Cosmic string formation from correlated fields

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We simulate the formation of cosmic strings at the zeros of a complex Gaussian field with a power spectrum $P(k) \propto k^n$, specifically addressing the issue of the fraction of length in infinite strings. We make two improvements over previous simulations: we include a nonzero random background field in our box to simulate the effect of long-wavelength modes, and we examine the effects of smoothing the field on small scales. The inclusion of the background field significantly reduces the fraction of length in infinite strings for $n < -2$. Our results are consistent with the possibility that infinite strings disappear at some $n = n_c$ in the range $-3 \leq n_c < -2.2$, although we cannot rule out $n_c = -3$, in which case infinite strings would disappear only at the point where the mean string density goes to zero. We present an analytic argument which suggests the latter case. Smoothing on small scales eliminates closed loops on the order of the lattice cell size and leads to a "lattice-free" estimate of the infinite string fraction. As expected, this fraction depends on the type of window function used for smoothing. [S0556-2821(97)04214-8]

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I. INTRODUCTION

Cosmic strings are effectively one-dimensional topological defects which may form at a phase transition in the early universe (see Ref. [1] for a review). Although much of the early interest in cosmic strings has centered on the possibility that they might have served as seeds for the formation of large-scale structure, cosmic strings are interesting physical objects in any case, and they have analogues in the study of condensed matter [2].

Any investigation of the evolution and cosmological consequences of cosmic strings must begin with the study of the initial cosmic string configuration, a study which was first undertaken by Vachaspati and Vilenkin [3], and reexamined by many others [4–9]. Although subsequent cosmic string evolution will erase many of the details of the initial configuration, one fundamental property of the initial conditions is crucial to the subsequent evolution: the existence of infinite strings. Without the existence of infinite strings, the cosmic strings produced at the phase transition may all decay via gravitational radiation long before they can have any interesting cosmological effects. Vachaspati and Vilenkin found in their simulation that roughly 80% of the string was in the form of infinite strings [3]. Using a very different type of simulation for the string formation process, Borrill [8] claimed that this fraction was, in fact, zero, while Robinson and Yates [9], in a study of the dependence of this fraction on the power spectrum of the initial field, argued that for power spectra of the form $P(k) \propto k^n$, the infinite string fraction f_∞ drops to zero for $n \leq -2$. In fact, this result is not obvious from their simulations; it is based on fitting f_∞ to an analytic function for $n > -2$.

In this paper, we extend the simulations of Robinson and Yates in two ways. First, we include the effects of long-wavelength modes which are absent from earlier simulations. Second, we examine the effect of smoothing the initial field

to remove lattice effects. We find that the former change has a dramatic effect on f_∞, sharply reducing f_∞ for $n < -2$. Smoothing also affects f_∞, leading to "lattice-free" estimates of the fraction in infinite strings. In the next section we present our numerical results, and in Sec. III we discuss briefly our main conclusions.

II. NUMERICAL SIMULATIONS

With one exception [8], simulations of cosmic string formation typically make use of a lattice. A value of the field ϕ associated with the string is assigned to each of the cells of the lattice, and the location of the cosmic string is then identified with edges of the lattice around which the field winds through 360°. The original simulations of Vachaspati and Vilenkin [3] were performed on a cubic lattice with no correlations between values of the field in different cells. Subsequent researchers investigated changing the probability of string formation by "biasing" the distribution of the values of the field [5–7], by allowing the field to be divided into domains of variable size [6], or by allowing long-range correlations between the field values [9]. All simulations on a cubic lattice suffer from the problem that an ambiguity exists with regard to the string assignments at the vertices of the cubes; this problem can be eliminated by going to more exotic lattices, such as the tetrakaidekahedral lattice, for which only four edges meet at every vertex [4,7].

In this paper, we simulate the formation of cosmic strings using a cubic lattice and a complex Gaussian field ϕ with long-range correlations, where the strings are taken to lie along the zeros of ϕ. A Gaussian field is completely characterized by its power spectrum, defined by

$$P(k) = \int d^3 r\, e^{i\mathbf{k} \cdot \mathbf{r}} \langle \phi(\mathbf{x}) \phi(\mathbf{x} + \mathbf{r}) \rangle. \tag{1}$$

We take ϕ to have a power-law power spectrum

$$P(k) \propto k^n. \tag{2}$$

Models of this sort were first examined by Vishniac, Olive, and Seckel [10], who proposed that cosmic strings with $n = -3$ could be produced due to quantum fluctuations of the field ϕ during inflation. Vishniac *et al.* showed that the mean string length per unit volume, L/V, is given by [10]

$$L/V = \frac{1}{3\pi}\langle k^2 \rangle, \tag{3}$$

where

$$\langle k^2 \rangle = \frac{\displaystyle\int P(k)k^4 dk}{\displaystyle\int P(k)k^2 dk}. \tag{4}$$

Note that some sort of smoothing or cutoff is required for convergence at large k, but this is provided automatically by the lattice cutoff in numerical simulations. As $n \to -3$, Eq. (3) gives $L/V \to 0$; this result is confirmed by our numerical simulations. Physically, what happens is that within arbitrarily large regions of space the field ϕ has $\text{Re}(\phi) > 0$ (for example). An expression similar to Eq. (3) is also given in Ref. [9].

In our model, we assign values of ϕ to the sites on a cubic periodic lattice, and the string is assumed to lie along the location of the zeros of this complex field. This model, as we have described it, is identical to that of Ref. [9], but we make two improvements in the model, both of which have to do with the limitations in dynamic range inherent in such a simulation.

Consider first the largest scales. Because our simulation volume is finite, the simulation loses power on all scales larger than the box size, a problem which was noted by Robinson and Yates [9]. In the simulations, this corresponds to the fact that the mean value of ϕ averaged over the entire box is zero. This problem can be resolved by adding a random uniform background field ϕ_b to every cell in the box; this field represents the contribution to the field value from all Fourier modes which are larger than the box size. The variance of ϕ_b can be determined from Eq. (2), and in our simulations it is chosen to have a Gaussian distribution with this variance.

Our second addition to the simulation involves the behavior of the field on small scales. The use of a lattice to simulate string formation is obviously unphysical, and numerical simulations clearly show that f_∞ depends on the lattice being used. (Compare, e.g., the results of Ref. [3] for a cubic lattice with those of [4] and [7] for a tetrakaidecahedral lattice.) On the other hand, we do expect the field ϕ to be correlated on the smallest scales, leading to some sort of domain structure. We resolve this problem by smoothing the field on a scale larger than the cell size to eliminate lattice effects.

Our simulations were performed on a cubic lattice of size 128^3. A Gaussian, complex-valued random field ϕ having the power-law power spectrum given in Eq. (2) was set down on the lattice. The strings were identified as vortices of the field ϕ. In tracing a closed path around the four cells bound-

ing each edge of the lattice, the field was assumed to change values by moving in the shorter of the two possible directions in the complex plane. A cosmic string was then placed along an edge if the field traced out a 360° circle in the complex plane as a path was traced around the four cells bounding that edge. We define an infinite string as one which crosses all the way from one end of the box to the other in at least one of the three directions. Note that this definition differs from that used by Robinson and Yates [9], who used a cutoff in the string length.

Consider first the simplest case, for which we have no smoothing, and the mean field value is set to zero. This corresponds exactly to the simulations of Ref. [9]. To find the value of f_∞ and its variance, we performed 32 simulations for each value of n, grouped into eight groups of four simulations. Within each group of four, we derived an average value for f_∞ by dividing the combined length of infinite strings in all four simulations by the total length of string in that group. This procedure corresponds to considering each of the four simulations in a group as sampling a different region of space. We then averaged f_∞ for the eight groups of simulations to derive a final mean f_∞ and variance. (Note that in deriving Figs. 3–5, we used four groups of four simulations rather than eight.)

In Fig. 1(a), we present our results for no smoothing with zero mean field. (All error bars are 1σ.) These results agree closely with those of Ref. [9], as they should. This indicates that f_∞ is relatively insensitive to the exact definition of infinite string, since this is the only difference between our simulations and those of Ref. [9].

Now we repeat the simulations, but add a random uniform background field ϕ_b to the entire box. This background field has a Gaussian distribution with a variance which can be determined, in principle, from Eq. (2). In practice, we use a simpler method: We embed our simulation volume in a larger 128^3 box with the same power spectrum, in which our entire simulation volume occupies a single cell of the larger box. The mean field ϕ_b in the simulation volume is then simply the value of ϕ in a single (randomly chosen) cell of the larger box. Note that this method still causes a loss of power at the largest scales (i.e., Fourier modes with wavelengths longer than the size of the larger box), but it provides an effective dynamic range of $128^2 = 16\,384$, which represents a considerable improvement over the zero mean field case.

In Fig. 1(b), we show our results for this case. The greatest difference between Figs. 1(a) and 1(b) occurs for $n \lesssim -2$: The addition of the background field produces a larger variance between runs, and it reduces f_∞ sharply. The graph in Fig. 1(b) is consistent with the suggestion of Ref. [9] that f_∞ goes to zero when n is less than some critical value n_c, where $n_c > -3$; i.e., the average string density is still nonzero when the infinite strings disappear. In Ref. [9], it was claimed that $n_c = -2$, but this is inconsistent with our results. If we exclude points for which f_∞ is more than 3σ from zero, then we find $n_c < -2.2$. Although our results are consistent with the possibility that $n_c > -3$, they do not rule out the possibility that $n_c = -3$ (i.e., the infinite strings do not disappear until all the strings disappear).

In Fig. 2, we plot the total length in closed loops and infinite strings as a function of n with the inclusion of the background field. [All of the results presented in this paper, with the exception of Fig. 1(a), include this background
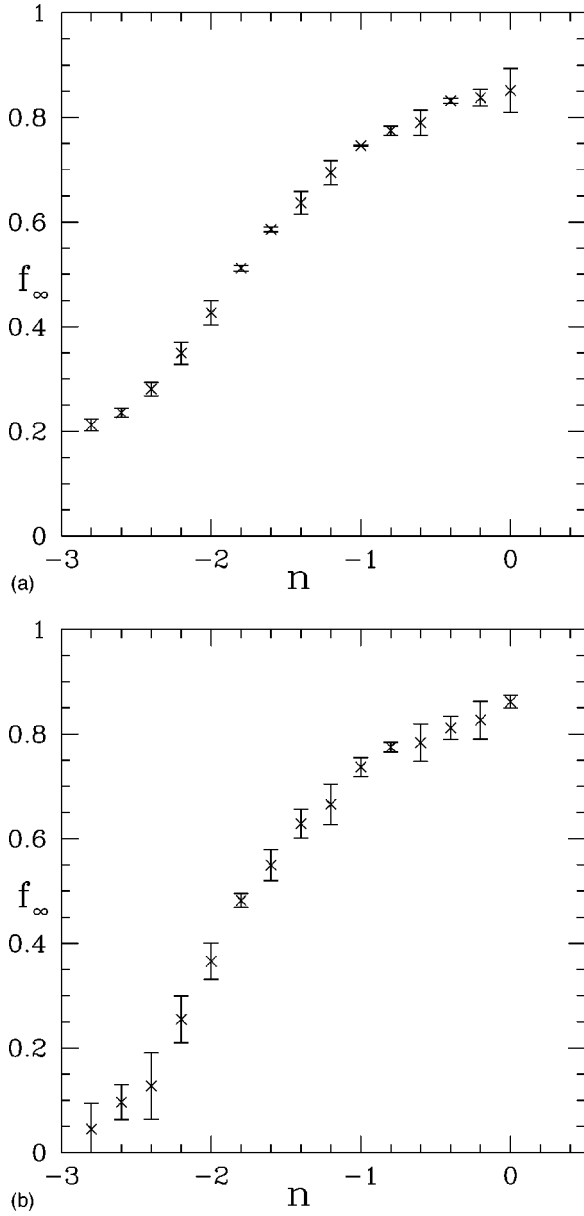
FIG. 1. The fraction of total string length in the form of infinite strings, f_∞, as a function of n, where $P(k) \propto k^n$, and $P(k)$ is the power spectrum of the complex Gaussian field which gives rise to the strings, where (a) the field in the box is forced to have zero mean, and (b) a background nonzero mean field is added to the box to simulate the effects of long-wavelength modes.

field.] Although the length in infinite strings is significantly reduced below that obtained without a background field (e.g., see [9]), we cannot tell with certainty at which value of n the infinite strings disappear.

In Fig. 3, we show the size distribution of closed loops for two representative cases ($n = 0$ and $n = -2.8$) when the background field is included. In both cases, $N(l)$ follows a power law in l, but the slope changes noticeably with n. Fitting the points with $10 \leq l \leq 100$, we find that

$$N(l) \propto l^{-2.62 \pm 0.01} \quad (n = 0), \tag{5}$$

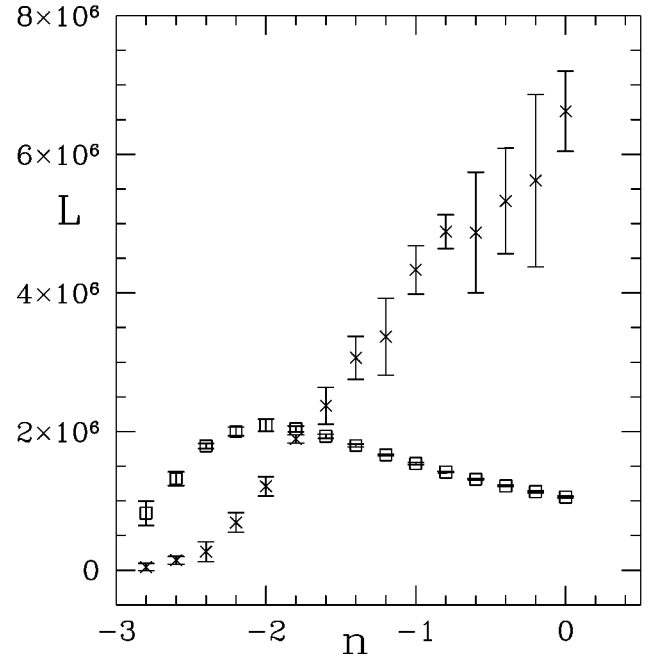$$N(l) \propto l^{-2.35 \pm 0.04} \quad (n = -2.8). \tag{6}$$

FIG. 2. The total length L in closed loops (open squares) and infinite strings (crosses) as a function of n, where $P(k) \propto k^n$, and $P(k)$ is the power spectrum of the complex Gaussian field which gives rise to the strings. A background nonzero mean field is added to the box to simulate the effect of long-wavelength modes.

These fits are shown as dashed lines in Fig. 3. Neither of these is consistent with the frequently assumed [3,4,9] behavior $N(l) \propto l^{-2.5}$, but even the original ($n = 0$) simulations of Vachaspati and Vilenkin [3] found that $N(l) \propto l^{-2.6 \pm 0.1}$.

Now consider the effect of smoothing the ϕ field on small scales. For a window function $W(\mathbf{r})$, the smoothed field $\phi_s(\mathbf{x})$ is the convolution of $\phi(\mathbf{x})$ with $W(\mathbf{r})$:

$$\phi_s(\mathbf{x}) = \int d^3 r\, \phi(\mathbf{x} + \mathbf{r}) W(\mathbf{r}). \tag{7}$$

The effect of smoothing is to reduce the magnitude of the small-scale fluctuations by averaging them out over the window function. We calculate f_∞ when ϕ is smoothed with three different window functions: the spherical Gaussian window function,

$$W(r) = \exp(-r^2 / 2r_0^2), \tag{8}$$

the spherical tophat

$$W(r) = 1 \quad (r < r_0),$$

$$W(r) = 0 \quad (r > r_0), \tag{9}$$

and a sharp cutoff in k space, which corresponds, in physical space, to the smoothing

$$W(r) = \frac{\sin(r/r_0)}{(r/r_0)^3} - \frac{\cos(r/r_0)}{(r/r_0)^2}, \tag{10}$$

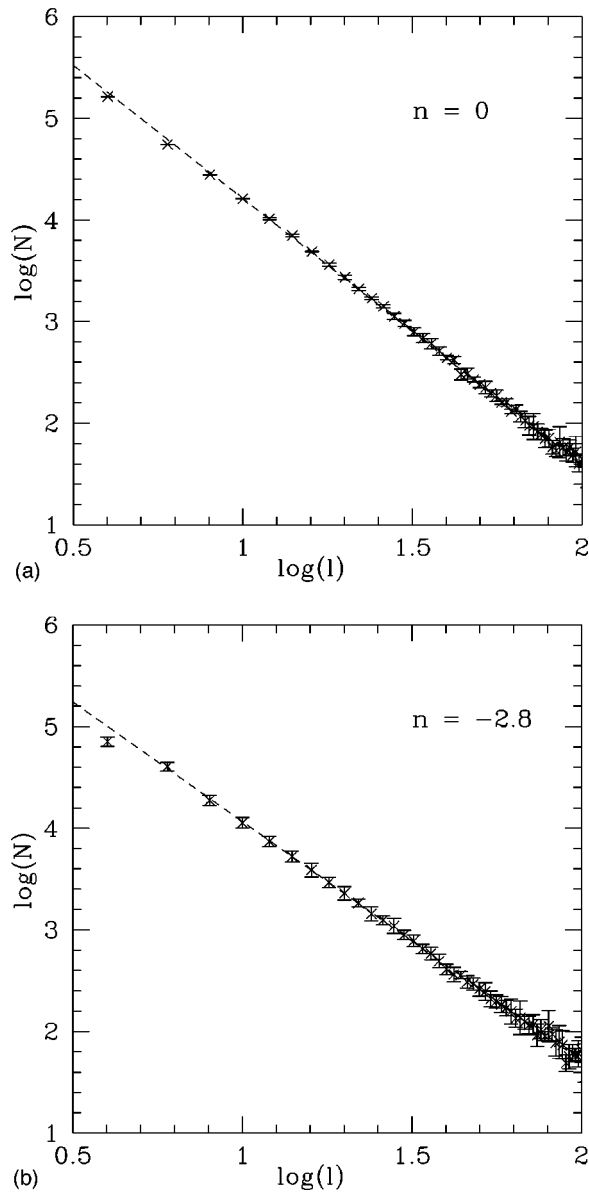where we then normalize each window function to give $\int W(r) d^3 r = 1$.

(a)

(b)

FIG. 3. The number of loops N with a given length l for (a) $n=0$ and (b) $n=-2.8$, where $P(k) \propto k^n$, and $P(k)$ is the power spectrum of the complex Gaussian field which gives rise to the strings. A background nonzero mean field is added to the box to simulate the effect of long-wavelength modes. The dashed line is the best-fit power law in each case. The logarithm is to base 10.

We now examine the variation of f_∞ with r_0. We consider only the case $n=0$ because it resembles the most likely scenario for string formation (i.e., no long-range correlations in the ϕ field), and for the range of n values we have examined, it has the most short-range power and should therefore show the largest sensitivity to smoothing. Our results are shown in Fig. 4(a) for the spherical Gaussian, in Fig. 4(b) for the spherical tophat, and in Fig. 4(c) for the sharp cutoff in k space.

Note that r_0 represents something different for each of our three window functions, and so it is meaningless to compare f_∞ for the same value of r_0 in each figure. However, the important point is whether there exists a range of values of r_0 for a given window function which produces a relatively constant set of values for f_∞. For the case of Gaussian
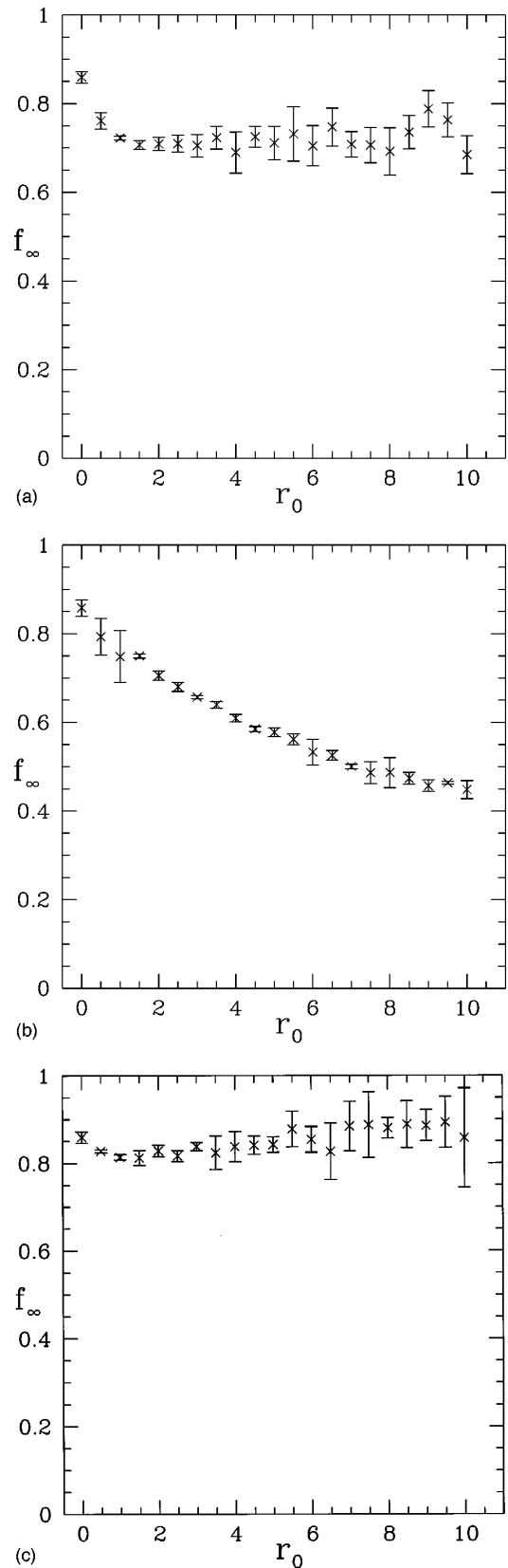

(a)

(b)

(c)

FIG. 4. The fraction of total string length in the form of infinite strings, f_∞, as a function of smoothing length r_0 for (a) Gaussian smoothing, (b) spherical tophat smoothing, (c) a sharp k-space cutoff, where the power-law index is $n=0$ (no field correlations), and a background nonzero mean field is added to the box to simulate the effect of long-wavelength modes.
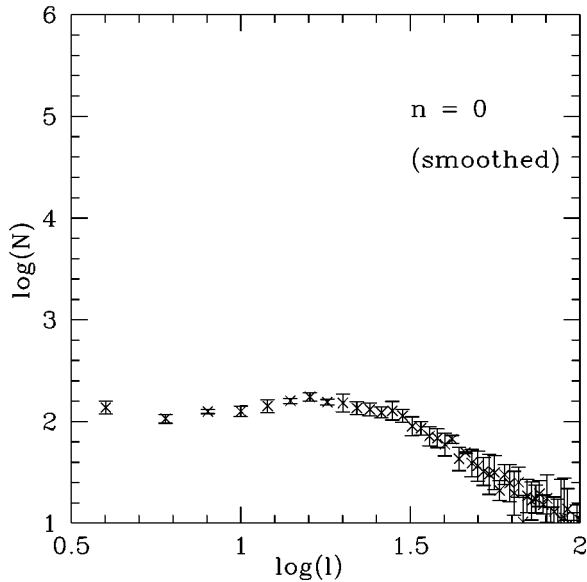
FIG. 5. The number of loops N with a given length l for a field with $n=0$ (no field correlations) smoothed with a Gaussian window function with smoothing length $r_0=3$. A background nonzero mean field is added to the box to simulate the effect of long-wavelength modes. The logarithm is to base 10.

smoothing, f_∞ decreases with r_0 but eventually "plateaus" for $r_0 \gtrsim 1.5$, suggesting that the value of f_∞ in this region is the "true" value of f_∞ for Gaussian smoothing. Averaging f_∞ for $1.5 \leq r_0 \leq 2.5$, we obtain $f_\infty = 0.71 \pm 0.01$. As r_0 is increased, the size of the smoothing volume grows compared to the size of the box, and we see larger fluctuations from one run to the next. However, the rms fluctuations in f_∞ remain within 10% of our mean value ($f_\infty = 0.71$) over the entire range of r_0 values we have examined ($r_0 \leq 10$). An average of f_∞ over the range $1.5 \leq r_0 \leq 10$ gives $f_\infty = 0.72 \pm 0.04$.

For spherical tophat smoothing, we again see that f_∞ decreases with r_0, but no obvious plateau is visible in Fig. 2(b). In fact, tophat smoothing fails to eliminate lattice effects. If we substitute the smoothed version of the power spectrum into Eq. (4), we find that for the spherical tophat window function with $n=0$, the integral in the numerator in Eq. (4) fails to converge at $k \to \infty$, indicating that it is the cubic lattice, rather than the spherical tophat, which provides the cutoff in this case. This is confirmed by the loop distribution for the spherical cutoff; unlike our other two window functions, the spherical tophat produces a loop distribution which retains a large number of loops with size of order the cell size.

The sharp k-space cutoff shows qualitatively similar behavior to the Gaussian window function. The value of f_∞ decreases slightly as a function of r_0, and reaches a plateau for $r_0 \gtrsim 1$. Averaging f_∞ for $1 \leq r_0 \leq 3$ and $1 \leq r_0 \leq 10$, we get $f_\infty = 0.82 \pm 0.01$ and $f_\infty = 0.85 \pm 0.05$, respectively.

In Fig. 5, we demonstrate the effect of smoothing on the size distribution of the loops, by showing $N(l)$ for the closed loops with Gaussian smoothing and $r_0=3$. Note that smoothing does not eliminate the closed loops smaller than the size of the window function; rather, it reduces the number of loops to a constant as a function of l for small l. Of course, the total length in closed loops is then dominated by

loops with size near the point at which the power-law behavior begins (in this case, $l \approx 30$).

III. DISCUSSION

Our results are consistent with the possibility that a transition occurs in the string network at some value $n = n_c$, with $-3 \leq n_c < -2.2$; when $n < n_c$ the infinite strings disappear from the network, leaving only closed loops. Robinson and Yates [9] argued that $n_c = -2$, but this value is not consistent with our results. Furthermore, we cannot rule out the possibility that $n_c = -3$, i.e., that the infinite strings do not disappear until the mean string density goes to zero.

In fact, the following argument suggests that infinite strings should not disappear for $n > -3$. The real and imaginary parts of ϕ are independent Gaussian fields, and the zeros of each of these fields form a set of two-dimensional surfaces. Consider first the surfaces defined by $\text{Re}(\phi) = 0$. Since the volumes of space occupied by the regions with positive and negative $\text{Re}(\phi)$ are equal, we expect both regions to percolate to an infinite distance. Hence, the boundaries dividing these regions should contain at least some infinite surfaces. This argument holds for both the surfaces defined by $\text{Re}(\phi) = 0$ and $\text{Im}(\phi) = 0$. The intersection of these two sets of surfaces gives us the location of the cosmic strings. Since infinite surfaces must exist in both sets of surfaces, it appears that there must also be infinite cosmic strings. Note that it is possible to imagine rather arcane distributions of the fields which violate this argument. For example, the regions with $\text{Re}(\phi) > 0$ and $\text{Re}(\phi) < 0$ could be nested inside of each other in larger and large finite volumes, producing a fractal distribution with arbitrarily large but finite surfaces of $\text{Re}(\phi) = 0$. However, it seems unlikely that a Gaussian field could lead to such a distribution. The crucial point in this argument is the fact that the distribution is symmetric with respect to positive and negative values of $\text{Re}(\phi)$ and $\text{Im}(\phi)$; if we relax this assumption and "bias" the distribution, the argument no longer holds. In fact, infinite strings are observed to disappear in simulations with such a "bias" [5–7].

The aim of our simulations with a smoothed field ϕ was to obtain a lattice-independent estimate of f_∞. We considered only the white noise spectrum, $n=0$, and found $f_\infty \approx 0.7$ for spherical Gaussian smoothing and $f_\infty \approx 0.8$ for a sharp cutoff in k space. These values are comparable to those obtained in earlier lattice simulations without smoothing [3,4]. The variation of f_∞ for different choices of smoothing is not surprising, since, for example, the total string density L/V in Eqs. (3) and (4) clearly depends on the short-wavelength behavior of the spectrum $P(k)$ (which is affected by the smoothing). This variation is again comparable to the variation between the values of f_∞ in simulations without smoothing on different types of lattice [3,4,7].

We find no evidence supporting the hypothesis of Borrill [8] that the presence of infinite strings is due entirely to the lattice and that they should disappear in lattice-free simulations. The smoothing length we used was sufficiently large to ensure that the loops making the largest contribution to the total string length had sizes much greater than the lattice cutoff, so that the lattice effects were minimal. Still, we found a substantial fraction of the total length to be in infinite

strings. We believe that this disagreement with Borrill's results may be due to the much smaller dynamic range in Borrill's simulations. On the other hand, our method for simulating cosmic string formation is very different from that used by Borrill, and so we would not necessarily expect the results to be the same.

We are currently investigating the formation of domain walls and monopoles with correlated fields. Given that a parallel literature on this subject exists in condensed matter physics [11], these results may have applications beyond the purely cosmological. It is conceivable, for example, that the decline of the infinite string density with the decrease of the spectral index n can be tested experimentally. What one needs is a condensed matter system (such as liquid ^{4}He) in which linear defects are formed at a second-order phase transition. Defect formation can then be observed by a rapid temperature (or pressure) quench from above to below the transition point [12,13]. Near the critical temperature T_c, the order parameter develops long-range fluctuations. At $T = T_c$, the fluctuation spectrum is a power law of the form

(2) with $n = -2 + \eta$, where the critical exponent η is typically a small number, $\eta \lesssim 0.05$ [14] ($\eta \approx 0.05$ for ^{4}He). If the system is allowed to equilibrate very close to the critical point and is then rapidly quenched to subcritical temperatures, one can expect the length in infinite strings to be suppressed compared to a quench from a temperature well above T_c (where the fluctuation spectrum is close to $n = 0$). Our Fig. 2 suggests a suppression roughly by an order of magnitude, while the value of $n_c = -2$ conjectured by Robertson and Yates [9] would give a much more dramatic suppression. It should be noted that a realistic quench is a rather complicated process, and its outcome can depend on a variety of physical effects (for a recent discussion see [12,15]).

ACKNOWLEDGMENTS

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