Abelian and non-Abelian induced parity-breaking terms at finite temperature

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We compute the exact canonically induced parity-breaking part of the effective action for 2+1 massive fermions, in particular, Abelian and non-Abelian gauge field backgrounds. The method of computation resorts to the chiral anomaly of the dimensionally reduced theory. [S0556-2821(97)01222-8]

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I. INTRODUCTION AND RESULTS

Three-dimensional gauge theories coupled to matter are relevant both in field theory and condensed matter physics. An important feature of these theories is that, apart from the usual Maxwell or Yang-Mills actions, there exists the possibility of considering a Chern-Simons (CS) term as a part [1] or as the entire [2] gauge field action. Moreover, even if the CS term is not included *ab initio*, it will be induced through fluctuation of Fermi fields [3,4], by the parity-violating fermion mass, and/or by the celebrated parity anomaly [1].

A fundamental property of the CS action is that its presence forces a quantization law: the (non-Abelian) CS term is noninvariant under ''large'' gauge transformations (i.e., gauge transformations carrying a nontrivial winding number), implying that the coefficient of the CS term should be quantized so that $\exp(iS_{\rm CS})$ remains single valued. Concerning the induced (through matter fluctuations) CS term, it is well established that any gauge invariant regularization of the massless fermionic determinant introduces a parity anomaly in the form of $\pm\frac{1}{2}S_{\rm CS}$ whose gauge noninvariance compensates the gauge noninvariance of the otherwise parity preserving effective action [3]. This parity anomalous contribution is also present in the case of massive fermions [5], when other canonical parity-violating terms associated to the fermion mass come into play.

The results above correspond to quantum field theory at zero temperature. What about $T\neq 0$? To our knowledge this question was first addressed in [6] where it was argued that the coefficient of the induced CS term remains unchanged at finite temperature. Contrasting with this analysis, perturbative calculations yielded effective actions with CS coefficients which are smooth functions of the temperature [7–16]. It is important to notice that these computations dealt with the fermion mass-dependent parity breaking and ignored the parity anomaly related to gauge invariant regularizations.

The issue of renormalization of the CS coefficient induced by fermions at $T\neq 0$ was reanalyzed in Refs. [17,18] where it was concluded that, in perturbation theory and on gauge invariance grounds, the effective action for the gauge field cannot contain the smoothly renormalized CS coefficient which was the answer of perturbative calculations. More recently, the exact result for the effective action of a 0+1 massive fermion system [19] as well as nonperturbative calculations of the effective action in the 2+1 Abelian case [20] and its explicit temperature-dependent parity-breaking part [21] have explicitly shown that although the perturbative expansion leads to a nonquantized T-dependent CS coefficient, the complete effective action can be seen to be gauge invariant under both small and large gauge transformations, the temperature-depending shift in the CS coefficient being just a byproduct of considering just the first term in the expansion of the effective action.

We extend in the present work the analysis presented in [21] for the Abelian model to the case of 2+1 massive fermions in a non-Abelian gauge background. By considering a particular class of gauge field background configurations we compute exactly the induced parity-breaking part of the effective action for three-dimensional massive fermions in the fundamental representation of SU(N).

To be precise, we are concerned with

$$\Gamma_{\rm odd}(A,M)=\frac{1}{2}[\Gamma(A,M)-\Gamma(A,-M)], \tag{1}$$

where

$$\exp[-\Gamma(A,M)]=\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\,\exp[-S_F(A,M)], \tag{2}$$

and $S_F(A,M)$ is the action for massive fermions (with mass M) in a gauge background A_μ. As mentioned above (see

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[20] for a discussion), the mass-dependent parity-violating term is not the only one arising in $\Gamma(A,M)$; there is also a local parity anomaly contribution in the form of half a CS term arising in any gauge invariant regularization. This term, first noticed at $T=0$ in [3] for massless fermions and in [5] for massive fermions, is mass and temperature independent and can be removed by a local counterterm at the price of breaking large gauge invariance. It is in fact not taken into account in most of the literature analyzing $(2+1)$-dimensional massive fermion models. To understand the interplay between the two contributions, one can regard the mass-dependent parity-violating term as naturally arising due to the fact that the Lagrangian already contains at the classical level a parity-violating mass term. Concerning the mass-independent contribution which comes from the parity anomaly, it can be seen as a necessary consequence of any gauge invariant regularization of the path integral fermionic measure. After these remarks, it is clear that our definition of $\Gamma_{\rm odd}$ excludes this last anomalous contribution but since it is temperature independent, it does not affect our analysis.

The calculation of Eq. (1) for the general case, namely, for *any* gauge field configuration is not something we can do exactly. Instead of making a perturbative calculation dealing with a small but otherwise arbitrary gauge field configuration, we shall consider a restricted set of gauge field configurations which can, however, be treated exactly.

In order to get an exact result we choose a particular gauge field background which corresponds to a vanishing color electric field and a time-independent color magnetic field

$$A_3=A_3(\tau), \tag{3}$$

$$A_j=A_j(x) \quad (j=1,2) \tag{4}$$

or any equivalent configuration by gauge transformations. In the non-Abelian case we further restrict A_3 to point in a fixed direction in the internal space

$$A_3=|A_3|\check{n}, \tag{5}$$

and A_j to commute with A_3,

$$[A_j,A_3]=0 \quad (j=1,2). \tag{6}$$

Although for SU(2) this implies that all of the components of A_μ commute, and can be thus seen as an ''Abelian-like'' configuration, for SU(N) with $N>2$ one can see that genuine non-Abelian effects are incorporated. The configurations under consideration are reminiscent of the ones treated in [4] for massless fermions at $T=0$; in that case Lorentz covariance of the local result allowed straightforward generalization to arbitrary backgrounds. Unfortunately, this will not be the case here.

Our main result can be presented through the formula we obtain for $\Gamma_{\rm odd}$

$$\Gamma_{\rm odd}=\frac{ig}{4\pi}{\rm tr}\left\{\arctan\left[\tanh\left(\frac{\beta M}{2}\right)\tan\left(\frac{g}{2}\int_0^\beta A_3 d\tau\right)\right]\right.\\ \left.\times\int d^2x\,\epsilon_{ij}F_{ij}\right\}, \tag{7}$$

where g is the coupling constant, $\beta=1/T$, and tr is an adequate trace in SU(N) (matrix functions are defined as usual as power series).

The paper is organized as follows. We give in Sec. II a more detailed description of the results presented in [21] for the Abelian case so as to clarify the method of computation, which relies on the ability to factorize the piece of the effective action depending on the sign of the fermion mass. The same method is applied in Sec. III to the analysis of the SU(N) case leading to formula (7). Finally, in Sec. IV we summarize and give a discussion of our results.

II. THE ABELIAN CASE

We are interested in evaluating the parity-odd piece of the effective action (1) which is induced by integrating out massive fermions coupled to an Abelian gauge field A_μ in $2+1$ dimensions at finite temperature.

The Euclidean action $S_F(A,M)$ is given by

$$S_F(A,M)=\int_0^\beta d\tau\int d^2x\,\bar\psi(\not\partial+ie\not A+M)\psi. \tag{8}$$

We are using Euclidean Dirac's matrices in the representation

$$\gamma_1=\sigma_1, \quad \gamma_2=\sigma_2, \quad \gamma_3=\sigma_3, \tag{9}$$

where σ_i are the familiar Pauli matrices and $\beta=1/T$ is the inverse temperature. The label 3 is used to denote the Euclidean time coordinate τ. The fermionic fields in Eq. (2) obey antiperiodic boundary conditions in the timelike direction

$$\psi(\beta,x)=-\psi(0,x), \quad \bar\psi(\beta,x)=-\bar\psi(0,x), \quad \forall x, \tag{10}$$

with x denoting the two space coordinates. The gauge field satisfies periodic boundary conditions instead:

$$A_\mu(\beta,x)=A_\mu(0,x), \quad \forall x. \tag{11}$$

We want to make a calculation which preserves an interesting property of the imaginary time formulation, namely, that there is room for gauge transformations with nontrivial winding around the time coordinate, and any approximation which assumes the smallness of A_3 may put the symmetry under those large transformations in jeopardy.

Let us first discuss the nontrivial gauge transformations at finite temperature. The set of allowed gauge transformations in the imaginary time formalism is defined in the usual way:

$$\psi(\tau,x)\to e^{-ie\Omega(\tau,x)}\psi(\tau,x), \quad \bar\psi(\tau,x)\to e^{ie\Omega(\tau,x)}\bar\psi(\tau,x),\\ A_\mu(\tau,x)\to A_\mu(\tau,x)+\partial_\mu\Omega(\tau,x), \tag{12}$$

where $\Omega(\tau,x)$ is a differentiable function vanishing at spatial infinity ($|x|\to\infty$), and whose time boundary conditions are chosen in order not to affect the fields' boundary conditions (10) and (11). It turns out that $\Omega(\tau,x)$ can wind an arbitrary number of times around the cyclic time dimension

$$\Omega(\beta,x)=\Omega(0,x)+\frac{2\pi}{e}k, \tag{13}$$

where k is an integer which labels the homotopy class of the gauge transformation.

Invoking gauge invariance of the fermionic determinant

$$\det(\not\partial+ie\not A+M)=\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\,\exp\left\{-\int_0^\beta d\tau\int d^2x\,\bar{\psi}(\not\partial+ie\not A+M)\psi\right\}, \tag{14}$$

we can always perform a gauge transformation of the fermionic fields in the functional integral (14),

$$\psi(\tau,x)=e^{-ie\Omega(\tau,x)}\psi'(\tau,x)\qquad \bar{\psi}(\tau,x)=e^{ie\Omega(\tau,x)}\bar{\psi}'(\tau,x), \tag{15}$$

in order to pass to an equivalent expression where the gauge field is traded for $A'_\mu=A_\mu+\partial_\mu\Omega$:

$$\det(\not\partial+ie\not A+M)=\int \mathcal{D}\psi'\mathcal{D}\bar{\psi}'\exp\left\{-\int_0^\beta d\tau\int d^2x\,\bar{\psi}'\times(\not\partial+ie\not A'+M)\psi'\right\}. \tag{16}$$

We consider the configurations given by Eqs. (3) and (4), namely, A_3 is only a function of τ, and A_j is independent of τ. Under these assumptions, we see that the only τ dependence of the Dirac operator comes from A_3. This dependence can however be erased by a redefinition of the integrated fermionic fields as in Eq. (15) if we take

$$\Omega(\tau)=-\int_0^\tau d\tau' A_3(\tau')+\left(\frac{1}{\beta}\int_0^\beta d\tau' A_3(\tau')+\frac{2\pi k}{e\beta}\right)\tau, \tag{17}$$

where k is the arbitrary integer labeling the homotopy class. Such a transformation renders A'_3 constant. The freedom to choose k could be used to further restrict the values of the constant A'_3

$$0\leqslant A'_3<\frac{2\pi}{e\beta}, \tag{18}$$

or any of the intervals obtained by a translation of this one by an integer number of $2\pi/e\beta$. In this sense, the value of the constant in such an interval is the only ''essential,'' i.e., gauge-invariant, $A_3(\tau)$-dependent information contained in the configurations (3),(4), describing the holonomy $\int_0^\beta d\tilde{\tau}A_3(\tilde{\tau})$ around the time direction (notice that the F_{3j} components of the field curvature tensor identically vanish for these configurations). However, we will limit ourselves to small gauge transformations ($k=0$) in order to avoid any assumption about large gauge invariance of the fermionic measure in Eq. (14) and safely discuss the effect of large gauge transformations on the final results. Thus the constant field A'_3 takes the mean value of $A_3(\tau)$:

$$\tilde{A}_3=\frac{1}{\beta}\int_0^\beta d\tau A_3(\tau). \tag{19}$$

Note that the spatial components of A_μ remain τ independent after this redefinition.

After redefining the fermionic fields according to this prescription, we see that the fermionic determinant we should consider is now

$$\det(\not\partial+ie\not A+M)=\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\,\exp[-S_F(A_j,\tilde{A}_3,M)], \tag{20}$$

where

$$S_F(A_j,\tilde{A}_3,M)=\int_0^\beta d\tau\int d^2x\,\bar{\psi}[\not\partial+ie(\gamma_jA_j+\gamma_3\tilde{A}_3)+M]\psi, \tag{21}$$

and we removed the primes for the sake of clarity.

Since the Dirac operator in the previous equation is invariant under imaginary time translations it is convenient to perform a Fourier transformation on the time variable for ψ and $\bar{\psi}$:

$$\psi(\tau,x)=\frac{1}{\beta}\sum_{n=-\infty}^{+\infty}e^{i\omega_n\tau}\psi_n(x),$$

$$\bar{\psi}(\tau,x)=\frac{1}{\beta}\sum_{n=-\infty}^{+\infty}e^{-i\omega_n\tau}\bar{\psi}_n(x), \tag{22}$$

where $\omega_n=(2n+1)\pi/\beta$ is the usual Matsubara frequency for fermions. Then the Euclidean action is written as an infinite series of decoupled actions, one for each Matsubara mode:

$$S_F(A_j,\tilde{A}_3,M)=\frac{1}{\beta}\sum_{n=-\infty}^{+\infty}\int d^2x\,\bar{\psi}_n(x)[\not d+M+i\gamma_3\times(\omega_n+e\tilde{A}_3)]\psi_n(x), \tag{23}$$

where $\not d$ is the $1+1$ Euclidean Dirac operator corresponding to the spatial coordinates and the spatial components of the gauge field

$$\not d=\gamma_j(\partial_j+ieA_j). \tag{24}$$

As the action splits up into a series and the fermionic measure can be written as

$$\mathcal{D}\psi(\tau,x)\mathcal{D}\bar{\psi}(\tau,x)=\prod_{n=-\infty}^{n=+\infty}\mathcal{D}\psi_n(x)\mathcal{D}\bar{\psi}_n(x) \tag{25}$$

the $2+1$ determinant is an infinite product of the corresponding $1+1$ Euclidean Dirac operators

$$\det(\not\partial+ie\not A+M)=\prod_{n=-\infty}^{n=+\infty}\det[\not d+M+i\gamma_3(\omega_n+e\tilde{A}_3)]. \tag{26}$$

Explicitly, the $1+1$ determinant for a given mode is a functional integral over $1+1$ fermions

$$\det[\not{d}+M+i\gamma_3(\omega_n+e\widetilde{A}_3)] = \int \mathcal{D}\chi_n \mathcal{D}\overline{\chi_n} \exp\left\{-\int d^2x \overline{\chi_n}(x)[\not{d}+M+i\gamma_3 \times(\omega_n+e\widetilde{A}_3)]\chi_n(x)\right\}. \tag{27}$$

In order to compute $\Gamma_{\rm odd}$ we factorize now these determinants in a piece which is sensitive to the sign of M and a piece which is not. The Euclidean action S_n corresponding to the mode n may be conveniently recasted in the following form:

$$S_n = \int d^2x \overline{\chi_n}(\not{d}+\rho_n e^{i\gamma_3\phi_n})\chi_n \tag{28}$$

with

$$\rho_n = \sqrt{M^2+(\omega_n+e\widetilde{A}_3)^2}; \quad \phi_n = \arctan\left(\frac{\omega_n+e\widetilde{A}_3}{M}\right). \tag{29}$$

We next realize that the change of fermionic variables

$$\chi_n(x) = e^{-i(\phi_n/2)\gamma_3}\chi'_n(x), \quad \overline{\chi_n}(x) = \overline{\chi'}_n(x) e^{-i(\phi_n/2)\gamma_3} \tag{30}$$

makes the action S_n independent of ϕ_n. This is not a gauge transformation but a global chiral rotation in the $1+1$ Euclidean fermionic variables. Correspondingly, the fermionic measure picks up an anomalous Fujikawa Jacobian [22] so that one ends with

$$\det[\not{d}+M+i\gamma_3(\omega_n+e\widetilde{A}_3)] = J_n[A,M]\ \det[\not{d}+\rho_n], \tag{31}$$

where

$$J_n[A,M] = \exp\left(-i\frac{e\phi_n}{2\pi}\int d^2x \epsilon_{jk}\partial_j A_k\right), \tag{32}$$

with ϵ_{jk} denoting the $1+1$ Euclidean Levi-Civita symbol.

Recalling the definition of $\Gamma_{\rm odd}$, we see that the second factor in expression (31) does not contribute to it, since it is invariant under $M\to -M$. The Jacobian (32), instead, changes to its inverse. As a consequence, the parity odd piece in the effective action is given in terms of the infinite set of n-dependent Jacobians

$$\exp[-\Gamma_{\rm odd}] = \prod_{n=-\infty}^{n=+\infty} J_n[A,M] \tag{33}$$

or

$$\Gamma_{\rm odd} = -\sum_{n=-\infty}^{n=+\infty} \ln J_n[A,M] = i\frac{e}{2\pi}\sum_{n=-\infty}^{n=+\infty} \phi_n \int d^2x \epsilon_{jk}\partial_j A_k. \tag{34}$$

There only remains to perform the summation over the ϕ_n's. This can be done by using standard techniques in finite temperature field Theory. We define

$$\mathcal{S} = \sum_{n=-\infty}^{n=+\infty} \arctan\left(\frac{\omega_n+e\widetilde{A}_3}{M}\right), \tag{35}$$

whose sign will obviously depend on the sign of M. We make this explicit by rewriting $\mathcal{S}$ as

$$\mathcal{S} = \frac{M}{|M|}\sum_{n=-\infty}^{n=+\infty} \arctan\left(\frac{\omega_n+e\widetilde{A}_3}{|M|}\right) \tag{36}$$

or, using the expression for ω_n,

$$\mathcal{S}(x,y) = \frac{M}{|M|}\sum_{n=-\infty}^{n=+\infty} \arctan\left(\frac{(2n+1)\pi+x}{y}\right), \tag{37}$$

where $x=e\beta\widetilde{A}_3$, and $y=\beta|M|$ are the two dimensionless parameters built from the original ones. This series must be regularized, and the standard technique consists in subtracting the zero-field value of each term; notice that the sum of these zero-field contributions conditionally converges to 0. Then

$$\mathcal{S}(x,y) = \frac{M}{|M|}\sum_{n=-\infty}^{n=+\infty} \int_0^x du \frac{d}{du}\arctan\left(\frac{(2n+1)\pi+u}{y}\right). \tag{38}$$

As the series now converges absolutely we can first perform the summation. The sum to be evaluated is then

$$\sum_{n=-\infty}^{n=+\infty} \frac{y}{y^2+[(2n+1)\pi+u]^2}, \tag{39}$$

which is solved by the summation formula

$$\sum_{n=-\infty}^{n=+\infty} \frac{1}{(n-x_1)(n-x_2)} = -\frac{\pi[\cot(\pi x_1)-\cot(\pi x_2)]}{x_1-x_2}. \tag{40}$$

After performing the integral we get

$$\mathcal{S} = \frac{M}{|M|}\arctan\left[\tanh\left(\frac{\beta|M|}{2}\right)\tan\left(\frac{1}{2}e\beta\widetilde{A}_3\right)\right]. \tag{41}$$

Thus the parity-odd part of Γ finally reads

$$\Gamma_{\rm odd} = i\frac{e}{2\pi}\frac{M}{|M|}\arctan\left[\tanh\left(\frac{\beta|M|}{2}\right) \times\tan\left(\frac{e}{2}\int_0^\beta d\tau A_3(\tau)\right)\right]\int d^2x \epsilon_{jk}\partial_j A_k. \tag{42}$$

There are several observations to be made about our result (42). First we observe that this result has the proper zero temperature limit

$$\lim_{T\to 0}\Gamma_{\rm odd} \to \frac{1}{2}\frac{M}{|M|}S_{\rm CS}, \tag{43}$$

where $S_{\rm CS}$ is the Chern-Simons action

$$S_{\mathrm{CS}}=i\frac{e^2}{4\pi}\int d^3x\,\epsilon_{\mu\nu\alpha}A_\mu\partial_\nu A_\alpha, \qquad (44)$$

which shows up in our particular configuration (3),(4) as $(e^2/2\pi)\int d\tau A_3(\tau)\int d^2x\epsilon_{ij}\partial_i A_j$. So we get the induced Chern-Simons term at zero temperature. As it is well known, in the zero temperature case the result is not invariant under large gauge transformations. The quantization of the spatial integral that measures the flux of the magnetic field through a spacelike manifold $\tau=$const in units of $2\pi/e$ shows that Eq. (43) changes by the addition of an odd multiple of $i\pi$ under a large gauge transformation with odd winding number when the magnetic flux is odd. This gauge noninvariance is compensated by the parity anomaly discussed in the Introduction when the complete result is regularized in a gauge invariant scheme.

The same situation occurs in the finite temperature result (42). A large gauge transformation with odd winding number $k=2p+1$ shifts the argument of the tangent in $(2p+1)\pi$. Although the tangent is not sensitive to such a change, one has to keep track of it by shifting the branch used for arctan definition. This amounts to the same result as in the $T\to 0$ limit: the gauge noninvariance of Γ_{odd} under large gauge transformations is compensated by the parity anomaly $\pm\frac{1}{2}S_{\mathrm{CS}}$.

Now we observe that a perturbative expansion in terms of e yields the usual perturbative result

$$\Gamma_{\mathrm{odd}}=\frac{1}{2}\frac{M}{|M|}\tanh\left(\frac{|M|\beta}{2}\right)S_{\mathrm{CS}}+O(e^4), \qquad (45)$$

where the coefficient of the Chern-Simons term acquires a smooth dependence on the temperature. Were we considering only the first nontrivial order in perturbation theory, we would find a clash between temperature dependence and gauge invariance [17,18]: the gauge noninvariance of the induced CS term is no longer compensated by the parity anomaly. Now we learn, as it was stressed in [19] in a (0+1)-dimensional example and in [20] in 2+1 dimensions, that one has to consider the full result in order to analyze gauge invariance.

Finally, we observe that the result (42) is not an extensive quantity in Euclidean time. It is, however, extensive in space, and that is indeed all one expects in finite temperature field theory. In contrast, the $T=0$ limit becomes an extensive quantity in space-time, as is expected from zero temperature field theory.

We shall now extend the previous results, obtained for space-independent A_3 and time-independent A_j to the somewhat more general situation of a smooth spatial dependence of A_3 besides the previous arbitrary time dependence.

The fermionic determinant we should calculate, after getting rid of the τ dependence of A_3 will have a form analogous to Eq. (20) with the only difference of having an x dependence in $\widetilde{A}_3$. As there is no explicit time dependence in the Dirac operator, we again pass to a Fourier description of the time coordinate. Defining the x-dependent fields $\rho_n(x)$ and $\phi_n(x)$,

$$\rho_n(x)=\sqrt{M^2+[\omega_n+e\widetilde{A}_3(x)]^2};$$

$$\phi_n(x)=\arctan\left(\frac{\omega_n+e\widetilde{A}_3(x)}{M}\right), \qquad (46)$$

we have for the complete fermionic determinant an expression equivalent to the previous case:

$$\det(\not\partial+ie\not A+M)=\prod_{n=-\infty}^{\infty}\det[\not d+\rho_n(x)e^{i\gamma_3\phi_n(x)}]. \qquad (47)$$

The determinant corresponding to the n-mode is again written as a functional integral over $(1+1)$-dimensional fields, but a transformation such as Eq. (30) is now a *local* chiral rotation of the $(1+1)$-dimensional fermions and gives rise to

$$\det[\not d+\rho_n(x)e^{i\gamma_3\phi_n(x)}]=J_n\det[\not d'+\rho_n(x)], \qquad (48)$$

where

$$\not d'=\not d-\frac{i}{2}\not\partial\phi_n\gamma_3 \qquad (49)$$

and the anomalous Jacobian reads

$$J_n=\exp\left\{-i\frac{e}{2\pi}\int d^2x\left[\phi_n(x)\epsilon_{jk}\partial_j A_k+\frac{1}{4}\phi_n(x)\Delta\phi_n(x)\right]\right\}. \qquad (50)$$

The x dependence of the phase factor ϕ_n affects the result in two ways. First, we see that the field redefinition changes the operator $\not d$ to $\not d'$ which depends on the sign of M, and so there will be a contribution to Γ_{odd} coming from the determinant of $\not d'+\rho_n(x)$. Second, the Jacobian is now a more involved function of ϕ_n, since the field redefinition affects the Dirac operator which is used to define the fermionic integration measure. In a first approximation, we shall only take into account the contribution coming from the Jacobian, since the one that follows from the determinant of the Dirac operator is of higher order in a derivative expansion (and we are assuming that the x dependence of $\widetilde{A}_3$ is smooth). The contribution which is quadratic in ϕ_n is irrelevant to the parity breaking piece, since it is even in M. Thus, neglecting the terms containing derivatives of $\widetilde{A}_3$, we have for Γ_{odd} a result which looks similar to a natural generalization of the previous case:

$$\Gamma_{\mathrm{odd}}=i\frac{e}{2\pi}\frac{M}{|M|}\int d^2x\ \arctan\left[\tanh\left(\frac{|M|\beta}{2}\right)\times\tan\left(\frac{e}{2}\int_0^\beta d\tau A_3(\tau,x)\right)\right]\epsilon_{jk}\partial_j A_k(x). \qquad (51)$$

It is not hard to check that the reliability of the approximation of neglecting derivatives of $\widetilde{A}_3$ is assured if the condition

$$|e\partial_j\widetilde{A}_3|\ll M^2 \qquad (52)$$

is fulfilled. To end with this example, let us point that all the remarks we made for the case of a space-independent A_3 also apply to this case.

III. THE NON-ABELIAN CASE

We extend in this section the previous analysis to the non-Abelian case. Although we shall consider, as in the Abelian case, particular background field configurations which allow us to make exact computations, the results will exhibit genuine non-Abelian effects through nontrivial commutators of spatial components of the gauge field. Our analysis will be valid for the SU(N) case although some points are made explicit for the particular $N=2,3$ cases. As we shall see, details arising in calculations are due to technicalities associated with handling the non-Abelian symmetry; once they are overcome, the results appear as a natural extension of the Abelian ones.

The Euclidean fermionic action which describes the system is now written as

$$S_F(A,M)=\int_0^\beta d\tau\int d^2x\,\overline{\psi}(\not{D}+M)\psi, \tag{53}$$

where the covariant derivative acting on the fermions in the fundamental representation of SU(N) is defined as

$$D_\mu=\partial_\mu+igA_\mu, \tag{54}$$

and the gauge connection A_μ is written as

$$A_\mu=A_\mu^a\tau_a \tag{55}$$

with τ_a denoting Hermitian generators of the Lie algebra $(a=1,\dots,N^2-1)$, verifying the relations

$$\tau_a^\dagger=\tau_a,\quad [\tau_a,\tau_b]=if_{abc}\tau_c,\quad \mathrm{tr}(\tau_a\tau_b)=\frac{1}{2}\delta_{ab}, \tag{56}$$

with f_{abc} the totally antisymmetric structure constants. For the particular case of SU(2), which we shall consider in more detail, we have $f_{abc}=\epsilon_{abc}$ since the generators will be taken to be the usual Pauli matrices.

We are concerned with the parity-odd piece of the effective action defined in Eq. (1). Fermionic (bosonic) fields satisfy again antiperiodic (periodic) boundary conditions in the timelike direction.

We shall in this case restrict the set of configurations for the gauge fields given by Eqs. (3)–(6) in order to be able to calculate Γ_{odd} exactly. Before doing so, let us clarify a point about the nature of the gauge group boundary conditions in imaginary time.

Non-Abelian gauge transformations are defined by their action on the fermionic and gauge fields:

$$\psi(\tau,x)\to\psi^U(\tau,x)=U(\tau,x)\psi(\tau,x),$$

$$\overline{\psi}(\tau,x)\to\overline{\psi}^U(\tau,x)=\psi(\tau,x)U^\dagger(\tau,x),$$

$$A_\mu(\tau,x)\to A_\mu^U(\tau,x)=U(\tau,x)A_\mu(\tau,x)U^\dagger(\tau,x)$$

$$-\frac{i}{g}U(\tau,x)\partial_\mu U^\dagger(\tau,x). \tag{57}$$

In order to decide the boundary conditions the gauge group element should satisfy in the timelike direction, one requires that the periodicity of the gauge field and the antiperiodicity of the fermions is unaltered under a gauge transformation. Concerning the gauge field, this only imposes on U the condition

$$U(\beta,x)=hU(0,x), \tag{58}$$

where h is an element of Z_N, the center of SU(N). Now, concerning fermions, the condition on U depends on whether they are in the fundamental or adjoint representation. In the fundamental one, it is easily seen that

$$U(\beta,x)=U(0,x), \tag{59}$$

while in the adjoint representation, condition (58) follows instead. As we assume fermions are in the fundamental representation, the group elements $U(\tau,x)$ are taken to be strictly periodic [a condition in fact analogous to the one used for the Abelian case in Eq. (13)]. One can then prove [23] that for compact groups

$$w(U)=\frac{1}{12\pi^2N}\mathrm{tr}\int d^3x\,\epsilon_{\mu\nu\alpha}U^{-1}\partial_\mu UU^{-1}\partial_\nu UU^{-1}\partial_\alpha U \tag{60}$$

is an integer number that labels homotopically equivalent gauge transformations. Thus the distinction between large and small gauge transformations has a different origin here than in the Abelian case.

We thus consider a class of configurations equivalent by gauge transformations to

$$A_3=|A_3|(\tau)\check{n}, \tag{61}$$

$$A_j=A_j(x),\quad [A_j,\check{n}]=0\,(j=1,2). \tag{62}$$

where $\check{n}$ is a fixed direction in the Lie algebra ($\check{n}=n^a\tau_a$, $n^an^a=1$).

We note that conditions (61) and (62) assure the vanishing of the color electric fields, as well as the time independence of the color magnetic fields. Regarding the condition (62), which requires the spatial gauge field components to commute with A_3, it is worth remarking that its consequences depend strongly on whether the group considered is SU(2) or SU(N) with $N>2$. In the former case, the only solution to Eq. (62) corresponds to a configuration with all the gauge field components pointing in the same direction $\check{n}$ in internal space, i.e., an ''Abelian-like'' configuration. In contrast, for $N>2$, configurations with $[A_1,A_2]\neq 0$ are indeed possible.

To make the point above more explicit let us analyze the simple specific example of SU(3) with the generators given by the standard Gell-Mann matrices; one can then take A_1 and A_2 as linear combinations of τ_1, τ_2 and τ_3 [generators of a SU(2) subgroup] and A_3 pointing in the direction of τ_8. This situation easily generalizes to $N>3$ since one can construct the set of generators for a higher N in such a way that it contains the generators corresponding to SU($N-1$) as a subset of block-diagonal matrices, and one of the extra generators can be always defined as to commute with them. Thus it is possible to take A_1 and A_2 as noncommuting vectors in the subalgebra corresponding to SU($N-1$) and A_3 commuting with them.

Coming back to the general case, let us point out that, as in the Abelian case, one can erase the τ dependence of A_3 component by considering a change of variables for the fermionic fields corresponding to a gauge transformation of the form

$$U(t)=e^{ig\Omega(\tau)\check{n}} \tag{63}$$

and

$$\Omega(\tau)=-\int_0^{\tau} d\tau' A_3^{(\check{n})}(\tau')+\left(\frac{1}{\beta}\int_0^{\beta} d\tau' A_3^{(\check{n})}(\tau')\right)\tau. \tag{64}$$

Now, because of condition (62) the space components of the gauge field remain unchanged under this transformation, while A_3 takes the constant value $\tilde{A}_3=(1/\beta)\int_0^{\beta} d\tau A_3(\tau) =|\tilde{A}_3|\check{n}$. After these remarks, we assume a gauge transformation has been made on the fermions in order to reach a constant $\tilde{A}_3$ and the rest of conditions (61) and (62) for the gauge field.

After a Fourier transformation on the time variable for ψ and $\overline{\psi}$ of the form (22) the Euclidean action can be written as an infinite series of decoupled actions:

$$S_F=\frac{1}{\beta}\sum_{n=-\infty}^{+\infty}\int d^2x\,\overline{\psi_n}(x)[\not{d}+M+i\gamma_3(\omega_n+g\tilde{A}_3^a\tau_a)]\psi_n(x), \tag{65}$$

where $\not{d}=\gamma_j(\partial_j+igA_j)$ is the non-Abelian Dirac operator corresponding to the spatial coordinates and the spatial components of the gauge field. Concerning the fermionic measure, we write it in the form

$$\mathcal{D}\psi(\tau,x)\mathcal{D}\overline{\psi(}\tau,x)=\prod_{n=-\infty}^{n=+\infty}\mathcal{D}\psi_n(x)\mathcal{D}\overline{\psi_n}(x), \tag{66}$$

so that again the $2+1$ determinant becomes an infinite product of the corresponding $1+1$ Euclidean Dirac operators

$$\det(\not{\partial}+ig\not{A}+M)=\prod_{n=-\infty}^{n=+\infty}\det[\not{d}+M+i\gamma_3(\omega_n+g\tilde{A}_3^a\tau_a)]. \tag{67}$$

We now show that the same trick which leads to the decoupling of parity-breaking and parity-conserving parts of the determinant for the Abelian case can be applied here. First, we use the property

$$M+i\gamma_3(\omega_n+g\tilde{A}_3^a\tau_a)=\rho_n e^{i\phi_n}, \tag{68}$$

where

$$\rho_n=\sqrt{M^2+(\omega_n+g\tilde{A}_3^a\tau_a)^2};\quad \phi_n=\arctan\left(\frac{\omega_n+g\tilde{A}_3^a\tau_a}{M}\right). \tag{69}$$

The usual definition of functions of matrices in terms of power series has been used above. It is important to realize that, being ϕ_n a nontrivial Hermitean function of a matrix in the Lie algebra, it will in general have components along the generators τ_a and also along the identity matrix, namely,

$$\phi_n=\phi_n^0 1+\phi_n^a\tau_a. \tag{70}$$

As an illustration, we consider the SU(2) case. A somewhat lenghty but otherwise straightforward calculation yields explicit expressions for these components of ϕ_n:

$$\phi_n^0=\frac{1}{2}\arctan\left(\frac{2M\omega_n}{M^2+\frac{g^2}{4}|\tilde{A}_3|^2-\omega_n^2}\right),$$

$$\phi_n^a=\arctan\left(\frac{gM|\tilde{A}_3|}{M^2-\frac{g^2}{4}|\tilde{A}_3|^2+\omega_n^2}\right)n^a. \tag{71}$$

The $1+1$ determinant for a given mode is a functional integral over $1+1$ fermions that using Eq. (68) can be written as

$$\det[\not{d}+M+i\gamma_3(\omega_n+g\tilde{A}_3^a\tau_a)] =\int \mathcal{D}\chi_n\mathcal{D}\overline{\chi_n}\exp\left\{-\int d^2x\,\overline{\chi_n}(x)(\not{d}+\rho_n e^{i\gamma_3\phi_n})\times\chi_n(x)\right\}. \tag{72}$$

We now perform the change of fermionic variables

$$\chi_n(x)=e^{-i(\phi_n/2)\gamma_3}\chi'_n(x),\quad \overline{\chi_n}(x)=\overline{\chi'}_n(x)e^{-i(\phi_n/2)\gamma_3}, \tag{73}$$

and verify that due to the last condition in Eq. (62) it indeed decouples the parity-violating part of the effective action. We find, including the anomalous Fujikawa Jacobian

$$\det[\not{d}+M+ig\gamma_3(\omega_n+\tilde{A}_3^a\tau_a)]=J_n\det[\not{d}+\rho_n]. \tag{74}$$

The Jacobian in Eq. (74) reads [22]

$$J_n[A,M]=\exp\left[-i\,\mathrm{tr}\frac{\phi_n}{2}\int d^2x\,\mathcal{A}\right], \tag{75}$$

with $\mathcal{A}=\mathcal{A}^a\tau^a$ denoting the $1+1$ Euclidean anomaly under an infinitesimal non-Abelian axial transformation. As this transformation is x independent, there is no difference between finite and infinitesimal transformations and one can just simply iterate the infinitesimal Fujikawa Jacobian [22] in order to get the finite answer (75). Also note that ϕ_n^0 (the component along the identity) does not contribute to the Jacobian since $\mathrm{tr}(\phi_n^0\mathcal{A})=0$. A standard calculation leads for the two-dimensional non-Abelian anomaly the answer (see, for example, [24])

$$\mathcal{A}=\frac{g}{2\pi}\epsilon_{ij}F_{ij} \tag{76}$$

so that the Jacobian finally takes the form

$$J_n[A,M]=\exp\left[-\frac{ig}{4\pi}\mathrm{tr}\left(\phi_n\int d^2x\,\epsilon_{ij}F_{ij}\right)\right]. \tag{77}$$

We see from Eqs. (67) and (74) that the parity odd piece of the effective action is again given in terms of the infinite set of n-dependent Jacobians:

$$\Gamma_{\mathrm{odd}}[A,M]=-\sum_{n=-\infty}^{n=+\infty}\ln J_n[A,M] =\frac{ig}{4\pi}\mathrm{tr}\left[\left(\sum_{n=-\infty}^{+\infty}\phi_n\right)\int d^2x\,\epsilon_{ij}F_{ij}\right]. \tag{78}$$

Now we have to perform the summation over the ϕ_n's. A careful analysis of the steps performed in the Abelian case shows that the result (41) is valid for matrix valued gauge fields. Thus we get

$$\Gamma_{\mathrm{odd}}=\frac{ig}{4\pi}\mathrm{tr}\left\{\arctan\left[\tanh\left(\frac{\beta M}{2}\right)\tan\left(\frac{g}{2}\beta\widetilde{A}_3\right)\right]\int d^2x\,\epsilon_{ij}F_{ij}\right\}. \tag{79}$$

This is the main result in this section, which extends Eq. (42) to SU(N) background fields.

We can check this result by doing explicit computations with the components ϕ_n^a given in Eq. (71) for the SU(2) case. From Eq. (78),

$$\Gamma_{\mathrm{odd}}[A,M]=\frac{ig}{8\pi}\sum_{n=-\infty}^{+\infty}\phi_n^a\int d^2x\,\epsilon_{ij}F_{ij}^a. \tag{80}$$

Using Eq. (71) we have to compute

$$\Sigma=\sum_{n=-\infty}^{\infty}\arctan\left(\frac{gM|\widetilde{A}_3|}{M^2-\frac{g^2}{4}|\widetilde{A}_3|^2+\omega_n^2}\right) \tag{81}$$

or, in terms of dimensionless variables,

$$m=\beta M,\quad x=\frac{g}{2}\beta|\widetilde{A}_3|, \tag{82}$$

$$\Sigma(x,m)=\sum_{n=-\infty}^{\infty}\arctan\left(\frac{2mx}{m^2-x^2+(2n+1)^2\pi^2}\right). \tag{83}$$

The sum is convergent, but in order to calculate Σ it will be convenient to write

$$\Sigma(x,m)=\int_0^x du\,\frac{\partial\Sigma}{\partial u}(u,m). \tag{84}$$

The implicit subtraction of a zero-field contribution vanishes term by term in this case.

After some calculations, one has

$$\frac{\partial\Sigma}{\partial x}(x,m)=2m\sum_{n=-\infty}^{\infty}\frac{m^2+(2n+1)^2\pi^2+x^2}{[m^2+(2n+1)^2\pi^2-x^2]^2+4m^2x^2}. \tag{85}$$

One could now arrange this expression to use the summation formula (40). With the purpose of illustration we use instead the standard Regge-type trick to rewrite Eq. (85) as a contour integral of the form

$$\frac{\partial\Sigma}{\partial x}(x,m)=-\frac{m}{2\pi i}\oint_C dz\tanh(z/2)\frac{m^2-z^2+x^2}{[m^2-z^2-x^2]^2+4m^2x^2}, \tag{86}$$

where C is a contour including all the poles of tanh(z/2). After continuing C into the upper and lower half-planes to pick up the four poles of the fraction only, we end with

$$\frac{\partial\Sigma}{\partial x}(x,m)=\frac{i}{2}\left[\tanh\left(\frac{x-im}{2}\right)-\tanh\left(\frac{x+im}{2}\right)\right]. \tag{87}$$

Using this expression in Eq. (84) we finally get

$$\Sigma(x,m)=2\arctan[\tanh(m/2)\tan(x/2)] \tag{88}$$

so that Γ_{odd} can be written as

$$\Gamma_{\mathrm{odd}}=\frac{ig}{4\pi}\arctan\left[\tanh\left(\frac{\beta M}{2}\right)\tan\left(\frac{g}{4}\beta|\widetilde{A}_3|\right)\right]n^a\int d^2x\,\epsilon_{ij}F_{ij}^a. \tag{89}$$

Finally, observing that $(n^a\tau_a)^{(2k+1)}=(1/2^{2k})n^a\tau_a$ and only odd powers enter the expansions of the functions involved, we see that the result is identical to Eq. (79).

In order to analyze the result (79) let us write it in the most general form

$$\Gamma_{\mathrm{odd}}=\frac{ig}{4\pi}\mathrm{tr}\left\{\arctan\left[\tanh\left(\frac{\beta M}{2}\right)\times\tan\left(\frac{g}{2}\int_0^\beta d\tau A_3(\tau)\right)\right]\int d^2x\,\epsilon_{ij}F_{ij}\right\}. \tag{90}$$

Then we note that in the zero-temperature limit one has

$$\lim_{T\to 0}\Gamma_{\mathrm{odd}}=\frac{ig^2}{8\pi}\frac{M}{|M|}\mathrm{tr}\left(\int_0^\beta d\tau A_3(\tau)\int d^2x\,\epsilon_{ij}F_{ij}\right). \tag{91}$$

This result is the usual one: namely,

$$\lim_{T\to 0}\Gamma_{\mathrm{odd}}=\frac{1}{2}\frac{M}{|M|}S_{\mathrm{CS}}, \tag{92}$$

restricted to the particular background we have considered. Here S_{CS} is the non-Abelian CS action

$$S_{\mathrm{CS}}=\frac{ig^2}{8\pi}\int d^3x\,\epsilon_{\mu\nu\alpha}\mathrm{tr}\left(F_{\mu\nu}A_\alpha-\frac{2}{3}A_\mu A_\nu A_\alpha\right), \tag{93}$$

which for a gauge field satisfying the restrictions (62) reads

$$S_{\rm CS}=\frac{ig^2}{4\pi}{\rm tr}\int d^3x A_3\epsilon_{ij}F_{ij}\,. \tag{94}$$

We thus recover the zero-temperature result first obtained in [3] by calculating the vacuum expectation value of the fermion current in a constant non-Abelian field strength tensor background or in [4] in a static non-Abelian magnetic background such as ours. We recall, however, that gauge invariance under large gauge transformations is obtained only when the parity anomaly $\pm\frac{1}{2}S_{\rm CS}$ is added to the mass- and temperature-dependent expression for $\Gamma_{\rm odd}$.

We finally note that a perturbative expansion in powers of the coupling constant g shows a smooth temperature dependence of the CS coefficient:

$$\Gamma_{\rm odd}=\frac{1}{2}\tanh\left(\frac{M\beta}{2}\right)S_{\rm CS}+O(e^4). \tag{95}$$

Concerning the gauge invariance of the finite temperature result we note that, in contrast to the Abelian case, there is no room for large gauge transformations preserving the conditions (61) and (62) under which our result (90) was obtained. We can only quote gauge invariance under small gauge transformations that do not mix spatial and time components. However, we expect that the large gauge invariance apparently broken by the perturbative expansion (95) should be recovered by the full result.

IV. SUMMARY AND DISCUSSION

We have been able to compute the exact form of the parity-violating contribution to the finite temperature effective action for $2+1$ massive fermions in a restricted set of gauge backgrounds, both for Abelian and non-Abelian gauge groups. Our computation reproduces the standard results both at zero temperature and/or perturbation theory.

The Abelian case allows for a complete analysis of the gauge invariance under large transformations; we have found that the mass and temperature-dependent contribution is not invariant but its variation is cancelled (modulo $2\pi i$) when the parity anomalous contribution $\pm\frac{1}{2}S_{\rm CS}$ is incorporated. We recall that in the zero temperature limit the gauge invariant result contains two contributions in the form of CS terms, one arising canonically from the fermion mass parity-violating term and the other coming from the necessary parity anomaly of the gauge invariant fermionic measure in odd dimensions. The present analysis gives a closed answer to the puzzle of gauge invariance of the effective action at finite temperature: the perturbative result in which the CS coefficient acquires a smooth dependence on the temperature is correct, but shows that any perturbative order is insufficient to maintain large gauge invariance.

The non-Abelian case follows the pattern described above in every detail. Although the restrictions imposed on the background fields do not allow the study of large gauge transformations, notice that the zero temperature limit shows the presence of two CS contributions with appropriate coefficients so as to cancel the gauge noninvariance of each other. This strongly suggests that the same behavior is to be expected concerning large gauge transformations at finite temperature.

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