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Propagation of light in a gravitational background

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We study the propagation of an electromagnetic field in a weak gravitational background generated by a rotating mass. The solution of the Maxwell equations beyond the geometrical optics shows, together with the well-known deflection and rotation of the polarization plane already present in the geometrical optics approximation, new classical dispersive effects. We analyze such effects at first order in the gravitational constant G. In the case of an incoming wave with linear polarization they consist in the development of a component of circular polarization, a breaking of the orthogonality of the electric and magnetic fields, and additional contributions to the deflection of the beam and the rotation of the polarization plane. [S0556-2821(97)06422-9]

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I. INTRODUCTION

Although the effects of a gravitational field on the propagation of light have been studied for many years, they have recently acquired a renewed significance. Usually, when we consider the propagation of electromagnetic fields, the interaction with matter dominates the interaction with gravitation. However, we can conceive astrophysical systems within or near our observational possibilities where the gravitational interaction is the main factor. This last situation can occur in the surroundings of a neutron star or a black hole without an accretion disk, or in general when we have a gravitational lensing effect. In these systems the deflection of light gives us information mainly on the mass distribution of the source. In addition to this, we could in principle obtain more detailed information by the study of other features of the outcoming light related to its wave character, such as the polarization of the different images of a given object. A polarization-dependent deflection of light or other changes induced by the gravitational field could give us interesting data on the angular momentum of the source of the gravitational lens. These phenomena would be very relevant in astrophysics, because they could provide additional information on objects of the uppermost interest such as dwarf stars, neutron stars, galactic nuclei, or black holes.

As is already known, the equations for the electromagnetic field in a gravitational background can be written in an analogous form to the Maxwell equations for a slow moving anisotropic and inhomogeneous medium, where the anisotropy, the inhomogeneity, and the velocity of the "medium" are related to features of the metric tensor [1]. The usual approach to study the changes of the polarization along the light path induced by the gravitational field is based on the approximation of the geometrical optics [2,3]. A more complete analysis has been developed only for cases with very simple metrics, for example, to study the scattering of an electromagnetic field by a Schwarzschild black hole [4].

In this paper we implement a different approach, which can be applied to a large variety of cases of phenomenological relevance. Instead of simply considering the geometrical optics approximation, we use the complete field equations for the electromagnetic field in a background given by a weak gravitational one. To simplify the physical interpretation we consider also a post-Newtonian approximation, applicable to a gravitational field with a nonrelativistic source and a dynamics due mainly to its gravitational interaction [6]. This is enough to give a reliable analysis of the gravitational effects on electromagnetic waves for most of the systems of astrophysical interest and to show several effects that the geometrical approach is unable to display.

In the following section we present a brief review of the Maxwell equations in a gravitational background, and in the next one we develop the general formal solution for the perturbation in the weak gravitational field approximation. Section IV focuses on polarized incoming plane waves and outlines the meaning of the different terms that appear in the solution. In Sec. V we construct the perturbative solution for a beam of light propagating in the gravitational field of a rotating mass and discuss its physical meaning, and in Sec. VI we show how this solution is seen in a locally inertial frame. Finally Sec. VII gives a brief summary and discussion.

II. FREE MAXWELL EQUATIONS IN A GRAVITATIONAL BACKGROUND

If there are no charges and currents present, the equations for an electromagnetic field in a gravitational background described by the metric $g_{\mu\nu}$ are [6]

$$H^{\mu\nu}_{,\nu}=0, \tag{1}$$

$$F_{\mu\nu,\rho}+F_{\nu\rho,\mu}+F_{\rho\mu,\nu}=0, \tag{2}$$

where $F_{\mu\nu}$ is the covariant electromagnetic field tensor and $H^{\mu\nu}\equiv\sqrt{-g}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}$ with $g\equiv\det\|g_{\mu\nu}\|$. From now on the Greek indices run from 0 to 3, whereas Latin indices run from 1 to 3. We use a system of units with the speed of light $c=1$. Expressing the components of $F_{\mu\nu}$ and $H^{\mu\nu}$ in terms of the electromagnetic vector fields,

$$F_{\mu\nu}\rightarrow(\mathbf{E},\mathbf{B}),\quad H^{\mu\nu}\rightarrow(-\mathbf{D},\mathbf{H}), \tag{3}$$

with

$$E_i \equiv F_{i0}, \quad D_i \equiv \sqrt{-g} g^{0\alpha} g^{i\beta} F_{\alpha\beta}, \tag{4}$$

$$B_i \equiv \tfrac{1}{2} \varepsilon_{ijk} F_{jk}, \quad H^i \equiv \tfrac{1}{2} \varepsilon^{ijk} \sqrt{-g} g_{j\alpha} g_{k\beta} F^{\alpha\beta}, \tag{5}$$

where ε_{ijk} is the three-dimensional (3D) complete antisymmetric tensor, Eqs. (1) and (2) can be rewritten

$$\nabla \times \mathbf{E} + \partial_0 \mathbf{B} = 0, \quad \nabla \cdot \mathbf{B} = 0, \tag{6}$$

$$\nabla \times \mathbf{H} - \partial_0 \mathbf{D} = 0, \quad \nabla \cdot \mathbf{D} = 0, \tag{7}$$

together with

$$B_i = -\sqrt{-g} \frac{g^{ik}}{g_{00}} H_k - \epsilon_{ijk} \frac{g_{0j}}{g_{00}} E_k, \tag{8}$$

$$D_i = -\sqrt{-g} \frac{g^{ik}}{g_{00}} E_k + \epsilon_{ijk} \frac{g_{0j}}{g_{00}} H_k. \tag{9}$$

The nabla operator that appears in Eqs. (6) and (7) is defined with respect to the coordinates of the $g_{\mu\nu}$ metric, $\nabla_i = \partial/\partial x^i$ and $(\nabla \times)^{ij} = \epsilon^{ijk} \partial/\partial x^k$, and therefore these equations remain covariant under a general transformation of coordinates, although their form is not explicitly covariant.

We can still handle Eqs. (8) and (9) so as to put them in a more suggestive form:

$$B_i = \epsilon_{ij} H_j + (\mathbf{G} \times \mathbf{E})_i, \tag{10}$$

$$D_i = \epsilon_{ij} E_j - (\mathbf{G} \times \mathbf{H})_i \ , \tag{11}$$

with $\epsilon_{ij} \equiv -\sqrt{-g} g^{ij} g_{00}^{-1}$ and $G_i \equiv -g_{0i} g_{00}^{-1}$. These equations play the role of constitutive equations, and have the form of the usual ones for an anisotropic medium moving with a low velocity in a flat space-time [5]. In this description the movement of the ''medium'' is given by the g^{0i} components of the metric tensor, related to the angular moment of the source of the gravitational field, and the remaining components define the characteristics of the ''medium'' at rest.

III. WEAK GRAVITATIONAL BACKGROUND: EFFECTIVE ELECTROMAGNETIC SOURCE

The covariant Maxwell equations are very difficult to solve in an arbitrary gravitational field and we thus have to resort to numerical solutions except for very special cases. However, we can state a general solution if we consider a weak gravitational field. For the sake of simplicity we will only develop here the first order approximation, which in fact is enough to show the most important effects due to the wave character of the light. To do this we will assume the metric to be close to the flat space-time one $\eta_{\mu\nu}$, such that we can write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$. The parameters that appear in the constitutive equations are now

$$\epsilon_{ij} = -(1 + \tfrac{1}{2}) \eta^{ij} \eta_{00} - \eta^{ij} h_{00} - \eta_{00} h^{ij}, \tag{12}$$

$$G_i = -\eta_{0i} \eta_{00} - \eta_{0i} h_{00} - \eta_{00} h_{0i}. \tag{13}$$

To be more specific we will consider the linear approximation for the gravitational field in the post-Newtonian framework and from here onwards we will use coordinate systems in which the metric tensor is nearly equal to the Minkowski metric. In this case we have

$$g_{00} = -1 - 2\phi, \quad g_{0i} = h_i, \quad g_{ij} = \delta_{ij}(1 - 2\phi), \tag{14}$$

$$g^{00} = -1 + 2\phi, \quad g^{0i} = h_i, \quad g^{ij} = \delta_{ij}(1 + 2\phi), \tag{15}$$

with

$$\phi(\mathbf{r}, t) = -G \int d^3 r' \frac{T^{00}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|},$$

$$h_i(\mathbf{r}, t) = -4G \int d^3 r' \frac{T^{i0}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|}. \tag{16}$$

T^{00} and T^{i0} are the components of the energy-momentum tensor, source of the gravitational field, which correspond to the density of energy and linear momentum. The scalar and vectorial potentials ϕ and $\mathbf{h}$ satisfy the harmonic condition $\nabla \cdot \mathbf{h} + 4\partial_0 \phi = 0$; i.e., we are working in the harmonic gauge. At lower order in $r \equiv |\mathbf{r}|$ and using a coordinate system with the origin at the energy center of the source, the gravitational fields can be written

$$\phi = -\frac{GM}{r}, \quad h_i = -\frac{2 G r^j J_{ji}}{r^3}, \tag{17}$$

where

$$M \equiv \int d^3 r T^{00}, \quad J_{ij} \equiv 2 \int d^3 r r^i T^{j0}. \tag{18}$$

A consistent expansion results from only considering the linear terms in ϕ and h_i, because the contributions of order ϕ^2 can be neglected compared with h_i. This fact is easy to see from the following argument. The magnitude of $\mathbf{h}$ is of order $|\mathbf{h}| \sim 2GJ/r^2 \sim 2MG\overline{v}/\overline{r}$, where J is the magnitude of the angular momentum, and $\overline{r}$ and $\overline{v}$ are the mean radius and the mean velocity, respectively. Besides, the virial theorem states that $GM^2/\overline{r} \sim M\overline{v^2}/2$, which, together with the expression for the magnitude of $|\mathbf{h}|$, implies that $\phi^2 = (GM/r)^2 \sim \frac{1}{4} |\mathbf{h}| \overline{v}$. Therefore, ϕ^2 is of order v/c with respect to $|\mathbf{h}|$.

The characterization of the gravitational interaction on the electromagnetic field up to linear terms in ϕ and $\mathbf{h}$ within this family of coordinates is given by

$$\epsilon_{ij} = (1 - 2\phi) \delta_{ij}, \quad G_i = h_i, \tag{19}$$

and hence the Maxwell equations reduce to

$$\nabla \times \mathbf{E} + \partial_0 \mathbf{B} = 0, \tag{20}$$

$$\nabla \times [(1 + 2\phi)\mathbf{B} - (\mathbf{h} \times \mathbf{E})] - \partial_0 [(1 - 2\phi)\mathbf{E} - (\mathbf{h} \times \mathbf{B})] = 0, \tag{21}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{22}$$

$$\nabla \cdot [(1 - 2\phi)\mathbf{E} - (\mathbf{h} \times \mathbf{B})] = 0, \tag{23}$$

where the nabla operators are defined with respect to the near flat space-time metric $g_{\mu\nu}$. Equations (20) and (23) come from a linearization on the fields ϕ and $\mathbf{h}$, and hence we can solve them perturbatively. Thus, we decompose the electric and magnetic fields according to

$$\mathbf{E}=\mathbf{E}^{(0)}+\mathbf{E}^{(1)}, \quad \mathbf{B}=\mathbf{B}^{(0)}+\mathbf{B}^{(1)}, \tag{24}$$

where $\mathbf{E}^{(0)}$ and $\mathbf{B}^{(0)}$ are fields with the form of a plane wave but in terms of the post-Newtonian coordinates, which satisfy

$$\nabla\times\mathbf{E}^{(0)}+\partial_0\mathbf{B}^{(0)}=0, \quad \nabla\cdot\mathbf{B}^{(0)}=0, \tag{25}$$

$$\nabla\times\mathbf{B}^{(0)}-\partial_0\mathbf{E}^{(0)}=0, \quad \nabla\cdot\mathbf{E}^{(0)}=0. \tag{26}$$

The $\mathbf{E}^{(1)}$ and $\mathbf{B}^{(1)}$ fields are the first order gravitational corrections to $\mathbf{E}^{(0)}$ and $\mathbf{B}^{(0)}$, such that $\mathbf{E}$ and $\mathbf{B}$ are solutions of Eqs. (20)–(23). Thus, they are given by

$$\nabla\times\mathbf{E}^{(1)}+\partial_0\mathbf{B}^{(1)}=0, \quad \nabla\cdot\mathbf{B}^{(1)}=0, \tag{27}$$

$$\nabla\times\mathbf{B}^{(1)}-\partial_0\mathbf{E}^{(1)}=4\pi\mathbf{J}_g, \quad \nabla\cdot\mathbf{E}^{(1)}=4\pi\rho_g, \tag{28}$$

with

$$\rho_g=\frac{1}{4\pi}\nabla\cdot(\mathbf{h}\times\mathbf{B}^{(0)}+2\phi\mathbf{E}^{(0)}), \tag{29}$$

$$\mathbf{J}_g=\frac{1}{4\pi}[\nabla\times(\mathbf{h}\times\mathbf{E}^{(0)}-2\phi\mathbf{B}^{(0)})-\partial_0(2\phi\mathbf{E}^{(0)}+\mathbf{h}\times\mathbf{B}^{(0)})] \ . \tag{30}$$

From the Maxwell equations (20)–(23) and the decomposition (24) it is easy to show that this source satisfies a continuity equation $\dot{\rho}_g+\boldsymbol{\nabla}\cdot\mathbf{J}_g=0$. The structure of these last equations is very interesting and suggestive. They are completely analogous to the Maxwell equations in a flat space-time, but with sources that depend on the gravitational and the free electromagnetic fields.

This approximation allows us to compute the gravitational contribution to the electromagnetic field using the formal solution already known from classical electrodynamics. In particular, to study the evolution of an incoming electromagnetic field it is convenient to use retarded potentials, which leads to

$$\mathbf{E}^{(1)}(\mathbf{r},t)=-\nabla\int d^3r'\,\frac{\rho_g(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|} -\partial_0\int d^3r'\,\frac{\mathbf{J}_g(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}, \tag{31}$$

$$\mathbf{B}^{(1)}(\mathbf{r},t)=\nabla\times\int d^3r'\,\frac{\mathbf{J}_g(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}. \tag{32}$$

IV. PROPAGATION OF AN INCOMING PLANE WAVE

In this section we will write in detail the propagation of an incoming asymptotically plane wave with linear polarization in interaction with a weak gravitational field $(\phi,\mathbf{h})$. To simplify the expressions we suppose that the propagation takes place in a region where there is no mass, so that $\nabla^2\phi=0$, and we assume that both ϕ and $\mathbf{h}$ potentials are time independent. Thus, given that the metric satisfies the harmonic condition we also have $\nabla\cdot\mathbf{h}=0$.

Let us consider the effect of a gravitational field on a monochromatic light beam, with frequency ω and direction $\hat{\varepsilon}_3$. By introducing the orthogonal frame $(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{\varepsilon}_3)$ we can write a solution for Eqs. (25) and (26) with linear polarization as

$$\mathbf{E}^{(0)}(\mathbf{r},t)=|\mathbf{E}|\mathrm{Re}(\hat{\varepsilon}_1 e^{i\omega[\hat{\varepsilon}_3\cdot(\mathbf{r}-\mathbf{r}_0)-t]}),$$

$$\mathbf{B}^{(0)}(\mathbf{r},t)=|\mathbf{E}|\mathrm{Re}(\hat{\varepsilon}_2 e^{i\omega[\hat{\varepsilon}_3\cdot(\mathbf{r}-\mathbf{r}_0)-t]}).$$

From Eqs. (29) and (30), the effective sources for the gravitational contribution to the electromagnetic field are

$$\rho_g=\frac{|\mathbf{E}|}{4\pi}\mathrm{Re}\{(2\nabla\phi\cdot\hat{\varepsilon}_1+i\omega\mathbf{h}\cdot\hat{\varepsilon}_1+\nabla\times\mathbf{h}\cdot\hat{\varepsilon}_2)e^{i\omega[\hat{\varepsilon}_3\cdot(\mathbf{r}-\mathbf{r}_0)-t]}\}, \tag{33}$$

$$\mathbf{J}_g=\frac{|\mathbf{E}|}{4\pi}\mathrm{Re}(\{[2(\nabla\phi-i\omega\mathbf{h})\cdot\hat{\varepsilon}_3+4i\omega\phi]\hat{\varepsilon}_1+[(-2\nabla\phi+i\omega\mathbf{h})\cdot\hat{\varepsilon}_1]\hat{\varepsilon}_3+(\hat{\varepsilon}_1\cdot\nabla)\mathbf{h}\}e^{i\omega[\hat{\varepsilon}_3\cdot(\mathbf{r}-\mathbf{r}_0)-t]}). \tag{34}$$

Hence from Eqs. (33), (34), (31), and (32) we have the following expressions for the gravitational contributions to the electric and magnetic fields, respectively:

$$\mathbf{E}^{(1)}(\mathbf{r},t)=\frac{|\mathbf{E}|}{4\pi}\mathrm{Re}\Bigg\{\int d^3r'\,\frac{e^{i\omega[\hat{\varepsilon}_3\cdot(\mathbf{r}'-\mathbf{r}_0)+|\mathbf{r}-\mathbf{r}'|-t]}}{|\mathbf{r}-\mathbf{r}'|}\{\hat{\varepsilon}_2\ \ (i\omega\nabla'\times\mathbf{h})\cdot\hat{\varepsilon}_3+\hat{\varepsilon}_1\ \ [\omega^2(2\mathbf{h}\cdot\hat{\varepsilon}_3-4\phi)+i\omega 2\hat{\varepsilon}_3\cdot\nabla'\phi] +\hat{\varepsilon}_3\ \ i\omega[-4\nabla'\phi\cdot\hat{\varepsilon}_1-2(\nabla'\times\mathbf{h})\cdot\hat{\varepsilon}_2]-\nabla'(\hat{\varepsilon}_2\cdot\nabla'\times\mathbf{h})-2\nabla'(\hat{\varepsilon}_1\cdot\nabla')\phi\}\Bigg\}, \tag{35}$$

and

$$\mathbf{B}^{(1)}(\mathbf{r},t)=\frac{|\mathbf{E}|}{4\pi}\mathrm{Re}\Bigg\{\int d^3r'\,\frac{e^{i\omega[\hat{\varepsilon}_3\cdot(\mathbf{r}'-\mathbf{r}_0)+|\mathbf{r}-\mathbf{r}'|-t]}}{|\mathbf{r}-\mathbf{r}'|}\{-\hat{\varepsilon}_1\ (i\omega\nabla'\times\mathbf{h})\cdot\hat{\varepsilon}_3+\hat{\varepsilon}_2\ [\omega^2(2\mathbf{h}\cdot\hat{\varepsilon}_3-4\phi)+i\omega(\hat{\varepsilon}_3\cdot\nabla')(6\phi-2\mathbf{h}\cdot\hat{\varepsilon}_3)]$$
$$+\hat{\varepsilon}_3\ i\omega[-4\nabla'\phi\cdot\hat{\varepsilon}_2+2(\hat{\varepsilon}_2\cdot\nabla')(\mathbf{h}\cdot\hat{\varepsilon}_3)]+\nabla'\times(\hat{\varepsilon}_1\cdot\nabla')\mathbf{h}-2\nabla'(\hat{\varepsilon}_2\cdot\nabla')\phi\}\Bigg\}. \tag{36}$$

In the above expressions we see that this contribution contains several terms. They are related to already known effects, such as the gravitational light deflection and the rotation of the polarization plane, the Rytov effect, and they are also responsible for new effects, which do not exist in the framework of the geometrical optics. All these effects will be analyzed in detail in the following sections.

V. EFFECTS ON A BEAM OF LIGHT

To warrant the assumption that the electromagnetic field propagates in a region where there is no mass, we will consider here a beam of light with an impact parameter R greater than the radius of the beam, propagating in a gravitational field of a rotating mass. The unperturbed electromagnetic field and its direction of propagation are the ones which have already been defined above. The details of the computation of the integrals of Eqs. (35) and (36) are discussed in the Appendix. The infinity range of ρ_g and $\mathbf{j}_g$ produces a divergent phase shift from $-\infty$ to z. To prevent a permanent handling of this contribution we refer the phase to a finite point z_0, where the phase is null at $t=0$. The final expressions in terms of the gravitational potentials are

$$\mathbf{E}(\mathbf{r},t)=|\mathbf{E}|\mathrm{Re}\Bigg(\exp\bigg[i\omega\bigg(\int_{z_0}^{z}(1-2\phi+\mathbf{h}\cdot\hat{\varepsilon}_3)dz'-t\bigg)\bigg]\hat{\varepsilon}_1\ (1-\phi)+\int_{-\infty}^{z}dz'\bigg\{-\frac{1}{2}\hat{\varepsilon}_2\ (\nabla\times\mathbf{h})\cdot\hat{\varepsilon}_3+\hat{\varepsilon}_3\ [2\nabla\phi\cdot\hat{\varepsilon}_1+(\nabla$$
$$\times\mathbf{h})\cdot\hat{\varepsilon}_2]+\frac{i}{2\omega}\bigg[-\nabla(\hat{\varepsilon}_2\cdot\nabla\times\mathbf{h})-2\nabla(\hat{\varepsilon}_1\cdot\nabla)\phi+\hat{\varepsilon}_1\ (\hat{\varepsilon}_3\cdot\nabla)^2\bigg(\frac{1}{2}\mathbf{h}\cdot\hat{\varepsilon}_3-\phi\bigg)\bigg]\bigg\}\Bigg) \tag{37}$$

and

$$\mathbf{B}(\mathbf{r},t)=|\mathbf{E}|\mathrm{Re}\Bigg(\exp\bigg[i\omega\bigg(\int_{z_0}^{z}(1-2\phi+\mathbf{h}\cdot\hat{\varepsilon}_3)dz'-t\bigg)\bigg]\hat{\varepsilon}_2\ (1+\mathbf{h}\cdot\hat{\varepsilon}_3-3\phi)+\int_{-\infty}^{z}dz'\bigg\{\frac{1}{2}\hat{\varepsilon}_1\ (\nabla\times\mathbf{h})\cdot\hat{\varepsilon}_3+\hat{\varepsilon}_3\ [2\nabla\phi\cdot\hat{\varepsilon}_2$$
$$-(\hat{\varepsilon}_2\cdot\nabla)(\mathbf{h}\cdot\hat{\varepsilon}_3)]+\frac{i}{2\omega}\bigg[\nabla\times(\hat{\varepsilon}_1\cdot\nabla)\mathbf{h}-2\nabla(\hat{\varepsilon}_2\cdot\nabla)\phi+\hat{\varepsilon}_2\ (\hat{\varepsilon}_3\cdot\nabla)^2\bigg(\frac{1}{2}\mathbf{h}\cdot\hat{\varepsilon}_3-\phi\bigg)\bigg]\bigg\}\Bigg). \tag{38}$$

To avoid excessively large expressions we have left without developing the integrals on z. We can also rewrite these solutions directly in terms of the angular momentum and mass of the source of the gravitational field, $\mathbf{J}$ and M, which perhaps allows a more straightforward reading of the effect on the light beam. Defining $\mathbf{r}=x\hat{\varepsilon}_1+y\hat{\varepsilon}_2+z\hat{\varepsilon}_3$ and $\mathbf{r}'=x\hat{\varepsilon}_1+y\hat{\varepsilon}_2+z'\hat{\varepsilon}_3$, we have

$$\mathbf{E}(\mathbf{r},t)=|\mathbf{E}|e^{i\alpha_+}\Bigg\{\bigg(1+\frac{GM}{r}\bigg)\hat{\varepsilon}_1+G\bigg[-\frac{\mathbf{J}\cdot\mathbf{r}}{r^3}+\frac{3i}{\omega}\int_{-\infty}^{z}\bigg(2y\frac{J_2}{r'^5}+\frac{\mathbf{J}\cdot\mathbf{r}'}{r'^7}(r'^2-5y^2)+\frac{Mxy}{r'^5}\bigg)dz'\bigg]\hat{\varepsilon}_2$$
$$+G\bigg[2\int_{-\infty}^{z}\bigg(\frac{J_2}{r'^3}-3y\frac{\mathbf{J}\cdot\mathbf{r}'}{r'^5}+\frac{Mx}{r'^3}\bigg)dz'+\frac{i}{\omega}\bigg(-\frac{J_2}{r^3}+3y\frac{\mathbf{J}\cdot\mathbf{r}}{r^5}-\frac{Mx}{r^3}\bigg)\bigg]\hat{\varepsilon}_3\Bigg\}, \tag{39}$$

and

$$\mathbf{B}(\mathbf{r},t)=|\mathbf{E}|e^{i\alpha_-}\Bigg\{\bigg[1-G\bigg(\frac{2(\mathbf{J}\times\mathbf{r})\cdot\hat{k}}{r^3}+\frac{3M}{r}\bigg)\bigg]\hat{\varepsilon}_2+G\bigg[\frac{\mathbf{J}\cdot\mathbf{r}}{r^3}+\frac{3i}{\omega}\int_{-\infty}^{z}\bigg[-2x\frac{J_1}{r'^5}-\frac{\mathbf{J}\cdot\mathbf{r}'}{r'^7}(r'^2-5x^2)+\frac{Mxy}{r'^5}\bigg]dz'\bigg]\hat{\varepsilon}_1$$
$$+G\bigg[2\int_{-\infty}^{z}\bigg(\frac{J_1}{r'^3}-3y\frac{(\mathbf{J}\times\mathbf{r}')\cdot\hat{k}}{r'^5}+\frac{My}{r'^3}\bigg)dz'+\frac{i}{\omega}\bigg(\frac{J_1}{r^3}-3x\frac{\mathbf{J}\cdot\mathbf{r}}{r^5}-\frac{My}{r^3}\bigg)\bigg]\hat{\varepsilon}_3\Bigg\}, \tag{40}$$

where the phase of each field is

$$\alpha_{\pm}=\omega\left[\int_{z_0}^{z}\left(1+\frac{2GM}{r'}-\frac{2G(\mathbf{J}\times\mathbf{r}')\cdot\hat{k}}{r'^3}\right)dz'-t\right]\pm\frac{3G}{\omega}\left[\int_{-\infty}^{z}\frac{J_1y+J_2x}{r'^5}dz'\mp\frac{1}{2}\frac{z(\mathbf{J}\times\mathbf{r})\cdot\hat{k}}{r^5}\right.$$
$$\left.-5xy\int_{-\infty}^{z}\frac{\mathbf{J}\cdot\mathbf{r}'}{r'^7}dz'+\frac{1}{2}M\int_{-\infty}^{z}\frac{x^2-y^2}{r'^5}dz'\right]. \tag{41}$$

We can see that the electric and magnetic fields have a different phase; this effect is analogous to the one that appears when an electromagnetic field propagates in an inhomogeneous medium [5]. Despite this, the phase velocity is the same up to order $\lambda=2\pi/\omega$ for both fields:

$$v_0=\frac{\partial\alpha_{\pm}}{\partial t}|\nabla\alpha_{\pm}|^{-1}=1+2\phi-h_3. \tag{42}$$

This is the well-known result for the velocity of propagation of an electromagnetic wave in a curved space-time. The path of the beam is a null geodesic; i.e., the interval between two points connected by a light signal is null.

To enlighten the physical meaning of the fields given by Eqs. (39) and (40) it is convenient to discriminate the terms of order ω and ω^0 from the ones of order ω^{-1}. The first ones lead to well-known effects, already given by the geometrical optics in curved spaces, and will first be discussed.

The direction of propagation $\hat{\mathbf{k}}$ of the wave is given by the gradient of the phase, which up to order λ^0 is

$$\mathbf{k}=\hat{\varepsilon}_1\int_{-\infty}^{z}\frac{\partial(-2\phi+h_3)}{\partial x}dz'+\hat{\varepsilon}_2\int_{-\infty}^{z}\frac{\partial(-2\phi+h_3)}{\partial y}dz'$$
$$+\hat{\varepsilon}_3(1-2\phi+h_3)\quad . \tag{43}$$

From here we see that the deflection of the beam depends only on the scalar potential ϕ and the component h_3 of the vectorial potential in the direction of the nonperturbed electromagnetic wave. In general the path is not a plane curve due to the contribution of h_3. If we assume that the h_3 component of the angular momentum of the source is null, we have a plane curve with a deflection (curvature of flection) given by

$$\theta_M(z)=\int_{-\infty}^{z}\frac{2GMR}{(R^2+s^2)^{3/2}}ds=\frac{2GM}{R}\left(1+\frac{z}{r}\right), \tag{44}$$

where $R^2\equiv x^2+y^2$. Hence, for $z=+\infty$ this expression leads to

$$\theta_M(\infty)=\frac{4MG}{R}, \tag{45}$$

which is exactly the value for the light deflection we can obtain from the geometrical optics for a weak gravitational field [6]. From the deflection angle (44) we can compute the total deviation of the light beam as a function of z:

$$d(z)=\int_{-\infty}^{z}\theta(s)ds=\frac{2GMz}{R}\left[1+\sqrt{1+\left(\frac{R}{z}\right)^2}\right], \tag{46}$$

where R is the impact parameter of the beam. This result coincides with the one directly obtained from the geometrical optics for a beam propagating in the gravitational field of a nonrotating mass [7]. To analyze the effect of the angular momentum in the deflection of the beam, we will use the frame defined by the tangent $\hat{t}$ as the direction of propagation of the beam, the principal normal $\hat{n}$ as the direction perpendicular to $\hat{t}$, in the osculator plane of the beam and orientated inside, and the binormal $\hat{b}$ as $\hat{b}\equiv\hat{t}\times\hat{n}$. Using this frame we have that the angular momentum contributes to the deflection already discussed with an angle $\theta_{J_b}(z)$,

$$\theta_{J_b}(z)=\int_{-\infty}^{z}\frac{2GJ_b}{(R^2+s^2)^{3/2}}ds-3\int_{-\infty}^{z}\frac{R^2 2GJ_b}{(R^2+s^2)^{5/2}}ds$$
$$=-\frac{2GJ_b}{R^2}\left(1+\frac{z}{r}+\frac{zR^2}{r^3}\right), \tag{47}$$

and pulls the path out of the osculator plane, in the direction of the binormal, by an angle $\theta_{J_n}(z)$,

$$\theta_{J_n}(z)=\int_{-\infty}^{z}\frac{2GJ_n}{(R^2+s^2)^{3/2}}ds=\frac{2GJ_n}{R^2}\left(1+\frac{z}{r}\right), \tag{48}$$

where J_n is the component of the angular momentum in the direction of $\hat{n}$, and J_b is the component in the direction of $\hat{b}$. For $z\to\infty$ both angles take the value $4GJ/R^2$ [8]. This effect is of order J_{pn}/MR with respect to the deflection due to the mass. For example, in the case of a gravitational lens effect produced by a spherical distribution of mass, $\mathbf{J}$ introduces a small correction, which breaks the dominant radial symmetry.

The most interesting consequence of the contributions that involve $\mathbf{h}$ at zero order in λ is the Rytov effect, i.e., a rotation of the plane of polarization of the incident beam in a gravitational field of a rotating mass [2]. This effect can be evaluated from Eqs. (39) and (40) for a propagation from $-\infty$ to z, and is characterized by the angle defined positive from the normal to the binormal given by

$$\theta_R(z)=-\frac{G\mathbf{J}\cdot\mathbf{r}}{r^3}\quad . \tag{49}$$

To make this effect clearer, we can consider two representative cases, i.e., the angular momentum parallel or orthogonal to the direction of the beam. In the first case the Rytov angle is $\theta_{R\parallel}=-GJz/r^3$. It reaches its maximum values at $z=\pm R/\sqrt{2}$ and becomes null at $z=0$ and $z=+\infty$. In the second case the maximum value corresponds to $z=0$, and is given by $\theta_R(0)=-GJ_n/R^2$, and from $0<z<\infty$ it develops the same angle but with an opposite sign. Thus, the total rotation of the plane of polarization between $-\infty<z<\infty$ is null.

At first order in λ new effects appear related to the mass and the angular momentum of the source of the gravitational field. From Eqs. (39) and (40), we see that the solution has components perpendicular to the unperturbed fields, which have a $\pi/2$ phase with respect to these fields. Naming these components according to

$$E_\lambda^2=\frac{3iG|\mathbf{E}|}{\omega}\int_{-\infty}^{z}\left(2y\frac{J_2}{r'^5}+\frac{\mathbf{J}\cdot\mathbf{r}'}{r'^7}(r'^2-5y^2)+M\frac{xy}{r'^5}\right)dz', \tag{50}$$

$$B_\lambda^1=\frac{3iG|\mathbf{E}|}{\omega}\int_{-\infty}^{z}\left[-2x\frac{J_1}{r'^5}-\frac{\mathbf{J}\cdot\mathbf{r}'}{r'^7}(r'^2-5x^2)+M\frac{xy}{r'^5}\right]dz', \tag{51}$$

we can analyze the physical meaning of these contributions by writing them as

$$E_\lambda^2=\Sigma_++\Sigma_-,\quad B_\lambda^1=\Sigma_+-\Sigma_-, \tag{52}$$

where

$$\Sigma_+=\frac{1}{2}(E_\lambda^1+B_\lambda^2)=i\frac{3G|\mathbf{E}|}{2}\int_{-\infty}^{z}\left[2\frac{yJ_2-xJ_1}{r'^5}+5\frac{\mathbf{J}\cdot\mathbf{r}'}{r'^7}(x^2-y^2)+2M\frac{xy}{r'^5}\right]dz', \tag{53}$$

$$\Sigma_-=\frac{1}{2}(E_\lambda^1-B_\lambda^2)=i\frac{3G|\mathbf{E}|}{2}\int_{-\infty}^{z}\left[\frac{\mathbf{J}\cdot\mathbf{r}'}{r'^7}(5z'^2-r'^2)-2\frac{z'J_3}{r'^5}\right]dz'. \tag{54}$$

The terms with opposite sign in $\mathbf{E}_\lambda$ and $\mathbf{B}_\lambda$ describe a gravitationally induced circular polarization, whereas the equal sign terms correspond to components that turn the electric and magnetic fields in opposite directions, and thus break the orthogonality of the unperturbed fields.

The circular polarization only appears due to the angular momentum. It has a contribution from the component of $\mathbf{J}$ parallel to the direction of propagation of the beam, which develops peak values of amplitude at $z=\pm\sqrt{3/2}R$ and $z=0$, and a contribution from the orthogonal component, with maximum values at $z=\pm R/2$. Both contributions nullify when $z\to\infty$:

$$\Sigma_-(\infty)=0. \tag{55}$$

In contrast with the circular polarization, the orthogonality breaking component is generated by both $\mathbf{J}$ and M and survives even for $z\to\infty$. That is,

$$\Sigma_+(\infty)=\frac{i4G|\mathbf{E}|}{\omega R^6}[xJ_1(x^2-3y^2)+yJ_2(3x^2-y^2)+MR^2xy]. \tag{56}$$

Another interesting feature of this solution is a different phase shift to order λ for the electric and magnetic fields, which produces a difference in the phase velocity of both fields. Using Eq. (41), the phase velocities for the electric and magnetic fields are

$$v_E=v_0+\Delta v_+,\quad v_B=v_0+\Delta v_-, \tag{57}$$

with

$$\Delta v_\pm=\pm\frac{3G}{\omega^2}\left(-\frac{J_1y+J_2x}{r^5}\pm\frac{1}{2}\frac{(\mathbf{J}\times\mathbf{r})\cdot\hat{k}}{r^7}(r^2-5z^2)+5xy\frac{\mathbf{J}\cdot\mathbf{r}}{r^7}+\frac{1}{2}M\frac{y^2-x^2}{r^5}\right). \tag{58}$$

This effect is analogous to the one that appears for an electromagnetic wave propagating in an inhomogeneous medium [5]. In our case the velocities depend on the angular momentum and mass of the gravitational source.

The λ-dependent contributions to the phase have a dispersive character, and remain non-null for the outcoming wave. In fact their value is

$$\Delta\alpha_\pm(z_{\to\infty})=\pm 2\frac{G}{\omega R^6}[2J_1y(y^2-3x^2)+2J_2x(x^2-3y^2)+R^2M(x^2-y^2)]. \tag{59}$$

Thus Eqs. (56) and (59) clearly show that the asymptotic outcoming wave is not a plane wave. This is due to the infinite range of the gravitational potentials. To close this section we can mention an interesting consequence of this dispersive behavior: a rainbow effect. The directions of the outcoming electric and magnetic fields depend on λ, as becomes clear by computing the gradient of the phases.

VI. SOLUTION IN LOCALLY INERTIAL COORDINATES

In order to unveil the most interesting characteristics of the solution encountered, we will write it in locally inertial coordinates. Up to first order in G a general expression for a tetrad $t^\mu_{\ \alpha}$ is

$$t^\mu_{\ \alpha}=\begin{pmatrix}1-\phi & \alpha_1 & \alpha_2 & \alpha_3\\ -h_1+\alpha_1 & +1+\phi & +\beta & +\gamma\\ -h_2+\alpha_2 & -\beta & +1+\phi & +\delta\\ -h_3+\alpha_3 & -\gamma & -\delta & +1+\phi\end{pmatrix}, \tag{60}$$

where α_1, α_2, α_3, β, γ, and δ parametrize the Lorentz-related family of locally inertial coordinates in a given neighborhood in space-time. If dx^μ are the differentials in the general coordinate system and $d\tilde{x}^\alpha$ are the corresponding ones in the locally coordinate system, they are related by the tetrad such that $dx^\mu=t^\mu_{\ \alpha}d\tilde{x}^\alpha$ and $\eta_{\alpha\beta}=t^\mu_{\ \alpha}g_{\mu\nu}t^\nu_{\ \beta}$, with $\eta_{\alpha\beta}$ the Minkowskian metric $(-1,1,1,1)$. From here the electromagnetic tensor $F_{\mu\nu}$,

$$F_{\mu\nu}=\begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}, \tag{61}$$

transforms into $\widetilde{F}_{\alpha\beta}$ in the locally inertial coordinates according to

$$\widetilde{F}_{\alpha\beta}(X^\lambda+d\widetilde{x}^\lambda)=t^\mu{}_\alpha t^\nu{}_\beta F_{\mu\nu}(X^\lambda+t^\lambda{}_\varepsilon d\widetilde{x}^\varepsilon), \tag{62}$$

where X^λ is the point around which we define the neighborhood, where the locally inertial coordinates are given by $X^\lambda+d\widetilde{x}^\lambda$. Therefore, up to first order in G the components of $\widetilde{F}_{\alpha\beta}$ are

$$\begin{aligned} \widetilde{E}_1&=E_1+(h_3-\alpha_3)B_2, \\ \widetilde{E}_2&=E_2+\beta E_1, \\ \widetilde{E}_3&=E_3+\gamma E_1+(\alpha_1-h_1)B_2, \\ \widetilde{B}_1&=B_1-\beta B_2, \\ \widetilde{B}_2&=(1+2\phi)B_2-\alpha_3 E_1, \\ \widetilde{B}_3&=B_3+\delta B_2+\alpha_2 E_1. \end{aligned} \tag{63}$$

We can now analyze the structure of the electromagnetic field in the locally inertial system. From these equations and Eqs. (39) and (40) we can verify that the modulus of both fields become equal, that is,

$$|\widetilde{\mathbf{E}}|=|\widetilde{\mathbf{B}}|=|\mathbf{E}|(1-\phi+h_3-\alpha_3), \tag{64}$$

where the parameter α_3 expresses the modulus dependence on boosts in the $\hat{\varepsilon}_3$ direction.

Considering now the fields in the original frame, the contribution of order λ^{-1} to the phase φ of the solution is

$$\varphi=\omega\left[\int_{z_0}^{z}\left(1+\frac{2GM}{r'}-\frac{2G(\vec{J}\times\vec{r}')\cdot\hat{k}}{r'^3}\right)dz'-t\right]. \tag{65}$$

Here we see that the wave propagates in a direction $\mathbf{k}$, given by the gradient of φ,

$$\begin{aligned} \mathbf{k}=&\hat{\varepsilon}_1 2G\omega\int_{-\infty}^{z}\left(\frac{J_2-Mx}{r'^3}-3\frac{(xJ_2-yJ_1)x}{r'^5}\right)dz' \\ &+\hat{\varepsilon}_2 2G\omega\int_{-\infty}^{z}\left(\frac{-J_1-My}{r'^3}-3\frac{(xJ_2-yJ_1)y}{r'^5}\right)dz' \\ &+\hat{\varepsilon}_3\omega\left(1+2G\frac{xJ_2-yJ_1+Mr^2}{r^3}\right), \end{aligned} \tag{66}$$

and that the magnetic field is orthogonal to $\mathbf{k}$,

$$\mathbf{k}\cdot\mathbf{B}=0, \tag{67}$$

whereas the electric field satisfies

$$\mathbf{k}\cdot\mathbf{E}=\omega h_1. \tag{68}$$

If we now transform to a locally inertial system, the differential of the phase becomes

$$\begin{aligned} d\varphi=&\omega[(1+h_3-\phi-\alpha_3)dz'-(1+h_3-\alpha_3-\phi)dt' \\ &+(-\alpha_1-\gamma)dx'+(-\alpha_2-\delta)dy'] \\ &+2\omega G\int_{-\infty}^{z}\left(\frac{J_2-Mx}{(R^2+s^2)^{3/2}}-3\frac{(xJ_2-yJ_1)x}{(R^2+s^2)^{5/2}}\right)ds\,dx' \\ &+2\omega G\int_{-\infty}^{z}\left(\frac{-J_1-My}{(R^2+s^2)^{3/2}}-3\frac{(xJ_2-yJ_1)y}{(R^2+s^2)^{5/2}}\right)ds\,dy'. \end{aligned} \tag{69}$$

So up to order λ^0, in this new system the wave propagates in the direction $\mathbf{k}'$ given by

$$\mathbf{k}'=\mathbf{k}-\omega[\hat{\varepsilon}_1(\alpha_1+\gamma)+\hat{\varepsilon}_2(\alpha_2+\delta)+\hat{\varepsilon}_3(\alpha_3-\phi)], \tag{70}$$

with a velocity

$$\begin{aligned} v_{\text{loc}}=&\frac{\omega(1+h_3-\phi-\alpha_3)}{|\mathbf{k}'|}=\frac{(1+h_3-\phi-\alpha_3)}{\sqrt{(1+h_3-\phi-\alpha_3)^2+O(G^2)}} \\ =&1+O(G^2). \end{aligned} \tag{71}$$

Here, both the magnetic and the electric fields are orthogonal to $\mathbf{k}'$:

$$\mathbf{k}'\cdot\mathbf{B}'=\mathbf{k}\cdot\mathbf{B}=0, \tag{72}$$

$$\mathbf{k}'\cdot\mathbf{E}'=\mathbf{k}\cdot\mathbf{E}-\omega h_1=0. \tag{73}$$

Therefore, we see that up to order zero in λ in the locally inertial frame the electromagnetic field has the properties expected for a plane wave. But when we analyze the terms of order λ novel characteristics appear. The new terms are linear in G and so are unaffected by the change of coordinates. In other words, they are the same in the original coordinates and in the locally inertial ones. Thus, even in the local system the presence of a gravitational field is apparent. The most conspicuous phenomena that these terms generate have already been discussed at the end of the preceding section.

VII. FINAL REMARKS

In this paper we have studied the effects of a gravitational field on the propagation of light. For such purposes we have developed a perturbative approach in powers of the gravitational constant G. Although for the sake of simplicity we have restricted here the exposition to a first order in G, the formalism can be extended without difficulty to higher orders. According to our approach the electromagnetic field in a curved space-time is described by a dominant contribution, solution of the Maxwell equations in a flat space-time, plus a metric-dependent perturbation. This last component is also a solution of the Maxwell equations in a flat space-time, but now with sources that depend both on the dominant solution and on the gravitational field. The perturbation can be computed as a power expansion in the wavelength of the electromagnetic field. It can be exactly evaluated for each order in G. The solution up to nth order in G is given by a polynomial of the same order in λ.

At first order in G we not only recover the well-known results of the geometrical optics, but we also identify new contributions, at first order in λ, which show that the gravitational field induces terms that change the polarization of the light, as well as anomalous terms that break the orthogonality of the electric and magnetic fields. The first terms are due to the angular momentum of the source of the gravitational field, whereas the second ones depend both on the angular momentum and the mass. In the particular case considered here, i.e., an incoming plane wave with linear polarization, both effects are present in the surroundings of the rotating mass, but only the second one remains in the asymptotic outcoming wave. This somehow unexpected behavior is possible because the effective sources of the equations for the perturbations extend on the whole space.

Another consequence is the difference of phases to order λ that the electric and magnetic fields develop. For this reason they acquire phase velocities that are both different in magnitude and in direction. Although these velocities become asymptotically equal, the phase shifts remain asymptotically non-null and different. Thus, the gravitational interaction induces a dispersive behavior on the propagation of the electromagnetic field, which produces effects of the type of an asymptotic rainbow, the smaller the impact parameter of the beam, the more noticeable the rainbow. Although very small, these dispersive phenomena are interesting candidates for providing additional information about the angular momentum of the source of the gravitational lensing effect.

To appreciate the magnitude of the asymptotic gravitational effects on the electromagnetic wave, we can consider some simple configurations. From Eq. (56), if the unperturbed electric or magnetic field has a radial direction, the outcoming fields describe ellipses with the same ratio between the minor and the major axes, $4GJ_n/\omega R^3$, which depends only on the component of the angular momentum in the osculator plane. The distinctive feature of this effect is that both fields rotate in opposite directions. In particular, for a black hole with maximum angular momentum we have $GJ_n/R_S^2=1$, with R_S the Schwarzschild radius, and thus this ratio is $4(R_S/R)^2(1/\omega R)$. If the unperturbed fields have any other directions the contribution of the mass M is non-null, and it acquires its maximum value when they form an angle of $\pi/4$ with the radial direction. If $\mathbf{J}=0$ and with this orientation for the electromagnetic fields, the ratio between the axes of the ellipses is $2(R_S/R)(1/\omega R)$.

The gravitational rainbow can be computed from Eq. (59). For a beam with the magnetic field in a radial direction the contribution $\mathbf{k}_\lambda$ of order λ to the outcoming wave vector for the electric field is

$$\mathbf{k}_\lambda=\frac{12G}{\omega R^4}\left[\left(\frac{MR}{12}-J_b\right)\hat{\varepsilon}_1+J_n\hat{\varepsilon}_2\right]. \tag{74}$$

The magnetic field has the same contribution but with opposite sign. This difference between both wave vectors is a manifestation of the inhomogeneity of the gravitational background.

The remaining effect at first order in λ is the induced circular polarization. It depends only on the angular momentum of the source of the gravitational field and is significant in the proximity of this source. For example, in the equatorial plane we have $|\mathbf{E}_{\mathrm{circ}}|/|\mathbf{E}|=GJ_t/2R^3$.

These examples are only given to visualize the order of magnitude of the different effects, but as is clear from Eqs. (53), (56), and (59) their spatial structure and values strongly depend on the orientation of the fields and of the angular momentum of the source.

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APPENDIX

In the Eqs. (35) and (36) for $\mathbf{E}^{(1)}$ and $\mathbf{B}^{(1)}$ we have to integrate expressions with the general form

$$I(\mathbf{r},t)=\int f(\mathbf{r}')\frac{e^{i\omega[\hat{\varepsilon}_3\cdot(\mathbf{r}'-\mathbf{r}_0)+|\mathbf{r}-\mathbf{r}'|-t]}}{|\mathbf{r}-\mathbf{r}'|}d^3r', \tag{A1}$$

where the $f(\mathbf{r})$ can be the functions ϕ, $\partial_i\phi$, $\partial_i\partial_j\phi$, h_k, $\partial_i h_k$, or $\partial_i\partial_j h_k$, which are

$$\phi=-\frac{GM}{|x|}, \quad h_k=\frac{2Gx^rJ_{kr}}{|x|^3},$$

$$\partial_i\phi=\frac{GMx^i}{|x|^3}, \quad \partial_i h_k=\frac{2GJ_{ki}}{|x|^3}-3\frac{2Gx^rx^iJ_{kr}}{|x|^5},$$

$$\partial_i\partial_j\phi=\frac{GM\delta_{ij}}{|x|^3}-3\frac{GMx^ix^j}{|x|^5},$$

$$\partial_i\partial_j h_k=-3\frac{2Gx^jJ_{ki}}{|x|^5}+15\frac{2Gx^rx^ix^jJ_{kr}}{|x|^7}-3\frac{2Gx^iJ_{kj}}{|x|^5}-3\frac{2Gx^r\delta_{ij}J_{kr}}{|x|^5}. \tag{A2}$$

To perform this type of integrals we can introduce spherical coordinates, such that

$$\mathbf{r}=x\hat{\varepsilon}_1+y\hat{\varepsilon}_2+z\hat{\varepsilon}_3, \tag{A3}$$

$$\mathbf{r}'-\mathbf{r}=\rho\sin\theta\cos\varphi\hat{\varepsilon}_1+{}_2\rho\sin\theta\sin\varphi\hat{\varepsilon}+\rho\cos\theta\hat{\varepsilon}_3. \tag{A4}$$

With this change of variables Eq. (A1) becomes

$$I(\mathbf{r},t)=\int e^{i\omega[\hat{\varepsilon}_3\cdot(\mathbf{r}-\mathbf{r}_0)-t]}f(\mathbf{r}')\rho\sin\theta e^{i\omega\rho(1+\cos\theta)}d\rho d\theta d\varphi. \tag{A5}$$

The integrand contains an oscillating factor $\exp[i\omega\rho(1+\cos\theta)]$. The surfaces of constant phase determined by this factor are paraboloids given by the equation

$$\rho = \frac{2n\lambda}{(1+\cos\theta)} \quad , \tag{A6}$$

where $\lambda = 2\pi/\omega$.

The characteristic lengths for the functions $f(\mathbf{r}')$ are given by the impact parameter R, the Schwarzschild radius $R_S = 2GM$, and the angular-momentum-related length $GJ/R = (R_S/R)J/2M$, while for the oscillating factor we have the wavelength λ. We can develop an expansion of $f(\mathbf{r}')$ around the axis of the paraboloids that gives an expansion of the solution in terms of powers of λ. To do this we write

$$\mathbf{r}' = \mathbf{r}' - \mathbf{r} + \mathbf{r} = (\rho\sin\theta\cos\varphi + x)\hat{\varepsilon}_1 + (\rho\sin\theta\sin\varphi + y)\hat{\varepsilon}_2 + (\rho\cos\theta + z)\hat{\varepsilon}_3, \tag{A7}$$

and hence we have

$$\mathbf{r}'^2 = R^2 + (z-\rho)^2 + 2\rho z(\cos\theta + 1) + 2\rho R\sin\theta\sin(\varphi - \beta), \tag{A8}$$

where $R \equiv x^2 + y^2$ and $\tan(\beta) \equiv -x/y$. For a point $\mathbf{r}'$ inside the nth paraboloid, the following relations are satisfied:

$$\sin\theta = \sqrt{1-\cos^2\theta} < \sqrt{\frac{2n\lambda}{\rho}\left(2 - \frac{2n\lambda}{\rho}\right)} < 2\sqrt{\frac{n\lambda}{\rho}}, \tag{A9}$$

$$2\rho R\sin\theta\sin(\varphi-\beta) < 4\rho\sqrt{\frac{n\lambda}{\rho}}R = 4\sqrt{\rho n\lambda}R, \tag{A10}$$

$$2\rho z(\cos\theta + 1) < 4zn\lambda, \tag{A11}$$

and so if $\lambda \ll R$, we have

$$2\rho z(\cos\theta+1) + 2\rho R\sin\theta\sin(\varphi-\beta) < 4(\sqrt{\rho n\lambda}R + n\lambda z) \ll R^2 + (z-\rho)^2, \tag{A12}$$

which allows us to perform the expansion

$$|\mathbf{r}'|^b = [R^2 + (z-\rho)^2 + 2\rho z(\cos\theta+1) + 2\rho R\sin\theta\sin(\varphi-\beta)]^{b/2} = [R^2+(z-\rho)^2]^{b/2}\left[1 + b\frac{\rho z(\cos\theta+1) + \rho R\sin\theta\sin(\varphi-\beta)}{R^2+(z-\rho)^2} + \frac{b(b-2)}{2}\left(\frac{\rho z(\cos\theta+1)+\rho R\sin\theta\sin(\varphi-\beta)}{R^2+(z-\rho)^2}\right)^2 + \cdots\right]. \tag{A13}$$

Using the expressions given by Eq. (A2) and the expansion (A13) we can perform analytically all the integrations to the desired order in λ.

In our case the sources of the Maxwell equations for the perturbations have a very simple dependence in ω, of the form

$$Ge^{i\omega(z-z_0-t)}(\omega A + B) \quad , \tag{A14}$$

were A and B are ω-independent functions. The Maxwell equations are linear differential equations, and so if we write the solution as an expansion in powers of ω,

$$Ge^{i\omega(z-z_0-t)}(\omega S_{-1} + S_0 + \omega^{-1}S_1 + \omega^{-2}S_2 + \cdots), \tag{A15}$$

we can easily see that S_n with $n>1$ vanishes identically; so the solution up to order λ is the exact one. The general scheme to perform the integrals can be summarized in the following simple recipe: Expand $|\mathbf{r}'|^b$ following Eq. (A13), keep all the terms that contribute up to first order in λ, and then perform the integrations in φ and θ. The integration over ρ can also be performed, but for the sake of simplicity we sometimes leave it expressed in terms of an integral over z.

Finally, a comment on the identification of the phases. The total solution, for the electric field for example, is written $\mathbf{E} = \mathbf{E}^{(0)} + \mathbf{E}^{(1)}$, where $\mathbf{E}^{(0)}$ and $\mathbf{E}^{(1)}$ are the unperturbed field and its correction, respectively. If we are analyzing the component in the direction of $\mathbf{E}^{(0)}$, the contribution of $\mathbf{E}^{(1)}$ in such a direction has the form $\mathbf{E}_1^{(1)} = G|\mathbf{E}|(a+ib)$, with a and b real, and so the total solution $\mathbf{E}_1$ in the $\hat{\varepsilon}_1$ direction can be written as

$$\mathbf{E}_1 = |\mathbf{E}|e^{i\omega[\hat{\varepsilon}_3\cdot(\mathbf{r}-\mathbf{r}_0)-t]}(1 + Ga + iGb), \tag{A16}$$

which to first order in G can be expressed as

$$\mathbf{E}_1 = |\mathbf{E}|(1+Ga)e^{i\omega[\hat{\varepsilon}_3\cdot(\mathbf{r}-\mathbf{r}_0)+Gb/\omega-t]}. \tag{A17}$$

This is the phase assignment in the perturbative results obtained to first order in G.

Taking into account all the above considerations we obtain the solutions given by Eqs. (39) and (40), which are exact in λ at first order in G. If we extend the computation to second order in G, we will also obtain contributions to the second order in λ. In general, the contributions to the nth order in λ (with $n \geq 0$) are given by series in G that begin with the nth order in this constant.

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