Gravitational excitons from extra dimensions

U. Günther* and A. Zhuk†
Department of Physics, University of Odessa, 2 Petra Velikogo Street, Odessa 270100, Ukraine
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Inhomogeneous multidimensional cosmological models with a higher-dimensional space-time manifold $M = M_0 \times \Pi_{i=1}^{n} M_i$ $(n \geqslant 1)$ are investigated under dimensional reduction to D_0-dimensional effective models. In the Einstein conformal frame, small excitations of the scale factors of the internal spaces near minima of an effective potential can be observed as massive scalar fields in the external space-time. Parameters of models that ensure minima of the effective potentials are obtained for particular cases and masses of gravitational excitons are estimated. [S0556-2821(97)03622-9]

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I. INTRODUCTION

The large-scale dynamics of the observable part of our present time Universe is well described by the Friedmann model with the four-dimensional Friedmann-Robertson-Walker (FRW) metric. However, it is possible that space-time at short (Planck) distances might have a dimensionality of more than 4 and possess a rather complex topology [1]. String theory [2] and its recent generalizations — p-brane, M-, and F-theory [3,4] — widely use this concept and give it a new foundation. The most consistent formulations of these theories are possible in space-times with critical dimensions $D_c > 4$; for example, in string theory there are $D_c = 26$ or 10 for the bosonic and supersymmetric version, respectively. Usually it is supposed that a D-dimensional manifold M undergoes a ''spontaneous compactification'' [5–8] $M \to M^4 \times B^{D-4}$, where M^4 is the four-dimensional external space-time and B^{D-4} is a compact internal space. So it is natural to consider cosmological consequences of such compactifications. With this in mind, we shall investigate multidimensional cosmological models (MCMs) with the topology

$$M = M_0 \times M_1 \times \cdots \times M_n , \tag{1.1}$$

where M_0 denotes the D_0-dimensional (usually $D_0 = 4$) external space-time and M_i $(i = 1, \ldots, n)$ are D_i-dimensional internal spaces. To make the internal dimensions unobservable at the present time these internal spaces have to be compact and reduced to scales near the Planck length $L_{\rm Pl} \sim 10^{-33}$ cm, i.e., scale factors a_i of the internal spaces should be of order $L_{\rm Pl}$. In this case we cannot move in extra dimensions and our space-time is apparently four dimensional. There is no problem in constructing compact spaces with a positive curvature [9,10]. (For example, every Einstein manifold with constant positive curvature is necessarily compact [11].) However, Ricci-flat spaces and negative curvature spaces also can be compact. This can be achieved by appropriate periodicity conditions for the coordinates [12]–[16] or, equivalently, through the action of discrete groups Γ of isometries related to face pairings and to the manifold's topology. For example, three-dimensional spaces of constant negative curvature are isometric to the open, simply connected, infinite hyperbolic (Lobachevsky) space H^3 [9,10]. However, there exist also an infinite number of compact, multiply connected, hyperbolic coset manifolds H^3/Γ, which can be used for the construction of FRW metrics with negative curvature [12,14]. These manifolds are built from a fundamental polyhedron (FP) in H^3 with faces pairwise identified. The FP determines a tessellation of H^3 into cells that are replicas of the FP, through the action of the discrete group Γ of isometries [14]. The simplest example of Ricci-flat compact spaces is given by D-dimensional tori $T^D = \mathbb{R}^D/\Gamma$. Thus internal spaces may have nontrivial global topology, being compact (i.e., closed and bounded) for any sign of spatial curvature.

In the cosmological context, internal spaces can be called compactified when they are obtained by a compactification [17] or factorization (''wrapping'') in the usual mathematical understanding (e.g. by replacements of the type $\mathbb{R}^D \to S^D$, $\mathbb{R}^D \to \mathbb{R}^D/\Gamma$, or $H^D \to H^D/\Gamma$) with additional contraction of the sizes to Planck scale. The physical constants that appear in the effective four-dimensional theory after dimensional reduction of an originally higher-dimensional model are the result of integration over the extra dimensions. If the volumes of the internal spaces would change, so would the observed constants. Because of limitation on the variability of these constants [18,19] the internal spaces are static or at least slowly variable since the time of primordial nucleosynthesis and, as we mentioned above, their sizes are of the order of the Planck length. Obviously, such compactifications have to be stable against small fluctuations of the sizes (the scale factors a_i) of the internal spaces. This means that the effective potential of the model obtained under dimensional reduction to a four-dimensional effective theory should have minima at $a_i \sim L_{\rm Pl}$ $(i = 1, \ldots, n)$. Because of its crucial role, the problem of stable compactification of extra dimensions was studied intensively in a large number of papers, [20–36]. As result certain conditions were obtained that ensure the stability of these compactifications. However, the position of a system at a minimum of an effective potential does not necessarily mean that extra dimensions are unobservable. As we shall show below, small excitations of a system near a minimum can be observed as massive scalar

*Electronic address: guenther@pool.hrz.htw-zittau.de
†Electronic address: zhuk@paco.odessa.ua

fields in the external space-time. In solid-state physics, excitations of electron subsystems in crystals are called excitons. In our case the internal spaces are an analog of the electronic subsystem and their excitations can be called gravitational excitons. If masses of these excitations are much less than the Planck mass $M_{\rm Pl}\sim 10^{-5}$ g, they should be observable, thus confirming the existence of extra dimensions. In the opposite case of very heavy excitons with masses $m\sim M_{\rm Pl}$ it is impossible to excite them at present time and extra dimensions are unobservable in this way.

The paper is organized as follows. In Sec. II we describe our model and obtain an effective theory in Brans-Dicke and Einstein conformal frames. In Sec. III it is shown that small excitations of the scale factors of the internal spaces near minima of an effective potential in the Einstein frame have a form of massive scalar fields in the external space-time. The masses of such scalar fields are evaluated for particular classes of effective potentials with minima in the case of one-internal-space models (Sec. IV) and two-internal-space models (Sec. V). In Sec. VI we show that conditions for the existence of stable configurations may be quite different for these two types of models.

II. MODEL

We consider a cosmological model with the metric

$$g=g^{(0)}+\sum_{i=1}^{n} e^{2\beta^i(x)}g^{(i)}, \qquad (2.1)$$

which is defined on manifold (1.1) where x are some coordinates of the D_0-dimensional manifold M_0 and

$$g^{(0)}=g^{(0)}_{\mu\nu}(x)dx^\mu\otimes dx^\nu. \qquad (2.2)$$

Let manifolds M_i be D_i-dimensional Einstein spaces with the metric $g^{(i)}$, i.e.,

$$R_{mn}[g^{(i)}]=\lambda^i g^{(i)}_{mn}, \quad m,n=1,\dots,D_i \qquad (2.3)$$

and

$$R[g^{(i)}]=\lambda^i D_i\equiv R_i. \qquad (2.4)$$

In the case of constant curvature spaces parameters λ^i are normalized as $\lambda^i=k_i(D_i-1)$ with $k_i=\pm 1,0$. We note that each of the spaces M_i can be split into a product of Einstein spaces: $M_i\to\prod_{k=1}^{n_i}M_i^k$ [37]. Here M_i^k are Einstein spaces of dimensions D_i^k with the metric $g^{(i)}_{(k)}$: $R_{mn}[g^{(i)}_{(k)}]=\lambda^i_k g^{(i)}_{(k)mn}$ $(m,n=1,\dots,D_i^k)$ and $R[g^{(i)}_{(k)}]=\lambda^i_k D_i^k$. Such a splitting procedure is well defined provided M_i^k are not Ricci flat [37,38]. If M_i is a split space, then for curvature and dimension we have, respectively [37], $R[g^{(i)}]=\sum_{k=1}^{n_i}R[g^{(i)}_{(k)}]$ and $D_i=\sum_{k=1}^{n_i}D_i^k$. Later on we shall not specify the structure of the spaces M_i. We require only M_i to be compact spaces with arbitrary sign of curvature.

With the total dimension $D=\sum_{i=0}^{n}D_i$, κ^2 a D-dimensional gravitational constant, Λ a D-dimensional cosmological constant, and $S_{\rm YGH}$ the standard York-Gibbons-Hawking boundary term [39,40], we consider an action of the form

$$S=\frac{1}{2\kappa^2}\int_M d^Dx\sqrt{|g|}\{R[g]-2\Lambda\}+S_{\rm add}+S_{\rm YGH}. \qquad (2.5)$$

The additional potential term

$$S_{\rm add}=-\int_M d^Dx\sqrt{|g|}\rho(x) \qquad (2.6)$$

is not specified and left in its general form, taking into account the Casimir effect [20], the Freund-Rubin monopole ansatz [6], a perfect fluid [41,42], or other hypothetical potentials [34,36]. In all these cases ρ depends on the external coordinates through the scale factors $a_i(x)=e^{\beta^i(x)}$ $(i=1,\dots,n)$ of the internal spaces. We did not include into the action (2.5) a minimally coupled scalar field with potential $U(\psi)$ because in this case there exist no solutions with static internal spaces for scalar fields ψ depending on the external coordinates [34].

After dimensional reduction the action reads

$$S=\frac{1}{2\kappa_0^2}\int_{M_0}d^{D_0}x\sqrt{|g^{(0)}|}\prod_{i=1}^{n}e^{D_i\beta^i}\times\Bigg\{R[g^{(0)}]-G_{ij}g^{(0)\mu\nu}\partial_\mu\beta^i\partial_\nu\beta^j+\sum_{i=1}^{n}R[g^{(i)}]e^{-2\beta^i}-2\Lambda-2\kappa^2\rho\Bigg\}, \qquad (2.7)$$

where $\kappa_0^2=\kappa^2/\mu$ is the D_0-dimensional gravitational constant, $\mu=\prod_{i=1}^n\mu_i=\prod_{i=1}^n\int_{M_i}d^{D_i}y\sqrt{|g^{(i)}|}$, and $G_{ij}=D_i\delta_{ij}-D_iD_j$ $(i,j=1,\dots,n)$ is the midisuperspace metric [43,44]. Here the scale factors β^i of the internal spaces play the role of scalar fields. Comparing this action with the tree-level effective action for a bosonic string it can be easily seen that the volume of the internal spaces $e^{-2\Phi}\equiv\prod_{i=1}^n e^{D_i\beta^i}$ plays the role of the dilaton field [37,44,45]. We note that sometimes all scalar fields associated with β^i are called dilatons. Action (2.7) is written in the Brans-Dicke frame. Conformal transformation to the Einstein frame

$$\hat g^{(0)}_{\mu\nu}=e^{-4\Phi/(D_0-2)}g^{(0)}_{\mu\nu}=\left(\prod_{i=1}^{n}e^{D_i\beta^i}\right)^{2/(D_0-2)}g^{(0)}_{\mu\nu} \qquad (2.8)$$

yields

$$S=\frac{1}{2\kappa_0^2}\int_{M_0}d^{D_0}x\sqrt{|\hat g^{(0)}|}\{\hat R[\hat g^{(0)}]-\overline{G}_{ij}\hat g^{(0)\mu\nu}\partial_\mu\beta^i\partial_\nu\beta^j-2U_{\rm eff}\}. \qquad (2.9)$$

The tensor components of the midsuperspace metric (target space metric on $\mathbb{R}^n_T$) $\overline{G}_{ij}(i,j=1,\dots,n)$, its inverse metric $\overline{G}^{ij}$, and the effective potential are, respectively,

$$\overline{G}_{ij}=D_i\delta_{ij}+\frac{1}{D_0-2}D_iD_j, \qquad (2.10)$$

$$\overline{G}^{ij}=\frac{\delta^{ij}}{D_i}+\frac{1}{2-D}, \qquad (2.11)$$

and

$$U_{\rm eff}=\left(\prod_{i=1}^{n} e^{D_i\beta^i}\right)^{-2/(D_0-2)}\left[-\frac{1}{2}\sum_{i=1}^{n} R_i e^{-2\beta^i}+\Lambda+\kappa^2\rho\right]. \tag{2.12}$$

We recall that ρ depends on the scale factors of the internal spaces: $\rho=\rho(\beta^1,\ldots,\beta^n)$. Thus we are led to the action of a self-gravitating σ-model with flat target space $(\mathbb{R}_T^n,\overline{G})$ (2.10) and self-interaction described by the potential (2.12).

Let us first consider the case of one internal space $n=1$. Redefining the dilaton field as

$$\varphi\equiv\pm\sqrt{\frac{D_1(D-2)}{D_0-2}}\beta^1, \tag{2.13}$$

we get, for the action and effective potential, respectively,

$$S=\frac{1}{2\kappa_0^2}\int d^{D_0}x\sqrt{|\hat{g}^{(0)}|}\{\hat{R}[\hat{g}^{(0)}]-\hat{g}^{(0)\mu\nu}\partial_\mu\varphi\partial_\nu\varphi-2U_{\rm eff}\} \tag{2.14}$$

and

$$U_{\rm eff}=\exp\left[2\varphi\left(\frac{D_1}{(D-2)(D_0-2)}\right)^{1/2}\right]\times\left\{-\frac{1}{2}R_1\exp\left[2\varphi\left(\frac{D_0-2}{D_1(D-2)}\right)^{1/2}\right]+\Lambda+\kappa^2\rho(\varphi)\right\}, \tag{2.15}$$

where in expression (2.15) we use for definiteness the minus sign in Eq. (2.13).

Returning to the general case $n>1$, we transform the midsuperspace metric (target space metric) (2.10) by a regular coordinate transformation

$$\varphi=Q\beta,\quad \beta=Q^{-1}\varphi \tag{2.16}$$

to a pure Euclidean form

$$\overline{G_{ij}}d\beta^i\otimes d\beta^j=\sigma_{ij}d\varphi^i\otimes d\varphi^j=\sum_{i=1}^{n} d\varphi^i\otimes d\varphi^i,$$

$$\overline{G}=Q'Q,\quad \sigma={\rm diag}(+1+1,\ldots,+1). \tag{2.17}$$

(The prime denotes the transposition.) An appropriate transformation $Q:\beta^i\mapsto\varphi^j=Q_i^j\beta^i$ is given, e.g., by [43]

$$\varphi^1=-A\sum_{i=1}^{n} D_i\beta^i,$$

$$\varphi^i=[D_{i-1}/\Sigma_{i-1}\Sigma_i]^{1/2}\sum_{j=i}^{n} D_j(\beta^j-\beta^{i-1}),\quad i=2,\ldots,n, \tag{2.18}$$

where $\Sigma_i=\Sigma_{j=i}^n D_j$,

$$A=\pm\left[\frac{1}{D'}\frac{D-2}{D_0-2}\right]^{1/2}, \tag{2.19}$$

and $D':=\Sigma_{i=1}^n D_i$. So we can write action (2.9) as

$$S=\frac{1}{2\kappa_0^2}\int_{M_0} d^{D_0}x\sqrt{|\hat{g}^{(0)}|}\{\hat{R}[\hat{g}^{(0)}]-\sigma_{ik}\hat{g}^{(0)\mu\nu}\times\partial_\mu\varphi^i\partial_\nu\varphi^k-2U_{\rm eff}\} \tag{2.20}$$

with the effective potential

$$U_{\rm eff}=\exp\left(\frac{2}{A(D_0-2)}\right)\varphi^1\times\left(-\frac{1}{2}\sum_{i=1}^{n} R_i e^{-2(Q^{-1})^i{}_k\varphi^k}+\Lambda+\kappa^2\rho\right). \tag{2.21}$$

III. GRAVITATIONAL EXCITONS

Let us suppose that the effective potential (2.12) has minima at points $\vec{\beta}_c=(\beta_c^1,\ldots,\beta_c^n)$,

$$\left.\frac{\partial U_{\rm eff}}{\partial\beta^i}\right|_{\vec{\beta}_c}=0,\quad c=1,\ldots,m, \tag{3.1}$$

and that its Hessian

$$a_{(c)ik}:=\left.\frac{\partial^2 U_{\rm eff}}{\partial\beta^i\partial\beta^k}\right|_{\vec{\beta}_c} \tag{3.2}$$

does not identically vanish at these points. For small fluctuations $\eta^i:=\beta^i-\beta_c^i$ we have then, up to second order in the Taylor expansion,

$$U_{\rm eff}=U_{\rm eff}(\vec{\beta}_c)+\frac{1}{2}\sum_{i,k=1}^{n} a_{(c)ik}\eta^i\eta^k. \tag{3.3}$$

As a sufficient condition for the existence of minima at $\vec{\beta}_c$ we choose in this paper the strong condition consisting in the positivity of the quadratic form

$$\eta' A_c\eta\equiv\sum_{i,k=1}^{n} a_{(c)ik}\eta^i\eta^k>0,\quad\forall\,\eta^1,\ldots,\eta^n, \tag{3.4}$$

with exception of the point $\eta^1=\eta^2=\cdots=\eta^n=0$. It is clear that for higher-order expansions of the effective potential inequality (3.4) can be weaken to a non-negativity condition $\eta' A_c\eta\geqslant 0$ with additional requirements on the multilinear forms occurring in this case. We note that, according to the Sylvester criterion, positivity of quadratic forms is ensured by the positivity of the principal minors of the corresponding matrix, in our case of the matrix A_c:

$$a_{(c)11}>0,\quad \begin{vmatrix} a_{(c)11} & a_{(c)12} \\ a_{(c)21} & a_{(c)22} \end{vmatrix} >0,\ldots$$

$$\ldots,\quad \begin{vmatrix} a_{(c)11} & \cdots & a_{(c)1n} \\ a_{(c)21} & \cdots & a_{(c)2n} \\ \cdots & \cdots & \cdots \\ a_{(c)n1} & \cdots & a_{(c)nn} \end{vmatrix} = \det A_c > 0. \tag{3.5}$$

Equation (3.1) and Hessian (3.2) are affected by the midsuperspace coordinate transformation (2.16) as follows:

$$\left.\frac{\partial U_{\rm eff}}{\partial \varphi^i}\right|_{\vec{\varphi}_c} = \left.\frac{\partial U_{\rm eff}}{\partial \beta^k}\right|_{\vec{\beta}_c} (Q^{-1})^k{}_i = 0,\quad \varphi_c = Q\beta_c, \tag{3.6}$$

$$a_{(c)ik} = \left.\frac{\partial^2 U_{\rm eff}}{\partial \beta^i \partial \beta^k}\right|_{\vec{\beta}_c} = \frac{\partial \varphi^j}{\partial \beta^i} \left.\frac{\partial^2 U_{\rm eff}}{\partial \varphi^j \partial \varphi^l}\right|_{\vec{\varphi}_c} \frac{\partial \varphi^l}{\partial \beta^k} \equiv Q^j_i \overline{a_{(c)jl}} Q^l_k. \tag{3.7}$$

This means that matrices A_c and $\overline{A_c}$ are congruent matrices [46] $A_c = Q' \overline{A_c} Q$ and hence their rank and signature coincide.

Taking into account that transformation (2.16) holds also for small fluctuations near the minima,

$$\xi = Q\eta,\quad \eta = Q^{-1}\xi,\quad \xi^i := \varphi^i - \varphi^i_c, \tag{3.8}$$

we conclude that the quadratic form (3.4) is invariant under this transformation

$$\eta' A_c \eta = (Q^{-1}\xi)' Q' \overline{A_c} Q (Q^{-1}\xi) = \xi' \overline{A_c} \xi. \tag{3.9}$$

Together with the coinciding rank and signature of the congruent matrices $\overline{A_c}$ and A_c, this implies that the positivity of the quadratic form (3.4) remains preserved and minima of $U_{\rm eff}$ in the β representation correspond to minima of $U_{\rm eff}$ in the φ representation.

To get masses of excitations we need to diagonalize the matrices $\overline{A_c}$, keeping at the same time the kinetic term $\hat{g}^{(0)\mu\nu}\sum_{i=1}^n \varphi^i_{,\mu}\varphi^i_{,\nu}$ in its diagonal form. One immediately checks that appropriate SO(n) rotations S_c: $S_c' = S_c^{-1}$ satisfy these requirements

$$\overline{A_c} = S_c' M_c^2 S_c,\quad M_c^2 = \mathrm{diag}(m^2_{(c)1}, m^2_{(c)2}, \ldots, m^2_{(c)n}) \tag{3.10}$$

and

$$\hat{g}^{(0)\mu\nu}\sum_{i=1}^n \varphi^i_{,\mu}\varphi^i_{,\nu} = \hat{g}^{(0)\mu\nu}\sum_{i=1}^n \phi^i_{,\mu}\phi^i_{,\nu}, \tag{3.11}$$

where $\phi = S_c\varphi$. Introducing the corresponding transformed fluctuation fields $\psi = S_c\xi$, we also verify that

$$\eta' A_c \eta = \xi' \overline{A_c} \xi = \psi' M_c^2 \psi. \tag{3.12}$$

It is clear from the Sylvester criterion that all diagonal elements of the matrix M_c^2 should be positive. From relations (2.17), (3.7), and (3.10) it follows that they are eigenvalues of matrix $\overline{A_c}$ as well as matrix $\overline{G^{-1}} A_c$.

So explicit calculations of the matrices S_c and M_c^2 go along standard lines [46] and give, e.g., in the case of two internal spaces ($n=2$),

$$S_c = \begin{pmatrix} \cos\alpha_c & -\sin\alpha_c \\ \sin\alpha_c & \cos\alpha_c \end{pmatrix} \tag{3.13}$$

with the angle of rotation

$$\tan 2\alpha_c = \frac{2\overline{a}_{(c)12}}{\overline{a}_{(c)22} - \overline{a}_{(c)11}} \tag{3.14}$$

and

$$m^2_{(c)1,2} = \frac{1}{2}\left[\mathrm{Tr}(B_c) \pm \sqrt{\mathrm{Tr}^2(B_c) - 4\det(B_c)}\right], \tag{3.15}$$

where

$$B_c = \overline{A_c} \quad \text{or} \quad B_c = \overline{G^{-1}} A_c. \tag{3.16}$$

It can be easily seen that $m^2_{(c)1}, m^2_{(c)2}$ are positive because $\overline{a}_{(c)11}, \overline{a}_{(c)22} > 0$ and $\overline{a}_{(c)11}\overline{a}_{(c)22} > \overline{a}^2_{(c)12}$. So the action functional (2.20) is equivalent to a family of action functionals for small fluctuations of the scale factors of internal spaces in the vicinity of the minima of the effective potential

$$S = \frac{1}{2\kappa_0^2}\int_{M_0} d^{D_0}x \sqrt{|\hat{g}^{(0)}|}\{\hat{R}[\hat{g}^{(0)}] - 2\Lambda_{(c)\rm eff}\} + \sum_{i=1}^n \frac{1}{2}\int_{M_0} d^{D_0}x \sqrt{|\hat{g}^{(0)}|}\{-\hat{g}^{(0)\mu\nu}\psi^i_{,\mu}\psi^i_{,\nu} - m^2_{(c)i}\psi^i\psi^i\},$$

$$c = 1,\ldots,m, \tag{3.17}$$

where $\Lambda_{(c)\rm eff} := U_{\rm eff}(\vec{\phi}_c)$ and the factor $\sqrt{\mu/\kappa^2}$ has been included in ψ for convenience: $\sqrt{\mu/\kappa^2}\psi \rightarrow \psi$.

Thus conformal excitations of the metric of the internal spaces behave as massive scalar fields developing on the background of the external space-time. By analogy with excitons in solid-state physics, where they are excitations of the electronic subsystem of a crystal, the excitations of the internal spaces may be called gravitational excitons.

To conclude this section we want to make a few remarks concerning the form of the effective potential. From the physical viewpoint it is clear that the effective potential should provide the following conditions:

$$\text{(i)}\quad a_{(c)i} = e^{\beta^i_c} \gtrsim L_{\rm Pl},$$
$$\text{(ii)}\quad m_{(c)i} \leqslant M_{\rm Pl},$$
$$\text{(iii)}\quad \Lambda_{(c)\rm eff} \rightarrow 0. \tag{3.18}$$

Condition (i) expresses the fact that the internal spaces should be unobservable at the present time and stable against quantum gravitational fluctuations. This condition ensures the applicability of the classical gravitational equations near positions of minima of the effective potential. Condition (ii) means that the curvature of the effective potential should be less than a Planckian one. Of course, gravitational excitons can be excited at the present time if $m_i \ll M_{\rm Pl}$. Condition (iii) reflects the fact that the cosmological constant at the present time is very small: $\Lambda \leqslant 10^{-54}\ \mathrm{cm}^{-2} \approx 10^{-120}\Lambda_{\rm Pl}$, where $\Lambda_{\rm Pl} = L_{\rm Pl}^{-2}$. Thus, for simplicity, we can demand that $\Lambda_{\rm eff} = U_{\rm eff}(\vec{\beta}_c) = 0$. (We used the abbreviation $\Lambda_{\rm eff} := \Lambda_{(c)\rm eff}$.) Strictly speaking, in the multi-minimum case ($c>1$) we can demand $a_{(c)i} \sim L_{\rm Pl}$ and $\Lambda_{(c)\rm eff}=0$ only for one of the minima, namely, the minimum that corresponds to the state of the present Universe. For all other minima it may be $a_{(c)i} \gg L_{\rm Pl}$ and $|\Lambda_{(c)\rm eff}| \gg 0$.

It can be easily seen that the conditions $\Lambda_{\rm eff}=0$ and $\rho \equiv 0$ are incompatible. In fact, the necessary extremum condition for the potential (2.21) reads

$$\tilde{B}^{-1}\frac{\partial U_{\rm eff}}{\partial \varphi^1} = \sum_{j=1}^{n} r_j (Q^{-1})^j{}_1 + \frac{\partial \rho}{\partial \varphi^1} + q_1 \tilde{B}^{-1} U_{\rm eff} = 0,$$

$$\tilde{B}^{-1}\frac{\partial U_{\rm eff}}{\partial \varphi^i} = \sum_{j=1}^{n} r_j (Q^{-1})^j{}_i + \frac{\partial \rho}{\partial \varphi^i} = 0, \quad i=2,\ldots,n, \tag{3.19}$$

where $r_i := R_i \exp[-2(Q^{-1})^i{}_k \varphi^k]$, $\tilde{B} := \exp q_1 \varphi^1$, and $q_1 = 2/A(D_0-2)$. For $U_{\rm eff}|_{\rm min}=0$ and $\rho\equiv 0$ this system has a nontrivial solution if and only if $\det(Q^{-1})=0$. However, transformation (2.16) is regular. Thus there are no solutions for $U_{\rm eff}|_{\rm min}=0$ and $\rho \equiv 0$ unless all internal spaces are Ricci-flat. Moreover, as follows from potential (2.12), the conditions $U_{\rm eff}|_{\rm min}=0$ and $\partial U_{\rm eff}/\partial \beta^i|_{\rm min}=0$ are compatible if and only if

$$\sum_{i=1}^{n} R_i e^{-2\beta_c^i} = 2[\Lambda + \kappa^2 \rho(\vec{\beta}_c)] \tag{3.20}$$

and

$$R_i e^{-2\beta_c^i} = -\kappa^2 \left.\frac{\partial \rho}{\partial \beta^i}\right|_{\vec{\beta}_c}, \quad i=1,\ldots,n. \tag{3.21}$$

If all internal spaces are Ricci-flat ($R_i\equiv 0$, $i=1,\ldots,n$) and $\rho \equiv 0$, there are no extrema at all.

With $U_{\rm eff}|_{\vec{\beta}_c}=0$, $\partial U_{\rm eff}/\partial \beta^i|_{\vec{\beta}_c}=0$, and Eq. (3.21), the Hessian (3.2) of potential (2.12) reads

$$a_{(c)ik} = \overline{B}\kappa^2 \left[2\delta_{ik} \left.\frac{\partial \rho}{\partial \beta^i}\right|_{\vec{\beta}_c} + \left.\frac{\partial^2 \rho}{\partial \beta^i \partial \beta^k}\right|_{\vec{\beta}_c} \right], \tag{3.22}$$

where $\overline{B} := \exp[-2/(D_0-2)\sum_{i=1}^n D_i \beta_c^i]$. The effective potential $U_{\rm eff}$ has minima at $\vec{\beta}_c$ if matrices $a_{(c)ik}$ satisfy the Sylvester criterion (3.4). Because of $\overline{B}>0$, it is sufficient to check this criterion for the matrix elements $h_{ij} = \overline{B}^{-1} a_{ij}$. For example, in the two-internal-space case ($n=2$) there will be minima if

$$2\left.\frac{\partial \rho}{\partial \beta^i}\right|_{\vec{\beta}_c} + \left.\frac{\partial^2 \rho}{\partial \beta^{i2}}\right|_{\vec{\beta}_c} > 0, \quad i=1,2 \tag{3.23}$$

and

$$\prod_{i=1}^{2}\left(2\left.\frac{\partial \rho}{\partial \beta^i}\right|_{\vec{\beta}_c} + \left.\frac{\partial^2 \rho}{\partial \beta^{i2}}\right|_{\vec{\beta}_c} \right) > \left(\left.\frac{\partial^2 \rho}{\partial \beta^1 \partial \beta^2}\right|_{\vec{\beta}_c} \right)^2. \tag{3.24}$$

Let us suppose a structure of ρ

$$\rho = \sum_{a=1}^{N} A_a \exp\left(\sum_{k=1}^{n} f^a{}_k \beta^k \right), \tag{3.25}$$

where $A_a, f^a{}_k$ are constants. This potential has very general form and includes, for example, a Freund-Rubin monopole ansatz [6], crude approximations of the Casimir effect due to nontrivial topology of the space-time [20,36] and multicomponent perfect fluids [41,42]. In the former case (monopole) the potential ρ reads [25]

$$\rho = \sum_{i=1}^{n} \frac{(f_i)^2}{a_i^{2D_i}} = \sum_{i=1}^{n} (f_i)^2 e^{-2D_i \beta^i}, \tag{3.26}$$

where f_i = const. So, for the matrix $f^i{}_k$ we have $f^i{}_k = -2D_i \delta_{ik}$, $i,k=1,\ldots,n$. In the case of the multicomponent perfect fluid the energy density reads [41,42]

$$\rho = \sum_{a=1}^{m} \rho^{(a)} = \sum_{a=1}^{m} A_a \exp\left(-\sum_{k=1}^{n} \alpha_k^{(a)} D_k \beta^k \right), \tag{3.27}$$

where A_a are constants. This formula describes the m-component perfect fluid with the equations of state $P_i^{(a)} = (\alpha_i^{(a)}-1)\rho^a$ in the internal space M_i ($i=1,\ldots,n$). In the external space each component corresponds to vacuum $\alpha_0^{(a)}=0$ ($a=1,\ldots,m$). For this example $f^a{}_k = -\alpha_k^{(a)} D_k$.

For potential (3.25) Eq. (3.21) can be rewritten as

$$r_k = -\kappa^2 \sum_{a=1}^{N} h_a f^a{}_k, \quad k=1,\ldots,n, \tag{3.28}$$

where $r_k := R_k \exp(-2\beta_c^k)$ and $h_a := A_a \exp(\sum_{k=1}^n f^a{}_k \beta_c^k)$. Now the minimum conditions (3.23) and (3.24), respectively, read

$$\sum_{a=1}^{N} h_a f^a{}_k (f^a{}_k + 2) > 0, \quad k=1,2 \tag{3.29}$$

and

$$\prod_{k=1}^{2}\left(\sum_{a=1}^{N} h_a f^a{}_k (f^a{}_k+2) \right) > \left(\sum_{a=1}^{N} h_a f^a{}_1 f^a{}_2 \right)^2. \tag{3.30}$$

For example, for the monopole potential (3.26) we obtain the extremum condition

$$R_k \exp[2(D_k-1)\beta_c^k] = 2D_k \kappa^2 (f_k)^2, \quad k=1,\ldots,n. \tag{3.31}$$

It follows from this expression that there exists an extremum if $\text{sgn}R_k>0$, $k=1,\ldots,n$. Conditions (3.29) and (3.30) show that this extremum is a minimum (for $D_k>1$).

IV. ONE INTERNAL SPACE

Here we consider the case of one internal space or, strictly speaking, the case where all internal spaces have one common scale factor. In the case under consideration the action and the effective potential are given by Eqs. (2.14) and (2.15), respectively. To get masses of the gravitational excitons it is necessary to specify the potential ρ. For this purpose we consider four particular examples.

A. Pure geometrical potential $\rho\equiv 0$

The necessary condition for the existence of an extremum gives

$$\frac{R_1}{D_1}e^{-2\beta_c}=\frac{2\Lambda}{D-2}, \tag{4.1}$$

where $\beta:=\beta^1$. It follows from this expression that $\text{sgn}\Lambda=\text{sgn}R_1$. From the minimum condition

$$a_{11}=\left.\frac{\partial^2 U_{\text{eff}}}{\partial\beta^2}\right|_{\beta_c}=-\frac{2(D-2)}{D_0-2}R_1(e^{-2\beta_c})^{(D-2)/(D_0-2)}>0 \tag{4.2}$$

we see that bare cosmological constant and curvature of the internal space should be negative $\Lambda,R_1<0$. The effective cosmological constant is

$$\Lambda_{\text{eff}}=\frac{D_0-2}{2D_1}R_1(e^{-2\beta_c})^{(D-2)/(D_0-2)} \tag{4.3}$$

and negative for $R_1<0$. The mass squared of the exciton reads

$$m^2=-\frac{4\Lambda_{\text{eff}}}{D_0-2}=\frac{2|R_1|}{D_1}(e^{-2\beta_c})^{(D-2)/(D_0-2)}. \tag{4.4}$$

If we assume, for example, that for a space-time configuration $M_0\times M_1$ with four-dimensional external space-time ($D_0=4$) and compact internal factor space $M_1=H^{D_1}/\Gamma$ with constant negative curvature $R_1=-D_1(D_1-1)$ there exists a minimum of the effective potential at $a_c=10^2L_{\text{Pl}}$, then we get $m^2=2(D_1-1)10^{-2(D_1+2)}M_{\text{Pl}}^2$ and $\Lambda_{\text{eff}}=-(D_1-1)10^{-2(D_1+2)}\Lambda_{\text{Pl}}$. Thus, according to observational data with $|\Lambda_{\text{eff}}|\leqslant 10^{-120}\Lambda_{\text{Pl}}$, there should be at least $D_1=59$ and the corresponding excitons would be extremely light particles with masses $m\leqslant 10^{-60}M_{\text{Pl}}\sim 10^{-55}$ g. If one uses a reduction of the effective cosmological constant holding $\Lambda=2R_1$ and R_1 fixed when $D_1\to\infty$ (this can be achieved by a conformal transformation $g^{(1)}\to D_1^2 g^{(1)}$ with fixed $\kappa_0^2=\kappa^2/\mu$), one gets $a_c\to L_{\text{Pl}}$ and $\Lambda_{\text{eff}}\to 0$. But at the same time the exciton mass vanishes ($m\to 0$) and the effective potential degenerates into a step function with infinite height: $U_{\text{eff}}\to\infty$ for $a<1$ and $U_{\text{eff}}=0$ for $a\geqslant 1$. Thus, in the limit $D_1\to\infty$ there is no minimum at all.

As it was shown in Sec. III, the effective cosmological constant is not equal to zero if $\rho\equiv 0$. To satisfy this condition we should consider the case $\rho\not\equiv 0$.

B. Casimir potential $\rho=Ce^{-D\beta}$

Because of a nontrivial topology of the space-time, vacuum fluctuations of quantized fields result in a nonzero energy density of the form [20,23,26,47,48]

$$\rho=Ce^{-D\beta}, \tag{4.5}$$

where C is a constant and its value depends strongly on the topology of the model. For example, for fluctuations of scalar fields the constant C was calculated to take the values $C=-8.047\times 10^{-6}$ if $M_0=\mathbb{R}\times S^3$, $M_1=S^1$ (with e^{β^0} a scale factor of S^3 and $e^{\beta^0}\gg e^{\beta^1}$) [23]; $C=-1.097$ if $M_0=\mathbb{R}\times\mathbb{R}^2$, $M_1=S^1$ [47]; and $C=3.834\times 10^{-6}$ if $M_0=\mathbb{R}\times S^3$, $M_1=S^3$ (with $e^{\beta^0}\gg e^{\beta^1}$) [23].

From Eqs. (3.20) and (3.21) (for $n=1$), i.e., conditions $\partial U_{\text{eff}}/\partial\beta|_{\text{min}}=0$ and $\Lambda_{\text{eff}}=0$, we immediately derive

$$R_1e^{-2\beta_c}=\frac{2D}{D-2}\Lambda \tag{4.6}$$

and

$$R_1e^{(D-2)\beta_c}=\kappa^2CD. \tag{4.7}$$

An extremum exists if $\text{sgn}R_1=\text{sgn}\Lambda=\text{sgn}C$. Expressions (4.6) and (4.7) provide fine-tuning for the parameters of the model. Similar fine-tuning was obtained by different methods in papers [26] (for one internal space) and [34] (for n identical internal spaces). The second derivative and mass squared read, respectively,

$$a_{11}=\left.\frac{\partial^2 U_{\text{eff}}}{\partial\beta^2}\right|_{\beta_c}=(D-2)R_1(e^{-2\beta_c})^{(D-2)/(D_0-2)}, \tag{4.8}$$

$$m^2=\frac{D_0-2}{D_1}R_1(e^{-2\beta_c})^{(D-2)/(D_0-2)}. \tag{4.9}$$

Thus the internal space should have positive curvature $R_1>0$ (or for split space M_1 the sum of the curvatures of the constituent spaces M_1^k should be positive).

Let us consider a manifold M with topology $M=\mathbb{R}\times S^3\times S^3$, where $e^{\beta^0}\gg e^{\beta^1}$. Then [23] $C=3.834\times 10^{-6}>0$. As $C,R_1>0$, the effective potential has a minimum provided $\Lambda>0$. Normalizing κ_0^2 to unity, we get $\kappa^2=\mu$, where $\mu=2\pi^{(d+1)/2}/\Gamma(\frac{1}{2}(d+1))$ for the d-dimensional sphere. For the model under consideration we obtain $a_c\approx 1.5\times 10^{-1}L_{\text{Pl}}$ and $m\approx 2.12\times 10^2 M_{\text{Pl}}$. Hence conditions (i) and (ii) are not satisfied for this topology. For other topologies this problem needs a separate investigation.

C. Monopole potential $\rho=f^2e^{-2D_1\beta}$

The monopole ansatz [6] consists in the proposal that an antisymmetric tensor field of rank D_1 is not equal to zero only for components corresponding to the internal space M_1. The energy density of this field reads [24,25]

$$\rho = f^2 e^{-2D_1\beta}, \tag{4.10}$$

where f is an arbitrary constant (free parameter of the model). Equations (3.20), (3.21), and (3.31) yield the zero extremum conditions

$$\Lambda = \frac{D_1 - 1}{2D_1} R_1 e^{-2\beta_c} \tag{4.11}$$

and

$$\frac{R_1}{2D_1\kappa^2 f^2} = e^{-2\beta_c(D_1-1)}, \tag{4.12}$$

which show that $R_1, \Lambda > 0$. The exciton mass squared reads

$$m^2 = \frac{2(D_0-2)(D_1-1)}{D_1(D-2)} R_1 (e^{-2\beta_c})^{(D-2)/(D_0-2)}. \tag{4.13}$$

Condition (i) is satisfied if

$$f^2 \gtrsim R_1/2\kappa^2 D_1. \tag{4.14}$$

Let M_1 be a three-dimensional sphere; then $R_1 = 6$ and $\kappa^2 = 2\pi^2$. To get a minimum of the effective potential for a scale factor $a_c = 10L_{\rm Pl}$ we should take $f^2 \approx 5\times 10^2$. For this value of a_c and for $D_0 = 4$ the mass squared is $m^2 = \frac{16}{5}\times 10^{-5} \ll M_{\rm Pl}^2$. Thus, all three conditions (i)–(iii) are satisfied.

D. Perfect-fluid potential $\rho = Ae^{-\alpha D_1\beta}$

The one-component perfect-fluid potential reads [41,42]

$$\rho = Ae^{-\alpha D_1\beta}, \tag{4.15}$$

where A is an arbitrary positive constant. It describes the vacuum in the external space and a perfect fluid with the equation of state $P = (\alpha - 1)\rho$ in the internal space M_1. Physical values of α are restricted to

$$0 \leqslant \alpha \leqslant 2. \tag{4.16}$$

It is easy to see that the case $\alpha = 0$ corresponds to the vacuum in the space M_1 and contributes to the bare cosmological constant Λ. Therefore, we shall not consider $\alpha = 0$ because in this case we return to Sec. IV A. The other limiting case with $\alpha = 2$ formally coincides here with the monopole potential (4.10).

For the perfect-fluid potential (4.15) a vanishing effective cosmological constant $\Lambda_{\rm eff} = 0$ [Eq. (3.20)] and extremum condition (3.21) yield

$$R_1 e^{(\alpha D_1 - 2)\beta_c} = \kappa^2 \alpha D_1 A \tag{4.17}$$

and

$$R_1 e^{-2\beta_c} = \frac{2\alpha D_1}{\alpha D_1 - 2}\Lambda. \tag{4.18}$$

For the second derivative of the effective potential in the minimum we obtain

$$a_{11} = \left.\frac{\partial^2 U_{\rm eff}}{\partial \beta^2}\right|_{\beta_c} = (\alpha D_1 - 2) R_1 (e^{-2\beta_c})^{(D-2)/(D_0-2)}. \tag{4.19}$$

Because $\alpha, A > 0$, Eq. (4.17) shows that the internal space M_1 should have a positive curvature $R_1 > 0$. From Eq. (4.19) we see that there exists a minimum if $\alpha > 2/D_1$. The corresponding mass squared of the exciton is given as

$$m^2 = \frac{(D_0-2)(\alpha D_1 - 2)}{D_1(D-2)} R_1 (e^{-2\beta_c})^{(D-2)/(D_0-2)}. \tag{4.20}$$

For the critical value of α at $\alpha = 2/D_1$ the model becomes degenerate $U_{\rm eff} \equiv 0$.

As an illustration, let M_1 be a three-dimensional sphere and $a_c = 10L_{\rm Pl}$. This minimum can be achieved for $A = (\alpha\pi^2)^{-1}\times 10^{\alpha D_1 - 2}$. Thus $3/2\pi^2 < A \leqslant 5\times 10^2$ and $0 < m^2 \leqslant \frac{16}{5}\times 10^{-5}$ for $2/D_1 < \alpha \leqslant 2$ and $D_0 = 4$. We see that all conditions (i)–(iii) are satisfied here.

In this section we considered four simple examples of the effective potential and showed that some of them satisfy conditions (i)–(iii).

V. INTERNAL SPACES WITH TWO SCALE FACTORS

In this section we extend the consideration of possible excitons from effective potentials satisfying conditions (3.18) to internal spaces with two scale factors. We analyze three potentials: the pure geometrical potential, the effective potential of a perfect fluid, and the monopole potential. Stability considerations for Casimir-like potentials can be found in our paper [36].

A. Pure geometrical potential $U_{\rm eff,0} \equiv U_{\rm eff}$ $(\rho \equiv 0)$

In this case the condition for the existence of an extremum $\partial U_{\rm eff,0}/\partial \beta^k = 0$ implies a fine-tuning

$$\frac{R_k}{D_k} e^{-2\beta_c^k} = \frac{2\Lambda}{D-2}, \quad k = 1,2, \quad e^{\beta_c^k} = \left[\frac{R_k D_i}{R_i D_k}\right]^{1/2} e^{\beta_c^i} \tag{5.1}$$

of the scale factors and $\mathrm{sgn}\Lambda = \mathrm{sgn} R_i$. From the Hessian

$$a_{(c)ik} \equiv \left.\frac{\partial^2 U_{\rm eff,0}}{\partial \beta^i \partial \beta^k}\right|_{\vec\beta_c} = -\frac{4\Lambda_{\rm eff}}{D_0-2}\left[\frac{D_i D_k}{D_0-2} + \delta_{ik} D_k\right]$$
$$= -\frac{4\Lambda}{D-2}\left[\frac{D_i D_k}{D_0-2} + \delta_{ik} D_k\right] \exp\left[-\frac{2}{D_0-2}\sum_{i=1}^{2} D_i \beta_c^i\right] \tag{5.2}$$

we see that, according to the Sylvester criterion where $a_{(c)11} > 0$, $a_{(c)22} > 0$, and $a_{(c)11} a_{(c)22} > a_{(c)12}^2$, there exist massive excitons for this effective potential in the case of a negative cosmological constant $\Lambda < 0$ and negative scalar curvatures $R_k < 0$. The masses of the excitons are easy calculated as eigenvalues of the matrix $\overline{G^{-1}} A_c$ [Eqs. (3.15) and (3.16)]. Because of

$$\overline{G^{-1}}A_c = \begin{pmatrix} \frac{a_{(c)11}}{D_1} - \frac{a_{(c)11}+a_{(c)12}}{D-2} & \frac{a_{(c)12}}{D_1} - \frac{a_{(c)22}+a_{(c)12}}{D-2} \\ \frac{a_{(c)12}}{D_2} - \frac{a_{(c)11}+a_{(c)12}}{D-2} & \frac{a_{(c)22}}{D_2} - \frac{a_{(c)22}+a_{(c)12}}{D-2} \end{pmatrix} = -\frac{4\Lambda_{\rm eff}}{D_0-2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.3}$$

they are given as

$$\begin{aligned} m_1^2 = m_2^2 &= -\frac{4\Lambda_{\rm eff}}{D_0-2} = -\frac{4\Lambda}{D-2}\exp\left[-\frac{2}{D_0-2}\sum_{i=1}^{2} D_i\beta_c^i\right] \\ &= 2\left|\frac{R_1R_2}{D_1D_2}\right|^{1/2}(e^{-\beta_c^1})^{2D_1/(D_0-2)+1}(e^{-\beta_c^2})^{2D_2/(D_0-2)+1} \\ &= 2\left|\frac{R_1}{D_1}\right|\left[\frac{R_2D_1}{R_1D_2}\right]^{-D_2/(D_0-2)}(e^{-2\beta_c^1})^{(D-2)/(D_0-2)}, \end{aligned} \tag{5.4}$$

where the last line follows immediately from the fine-tuning condition (5.1). From Eq. (5.4) we see that the exciton masses m_1, m_2 of the two-scale-factor model are degenerate and related to the corresponding effective cosmological constant $\Lambda_{\rm eff}$ in the same way as in the one-scale-factor case (4.4). As in the one-scale-factor model, for specific space configurations the two-scale-factor model allows the existence of excitons satisfying physical conditions (3.18).

Let us illustrate this situation with an extended version of the example of Sec. IV A. Suppose that $D_0=4$; $M_1 = H^{D_1}/\Gamma_1, R_1 = -D_1(D_1-1)$, $D_1=2$, and $a_{(c)1}=10^2 L_{\rm Pl}$; $M_2 = H^{D_2}/\Gamma_2, R_2 = -D_2(D_2-1)$. Mass formula (5.4), effective cosmological constant and fine-tuning condition (5.1) read, in this case,

$$\begin{aligned} & m_1^2 = m_2^2 = 2(D_2-1)^{-D_2/2}\times 10^{-2(D_2+4)}M_{\rm Pl}^2, \\ & \Lambda_{\rm eff} = -(D_2-1)^{-D_2/2}\times 10^{-2(D_2+4)}\Lambda_{\rm Pl}, \\ & a_{(c)2} = (D_2-1)^{1/2}a_{(c)1} = (D_2-1)^{1/2}10^2 L_{\rm Pl}. \end{aligned} \tag{5.5}$$

Thus conditions (3.18) are satisfied for internal spaces M_2 with dimensions $D_2 \geqslant D_{2,\rm crit}=40$. Indeed, in the case of $D_2 = 40$ we have $m_i^2 \simeq 2\times 10^{-120}M_{\rm Pl}^2$, $\Lambda_{\rm eff} \simeq -10^{-120}\Lambda_{\rm Pl}$, $a_{(c)2} \simeq 6\times 10^2 L_{\rm Pl}$ and hence for $D_2 > 40$ the relations $m_i \ll M_{\rm Pl}$, $|\Lambda_{\rm eff}| < 10^{-120}\Lambda_{\rm Pl}$, and $a_{(c)i} \gtrsim L_{\rm Pl}$ hold, as required in conditions (3.18).

B. Perfect fluid

For a multicomponent perfect fluid with energy density (3.27) the effective potential reads

$$U_{\rm eff} = \left(\prod_{i=1}^{2} e^{D_i\beta^i}\right)^{-2/(D_0-2)}\left[-\frac{1}{2}\sum_{i=1}^{2} R_i e^{-2\beta^i} + \Lambda + \kappa^2\sum_{a=1}^{m} A_a \exp\left(-\sum_{k=1}^{2}\alpha_k^{(a)}D_k\beta^k\right)\right]. \tag{5.6}$$

Following the same scheme as in the previous considerations we first calculate the extremum condition, the Hessian, and exciton masses in their general form and then analyze some concrete subclasses of potentials.

For brevity we introduce the abbreviations

$$u_k^{(a)} := \alpha_k^{(a)} + \frac{2-\sum_{i=1}^{2}\alpha_i^{(a)}D_i}{D-2}, \quad v_k^{(a)} := \tilde{h}_a\alpha_k^{(a)},$$

$$c_k := \frac{2\Lambda D_k}{D-2}, \quad h_a := \kappa^2 A_a e^{-\alpha_1^{(a)}D_1\beta_c^1}e^{-\alpha_2^{(a)}D_2\beta_c^2} > 0,$$

$$\tilde{h}_a := h_a \exp\left[-\frac{2}{D_0-2}\sum_{i=1}^{2} D_i\beta_c^i\right]. \tag{5.7}$$

The extremum condition and Hessian read then

$$\frac{\partial U_{\rm eff}}{\partial \beta^k} = 0, \quad k=1,2,$$

$$\begin{aligned} I_k := c_k + D_k\kappa^2\sum_{a=1}^{m} A_a u_k^{(a)} e^{-\alpha_1^{(a)}D_1\beta_c^1}e^{-\alpha_2^{(a)}D_2\beta_c^2} \\ - R_k e^{-2\beta_c^k} = 0, \quad k=1,2 \end{aligned} \tag{5.8}$$

$$\begin{aligned} a_{(c)ik} \equiv \left.\frac{\partial^2 U_{\rm eff}}{\partial\beta^i\partial\beta^k}\right|_{\vec{\beta}_c} &= -\frac{4\Lambda_{\rm eff}}{D_0-2}\left[\frac{D_iD_k}{D_0-2} + \delta_{ik}D_k\right] \\ &+ \sum_{a=1}^{m}\tilde{h}_a\alpha_k^{(a)}D_k(\alpha_i^{(a)}D_i - 2\delta_{ik}) \end{aligned} \tag{5.9}$$

and from the auxiliary matrix

$$[\overline{G^{-1}}A_c]_{ik} = -\frac{4\Lambda_{\rm eff}}{D_0-2}\delta_{ik} + J_{ik},$$

$$J_{ik} := \sum_{a=1}^{m} v_k^{(a)}(D_k u_i^{(a)} - 2\delta_{ik}) \tag{5.10}$$

we calculate the exciton masses squared as

$$m_{1,2}^2 = -\frac{4\Lambda_{\rm eff}}{D_0-2} + \frac{1}{2}\left[\mathrm{Tr}(J) \pm \sqrt{\mathrm{Tr}^2(J) - 4\det(J)}\right]. \tag{5.11}$$

From Eq. (5.8) we see that the extremum condition has the form of a system of equations in variables $z_1 = e^{-\beta_c^1}$ and $z_2 = e^{-\beta_c^2}$,

$$I_k = c_k + D_k \kappa^2 \sum_{a=1}^{m} A_a u_k^{(a)} z_1^{\alpha_1^{(a)} D_1} z_2^{\alpha_2^{(a)} D_2} - R_k z_k^2 = 0, \quad k=1,2, \tag{5.12}$$

and for a given point $p = \{\Lambda, R_1, R_2, A_1, \ldots, A_m, \alpha_1^{(1)}, \ldots, \alpha_2^{(m)}\}$ in parameter space $\mathbb{R}_{\rm par}^{3(m+1)}$ positions of extrema should be found as solutions of this system. In the general case of $m>1$ and $\alpha_i^{(a)}$ real ($\alpha_i^{(a)} \in \mathbb{R}$) this can be done most efficiently by numerical methods. Partially analytical methods can be applied, e.g., for $\alpha_i^{(a)}$ rational ($\alpha_i^{(a)} \in \mathbb{Q}$). In this case the representation $\alpha_i^{(a)} D_i = n_i^{(a)}/d_i^{(a)}$ holds with natural numerator $n_i^{(a)} \in \mathbb{N}$ and denominator $d_i^{(a)} \in \mathbb{N}^+$, where $n_i^{(a)}, d_i^{(a)}$ are relatively prime, and $\mathcal{G}\,(n_i^{(a)}, d_i^{(a)}) = 1$ (where $\mathcal{G}$ denotes the greatest common denominator). Introducing the least common multiple $\mathcal{L}$ of the denominators $l = \mathcal{L}(d_1^{(1)}, \ldots, d_2^{(m)})$ and the natural numbers $\vartheta_i^{(a)} := (l/d_i^{(a)})\, n_i^{(a)}$ one has $\alpha_i^{(a)} D_i = \vartheta_i^{(a)}/l$. Equations (5.12) transform then to a system of polynomials

$$I_k = c_k + D_k \kappa^2 \sum_{a=1}^{m} A_a u_k^{(a)} y_1^{\vartheta_1^{(a)}} y_2^{\vartheta_2^{(a)}} - R_k y_k^{2l} = 0, \quad k=1,2, \tag{5.13}$$

in the new variables $y_k = z_k^{1/l}$, which can be analyzed by algebraic methods (resultant techniques [49] and techniques of algebraic geometry [50]) and for rational parameters by methods of number theory [51]. So, for common roots of equations $I_1=0$ and $I_2=0$ the resultants [49] $R_{y_1}[I_1, I_2]$ and $R_{y_2}[I_1, I_2]$ must necessarily vanish,

$$R_{y_1}[I_1, I_2] = w(y_2) = 0, \quad R_{y_2}[I_1, I_2] = w(y_1) = 0, \tag{5.14}$$

and the analysis of Eqs. (5.12) can be reduced to an analysis of the polynomials $w(y_1), w(y_2)$ of degree

$$\deg[w(y_1)], \deg[w(y_2)] \le [l \max_a (\alpha_1^{(a)} D_1 + \alpha_2^{(a)} D_2, 2)]^2 \tag{5.15}$$

in only one of the variables y_1 and y_2, respectively. For explicit considerations of extremum positions with the help of algebraic methods in the case of Casimir-like potentials we refer to [36].

We now turn to the consideration of some concrete subclasses of perfect fluids.

1. m-component perfect fluid with $\alpha_i^{(a)} = \alpha^{(a)}$

In this case there exist no massive excitons for vanishing effective cosmological constants $\Lambda_{\rm eff}=0$. Indeed, $m_{1,2}^2>0$ and Eq. (5.11) imply $\mathrm{Tr}(J)>0$ and $\det(J)>0$, which with

$$J_{ik} = D_k W_1 - 2\delta_{ik} W_2, \quad W_1 := \sum_{a=1}^{m} u^{(a)} v^{(a)},$$

$$W_2 := \sum_{a=1}^{m} v^{(a)} \tag{5.16}$$

read $\mathrm{Tr}(J) = D' W_1 - 4W_2 > 0$ and $\det(J) = 2W_2(2W_2 - D' W_1) > 0$. However, because $v^{(a)} = \tilde{h}_a \alpha^{(a)} > 0$ and hence $W_2>0$, this leads to a contradiction. Thus, for the existence of massive excitons $m_{1,2}^2>0$ the effective cosmological constant must be negative $\Lambda_{\rm eff}<0$.

2. One-component perfect fluid with $\alpha_1 \neq \alpha_2$

Again massive excitons are possible for negative effective cosmological constants $\Lambda_{\rm eff}<0$ only. Here, on the one hand, we have in the case of $\Lambda_{\rm eff}=0$ $\det(J) = -2\delta v_1 v_2 (D_0-2)/(D-2) > 0$, $\delta := D_1\alpha_1 + D_2\alpha_2 - 2$, and hence $\delta<0$. On the other hand, from $\mathrm{Tr}(J)>0$ it follows that $[\alpha_1+\alpha_2 - (\delta+2)/(D-2)]\delta>0$ and hence $0>(D_0-2)(\alpha_1+\alpha_2) + D_1\alpha_1 + D_2\alpha_2$. Because $\alpha_k>0$ this is impossible.

3. One-component perfect fluid with $\alpha_1 = \alpha_2 = \alpha$

For this subclass extremum conditions (5.8) can be considerably simplified to yield

$$h = \kappa^2 A e^{-\alpha(D_1\beta_c^1 + D_2\beta_c^2)} = \frac{1}{(D_0-2)\alpha+2}\left(\frac{D-2}{D_k} R_k e^{-2\beta_c^k} - 2\Lambda\right) \tag{5.17}$$

and the same fine-tuning condition as in the case of a pure geometrical potential

$$\tilde{C} = \frac{R_1}{D_1} e^{-2\beta_c^1} = \frac{R_2}{D_2} e^{-2\beta_c^2}. \tag{5.18}$$

An explicit estimation of exciton masses and effective cosmological constant can be easily done. Using Eqs. (5.7), (5.11), and (5.16), we rewrite the exciton masses squared as

$$\begin{pmatrix} m_1^2 \\ m_2^2 \end{pmatrix} = \frac{1}{D-2}\left\{-4\Lambda + h[(D_0-2)\alpha+2] \times \left[\begin{pmatrix} D'\alpha \\ 0 \end{pmatrix} - 2\right]\right\} \exp\left[-\frac{2}{D_0-2}\sum_{i=1}^{2} D_i \beta_c^i\right] \tag{5.19}$$

and transform with Eq. (5.17) inequalities $m_{1,2}^2>0$ and $h>0$ to the equivalent condition

$$\frac{2}{D-2}\Lambda < \tilde{C} < 0. \tag{5.20}$$

Hence stable space configurations with massive excitons are only possible for internal spaces with negative curvature $R_k<0$. Reparametrizing Λ according to Eq. (5.20) as

$$\Lambda=\frac{D-2}{2}(\tilde{C}-\tau), \quad (5.21)$$

with $\tau>0$ a new parameter, we get for exciton masses squared and the effective cosmological constant

$$\begin{pmatrix} m_1^2 \\ m_2^2 \end{pmatrix}=\left[\begin{pmatrix} D'\alpha\tau \\ 0 \end{pmatrix}-2\tilde{C}\right]\exp\left[-\frac{2}{D_0-2}\sum_{i=1}^{2} D_i\beta_c^i\right], \quad (5.22)$$

$$\Lambda_{\rm eff}=-\frac{D_0-2}{2}\left[\tau\frac{(D-2)\alpha}{(D_0-2)\alpha+2}-\tilde{C}\right] \times\exp\left[-\frac{2}{D_0-2}\sum_{i=1}^{2} D_i\beta_c^i\right]. \quad (5.23)$$

According to definition (5.21) and Eqs. (5.17) and (5.18), the parameter τ can be expressed in terms of $\tilde{C}$ and R_k as

$$\tau=\kappa^2 A\frac{(D_0-2)\alpha+2}{D-2}|\tilde{C}|^{D'\alpha/2}\prod_{k=1}^{2}\left|\frac{D_k}{R_k}\right|^{D_k\alpha/2}. \quad (5.24)$$

A comparison of Eqs. (5.22) and (5.23) with formula (5.4) shows that for

$$\tau\ll\tau_0\equiv|\tilde{C}|\min\left(\frac{2}{D'\alpha},\frac{(D_0-2)\alpha+2}{(D-2)\alpha}\right)$$

we return to the pure geometrical potential considered in Sec. V A. So physical conditions (3.18) are fulfilled for internal space configurations with sufficiently high dimensions greater than some critical dimension $D_{\rm crit}$. From Eqs. (5.22) and (5.23) we see that, depending on the value of τ, this critical dimension $D_{\rm crit}$ can only be larger than that for the pure geometrical model. According to Eq. (5.24), there exist excitons for any positive and finite values of the fluid parameter A, but the larger A for fixed α, the larger would be the critical dimension $D_{\rm crit}$. (Here we take into account that $\kappa^2=\mu$ and the volume μ of the compact internal factor spaces with constant negative curvature is finite.)

Comparing the results of this subsection with the results of Sec. IV D, we see that there exists a different behavior of the perfect fluid models in the case of vanishing effective cosmological constant $\Lambda_{\rm eff}=0$. For the one-scale-factor model massive excitons are allowed for $\Lambda_{\rm eff}=0$, whereas in the two-scale-factor model they cannot occur. An explanation of this situation will be given in Sec. VI.

C. Monopole potential $\rho=\Sigma_{k=1}^2(f_k)^2e^{-2D_k\beta^k}$

For the monopole potential the extremum condition (3.1) leads in the case of vanishing effective cosmological constant $\Lambda_{\rm eff}=0$ to a fine-tuning of the scale factors

$$\frac{R_k}{2D_k\kappa^2(f_k)^2}=e^{-2\beta^k(D_k-1)} \quad (5.25)$$

and

$$\Lambda=\frac{1}{2}\sum_{k=1}^{2} R_k e^{-2\beta^k}\frac{D_k-1}{D_k}, \quad (5.26)$$

so that, as for the one-scale-factor model, extrema are only possible if and only if $R_k>0$ and $\Lambda>0$. Because the monopole potential formally coincides with the potential of a perfect fluid with parameters $\alpha_k^{(a)}=2\delta_{ak}$, the exciton masses are given by Eq. (5.11)

$$m_{1,2}^2=\frac{1}{2}[{\rm Tr}(J)\pm\sqrt{{\rm Tr}^2(J)-4\det(J)}], \quad (5.27)$$

where in terms of abbreviations (5.7) matrix J reads

$$J_{ik}=4\tilde{h}_k(D_k-1)\left[\delta_{ik}-\frac{D_k}{D-2}\right]. \quad (5.28)$$

One immediately verifies that ${\rm Tr}(J)>0$, $\det(J)>0$, ${\rm Tr}^2(J)-4\det(J)\geq0$ for dimensions $D_1>1$ and $D_2>1$, and hence $0<m_2^2\leq\frac{1}{2}{\rm Tr}(J)\leq m_1^2<{\rm Tr}(J)$. This means that physical conditions (3.18) are satisfied if ${\rm Tr}(J)\leq M_{\rm Pl}^2$ and $e^{\beta_c^k}\gtrsim L_{\rm Pl}$. Substituting

$$\tilde{h}_k=\frac{R_k}{2D_k}e^{-2\beta_c^k}\exp\left[-\frac{2}{D_0-2}\sum_{i=1}^{2} D_i\beta_c^i\right] \quad (5.29)$$

into Eq. (5.28), we get the matrix trace as

$${\rm Tr}(J)=\frac{2}{D-2}\left[\sum_{k=1}^{2}\frac{(D_k-1)}{D_k}R_k(D-2-D_k)e^{-2\beta_c^k}\right] \times\exp\left[-\frac{2}{D_0-2}\sum_{i=1}^{2} D_i\beta_c^i\right]. \quad (5.30)$$

With this formula at hand we have, e.g., for an internal space configuration $M_1\times M_2, M_1=S^3$, $a_{(c)1}=10L_{\rm Pl}$; and $M_2=S^5$, $a_{(c)2}=10^2L_{\rm Pl}$ the estimate ${\rm Tr}(J)\approx56\times10^{-14}M_{\rm Pl}^2\ll M_{\rm Pl}^2$ and all conditions (i)–(iii) of Eq. (3.18) are satisfied.

VI. EXCITON MASSES AND SCALE FACTOR CONSTRAINTS

In this section we derive a relation between the exciton masses $m_{(c)1}, m_{(c)2}$ of a model with two independently varying scale factors β^1,β^2 and the effective mass $m_{(c)0}$ of the exciton that occurs under scale factor reduction, i.e., when the scale factors of the model are connected by a constraint $\beta=\beta^1=\beta^2$. In order to simplify our calculation we introduce the projection operator P on the constraint subspace $\mathbb{R}_P^1=\{\bar{\beta}=(\beta^1,\beta^2)|\beta^1-\beta^2=\bar{a}\cdot\bar{\beta}=0,\ \ \bar{a}=(1,-1)\}$ of the two-dimensional target space $\mathbb{R}_T^2$ of the σ-model:

$$P\mathbb{R}_T^2=\mathbb{R}_P^1\subset\mathbb{R}_T^2. \quad (6.1)$$

Explicitly, this projection operator can be constructed from the normalized base vector $\bar{e}$ of the subspace $\mathbb{R}_P^1$. With $\bar{e}=(1/\sqrt{2})\binom{1}{1}$ we have

$$P=\overline{e}\otimes\overline{e'}=\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}\otimes\begin{pmatrix}1 & 1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}, \qquad (6.2)$$

$P^2=P$, and $P\overline{a}=0$.

Let us now calculate the exciton mass $m_{(c)0}$ for the reduced model. For this purpose we introduce the exciton Lagrangian, written according to Sec. III in terms of the fluctuation fields $\overline{\eta}=(\eta^1,\eta^2)$, where $\eta^i\equiv\beta^i-\beta_c^i$:

$$\mathcal{L}_{\rm exci}=-[\overline{\eta}\overline{G}\hat{K}\overline{\eta}+\overline{\eta}A_{(c)}\overline{\eta}]. \qquad (6.3)$$

$\hat{K}:=\bar{\partial}_\mu\hat{g}^{(o)\mu\nu}\vec{\partial}_\nu$ denotes the pure kinetic operator. Under scale factor reduction $\overline{\eta}=(\eta,\eta)$ this Lagrangian transforms to

$$\mathcal{L}_{\rm exci}=-[\gamma_1\eta\hat{K}\eta+\gamma_{(c)2}\eta^2], \qquad (6.4)$$

$$\gamma_1:=2\overline{e'}\overline{G}\overline{e}=\sum_{i,j}\overline{G}_{ij}, \quad \gamma_{(c)2}:=2\overline{e'}A_{(c)}\overline{e}=\sum_{i,j}A_{(c)ij}, \qquad (6.5)$$

so that the substitution $\eta=\gamma_1^{-1/2}\psi$ yields the effective one-scale-factor Lagrangian $\mathcal{L}_{\rm exci}=-[\psi\hat{K}\psi+\psi m_{(c)0}^2\psi]$ with exciton mass $m_{(c)0}^2=\gamma_{(c)2}/\gamma_1$. Taking into account that $\overline{e'}A_{(c)}\overline{e}={\rm Tr}[PA_{(c)}]$, $A_{(c)}=Q'S_c'M_{(c)}^2S_cQ$, and $M_{(c)}^2={\rm diag}(m_{(c)1}^2,m_{(c)2}^2)$, the needed relation between the exciton masses of the reduced and unreduced two-scale-factor models is now easily established as

$$m_{(c)0}^2=2\gamma_1^{-1}{\rm Tr}[QPQ'S_c'M_{(c)}^2S_c]. \qquad (6.6)$$

With the use of

$$QPQ'=\frac{1}{2}D'\frac{D-2}{D_0-2}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \quad \gamma_1=D'\frac{D-2}{D_0-2} \qquad (6.7)$$

and the SO(2) rotation matrix S_c from Eqs. (3.13) and (3.14), this formula can be considerably simplified to give the final relation

$$m_{(c)0}^2=\cos^2(\alpha_c)m_{(c)1}^2+\sin^2(\alpha_c)m_{(c)2}^2. \qquad (6.8)$$

In its compact form this mass formula implicitly reflects the behavior of the effective potential $U_{\rm eff}$ in the vicinity $\Omega_{\vec{\beta}_c}\subset\mathbb{R}_T^2$ of the extremum point $\vec{\beta}_c$. So the exciton masses squared $m_{(c)1}^2, m_{(c)2}^2$ describe the potential as a function over the two-dimensional $\vec{\beta}_c$ vicinity $\Omega_{\vec{\beta}_c}$, whereas $m_{(c)0}^2$ characterizes $U_{\rm eff}$ as a function over the line interval $\Omega_{\vec{\beta}_c}\cap\mathbb{R}_P^1$ only. A comparison of the minimum conditions of the unreduced and reduced two-scale-factor models

$$m_{(c)1,2}^2>0: \quad a_{(c)11}>0, \quad a_{(c)22}>0, \quad a_{(c)11}a_{(c)22}>(a_{(c)12})^2 \qquad (6.9)$$

and

$$m_{(c)0}^2>0: \quad (a_{(c)11}+a_{(c)22}+2a_{(c)12})>0 \qquad (6.10)$$

shows that stable configurations of reduced models with $m_{(c)0}^2>0$ are not only possible for stable configurations of the unreduced model $m_{(c)1}^2>0$, $m_{(c)2}^2>0$, but even in cases when the potential $U_{\rm eff}$ has a saddle point at $\vec{\beta}_c$ and the unreduced model is unstable. For the masses we have in these cases $m_{(c)1}^2>0$, $m_{(c)2}^2<0$ or $m_{(c)1}^2<0$, $m_{(c)2}^2>0$ and massive excitons in the reduced model correspond to exciton-tachyon configurations in the unreduced model.

VII. CONCLUSIONS

This paper was devoted to the problem of stable compactification of internal spaces. This is one of the most important problems in multidimensional cosmology because via stable compactification of the internal dimensions near Planck length we can explain unobservability of extra dimensions. With the help of dimensional reduction we obtained an effective four-dimensional theory in Brans-Dicke and Einstein frames. The Einstein frame was considered here as a physical one [52]. In this frame we derived an effective potential. It was shown that small excitations of the scale factors of internal spaces near minima of the effective potential take the form of massive scalar particles (gravitational excitons) developing in the external space-time. Detection of these excitations can prove the existence of extra dimensions. Particular examples of effective potentials were investigated in the one- and two-internal-space cases. Parameters of the models that ensure a minimum were obtained and masses of the excitons were estimated. The solutions at the minima of the potential are stable against small perturbations of the scale factor(s) of the expanding external Universe [26]. We would like to note that the problem of stable compactification in MCMs with more than one internal scale factor was considered first for pure geometrical models in Refs. [28,29]. However, the analysis of the effective potential minima existence was not complete there.

Our analysis shows that conditions for the existence of stable configurations may be quite different for one- and two-scale-factor models. For example, in the case of a one-scale-factor model that is filled with a one-component perfect fluid stable compactifications are possible for vanishing effective cosmological constant $\Lambda_{\rm eff}=0$ and parameters α from the restricted interval $2/D_1<\alpha\leqslant 2$ determining the equation of state in the internal space $P_1=(\alpha-1)\rho$. In the case of two-scale-factor models stable compactifications can exist for negative effective cosmological constants $\Lambda_{\rm eff}<0$ only, but for values of the parameter α from the usual interval $0\leqslant\alpha\leqslant 2$ [here α determines the equations of state in both internal spaces $P_1=(\alpha-1)\rho$ and $P_2=(\alpha-1)\rho$]. At first sight the difference in the behavior of these two models looks a bit strange because the one-scale-factor model can be obtained by reduction of the two-scale-factor model with the help of the constraint $\beta_1=\beta_2\equiv\beta$. As it was shown in Sec. VI, such a different behavior may take place because stable configurations of reduced models are possible not only for stable configurations of unreduced models, but even in cases when the effective potential $U_{\rm eff}$ of the unreduced model has a saddle point. In the case of our two-scale-factor model with one-component perfect fluid we get such a saddle point for configurations with $\Lambda_{\rm eff}=0$ and $2/(D_1+D_2)<\alpha\leqslant 2$.

In the present paper we did not consider the case of degenerate minima of the effective potential, for example, self-interaction-type potentials or ''sombrero-type'' potentials. In the former case one obtains massless fields with self-interaction. In the latter case one gets massive fields together with massless ones. Here massless particles can be understood as the analog of Goldstone bosons. This type of the potential was described in [33].

Another possible generalization of our model consists in the proposal that the additional potential ρ may depend also on the scale factor of the external space. It would allow, for example, the consideration of a perfect fluid with arbitrary equation of state in the external space.

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