequilibrium quantum field dynamics in the regime of strong quasiclassical field amplitude [47,48,36,49]. In the large-N approach, the large-amplitude quasiclassical field is modeled by N fields, and the quantum-field-theoretic generating functional is expanded in powers of $1/N$. In this sense the method is a controlled expansion in a small parameter. Unlike perturbation theory in the coupling constant, which corresponds to an expansion of the theory around the vacuum, the large-N approximation corresponds to an expansion of the field theory about a strong quasiclassical field configuration [36]. At a particular order in the $1/N$ expansion, the approximation yields truncated Schwinger-Dyson equations which are gauge and renormalization-group invariant, unitary, and (in Minkowski space) energy conserving [36]. In contrast, the Hartree-Fock approximation cannot be systematically improved beyond leading order and (in the case of spontaneous symmetry breaking) violates Goldstone's theorem [49].

Let us implement the leading order large-N approximation in the two-loop, 2PI mean-field and gap equations (5.25) and (5.24) derived above. This amounts to computing the leading-order part of Γ in the limit of large N, which is O(N). In the unbroken-symmetry case, this is easily carried out by scaling $\hat\phi$ by $\sqrt{N}$ and leaving G unscaled [32],

$$\hat\phi^i_a(x) \to \sqrt{N}\hat\phi_a(x), \tag{5.27a}$$

$$G^{ij}_{ab}(x,x') \to G_{ab}(x,x')\delta^{ij}, \tag{5.27b}$$

$$\mathcal{A}^{ab}_{ij}(x,x') \to \mathcal{A}^{ab}(x,x')\delta_{ij}, \tag{5.27c}$$

$$\varphi^i_a(x) \to \varphi_a(x), \tag{5.27d}$$

for all i,j. The Heisenberg field operator $\varphi^i_{\rm H}$ scales like φ^i_a in Eq. (5.27d). In the above equations, the connection between the large-N limit and the strong mean-field limit is clear.

The truncation of the $1/N$ expansion should be carried out in the 2PI effective action, where it can be shown that the three-loop and higher-order diagrams do not contribute (at leading order in the $1/N$ expansion). Let us now also allow the metric $g_{\mu\nu}$ to be specified independently on the $+$ and $-$ time branches. We find, for the classical action,

$$\mathcal{S}^{\rm F}[\phi, g^{\mu\nu}] = S^{\rm F}[\phi_+, g^{\mu\nu}_+] - S^{\rm F}[\phi_-, g^{\mu\nu}_-], \tag{5.28}$$

where

$$S^{\mathrm{F}}[\phi,g^{\mu\nu}]=-\frac{N}{2}\int_M d^4x\sqrt{-g}\left[\phi(\Box+m^2+\xi R)\phi+\frac{\lambda}{2}\phi^4\right]. \tag{5.29}$$

The inverse of the one-loop propagator is[8]

$$i\mathcal{A}^{ab}(x,x')=-\left[c^{abc}[\Box_c^x+m^2+\xi R_c(x)]+\frac{\lambda}{2}c^{abcd}\hat{\phi}_c(x)\hat{\phi}_d(x)\right]\delta^4(x-x')\frac{1}{\sqrt{-g'}_b}. \tag{5.30}$$

Finally, for the CTP-2PI effective action at leading order in the $1/N$ expansion, we obtain

$$\begin{aligned}\Gamma[\hat{\phi},G,g^{\mu\nu}]=&\mathcal{S}^{\mathrm{F}}[\hat{\phi},g^{\mu\nu}]-\frac{i\hbar N}{2}\ln\det[G_{ab}]\\&+\frac{i\hbar N}{2}\int_M d^4x\sqrt{-g}_a\int_M d^4x'\sqrt{-g'}_b\\&\times\mathcal{A}^{ab}(x',x)G_{ab}(x,x')\\&-\frac{\lambda\hbar^2N}{8}c^{abcde}\int_M d^4x\sqrt{-g}_e G_{ab}(x,x)\\&\times G_{cd}(x,x)+O(1).\end{aligned} \tag{5.31}$$

Applying Eq. (4.13b) and taking the limits $\hat{\phi}_+=\hat{\phi}_-=\hat{\phi}$ and $g_+^{\mu\nu}=g_-^{\mu\nu}=g^{\mu\nu}$, we obtain the gap equation for G_{ab} at leading order in the $1/N$ expansion,

$$(G^{-1})^{ab}(x,x')=\hat{\mathcal{A}}^{ab}(x,x')+\frac{i\hbar\lambda}{2}c^{abcd}G_{cd}(x,x)\delta^4(x-x')\times\frac{1}{\sqrt{-g'}}+O\left(\frac{1}{N}\right), \tag{5.32}$$

where

$$i\hat{\mathcal{A}}^{ab}(x,x')\equiv-\left[c^{ab}[\Box+m^2+\xi R(x)]+\frac{\lambda}{2}c^{abcd}\hat{\phi}_c(x)\hat{\phi}_d(x)\right]\delta^4(x-x')\frac{1}{\sqrt{-g'}}. \tag{5.33}$$

Similarly, we obtain the mean-field equation for $\hat{\phi}$ at leading order in the $1/N$ expansion,

$$\left(\Box+m^2+\xi R+\frac{\lambda}{2}\hat{\phi}^2+\frac{\hbar\lambda}{2}G_{++}(x,x)\right)\hat{\phi}(x)+O\left(\frac{1}{N}\right)=0, \tag{5.34}$$

where we note that $G_{++}(x,x)=G_{ab}(x,x)$ for all a,b, which can be seen from Eq. (5.32); therefore, to get a consistent set of dynamical equations, we need only consider the $++$ component of Eq. (5.32). It should also be noted that $G_{ab}(x,x)$ is formally divergent. Regularization of the coincidence limit of the two-point function and the energy-momentum tensor is necessary. Multiplying Eq. (5.32) by G and integrating over spacetime, we obtain a differential equation for the $++$ CTP Green function,

$$\left(\Box_x+m^2+\xi R(x)+\frac{\lambda}{2}\hat{\phi}^2(x)+\frac{\hbar\lambda}{2}G_{++}(x,x)\right)G_{++}(x,x')+O\left(\frac{1}{N}\right)=\delta^4(x-x')\frac{-i}{\sqrt{-g'}}, \tag{5.35}$$

where boundary conditions must be specified on G_{++}.

Equations (5.34) and (5.35) are the covariant evolution equations for the mean field $\hat{\phi}$ and the two-point function G_{++} at leading order in the $1/N$ expansion. Following Eq. (5.13), we denote the coincidence limit of $\hbar G_{++}(x,x)$ by $\langle\varphi_{\mathrm{H}}^2\rangle$. With the inclusion of the semiclassical gravity field equation (4.14), these equations form a consistent, closed set of dynamical equations for the mean field $\hat{\phi}$, the time-ordered fluctuation-field Green function G_{++}, and the metric $g_{\mu\nu}$.

The one-loop equations for $\hat{\phi}$ and G can be obtained from the leading-order equations by setting $\hbar=0$ in Eq. (5.35), while leaving the mean-field equation (5.34) unchanged. In the Hartree approximation, the gap equation is unchanged from Eq. (5.32), and the mean-field equation is obtained from Eq. (5.34) by changing $\hbar\rightarrow3\hbar$ [48]. The principal limitation of the leading-order large-N approximation is that it neglects the setting-sun diagram which is the lowest-order contribution to collisional thermalization of the system [34]. The system therefore does not thermalize at leading order in $1/N$, and the approximation breaks down on a time scale τ_2 which is on the order of the mean free time for binary scattering [49] (collisional thermalization processes in reheating the post-inflationary universe is discussed in a subsequent paper [66]).

Let us now use Eq. (4.13a) to derive the bare semiclassical Einstein equation for the O(N) theory at leading order in $1/N$. This equation contains two parts $\delta\mathcal{S}^{\mathrm{G}}/\delta g_+^{\mu\nu}$ and $\delta\Gamma/\delta g_+^{\mu\nu}$. The latter part is related to the bare energy-momentum tensor $\langle T_{\mu\nu}\rangle$ by Eq. (4.15). At leading order in $1/N$, $\langle T_{\mu\nu}\rangle$ is given by a sum of classical and quantum parts,

$$\langle T_{\mu\nu}\rangle=T_{\mu\nu}^{\mathrm{C}}+T_{\mu\nu}^{\mathrm{Q}}-\frac{\lambda N}{8}\langle\varphi_{\mathrm{H}}^2\rangle^2 g_{\mu\nu}, \tag{5.36}$$

where we define the classical part of $\langle T_{\mu\nu}\rangle$ by

$$\begin{aligned}T_{\mu\nu}^{\mathrm{C}}=N\Big[&(1-2\xi)\hat{\phi}_{;\mu}\hat{\phi}_{;\nu}+\left(2\xi-\frac{1}{2}\right)g_{\mu\nu}g^{\rho\sigma}\hat{\phi}_{;\rho}\hat{\phi}_{;\sigma}\\&-2\xi\hat{\phi}_{;\mu\nu}\hat{\phi}+2\xi g_{\mu\nu}\hat{\phi}\Box\hat{\phi}-\xi G_{\mu\nu}\hat{\phi}^2\\&+\frac{1}{2}g_{\mu\nu}\left(m^2+\frac{\lambda}{4}\hat{\phi}^2\right)\hat{\phi}^2\Big]\end{aligned} \tag{5.37}$$

[8]Note that the index b is not to be summed in the right-hand side of Eq. (5.30), and the c subscript on $\Box$ and R is a CTP index.

and the quantum part of $\langle T_{\mu\nu}\rangle$ by

$$T^{\rm Q}_{\mu\nu}=N\hbar\lim_{x'\to x}\Bigg\{\Bigg[\Bigg[(1-2\xi)\nabla_\mu\nabla'_\nu+\left(2\xi-\frac{1}{2}\right)g_{\mu\nu}g^{\rho\sigma}\nabla_\rho\nabla'_\sigma -2\xi\nabla_\mu\nabla_\nu+2\xi g_{\mu\nu}g^{\rho\sigma}\nabla_\rho\nabla_\sigma-\xi G_{\mu\nu} +\frac{1}{2}g_{\mu\nu}\left(m^2+\frac{\lambda}{2}\hat{\phi}^2+\frac{\hbar\lambda}{4}G_{++}(x,x')\right)\Bigg]G_{++}(x,x')\Bigg\} +O(1). \tag{5.38}$$

The above expression for $T^{\rm Q}_{\mu\nu}$ is divergent in four spacetime dimensions, and needs to be regularized or renormalized. The energy-momentum tensor in the one-loop approximation is obtained by neglecting the $O(\hbar^2)$ terms in Eq. (5.38). It can be shown using Eq. (5.35) that the energy-momentum tensor at leading order in the $1/N$ expansion is covariantly conserved, up to terms of order $O(1)$ (next-to-leading order). The bare semiclassical Einstein equation is then given (in terms of $\langle T_{\mu\nu}\rangle$ shown above) by Eq. (4.14).

At this point we formally set $N=1$ since we are not including next-to-leading-order diagrams in the $1/N$ expansion. This can be envisioned as a simple rescaling of the Planck mass by $\sqrt{N}$, since the matter field effective action Γ is entirely O(N). We now turn to the issue of renormalization.

D. Renormalization

To renormalize the leading-order large-N CTP effective action in a general curved spacetime, one can use dimensional regularization [67], which requires formulating effective action in n spacetime dimensions. This necessitates the introduction of a length parameter μ^{-1} into the classical action, $\lambda\to\lambda\mu^{4-n}$, in order for the classical action to have consistent units. As above, we maintain the restriction $g_+^{\mu\nu}=g_-^{\mu\nu}=g^{\mu\nu}$, and we suppress indices inside functional arguments.

Making a substitution of the gap equation into the leading-order large-N 2PI effective action, we obtain

$$\Gamma[\hat{\phi},g^{\mu\nu}]=\mathcal{S}^{\rm F}[\hat{\phi},g^{\mu\nu}]+\frac{i\hbar N}{2}\,{\rm tr}\ln[(G^{-1})^{ab}]+\frac{\hbar^2N\lambda\mu^{4-n}}{8}\times\int_M d^nx\sqrt{-g}c^{abcd}[G_{ab}(x,x)G_{cd}(x,x)], \tag{5.39}$$

in terms of the CTP propagator $G_{ab}(x,x')$ which satisfies the gap equation

$$(G^{-1})^{ab}=i(\Box_x c^{ab}+\chi^{ab}(x))\delta(x-x')\frac{1}{\sqrt{-g'}}, \tag{5.40}$$

in terms of a four-component "effective mass"

$$\chi^{ab}(x)=(m^2+\xi R)c^{ab}+\frac{\lambda\mu^{4-n}}{2}c^{abcd}[\hat{\phi}_c\hat{\phi}_d+\hbar G_{cd}(x,x)]. \tag{5.41}$$

The divergences in the effective action can be made explicit with the use of the heat kernel $K^a_b(x,y;s)$ [68,67,69]. Let us define $K^a_b(x,y,s)$ which satisfies

$$\frac{\partial K^a_b(x,y;s)}{\partial s}+\int_M d^nz\sqrt{-g_z}c_{cd}(G^{-1})^{ac}(x,z)K^d_b(z,y;s)=0, \tag{5.42}$$

with boundary conditions

$$K^a_b(x,y;0)=\delta^a_b\delta(x-y)\frac{1}{\sqrt{-g_y}} \tag{5.43}$$

at $s=0$ [59]. From Eqs. (5.42) and (5.39) it follows that $K^+_-=K^-_+=0$ for all x, y, and s, and that K^+_+ (K^-_-) is a functional of $\hat{\phi}_+$ ($\hat{\phi}_-$) only. The CTP effective action can then be expressed as

$$\Gamma[\hat{\phi},g^{\mu\nu}]=\Gamma^+_{\rm IO}[\hat{\phi}_+,g^{\mu\nu}]-\Gamma^-_{\rm IO}[\hat{\phi}_-,g^{\mu\nu}], \tag{5.44}$$

in terms of a functional $\Gamma_{\rm IO}$ on M defined by

$$\Gamma^+_{\rm IO}[\hat{\phi}_+,g^{\mu\nu}]=S^{\rm F}[\hat{\phi}_+,g^{\mu\nu}]-\frac{i\hbar N}{2}\int_M d^nx\sqrt{-g}\times\int_0^\infty\frac{ds}{s}K^+_+(x,x;s)+\frac{\hbar^2N\lambda\mu^{4-n}}{8}\times\int_M d^nx\sqrt{-g}\left[\int_0^\infty ds K^+_+(x,x;s)\right]^2, \tag{5.45}$$

and similarly for $\Gamma^-_{\rm IO}$. It follows from Eq. (5.42) that $K^+_+(x,x;s)[\hat{\phi}_+]$ is exactly the same functional of $\hat{\phi}_+$ as $K^-_-(x,x;s)[\hat{\phi}_-]$ is of $\hat{\phi}_-$; we denote it by $K(x,x;s)[\hat{\phi}]$, where $\hat{\phi}$ is a function on M.

The divergences in the effective action arise in the small-s part of the integrations, so that in the equation

$$\int_0^\infty\frac{ds}{s}K(x,x;s)=\int_0^{s_0}\frac{ds}{s}K(x,x;s)+\int_{s_0}^\infty\frac{ds}{s}K(x,x;s) \tag{5.46}$$

only the first term on the right-hand side is divergent. Using the $s\to0^+$ asymptotic expansion for $K(x,x;s)$ [59], one has (for a scalar field, such as the unbroken-symmetry large-N limit of the O(N) model)

$$K(x,x;s)\sim(4\pi s)^{-n/2}\sum_{m=0}^{\infty}s^m a_m(x), \tag{5.47}$$

where the $a_n(x)$ are the well-known "Hamidew coefficients" made up of scalar invariants of the spacetime curvature [1,68]. The divergences then show up as poles in $1/(n-4)$ after the s integrations are performed. They have been evaluated for the $\lambda\Phi^4$ theory in a general spacetime by many authors (see, e.g., [59,70,71]) and in the large-N limit of the O(N) model [72]. At leading order in the $1/N$ expansion, the renormalization of λ, ξ, m, G, Λ, b, and c is required, but no field amplitude renormalization is required.

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