

“Nernst theorem” and black hole thermodynamics

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The Nernst formulation of the third law of ordinary thermodynamics (often referred to as the “Nernst theorem”) asserts that the entropy S of a system must go to zero (or a “universal constant”) as its temperature T goes to zero. This assertion is commonly considered to be a fundamental law of thermodynamics. As such, it seems to spoil the otherwise perfect analogy between the ordinary laws of thermodynamics and the laws of black hole mechanics, since rotating black holes in general relativity do not satisfy the analogue of the “Nernst theorem.” The main purpose of this paper is to attempt to lay to rest the “Nernst theorem” as a law of thermodynamics. We consider a boson (or fermion) ideal gas with its total angular momentum J as an additional state parameter, and we analyze the conditions on the single-particle density of states, $g(\epsilon, j)$, needed for the Nernst formulation of the third law to hold. (Here, ϵ and j denote the single-particle energy and angular momentum.) Although it is shown that the Nernst formulation of the third law does indeed hold under a wide range of conditions, some simple classes of examples of densities of states which violate the “Nernst theorem” are given. In particular, at zero temperature, a boson (or fermion) gas confined to a circular string (whose energy is proportional to its length) not only violates the “Nernst theorem” also but reproduces some other thermodynamic properties of an extremal rotating black hole. [S0556-2821(97)01122-3]

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I. INTRODUCTION

Nearly 25 years ago a remarkable relationship was established [1] between the ordinary laws of thermodynamics and certain laws of black hole physics. This relationship was then greatly enhanced by the discovery [2] that black holes radiate as perfect black bodies, and by strong evidence for the validity of the “generalized second law” [3–6]; see, e.g., [7,8] for comprehensive reviews.

However, one apparent blemish has existed on this otherwise seemingly perfect relationship. The Nernst formulation of the third law of thermodynamics asserts that the entropy S of a system must go to zero (or a “universal constant”) as its temperature T approaches absolute zero. On the other hand, for Kerr black holes in general relativity, the entropy is given by

$$S = A/4 = 2\pi[M^2 + (M^4 - J^2)^{1/2}], \quad (1)$$

and the temperature is given by

$$T = \kappa/2\pi = \frac{(M^4 - J^2)^{1/2}}{4\pi M[M^2 + (M^4 - J^2)^{1/2}]}, \quad (2)$$

where M and J denote, respectively, the mass and angular momentum of the black hole. (Here and throughout this paper, we use units where $G = c = \hbar = k = 1$.) Thus, absolute zero temperature corresponds to the “extremal limit”

$$|J| = M^2. \quad (3)$$

The entropy at absolute zero temperature is thus

$$S = 2\pi|J|, \quad (4)$$

which is nonvanishing and, furthermore, has a functional dependence on the state parameter J , so it does not approach a

“universal constant.” Thus, the Kerr black holes stand in blatant violation of the black hole mechanics analogue of the “Nernst theorem.”

This failure of the “Nernst theorem” to hold in black hole mechanics has not generally been viewed with alarm by most researchers because it is clear that the Nernst formulation of the third law does not have the same fundamental status in thermodynamics as the first or second laws (see, e.g., Sec. 9.4 of the standard text of Huang [9] for a clear statement of this view). Indeed, the Nernst formulation of the third law does not hold at all in classical physics, failing even for a classical ideal gas. In quantum statistical physics, the “Nernst theorem” corresponds to a claim about the behavior of the density of states, $n(E)$, as the total energy of the system goes to its minimum possible value. More precisely, it is a statement about the extrapolation to minimum energy of the higher energy, continuum approximation to $n(E)$ (see [9]); in other words, the thermodynamic limit should be taken prior to the limit $T \rightarrow 0$. It is not difficult to concoct examples where $n(E)$ is such that the Nernst formulation of the third law is violated. For example, a system comprised by particles with spin but having no spin interaction energy — so that the ground state is highly degenerate — will violate the “Nernst theorem.” Violations of the “Nernst theorem” also occur for systems in which the ground state is nondegenerate at finite volume (see [10] and references cited therein).

Nevertheless, most such counterexamples to the Nernst formulation of the third law seem rather contrived, and the fact that it has been empirically found to hold for all systems studied in the laboratory provides evidence that it might hold for all “physically reasonable” systems. If so, this would suggest that there might be something “exotic” about the thermodynamic properties of extremal rotating black holes.

In this paper we shall investigate this issue by studying the Nernst formulation of the third law for a very nonexotic

class of thermodynamic systems: ideal boson gases. To keep the system as simple as possible — and, in particular, to avoid complications resulting from Bose-Einstein condensation — we shall assume that, as in the case of the photon gas, particle number is not conserved; equivalently, the chemical potential of the gas will be assumed to vanish. However, we will assume that the gas is confined by an axially symmetric box (or potential), so that its total angular momentum, J , is conserved, and we will take J and the total energy, E , to be the state parameters of the system. In studying systems with nonzero angular momentum, it does not make sense to consider an infinite volume limit, so we will restrict attention to finite volume systems.

The thermodynamic properties of the ideal boson gas are determined by its single-particle density of states, $g(\epsilon, j)$, where ϵ and j denote, respectively, the single-particle energy and angular momentum. In order to facilitate our calculations, we shall further assume that $g(\epsilon, j)$ is sufficiently “nonexotic” that the appropriate canonical ensemble — modified to include angular momentum — can be defined (at least at low temperatures). This requires that $g(\epsilon, j)$ not grow more rapidly than exponentially in ϵ , and that the single-particle angular momentum to energy ratio be bounded, i.e., that $\Omega_{\pm} \neq 0$, where

$$(\Omega_{\pm})^{-1} \equiv \sup(\pm j/\epsilon). \quad (5)$$

Thus, we have $g(\epsilon, j) = 0$ unless

$$-\epsilon/\Omega_{-} \leq j \leq \epsilon/\Omega_{+}. \quad (6)$$

[Note that this condition holds for a system of free particles confined to within a (cylindrical) radius R of the symmetry axis, with $\Omega_{\pm} = 1/R$.] We then pose the following two questions: (i) What properties of $g(\epsilon, j)$ are required in order that the Nernst formulation of the third law be violated, i.e., so that $S(T, J)$ approaches a nonzero limit (which depends upon J) as $T \rightarrow 0$? (ii) Can these conditions be achieved for any classes of “physically reasonable” ideal gas systems?

Of course, even if the answer to (ii) were “no,” this would not mean that extremal Kerr black holes necessarily display any “unphysical” or “exotic” thermodynamic behavior, since there is no reason to expect that their behavior could be properly modeled by an ideal boson gas. Indeed, with the restrictions placed on the density of states needed to define the ordinary canonical ensemble, it is impossible to get negative heat capacities, as occurs for black holes with sufficiently small angular momentum. There is nothing “unphysical” or “exotic” about systems with negative heat capacities; for example, ordinary self-gravitating stars in Newtonian gravity have negative heat capacities. However, the simple ideal gas systems we consider here are not adequate to model this behavior. There is no reason, *a priori*, to believe that they should be adequate to model the violations of “Nernst’s theorem” displayed by extremal Kerr black holes. Nevertheless, it is of interest to see how close one can come to modeling the thermodynamic behavior of extremal Kerr black holes with ideal boson gas systems.

As we shall see in the next section, for a violation of “Nernst’s theorem,” it is sufficient (and, as explained there, “nearly necessary”) that there exist single-particle states which achieve the bound (6), i.e., that (for positive J) there

exist states which satisfy $\epsilon = \Omega_{+}j$ exactly.¹ No such states exist for a free boson gas confined by a spherical box in two or higher spatial dimensions, and such systems satisfy the Nernst formulation of the third law even when they are rotating. (We will explicitly calculate the low temperature behavior of a rotating gas in the next section.) However, massless ideal gases in one spatial dimension and ideal gases in “zero spatial dimensions” (i.e., spin systems) do have states for which $\epsilon = \Omega_{+}j$, and they violate the “Nernst theorem” when angular momentum is taken into account. Thus, violations of the “Nernst theorem” — which are qualitatively very similar to the violations of the “Nernst theorem” for Kerr black holes — do occur for some simple systems comprised by ideal gases with angular momentum, although the one (or zero) dimensionality of such systems seems essential.

Encouraged by this result, we may ask if the detailed thermodynamic properties of extremal Kerr black holes given by Eqs. (3) and (4) also can be modeled by ideal gas systems. As we shall see in the next section, for $J > 0$ the ideal gas systems will automatically satisfy $E = \Omega_{+}J$ at zero temperature, rather than $E \propto J^{1/2}$, as in Eq. (3). However, if we modify the model of a one-dimensional boson gas confined to a ring of radius R by simply treating R itself as an additional classical dynamical variable, and if we also attribute an additional energy proportional to R (due to “string tension”) to the total energy E , then the behavior $E \propto J^{1/2}$ is obtained — in agreement with Eq. (3). However, the behavior $S \propto J$ at zero temperature [see Eq. (4)] seems much more difficult to model, as it appears to require the density of states, $n(j)$, at $\epsilon = \Omega_{+}j$ to grow exponentially with j . (A collection of massless boson gases would have a constant $n(j)$, which leads to the behavior $S \propto J^{1/2}$ at zero temperature.) Nevertheless, it seems remarkable that such a simple model can come so close to mimicking the thermodynamic behavior of extremal Kerr black holes.

This investigation was stimulated by the recent success in modeling the thermodynamic behavior of certain extremal charged black holes (namely, those which saturate the “Bogomol’nyi-Prasad-Sommerfield (BPS) bound”) in string theory [11]. These results already provide a counterexample to the “Nernst theorem” for a particular system in the class considered here, since the degrees of freedom which contribute to the entropy in the weak coupling string model correspond to that of a free, one-dimensional gas. In the present investigation, we consider general ideal boson gas systems — not restricted by any models arising from string theory.² The one (or zero) dimensionality of the models we find

¹Note that this condition implies that if we define a new notion of “time translation” by $t'^{\mu} = t^{\mu} + \Omega_{+}\phi^{\mu}$ — with t^{μ} and ϕ^{μ} being the generators of ordinary time translations and rotations, respectively — then the “ground state” of the system relative to t'^{μ} is highly degenerate.

²The philosophy of the present paper bears some similarity with the philosophy adopted in a recent paper of Maldecena and Strominger [12], who study the emission properties of nearly BPS, slowly rotating black holes and deduce from those properties some aspects of the effective string theory description of such black holes. However, there does not appear to be any overlap in the contents of that paper and the present paper.

which violate the “Nernst theorem” is a conclusion, rather than an input, of our analysis.

Finally, we note that, for definiteness, we shall consider an ideal boson gas at zero chemical potential in our analysis. However, the analysis of an ideal fermion gas (at zero chemical potential) would proceed in complete parallel — with merely some sign changes in various expressions — and the conclusions in the fermion case would be unaltered.

II. THE THERMODYNAMICAL PROPERTIES OF A ROTATING BOSON GAS AT LOW TEMPERATURES

Consider an ideal boson gas, confined by a potential (or “box”) which is axially symmetric. Then the angular momentum about the symmetry axis is conserved, and the single-particle states of the gas can be labeled by their energy, ϵ , and angular momentum, j , about the symmetry axis. We shall assume that the single-particle Hamiltonian is positive, and that the minimum energy single-particle state has energy $\epsilon_0 > 0$. (This ensures that the “vacuum state” is the unique ground state of the system. If there existed any single-particle states with $\epsilon = 0$, the ground state of system would be highly degenerate and the Nernst formulation of the third law would be trivially violated even when the total angular momentum vanishes.) Let $G(\epsilon, j)$ denote the number of states with energy $\leq \epsilon$ and angular momentum $\leq j$. Thus, G is non-negative, is a monotone increasing function of ϵ and j , and satisfies $G(0, j) = 0$. The density of states, $g(\epsilon, j)$, is defined by

$$g(\epsilon, j) = \frac{\partial^2 G}{\partial \epsilon \partial j}. \quad (7)$$

In reality, on account of the discreteness of states, $G(\epsilon, j)$ is a piecewise constant function and, correspondingly, g is a sum of δ functions, but (following standard practice) in our expressions we will treat both of them as “continuum” (though *not* necessarily continuous) variables, i.e., we will write down integral expressions rather than sums in our formulas below. However, all of our formulas will continue to make sense if g is taken to be a sum of δ functions (or has δ -function contributions in addition to contributions which are treated as being continuous).

We will assume that, as for the case of a photon gas, particle number in our boson gas is not conserved, i.e., that particles can be created freely, at no “cost” other than the energy and angular momentum required to create them. (This corresponds to a vanishing chemical potential of the gas.) Thus, the state variables will not include the number of particles and will be taken to be simply E and J . Given only that $G(\epsilon, j)$ is bounded in j at each ϵ (i.e., that for each ϵ there are only a finite number of single-particle states with energy $< \epsilon$), the microcanonical ensemble appropriate to fixing the total energy, E , and total angular momentum, J , is well defined. The entropy, $S(E, J)$, may then be defined as $S(E, J) = \ln N(E, J)$, where $N(E, J)$ denotes the number of states of the total system (*not* single-particle states) with total energy between E and $E + \Delta E$ and total angular momentum between J and $J + \Delta J$. However, use of the microcanonical

ensemble is not very convenient for most calculations, and the entropy of systems is usually computed in the context of the canonical ensemble.

To obtain the appropriate canonical ensemble in the present case, we proceed in close parallel to the derivation of the grand canonical ensemble. We imagine that our system is able to exchange energy and angular momentum with a “heat bath and angular momentum reservoir” (rather than a “heat bath and particle reservoir”) characterized by temperature $T = 1/\beta$ and angular velocity Ω . (Here T and Ω are defined by their appearance in the first law of thermodynamics for the reservoir, namely $dE = TdS + \Omega dJ$.) In order that our ideal gas system be able to “come to equilibrium” with the reservoir (so that the canonical ensemble can be defined) it is necessary to impose two additional restrictions on $G(\epsilon, j)$: First, in the usual manner, we must have $G(\epsilon, j) \leq C \exp(\alpha \epsilon)$ for some constants C and α , since otherwise the system could indefinitely soak up energy from the reservoir. Second, we must have $\Omega_+ \neq 0$ and $\Omega_- \neq 0$ [where Ω_+ and Ω_- were defined by Eq. (5) above], since otherwise the system could indefinitely soak up angular momentum from the reservoir. In the following, we shall assume that these conditions are satisfied — so that the canonical ensemble is well defined for $T < 1/\alpha$ and $-\Omega_- < \Omega < \Omega_+$. We then shall use canonical ensemble methods to compute $S(T, J)$. As usual, the canonical ensemble is equivalent to the microcanonical ensemble for the purposes of computing the entropy and other thermodynamic quantities for the system provided that the energy and angular momentum fluctuations in the canonical ensemble are sufficiently small.³

In exact parallel with the grand canonical ensemble, in our “angular momentum modified canonical ensemble,” all thermodynamic quantities can be derived in a straightforward manner from a partition function $Z(\beta, \Omega)$. For an ideal boson gas, Z is given by

$$\ln Z = - \int d\epsilon dj g(\epsilon, j) \ln[1 - \exp(-\beta[\epsilon - \Omega j])]. \quad (8)$$

The (expected) angular momentum J is then given by

$$J = \frac{1}{\beta} \frac{\partial \ln Z}{\partial \Omega} = \int d\epsilon dj g(\epsilon, j) \frac{j}{\exp(\beta[\epsilon - \Omega j]) - 1}. \quad (9)$$

The (expected) energy, E , is determined by

$$E - \Omega J = - \frac{\partial \ln Z}{\partial \beta} = \int d\epsilon dj g(\epsilon, j) \frac{\epsilon - \Omega j}{\exp(\beta[\epsilon - \Omega j]) - 1}. \quad (10)$$

Finally, the entropy S is given by

³At extremely low temperatures, the microcanonical and canonical ensembles need not be equivalent. However, as emphasized in [9], the Nernst formulation of the third law really refers to the extrapolation to $T = 0$ of the formula for the entropy which applies at temperatures which are sufficiently high that the two ensembles should be equivalent. Thus, it is appropriate to use the canonical ensemble for our calculations here even if the two ensembles are not equivalent at $T = 0$.

$$S = \ln Z + \beta(E - \Omega J) = \ln Z - \beta \frac{\partial \ln Z}{\partial \beta}. \quad (11)$$

Equation (11) yields the entropy as a function of β and Ω . To obtain $S(\beta, J)$, we must solve Eq. (9) to express Ω as a function of β and J . Our task is to find conditions on the density of states, $g(\epsilon, j)$, so that $S(\beta, J)$ does not approach zero (or a ‘‘universal constant’’) when $\beta \rightarrow \infty$ at fixed J .

In the following, we shall restrict attention to analyzing the case where $J > 0$. (In particular, the case $J = 0$ will be excluded from our analysis.) The states with j near its maximal value ϵ/Ω_+ will then play an important role in the behavior of the gas as $\beta \rightarrow \infty$, and it useful to replace the variable ϵ with the variable

$$y \equiv \epsilon - \Omega_+ j. \quad (12)$$

The allowed ranges of y and j corresponding to the restrictions (6) are then

$$y \geq 0, \quad j \geq -y/(\Omega_+ + \Omega_-). \quad (13)$$

In addition, the condition $\epsilon \geq \epsilon_0$ yields

$$j \geq (\epsilon_0 - y)/\Omega_+. \quad (14)$$

We define $H(y, j)$ to be the total number of states labeled by (y', j') , such that $y' \leq y$ and $j' \leq j$. We define the corresponding density of states $h(y, j)$ by

$$h(y, j) = \frac{\partial^2 H}{\partial y \partial j}. \quad (15)$$

Then, we have $h(\epsilon - \Omega_+ j, j) = g(\epsilon, j)$, although the relationship between H and G is not quite as straightforward, since the state counting in the two cases is being done over different regions of single-particle state space. In terms of our new variables, the above formula (8) for $\ln Z$ becomes

$$\begin{aligned} \ln Z &= - \int dy dj h(y, j) \ln[1 - \exp(-\beta y - \beta[\Omega_+ - \Omega]j)] \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \int dy dj \frac{\partial^2 H}{\partial y \partial j} e^{-n\beta y} e^{-n\sigma j}, \end{aligned} \quad (16)$$

where, in the second line, we have made use of the series expansion

$$\ln[1 - e^{-x}] = - \sum_{n=1}^{\infty} \frac{1}{n} e^{-nx} \quad (17)$$

and we have written

$$\sigma \equiv \beta(\Omega_+ - \Omega). \quad (18)$$

(Note that $\sigma > 0$ in order for the canonical ensemble to be defined.) The corresponding series expanded formulas for J and S in our new variables are

$$J = \sum_{n=1}^{\infty} \int dy dj \frac{\partial^2 H}{\partial y \partial j} j e^{-n\beta y} e^{-n\sigma j} \quad (19)$$

and

$$\begin{aligned} S &= \sigma J + \sum_{n=1}^{\infty} \frac{1}{n} \int dy dj \frac{\partial^2 H}{\partial y \partial j} e^{-n\beta y} e^{-n\sigma j} \\ &\quad + \sum_{n=1}^{\infty} \beta \int dy dj \frac{\partial^2 H}{\partial y \partial j} y e^{-n\beta y} e^{-n\sigma j}. \end{aligned} \quad (20)$$

We now integrate Eqs. (19) and (20) by parts with respect to both y and j (taking the ranges of both of these integrals to be $-\infty$ to ∞). When we do so, no boundary terms arise from the upper limits on account of the exponentially decaying terms $e^{-n\beta y}$ and $e^{-n\sigma j}$, and no boundary terms arise from the lower limits on account of the vanishing of $H(y, j)$ outside of the range defined by Eq. (13). We obtain

$$J = \sum_{n=1}^{\infty} n^2 \beta \sigma \int dy dj H(y, j) \left(j - \frac{1}{n\sigma} \right) e^{-n\beta y} e^{-n\sigma j} \quad (21)$$

and

$$S = \sigma J + \sum_{n=1}^{\infty} n^2 \beta^2 \sigma \int dy dj H(y, j) y e^{-n\beta y} e^{-n\sigma j}. \quad (22)$$

Finally, we introduce the new variables

$$w = n\sigma j, \quad z = n\beta y \quad (23)$$

to convert these expressions to the form

$$\begin{aligned} J &= \frac{1}{\sigma} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} dz \int_{-z\sigma/\beta(\Omega_- + \Omega_+)}^{\infty} dw H\left(\frac{z}{n\beta}, \frac{w}{n\sigma}\right) \\ &\quad \times (w - 1) e^{-z} e^{-w} \end{aligned} \quad (24)$$

and

$$\begin{aligned} S &= \sigma J + \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} dz \int_{-z\sigma/\beta(\Omega_- + \Omega_+)}^{\infty} dw H\left(\frac{z}{n\beta}, \frac{w}{n\sigma}\right) \\ &\quad \times z e^{-z} e^{-w}, \end{aligned} \quad (25)$$

where we have now explicitly inserted lower limits on the integrals to remind the reader that H vanishes outside the range defined by Eq. (13). Note that since the second term on the right side of Eq. (25) is non-negative, we have

$$S \geq \sigma J. \quad (26)$$

We now show that for any fixed $J > 0$, σ must remain bounded from above when $\beta \rightarrow \infty$, i.e., Ω must approach Ω_+ at least as rapidly as $1/\beta$. Equivalently, we have $\sigma_0 < \infty$ where

$$\sigma_0 \equiv \limsup_{\beta \rightarrow \infty} \sigma. \quad (27)$$

To see this, we note that by Eq. (24) we have

$$J \leq \frac{1}{\sigma} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} dz \int_0^{\infty} dw H\left(\frac{z}{n\beta}, \frac{w}{n\sigma}\right) (w - 1) e^{-z} e^{-w}. \quad (28)$$

If there were a sequence $\beta_i \rightarrow \infty$ such that $\sigma_i \rightarrow \infty$, then, due to the factor of $1/\sigma$ together with the fact that H is a monotone increasing function of both of its arguments (and, hence, is monotone decreasing along this sequence), the right side of Eq. (28) would converge to zero, in contradiction with the fact that $J > 0$.

A crucial factor in the behavior of S at $T=0$ is whether or not $\sigma_0=0$. If $\sigma_0 > 0$, then by Eq. (26) we have

$$\limsup_{\beta \rightarrow \infty} S \geq \sigma_0 J > 0, \quad (29)$$

and the Nernst formulation of the third law will fail. On the other hand, suppose that $\sigma_0=0$ (so that $\sigma \rightarrow 0$ as $\beta \rightarrow \infty$, i.e., Ω approaches Ω_+ more rapidly than $1/\beta$). Then σJ converges to zero, so we only need worry about the second term on the right side of Eq. (25). However, in order to keep the right side of Eq. (24) from diverging as $\beta \rightarrow \infty$, it is necessary that $H(z/\beta, w/\sigma)$ converge pointwise to zero for all $z, w \geq 0$. [If not, then using the monotonicity and positivity of H , the integrals on the right side of Eq. (24) would remain finite, but the $1/\sigma$ factor would diverge.] If we knew, in addition, that for all β we had $H(z/\beta, w/\sigma) \leq F(z, w)$ where F is such that $\int dz dw F(z, w) z e^{-z} e^{-w}$ converges, then we could use the dominated convergence theorem to conclude that $S \rightarrow 0$ as $\beta \rightarrow \infty$. I have not attempted to give a complete analysis of the conditions on H which are necessary and sufficient for the Nernst behavior to occur when $\sigma_0=0$, but it seems clear that this “normally” will be the case (and possibly always is the case, since I do not know of any counterexamples to the Nernst behavior when $\sigma_0=0$).

What conditions on H are necessary and sufficient to ensure that $\sigma_0 > 0$, so that the Nernst formulation of the third law will be violated? A sufficient condition is that $H(0, j) > 0$ for some j , i.e., that there exists at least one single particle state which actually achieves the limiting angular momentum $j = \epsilon/\Omega_+$. To see this, we note that if we assume that $H(0, j) > 0$ for some j but that $\sigma_0=0$, it follows immediately that $H(z/\beta, w/\sigma)$ cannot converge pointwise to zero. However, as in the arguments of the previous paragraph, this yields a contradiction, since it implies that $J \rightarrow \infty$ as $\beta \rightarrow \infty$.

On the other hand, for a wide class of H 's, the condition that $H(0, j) > 0$ for some j also is necessary to have $\sigma_0 > 0$. In particular, suppose that $H(y, j)$ is polynomially bounded in j at each y in such a way that for $j \geq 0$ we have

$$H(y, j) \leq F(y)(1 + j^k), \quad (30)$$

where $F(y)$ is continuous, is exponentially bounded at large y (so that the canonical ensemble is well defined at large β), and satisfies $F(0)=0$. This behavior encompasses a very wide class of H 's such that $H(0, j)=0$ for all j . Since H is a monotone increasing function of y , we may assume, without loss of generality, that F also is a monotone increasing function. By Eq. (28), we have

$$\begin{aligned} J &\leq \frac{1}{\sigma} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} dz \int_0^{\infty} dw F\left(\frac{z}{n\beta}\right) \left[1 + \left(\frac{w}{n\sigma}\right)^k\right] \\ &\quad \times (w-1)e^{-z}e^{-w} \\ &\leq \frac{1}{\sigma^{(k+1)}} \Gamma(k+2) \sum_{n=1}^{\infty} \frac{1}{n^{(k+1)}} \int_0^{\infty} dz F\left(\frac{z}{n\beta}\right) e^{-z} \\ &\leq \frac{C}{\sigma^{(k+1)}} \int_0^{\infty} dz F\left(\frac{z}{\beta}\right) e^{-z}, \end{aligned} \quad (31)$$

where the monotone property of F was used in the last line to obtain $F(z/n\beta) \leq F(z/\beta)$. However, as $\beta \rightarrow \infty$, the functions $f_\beta(z) \equiv F(z/\beta)$ converge pointwise to 0 and are “dominated” by $F(z)$, so, by the dominated convergence theorem, the integral on the right side of Eq. (31) converges to 0. Consequently, we must have $\sigma_0=0$ in this case, as we desired to show.

If $H(y, j)$ is not polynomially bounded in j , then it is possible to have $\sigma_0 > 0$ even if $H(0, j)=0$ for all j . Indeed, if $H(y, j) = F(y)e^{\lambda j}$ where $\lambda > 0$ and F is as in the previous paragraph, then it is not difficult to see from Eq. (24) that $\sigma_0 = \lambda > 0$. However, I am not aware of any circumstances under which $\sigma_0 > 0$ when $H(0, j)=0$ for all j and $H(y, j)$ is such that at fixed y , $H(y, j)e^{-\alpha j}$ is bounded in j for all $\alpha > 0$.

We now summarize our results. We have considered ideal boson gases whose single-particle states satisfy the restriction (6). We have shown above that *if there exist any single-particle states which actually achieve the maximal ratio of angular momentum to energy — namely $j/\epsilon = 1/\Omega_+$ — then the Nernst formulation of the third law will fail for $J > 0$* . In a limited class of other circumstances — in particular, when $H(y, j)$ grows exponentially with j — the Nernst formulation of the third law also may fail even if no single particle states satisfy $j/\epsilon = 1/\Omega_+$. However, it appears that in the “vast majority of cases” — and conceivably all cases where $H(y, j)e^{-\alpha j}$ bounded in j for all $\alpha > 0$ — the Nernst formulation of the third law holds when no single-particle states satisfy $j/\epsilon = 1/\Omega_+$.

A few simple examples are useful to illustrate these general results and to gain insight into the conditions under which there are states with $j/\epsilon = 1/\Omega_+$, so that the Nernst formulation of the third law is violated. As a first example, consider a gas of particles of a free, massless, scalar field in three dimensions, confined by a spherical box of radius R , with Dirichlet boundary conditions on the walls of the box. The spatial mode functions for the particles are then of the form

$$\phi_{nlm} = j_l(k_{ln}r) Y_{lm}(\theta, \varphi), \quad (32)$$

where k_{ln} is the n th value of k such that $j_l(kR)=0$. The energy of the mode ϕ_{nlm} is k_{ln} and its z -angular momentum is m . (Recall that we are using units in which $\hbar=1$.) Since $k_{ln} > (l+1/2)/R$ (see, e.g., [13]), we have $j/\epsilon < 1/R$ for all single-particle states. However, since the first zero, k_{l1} , satisfies [13]

$$\lim_{l \rightarrow \infty} \frac{k_{l1}}{l} = 1 \quad (33)$$

we see that $\Omega_+ = 1/R$, and no single-particle state actually achieves the maximal angular momentum to energy ratio.

By the above arguments, the Nernst formulation of the third law should hold in this example. To see this explicitly, we note that for large l , the density of zeros of $j_l(x)$ is given by

$$\rho = \frac{1}{\pi} \left(1 - \frac{(l+1/2)^2}{x^2} \right)^{1/2}. \quad (34)$$

(This result can be derived from formulas given in Sec. 15.81 of [13].) Each l contributes one state of z -angular momentum j (for integer j) if $l \geq |j|$ and zero states otherwise. Hence, the density of states, $g(\epsilon, j)$, is given by

$$\begin{aligned} g(\epsilon, j) &= \frac{R}{\pi} \int_{|j|}^{\epsilon R - 1/2} \left(1 - \frac{(l+1/2)^2}{(\epsilon R)^2} \right)^{1/2} dl \\ &= \frac{R^2 \epsilon}{2\pi} \{ \arccos(|j|/R\epsilon) - (|j|/R\epsilon) [1 - (j/R\epsilon)^2]^{1/2} \}. \end{aligned} \quad (35)$$

In terms of the variable $y = \epsilon - j/R$, the density of states $h(y, j)$ is

$$\begin{aligned} h(y, j) &= \frac{R}{2\pi} (j + Ry) \left\{ \arccos\left(\frac{|j|}{j + Ry}\right) - \frac{|j|}{(j + Ry)^2} \right. \\ &\quad \left. \times [2Rjy + R^2y^2]^{1/2} \right\}. \end{aligned} \quad (36)$$

Taking into account the restriction (13), we see that

$$h(y, j) \leq C(|j| + Ry) \leq C'(1 + y)(1 + |j|), \quad (37)$$

from which it follows immediately that

$$H(y, j) \leq C''(y + y^2)(1 + j^2), \quad (38)$$

which is of the form (30). Thus, we have $\sigma_0 = 0$ in this case.

The explicit behavior of the entropy of the rotating gas at low temperatures can be calculated as follows. From Eq. (36), we see that for small y , we have

$$h(y, j) \approx \frac{2\sqrt{2}}{3\pi} R^{5/2} j^{-1/2} y^{3/2}. \quad (39)$$

Substituting this into Eq. (19), we find that, for $J > 0$ and large β ,

$$\begin{aligned} J &\approx \frac{2\sqrt{2}}{3\pi} R^{5/2} \sum_{n=1}^{\infty} \int dy dj j^{1/2} y^{3/2} e^{-n\beta y} e^{-n\sigma j} \\ &\approx \frac{2\sqrt{2}}{3\pi} R^{5/2} \Gamma(5/2) \Gamma(3/2) \zeta(4) \beta^{-5/2} \sigma^{-3/2} \\ &= \frac{\sqrt{2}\pi^4}{360} R^{5/2} \beta^{-5/2} \sigma^{-3/2}. \end{aligned} \quad (40)$$

[Here ζ denotes the Riemann zeta function, and we have used the values $\zeta(4) = \pi^4/90$, $\Gamma(3/2) = \sqrt{\pi}/2$, and $\Gamma(5/2) = 3\sqrt{\pi}/4$.] Thus, at large β , we have

$$\sigma \approx \left\{ \frac{\sqrt{2}\pi^4}{360} \right\}^{2/3} \frac{R^{5/3}}{\beta^{5/3} J^{2/3}}. \quad (41)$$

Substituting this into Eq. (20), we find that as $T \rightarrow 0$ at fixed $J > 0$, we have

$$S \propto R^{5/3} J^{1/3} T^{5/3} \rightarrow 0. \quad (42)$$

Thus, the Nernst formulation of the third law does indeed hold, although S goes to zero more slowly than in the case where the angular momentum of the gas is not constrained (in which case $S \propto R^3 T^3$ at all temperatures).

I have not succeeded in finding any simple examples of systems violating the Nernst formulation of the third law which, like the case of a free boson gas in a spherical box, satisfy the properties that (i) the angular momentum carried by the particles is primarily ‘‘orbital’’ (as opposed to ‘‘spin’’) in character, and (ii) the particles are not constrained to move exclusively in the φ direction. However, it is easy to find simple examples of ‘‘zero-dimensional systems’’ (i.e., spin systems) and one-dimensional systems which violate the Nernst formulation of the third law.

As a simple example of a spin system which violates the Nernst formulation of the third law, suppose that we have bosonic particles of mass M and spin s , which can be located on any one of N ‘‘lattice sites.’’ (Again, the total number of such particles is taken to be unconstrained.) Then the maximal angular momentum to energy ratio for single-particle states is s/M (i.e., $\Omega_+ = M/s$), which is attained by a particle whose spin is aligned along the z axis. In this case, we clearly have $H(0, j) = 0$ for $j < s$, whereas $H(0, j) = N$ for $j \geq s$. The states with $y = 0$ (i.e., $j = s$) will dominate the low temperature behavior of the gas when $J = 0$. Thus, taking the limit as $\beta \rightarrow \infty$ in Eqs. (24) and (25) and performing the z integrals, we find that, at $T = 0$,

$$J = \frac{1}{\sigma_{n=1}} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} dw H\left(0, \frac{w}{n\sigma}\right) (w-1) e^{-w} \quad (43)$$

and

$$S = \sigma J + \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} dw H\left(0, \frac{w}{n\sigma}\right) e^{-w}. \quad (44)$$

Consequently, in the present case, we have

$$\begin{aligned} J &= N \frac{1}{\sigma_{n=1}} \sum_{n=1}^{\infty} \frac{1}{n} \int_{s\sigma}^{\infty} dw (w-1) e^{-w} \\ &= N s \sum_{n=1}^{\infty} e^{-s n \sigma} \\ &= N s \frac{1}{e^{s\sigma} - 1}. \end{aligned} \quad (45)$$

Similarly, we get

$$S = \sigma J - N \ln(1 - e^{-s\sigma}). \quad (46)$$

Eliminating σ , we find that at $T=0$, we have

$$S = \frac{J}{s} \ln \left[1 + \frac{Ns}{J} \right] + N \ln \left[1 + \frac{J}{Ns} \right], \quad (47)$$

which violates the Nernst formulation of the third law. Note that a similar behavior of the entropy at $T=0$ also should hold for any system in which the angular momentum of the system is carried in discrete “vortex structures,” such as occurs in superfluid helium. (Here, N should correspond roughly to the number of vortex structures that could occur in the superfluid helium without overlapping. Presumably, we would need $J/s \ll N$ in order to have the vortex structures present.) Thus, if the vortex structures in superfluid helium persist to absolute zero temperature and can be treated as noninteracting, that system should violate the Nernst formulation of the third law. However, the entropy contributed by the vortex structures should be negligible at temperatures achievable in the laboratory.

Another simple example of a system which violates the Nernst formulation of the third law is provided by a free, massless, gas of scalar particles, which is confined to a one-dimensional ring of radius R . The states in this case decompose into “right movers” and “left movers,” and the density of states is simply

$$g(\epsilon, j) = \delta(\epsilon - j/R) + \delta(\epsilon + j/R). \quad (48)$$

Thus, $\Omega_{\pm} = 1/R$, and, in terms of the variables (y, j) , we have

$$h(y, j) = \delta(y) + \delta(y + 2j/R). \quad (49)$$

Again, for $J > 0$ the states with $y=0$ dominate the low temperature behavior. Since $H(0, j) = j$, Eqs. (43) and (44) yield

$$\begin{aligned} J &= \frac{1}{\sigma} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} dw \frac{w}{n\sigma} (w-1) e^{-w} \\ &= \frac{1}{\sigma^2} \zeta(2) \int_0^{\infty} dw w (w-1) e^{-w} \\ &= \frac{\pi^2}{6\sigma^2}, \end{aligned} \quad (50)$$

and, similarly,

$$S = \frac{\pi^2}{3\sigma}. \quad (51)$$

Thus, we find that at $T=0$,

$$S = \frac{2\pi}{\sqrt{6}} J^{1/2}, \quad (52)$$

in violation of the Nernst formulation of the third law. Note that this example is essentially the same system as considered in the string theory models of charged black holes which saturate the BPS bound [11].

Encouraged by the ability to violate the Nernst formulation of the third law in the simple examples above, we may ask whether it is possible to reproduce the relations (3) and (4) with an ideal boson gas at absolute zero temperature. However, it is easy to see that if $S(T, J)$ remains finite as $T \rightarrow 0$, then Eq. (3) *cannot* be satisfied by any ideal boson gas at $T=0$. Namely, it follows immediately from Eqs. (10) and (11) that as $T \rightarrow 0$, we have $(E - \Omega J) \rightarrow 0$. However, since $\sigma = \beta(\Omega_+ - \Omega)$ always remains bounded as $T \rightarrow 0$ at fixed $J > 0$ [see Eq. (27) above], we also have $\Omega \rightarrow \Omega_+$ as $T \rightarrow 0$. Thus, provided only that S is finite at $T=0$, the relation

$$E = \Omega_+ J \quad (53)$$

always holds at $T=0$, rather than $E \propto J^{1/2}$ as in Eq. (3).

However, a simple and natural modification of the model of a boson gas confined to a ring does yield the desired behavior $E \propto J^{1/2}$. Suppose that we take the ring radius, R , to be an additional dynamical degree of freedom of the system (which we treat classically). In addition, suppose that, due to tension, this ring has an energy $E_R = \lambda R$ with λ a constant. In other words, suppose that the “ring” is actually a “string.” [The “massless boson gas confined to the ring” could then arise naturally as certain (quantized) degrees of freedom describing deviations of the string from circularity.] The total energy of the system would then be

$$E = E_G + E_R = E_G + \lambda R, \quad (54)$$

where E_G denotes the energy of the boson gas. By Eq. (53), at $T=0$ we have $E_G = \Omega_+ J = J/R$, and R will be determined by minimizing the total energy. We obtain

$$R = \sqrt{J/\lambda} \quad (55)$$

and, thus

$$E = 2\sqrt{\lambda} J^{1/2}, \quad (56)$$

in agreement with the behavior in Eq. (3).⁴

Can Eq. (4) also be satisfied in this model? As calculated above, for a free, massless boson gas (or a collection of such gases), we have $S \propto J^{1/2}$ [see Eq. (52)], rather than $S \propto J$, as required by Eq. (4). Indeed, for any system for which Eqs. (43) and (44) hold at $T=0$ and for any polynomial behavior of $H(0, j)$ such that $H(0, 0) = 0$ [see Eq. (14)], it is easy to check that $S/J \rightarrow 0$ as $J \rightarrow \infty$. What seems to be required to obtain the behavior (4) in any model where Eqs. (43) and (44) hold at $T=0$ is to have exponential growth of $H(0, j)$ at large j . I know of no physically reasonable model involving an ideal boson gas in which this behavior occurs.

Nevertheless, one possibility is worth analyzing further with regard to whether the behavior (4) at $T=0$ can be obtained in the above simple “string model.” Suppose we allow the string to have a spectrum of massive particles which rises exponentially in M , i.e., $n(M) \propto e^{\alpha M}$. (Such an expo-

⁴Note that the above argument is similar in nature to the argument leading to the identical formula, $J \propto M^2$, for a meson modeled as a QCD flux tube (i.e., a string) with (massless) quarks at its ends; see, e.g., p. 311 of [14].

nentially rising spectrum occurs in string theory [15].) Although for a massive particle, no single-particle states satisfy $j/\epsilon=R$, a sufficiently rapidly growing density of states—in particular, as discussed above, exponential growth of the density of states in j at fixed y —could allow states with $j/\epsilon<R$ to contribute to the thermodynamic properties of the system at $T=0$, thus invalidating Eqs. (43) and (44). Since each particle of mass M contributes a density of states $g_M(\epsilon,j)=\delta(\epsilon-\sqrt{M^2+j^2/R^2})$, the density of states for an exponentially rising spectrum behaves as

$$g(\epsilon,j)\sim e^{\alpha\sqrt{\epsilon^2-j^2/R^2}} \quad (57)$$

or, equivalently,

$$h(y,j)\sim e^{\alpha\sqrt{y^2+2yj/R}}. \quad (58)$$

The leading order behavior of $H(y,j)$ (at large values of y^2+2yj/R) is similar. Thus, $H(y,j)$ does indeed grow more

rapidly than polynomially in j at fixed y , but it also grows more slowly than exponentially in j . If any massless particles are present [so that $H(0,j)>0$ for some j], then $\sigma_0>0$, and it is not difficult to see that the massive states will not, in fact, contribute to the thermodynamic behavior of the system at $T=0$. On the other hand, if no massless particles are present, then the growth of states with j is not rapid enough to avoid having $\sigma_0=0$, and the Nernst formulation of the third law should hold. Thus, I see no natural way of obtaining the behavior (4) at $T=0$ in the context of this simple “string model.” Of course, as emphasized in the Introduction, we have little right to expect to be able to obtain all of the thermodynamic properties of extremal rotating black holes with such a naive model.

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