

Stationary perturbations and infinitesimal rotations of static Einstein-Yang-Mills configurations with bosonic matter

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(Received 19 June 1997)

Using the Kaluza-Klein structure of stationary spacetimes, a framework for analyzing stationary perturbations of static Einstein-Yang-Mills configurations with bosonic matter fields is presented. It is shown that the perturbations giving rise to a nonvanishing ADM angular momentum are governed by a self-adjoint system of equations for a set of gauge-invariant scalar amplitudes. The method is illustrated for SU(2) gauge fields, coupled to a Higgs doublet or a Higgs triplet. It is argued that slowly rotating black holes arise generically in self-gravitating non-Abelian gauge theories with bosonic matter, whereas, in general, soliton solutions do not have rotating counterparts. [S0556-2821(97)04122-2]

PACS number(s): 04.20.Cv, 04.40.Nr

I. INTRODUCTION

In the presence of a (stationary) Killing symmetry, the Einstein-Maxwell (EM) equations reduce to a σ -model coupled to three-dimensional gravity [1]. This property is, in fact, shared by a large class of theories with scalar fields and Abelian vector fields (see [2] for a classification and [3,4] for some recent applications and references). If spacetime admits an additional (axial) Killing symmetry, then the σ -model structure gives rise to total integrability of the field equations, provided that the target space is a symmetric space. This has been known for quite some time for the EM system [5], and was recently demonstrated by Gal'tsov for EM-dilaton-axion models [6].

Since *scalar* magnetic potentials fail to exist in *non-Abelian* gauge theories, the σ -model structure — and, in particular, the property of integrability — are spoiled for self-gravitating Yang-Mills fields. Moreover, the circularity theorem [7] (which guarantees that spacetime admits a foliation by two-surfaces orthogonal to the integral trajectories of the two Killing fields) does not extend to the Einstein-Yang-Mills (EYM) system [8] (see also [9]). The familiar Papapetrou metric [10] does, therefore, not take account of *all* stationary and axisymmetric degrees of freedom of the EYM equations.

In view of these problems, an analytic approach to the full EYM equations with two Killing fields is likely to be extremely difficult. Motivated by recent work of Volkov and Straumann [11], we pursue a more modest aim in this paper; that is, we consider stationary *deviations* of static EYM configurations with bosonic matter fields. For the pure SU(2) EYM system, Volkov and Straumann [11] were able to reduce the relevant perturbation equations to a three-dimensional set. In this paper, we present a systematic investigation of stationary perturbations, which reveals that the decoupling of a specific set of perturbation amplitudes is a *general feature* of a large class of bosonic matter fields coupled to the EYM system with an arbitrary gauge group. We argue that stationary perturbations are most appropriately handled by means of a “3 + 1” — rather than a “2 + 2” — decomposition of spacetime. (This does, in particular, avoid

the circularity issue, since the metric is not required to be axially symmetric in the first place.) Hence, we use the Kaluza-Klein (KK) structure of a stationary spacetime to analyze *arbitrary stationary* perturbations of static configurations. Within this approach, the *nonstatic* deviations are encoded in the KK connection, which is related to (the dual of) the twist of the stationary Killing field.

The KK reduction of the Einstein-Hilbert action yields a three-dimensional gravitational theory coupled to the KK scalar field and the the KK connection [12]. The latter is described by a gauge potential, which enters the effective action only quadratically and only via the field strength. Using a suitable KK decomposition of the YM gauge potential, these features are preserved if gauge fields and additional bosonic matter are coupled to gravity. More precisely, it turns out that the linear terms in the KK connection enter the reduced action via a (nonminimal) coupling to the electric components of the YM field. These observations imply the following two conclusions: (i) The stationary EYM equations (coupled to bosonic fields) admit a generalized scalar twist potential, and (ii) the nonstatic, nonmagnetic deviations of a static, purely magnetic solution to the EYM equations form a consistent subset of all stationary perturbations. Moreover, it is exactly this subset of perturbations, henceforth called *purely stationary* perturbations, which gives rise to a nonvanishing Arnowitt-Deser-Misner (ADM) angular momentum.

By virtue of the crucial features (i) and (ii), the relevant perturbations (as far as angular momentum is concerned) of a static, purely magnetic EYM-Higgs configuration form a formally *self-adjoint* system for a set of *gauge-invariant* scalar amplitudes. For a spherically symmetric SU(2) background, these amplitudes, consisting of the generalized twist potential and the (Lie-algebra-valued) electric YM potential, can be expanded in terms of “isospin” harmonics C_{jm}^{ℓ} . Since only $j=1$ contributes to the ADM angular momentum, one finally obtains a standard Sturm-Liouville problem for three radial functions. For the twist channel one has $j=\ell=1$, whereas the orbital angular momenta in the two YM channels are $\ell=0$ and $\ell=2$. The Higgs fields enter the perturbation equations only via a background potential, which

gives mass to either one (triplet) or both (doublet) YM perturbations.

For a stationary background, the horizon is a regular singular point of the perturbation equations, which admit four acceptable solutions, whereas the corresponding number is three in the asymptotic regime. The fact that the perturbation equations admit a six-dimensional fundamental system then yields the conclusion that slowly rotating black hole solutions to the EYM-Higgs equations do exist. The corresponding solutions for the pure EYM system were recently discovered by Volkov and Straumann [11], who also argued that these configurations cannot be electrically neutral. The perturbation equations show that the coupling of isospin and orbital momentum, which is responsible for the ‘‘charging up’’ due to rotation, does not need to be effective if bosonic matter is coupled to the EYM equations.

For solitonic background solutions the origin is a regular singular point of the perturbation equations. The number of physically acceptable modes at the center is, however, not sufficiently large to allow for ‘‘generic’’ rotational degrees of freedom of self-gravitating bosonic matter coupled to non-Abelian gauge fields. The situation does, however, change in the *absence* of bosonic fields [13].

II. KALUZA-KLEIN REDUCTION

We consider the action for self-gravitating non-Abelian gauge fields coupled to bosonic matter:

$$S = -\frac{1}{16\pi G} \int [\mathcal{L}_G + \kappa(\mathcal{L}_{\text{YM}} + \mathcal{L}_B)], \quad (1)$$

where $\kappa = 8\pi G/g^2$, G is Newton’s constant, and g is the gauge coupling. The four-forms \mathcal{L}_G and \mathcal{L}_{YM} are the Einstein-Hilbert and the YM Lagrangians, respectively:

$$\mathcal{L}_G = {}^{*(4)}R^{(4)}, \quad \mathcal{L}_{\text{YM}} = 2 \text{Tr}\{F^{(4)} \wedge {}^{*(4)}F^{(4)}\}. \quad (2)$$

Here, $R^{(4)}$ and ${}^{*(4)}$ denote the Ricci scalar and the Hodge dual with respect to the spacetime metric $\mathbf{g}^{(4)}$. The one-form $A^{(4)}$ is the Lie-algebra-valued YM gauge potential with field strength $F^{(4)} = dA^{(4)} + A^{(4)} \wedge A^{(4)}$. For the bosonic matter we shall, for instance, consider a Higgs field H [with potential $P(H)$] which transforms according to some representation U of the gauge group, $D^{(4)}H = dH + U_*(A^{(4)})H$. In particular,

$$\mathcal{L}_B = -2 \text{Tr}\{(D^{(4)}H)^\dagger \wedge {}^{*(4)}D^{(4)}H\} - {}^{*(4)}P(H) \quad (3)$$

for a Higgs doublet or a triplet in matrix representation [see also Eq. (21)].

Our first aim is to perform the KK reduction of the above action (1). At least locally, a stationary spacetime $(M, \mathbf{g}^{(4)})$ [with Killing field ∂_t and corresponding one-form $k = -\sigma(dt + a)$] has the structure $\mathbb{R} \times \Sigma$ and admits a metric of KK type:

$$\mathbf{g}^{(4)} = -\sigma(dt + a) \otimes (dt + a) + \sigma^{-1} \mathbf{g}. \quad (4)$$

Here, σ and a are, respectively, a scalar field and a one-form on the three-dimensional Riemannian space (Σ, \mathbf{g}) . Under coordinate transformations the one-form a transforms like an Abelian gauge potential. The corresponding field strength da

is proportional to the dual of the twist one-form, $\omega \equiv 1/2 {}^{*(4)}(k \wedge dk) = -\sigma^2/2 {}^*da$. (Here and in the following * denotes the Hodge dual with respect to the Riemannian metric \mathbf{g} .) The canonical decomposition of the gauge field $A^{(4)}$ in terms of a stationary function ϕ and a stationary one-form A (both Lie algebra valued) on (Σ, \mathbf{g}) is

$$A^{(4)} = \phi(dt + a) + A. \quad (5)$$

In the following it will be crucial that $A^{(4)}$ is decomposed with respect to the orthonormal tetrad field $\theta^0 = \sqrt{\sigma}(dt + a)$ (rather than $\sqrt{\sigma}dt$). The reduction of the Einstein-Hilbert action with respect to the stationary metric (4) gives $\int \mathcal{L}_G = \int (dt \wedge L_G)$, where the three-form L_G is the Lagrangian for the KK scalar field σ and the Abelian gauge field a , effectively coupled to three-dimensional gravity. Up to an exact differential, one finds

$$L_G = {}^*R^{(g)} - \frac{1}{2\sigma^2} d\sigma \wedge {}^*d\sigma + \frac{\sigma^2}{2} da \wedge {}^*da. \quad (6)$$

The dimensional reduction of the YM action yields an effective YM-Higgs theory, with effective Higgs field ϕ and YM field strength $F \equiv dA + A \wedge A$. With $\int \mathcal{L}_{\text{YM}} = \int (dt \wedge L_{\text{YM}})$ one has

$$L_{\text{YM}} = 2 \text{Tr} \left\{ \sigma(F + \phi da) \wedge {}^*(F + \phi da) - \frac{1}{\sigma} D\phi \wedge {}^*D\phi \right\}, \quad (7)$$

where D denotes the gauge-covariant exterior derivative with respect to the one-form A on Σ . Introducing a field strength vector with components da and F , the above formulas imply that the stationary EYM system reduces to a three-dimensional EYM theory which is nonminimally coupled to a two-component vector of scalar fields (comprising combinations of the KK scalar σ and the YM scalar ϕ). Finally, the evaluation of the Higgs action with respect to the gauge potential (5) results in an additional potential term, involving the coupling between the actual Higgs field H and the effective Higgs field ϕ :

$$L_B = -2 \text{Tr} \left\{ (DH)^\dagger \wedge {}^*DH - \frac{1}{\sigma^2} [U_*(\phi)H]^\dagger \wedge {}^*U_*(\phi)H \right\} - \frac{1}{\sigma} {}^*P[H]. \quad (8)$$

The vacuum Einstein equations are obtained from variations of $\int L_G$ with respect to \mathbf{g} , σ , and a . Since L_G is a quadratic expression in terms of da , both the effective three-dimensional Einstein equation for \mathbf{g} and the equation for σ contain no linear terms in da . In the presence of YM and Higgs fields this property generalizes in the sense that the effective action continues to be quadratic in combinations of da and ϕ . Hence, the only equations which contain linear terms in da and/or ϕ are those which are obtained from variations of the effective action, $\int_\Sigma [L_G + \kappa(L_{\text{YM}} + L_B)]$, with respect to these quantities:

$$d^*[\sigma^2 da + 4\kappa\sigma \text{Tr}\{\phi(F + \phi da)\}] = 0, \quad (9)$$

$$D^*[\sigma^{-1}D\phi] + \sigma da \wedge^*(F + \phi da) = \sigma^{-2} J_B(\phi), \quad (10)$$

where $J_B(\phi)$ is the bosonic current (zero-form). In particular, one has

$$J_B(\phi) = -[H, [H, \phi]] \quad \text{and} \quad J_B(\phi) = \frac{1}{2}(\phi H^\dagger H + H^\dagger H \phi) \quad (11)$$

for a Higgs triplet and a Higgs doublet (in matrix representation), respectively, provided that the latter transforms by left multiplication under the action of $SU(2)$.

Equation (10) is the electric part of the YM equation. The twist Eq. (9) assumes the form of a differential conservation law. This is due to the fact that the connection a is an Abelian gauge field which — for reasons of gauge invariance — enters the effective action only via the field strength da . All stationary self-gravitating matter models give, therefore, rise to a generalized twist potential χ , say. It is well known that the twist potential for the Einstein-Maxwell system involves the electric *and* the magnetic potential. The above reasoning implies that the twist potential continues to exist in the EYM system, although scalar magnetic potentials cease to do so in non-Abelian gauge theories. In fact, Eq. (9) implies the existence of a function χ , such that

$$(1 + 4\kappa\sigma^{-1}\text{Tr}\{\phi^2\})da = \sigma^{-2}d\chi - 4\kappa\sigma^{-1}\text{Tr}\{\phi F\}. \quad (12)$$

(It may be worthwhile mentioning that an explicit expression for the twist potential does not exist for a rotating boson star. This is a consequence of the fact that the effective action does contain terms in a itself, since the model is not stationary in the strict sense and is, therefore, only gauge invariant under a combined transformation involving a *and* the time coordinate.)

III. STATIONARY PERTURBATIONS OF STATIC SPACETIMES

Let us now consider stationary perturbations of a static (i.e., $a=0$) EYM configuration. The above reasoning implies that the perturbations δa and $\delta\phi$ do *not* couple to the remaining metric and matter perturbations, provided that the static configuration is *purely magnetic*. (In this case both a and ϕ are first order quantities.) The stationary perturbations of a static, purely magnetic spacetime therefore fall into two complementary sets, henceforth called *static* perturbations and *purely stationary* perturbations. The static set involves only perturbations of fields (metric and matter) which are already present in the equilibrium configuration. It is obvious that the restriction to perturbations of this kind gives rise to a consistent set of first order equations. The purely stationary perturbations involve those fields which vanish for static, purely magnetic configurations. It is an interesting consequence of the above KK reduction that the purely stationary perturbations form a consistent subset as well; that is, the twist channel and the electric channel do *not* cause perturbations of the remaining fields.

It is very intuitive (and will be shown below) that it is precisely the set of purely stationary perturbations which gives rise to angular momentum. Hence, we shall now focus on these perturbations; that is, we consider

$$\delta g = 0, \quad \delta\sigma = 0, \quad \delta A = 0, \quad \delta H = 0 \quad (13)$$

and

$$a = \delta a, \quad \phi = \delta\phi. \quad (14)$$

The arguments presented above imply that the static equations for g , σ , A , and H remain unchanged in first order perturbation theory. The perturbation equations for δa and $\delta\phi$ are obtained from Eqs. (9) and (10), respectively. However, it turns out to be more convenient to use the linearized *twist potential* $\delta\chi$, rather than δa itself. The perturbation equation for $\delta\chi$ is derived from Eq. (12) by linearizing the integrability condition $d(da)=0$, whereas the perturbation equation for $\delta\chi$ is obtained from Eqs. (10) and (12). One easily finds (to first order in $\delta\chi$ and $\delta\phi$)

$$-\frac{1}{4\kappa}d\left(\frac{1}{\sigma^2}d\delta\chi\right) + d\left(\frac{1}{\sigma}\text{Tr}\{F\delta\phi\}\right) = 0, \quad (15)$$

$$D\left(\frac{1}{\sigma}D\delta\phi\right) + \frac{1}{\sigma}F\wedge d\delta\chi = 4\kappa\text{Tr}\{F\delta\phi\}\wedge^*F + \frac{1}{\sigma^2}J_B(\delta\phi). \quad (16)$$

The above equations for the scalar perturbations $\delta\chi$ and $\delta\phi$ form a formally *self-adjoint* system. This is manifest for the second order differential operators and for the diagonal potential terms on the right-hand side of Eq. (16). The two off-diagonal parts on the left-hand sides are easily seen to be symmetric as well. Moreover, $\delta\chi$ and $\delta\phi$ are *gauge-invariant* perturbation amplitudes: This is obvious for $\delta\chi$, since it is obtained from the Abelian field strength $\delta(da)$. The invariance of $\delta\phi$ follows from the infinitesimal transformation law $\delta\phi \rightarrow \delta\phi + U_\star(\phi)\delta f$ and the fact that ϕ vanishes for the background solution. [We recall that under an infinitesimal gauge transformation δf one has $\delta A^{(4)} \rightarrow \delta A^{(4)} + D^{(4)}(\delta f)$.] Before we proceed, we shall argue that the angular momentum of a stationary spacetime involves only the purely stationary set of perturbations, governed by Eqs. (15) and (16).

Apart from stationarity, no symmetry requirements have been imposed so far. We shall now assume that spacetime admits a second, axial Killing field, ∂_φ , and compute the Komar expression for the angular momentum \mathcal{J} . Asymptotic flatness implies that only the terms which are linear in a contribute to the Komar integral:

$$\mathcal{J} = \frac{1}{16\pi G} \int_{S_\infty^2} \star^{(4)} d\psi^{(4)} = \frac{1}{16\pi G} \int_{S_\infty^2} [a \wedge \star d\psi - \psi \wedge \star da]. \quad (17)$$

Here, $\psi^{(4)} = g_{\varphi\mu}^{(4)} dx^\mu$ is the axial Killing one-form and ψ its projection on Σ . Since in the asymptotic regime $\sigma \rightarrow 1$ and $\psi \rightarrow r^2 \sin^2 \vartheta d\varphi$, the first integrand in Eq. (17) becomes equal to -2 times the second one. Hence, the angular momentum becomes

$$\mathcal{J} = -\frac{3}{16\pi G} \int_{S^2_\infty} \psi \wedge *da = \frac{3}{16\pi G} \int_{S^2_\infty} ra \wedge d(\cos\vartheta). \quad (18)$$

Let us now consider arbitrary stationary, axisymmetric perturbations of a static and axisymmetric spacetime. In this case, a is a first order quantity, and the expression for \mathcal{J} involves neither perturbations of the three-metric \mathbf{g} nor of the KK scalar field σ . Hence, only the purely stationary modes, governed by Eqs. (15) and (16), contribute to the angular momentum.

IV. MULTIPOLE EXPANSION

We now restrict ourselves to spherically symmetric background configurations and perform a multipole expansion of the relevant first order quantities (which, for simplicity, are assumed to be axisymmetric). In the unperturbed spacetime ($\mathbb{R} \times \Sigma, \mathbf{g}^{(4)}$), we use standard Schwarzschild coordinates and parametrize the metric $\mathbf{g}^{(4)} = -\sigma dt^2 + \sigma^{-1} \mathbf{g}$ in the familiar form

$$\sigma = NS^2, \quad \sigma^{-1} \mathbf{g} = N^{-1} dr^2 + r^2 d\Omega^2, \quad (19)$$

where N and S are functions of the coordinate r . In the ‘‘canonical gauge,’’ the static, spherically symmetric, purely magnetic background YM potential assumes the form

$$A = [1 - w(r)]^{\hat{*}} d\tau_r, \quad (20)$$

where $\hat{*}$ denotes the Hodge dual with respect to the standard metric on S^2 , and τ_r, τ_ϑ , and τ_φ are the spherical generators of SU(2) (normalized such that $[\tau_\vartheta, \tau_\varphi] = \tau_r$). (See also [14] for a discussion of symmetric gauge fields with a higher rank gauge group.) For a static, spherically symmetric Higgs field we have

$$H^{(3)} = h(r) \tau_r, \quad H^{(2)} = \frac{1}{2} h(r), \quad (21)$$

where, as before, $H^{(3)}$ and $H^{(2)}$ denote a Higgs field in the adjoint (triplet) and the fundamental (doublet) representation of SU(2), respectively. [We recall that the general spherically symmetric ansatz for a Higgs doublet is $H^{(2)} = 1/2h(r) \mathbf{1} - g(r) \tau_r$, and that the magnetic gauge potential A involves the additional term $\tilde{w}(r) d\tau_r$. However, in the *static* case, the field equations imply that one may consistently set $g(r) = \tilde{w}(r) = 0$; see, e.g., [15].]

Let us now consider the multipole expansion for the perturbations. We first observe that the perturbations of the metric potential a which contribute to the ADM angular momentum belong to the sector with (total) angular momentum $j=1$. In fact, as δa is an axisymmetric one-form on the spherically symmetric manifold Σ , this has an expansion of the form

$$\delta a = \sum_j [\alpha_j \hat{*} dY_j + \beta_j Y_j dr + \gamma_j dY_j], \quad (22)$$

where the coefficients are functions of the radial coordinate r , and Y_j is shorthand for the spherical harmonics Y_{j0} . Since the integrand in the Komar expression (18) is proportional to

dY_1 , the orthogonality of the spherical harmonics implies that only the term proportional to $\hat{*} dY_1$ in the expansion for δa gives a nontrivial contribution. Hence, as claimed, the sector describing infinitesimal rotations consists of the purely stationary perturbations with total angular momentum $j=1$.

Next, we evaluate the perturbation equations (15) and (16) for the background fields A and H [given in Eqs. (20) and (21), respectively], which are easily seen to be symmetric under parity. To this end, we first expand the electric YM perturbation $\delta\phi$ in terms of the ‘‘isospin’’ harmonics C_{jm}^ℓ , which, after suitable identifications, are proportional to the standard vector harmonics Y_{jm}^ℓ :

$$C_{jm}^j = \tau_A \varepsilon^{AB} \hat{\nabla}_B Y_{jm},$$

$$C_{jm}^{j\pm 1} = \mp \frac{1}{2} (2j+1 \pm 1) \tau_r Y_{jm} + \tau_A \delta^{AB} \hat{\nabla}_B Y_{jm}, \quad (23)$$

where capital Latin letters refer to indices with respect to the orthonormal frame $\theta^\vartheta = d\vartheta$, $\theta^\varphi = \sin\vartheta d\varphi$ on S^2 . The harmonics C_{jm}^ℓ have parity $(-1)^\ell$ and are, of course, eigenfunctions of the Laplacian $\hat{\Delta} = \hat{*} d \hat{*} d$ on S^2 with eigenvalues $-\ell(\ell+1)$. It is not hard to see that the symmetry under a parity transformation implies that the odd parity component of $\delta\phi$ decouples. Moreover, this does not contribute to the ADM angular momentum, since the parity of the corresponding variation of a is also odd [see Eq. (12)]. Thus, the axial perturbations which are relevant to infinitesimal rotations can be parametrized in terms of three scalar functions $\mathbf{x}(r)$, $\mathbf{y}(r)$, and $\mathbf{z}(r)$:

$$\delta\chi = \sqrt{2\kappa} \mathbf{x}(r) Y_1, \quad \delta\phi = \mathbf{y}(r) \tau_r Y_1 + \mathbf{z}(r) \frac{1}{\sqrt{2}} \tau_\vartheta \partial_\vartheta Y_1. \quad (24)$$

At this point, it is a straightforward task to derive the perturbation equations for the vector-valued function $\mathbf{v} = (\mathbf{x}, \mathbf{y}, \mathbf{z})^T$ from Eqs. (15) and (16). The rotational deviations are governed by the following Sturm-Liouville equation:

$$\{-\partial r^2 \mathbf{A} \partial + \mathbf{J} + \mathbf{B} \partial - \partial \mathbf{B}^T + \mathbf{P}\} \underline{\mathbf{v}} = 0, \quad (25)$$

where ∂ denotes the differential operator,

$$\partial f \equiv f' \equiv \frac{1}{S} \frac{df}{dr}, \quad (26)$$

and S is defined in Eq. (19). The first two terms originate from the differential operators $D(\sigma^{-1} * D \delta\phi)$ and $d(\sigma^{-2} * d \delta\chi)$, which give rise to the matrix-valued background functions

$$\mathbf{A} = \begin{pmatrix} -\sigma^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (27)$$

$$\mathbf{J} = \frac{1}{\sigma} \begin{pmatrix} -2\sigma^{-1} & 0 & 0 \\ 0 & 2(w^2+1) & -2\sqrt{2}w \\ 0 & -2\sqrt{2}w & w^2+1 \end{pmatrix}.$$

(Note that for $w \rightarrow 1$ and $\sigma \rightarrow 1$ the eigenvalues of \mathbf{J} become $-2, 0$, and 6 , which reflects the fact that the twist channel has angular momentum $j = \ell = 1$, whereas the orbital angular momentum of the YM perturbations is 0 and 2 .) For the differential coupling between the twist potential and the gauge fields we obtain (in units with $\kappa/2 = 4\pi G/g^2 = 1$)

$$\mathbf{B} \partial - \partial \mathbf{B}^T = 2 \begin{pmatrix} 0 & \partial \sigma^{-1}(w^2-1) & 0 \\ -\sigma^{-1}(w^2-1)\partial & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (28)$$

Finally, the potential matrix \mathbf{P} is given by

$$\mathbf{P} = -\frac{2}{\sigma} \begin{pmatrix} 0 & 0 & \sqrt{2}w' \\ 0 & 2(w^2-1)^2 r^{-2} & 0 \\ \sqrt{2}w' & 0 & 2\sigma w'^2 \end{pmatrix} + \mathbf{P}_h, \quad (29)$$

where the background Higgs field enters the perturbation equations only via the matrix \mathbf{P}_h , which becomes

$$\mathbf{P}_h^{(3)} = \frac{r^2}{\sigma} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h^2 \end{pmatrix} \quad \text{and} \quad \mathbf{P}_h^{(2)} = \frac{r^2}{4\sigma} \begin{pmatrix} 0 & 0 & 0 \\ 0 & h^2 & 0 \\ 0 & 0 & h^2 \end{pmatrix} \quad (30)$$

for a Higgs triplet and a Higgs doublet, respectively.

In order to discuss the pulsation equations one needs the behavior of the background quantities N , S , w , and h . These are subject to the static, spherically symmetric EYM-Higgs equations, which are most conveniently obtained from the effective Lagrangian. For the gravitational part one finds (up to an exact differential) $*^{(4)}R^{(4)} = 4S(dm/dr)dt \wedge dr \wedge d\Omega$, where $2m(r) = r[1 - N(r)]$; see, e.g., [16]. Also evaluating the effective Lagrangians (7) and (8) (with $a=0$ and $\phi=0$) immediately gives the static, spherically symmetric action (using again $\kappa/2 = 1$)

$$\mathcal{S} = \frac{1}{G} \left[-\frac{dm}{dr} + N \left(\frac{dw}{dr} \right)^2 + N \frac{r^2}{2} \left(\frac{dh}{dr} \right)^2 + \frac{(w^2-1)^2}{2r^2} + \frac{r^2}{2} P(h) + Q(w, h) \right] S dr, \quad (31)$$

where $P(h)$ denotes the Higgs potential, and the interaction potential $Q(w, h)$ is given by

$$Q^{(3)}(w, h) = h^2 w^2 \quad \text{and} \quad Q^{(2)}(w, h) = \frac{1}{4} h^2 (1-w)^2 \quad (32)$$

for a Higgs triplet and a Higgs doublet, respectively. Variation of \mathcal{S} with respect to m and S yields the relevant Einstein equations, whereas variation with respect to w and h gives the magnetic YM-Higgs equations. Using the background equations enables one now to analyze the perturbation equations in the vicinity of the origin, the horizon, and in the asymptotic regime. In the following section we present the results of a systematic discussion.

V. ROTATING BLACK HOLES

We start by discussing the behavior of perturbations near the horizon, r_H , of a given black hole background. If the unperturbed solutions are analytic in a neighborhood of the horizon, then r_H is a regular singular point of the perturbation equations. Local properties of the solutions can, therefore, be analyzed by means of standard techniques. In particular, the number of physically acceptable solutions is easily determined: The perturbation equations for the EYM system coupled to a Higgs doublet or a Higgs triplet admit precisely *four* independent solutions which are admissible near the horizon (provided that the unperturbed black hole is not extreme).

Next we consider the asymptotic regime $r \rightarrow \infty$. Near infinity, the background solutions with a Higgs field in the adjoint representation approach the embedded Reissner-Nordström solution with magnetic charge $P^2 = 1$: $w \approx 0$ and $|h| \approx v$, where v is the vacuum expectation value of the Higgs field. Similarly, the unperturbed solutions with a Higgs field in the fundamental representation approach the embedded Schwarzschild solution: $|w| \approx 1$ and $|h| \approx v$. (The Abelian nature of the matter fields becomes manifest after a suitable gauge transformation.) It is straightforward to verify that the leading asymptotic behavior of the perturbations remains unchanged if a given background solution is replaced by its ‘‘asymptotic Abelian part.’’ Within this approximation, the perturbation equations simplify considerably in the asymptotic regime: For a Higgs triplet, the ‘‘massive’’ perturbation channel decouples, and the remaining two equations have a regular singular point at infinity. For a Higgs doublet, the asymptotic system can even be decoupled completely. For both types of Higgs fields it is, therefore, readily verified that precisely *three* independent solutions exist which are physically acceptable near infinity.

Since the background configurations are continuous for $r_H < r < \infty$, the above-defined local solutions have extensions with a range of definition containing the whole interval $r_H < r < \infty$. By construction, these extensions span the subspaces of global solutions which are acceptable near the inner and the outer boundary points, respectively. Since these solution subspaces have dimension 3 and 4, respectively, and since the dimension of the total solution space is 6, the intersection of the subspaces is (at least) one dimensional. Thus, in the black hole case, physically acceptable global solutions of the perturbation equations always exist for the EYM-Higgs system.

VI. ROTATING SOLITONS

Like in the black hole case, the perturbation equations for soliton background solutions have a regular singular point at the inner boundary point, $r=0$, provided that the unperturbed solutions are analytic in a neighborhood of the origin. In the vicinity of this point, the leading behavior of perturbations is completely fixed by the ‘‘centrifugal barrier,’’ $\mathcal{J}r^{-2}$. It is, therefore, straightforward to verify that precisely *three* independent solutions exist which are globally defined and physically acceptable near the origin. In the asymptotic regime, $r \rightarrow \infty$, the behavior of perturbations is the same as in the black hole case. Hence, the global solutions of the perturbation equation which are admissible near both boundary points are given by the intersection of two solution sub-

spaces, each of which is *three* dimensional. Since the intersection of two three-dimensional subspaces of a six-dimensional linear space generically is trivial, we are led to the conclusion that soliton solutions of the EYM-Higgs system generically do not admit rotational excitations.

VII. CONCLUDING REMARKS

Both the general structure and the main features of the perturbation equations are dominated by the EYM part of the system. It is, therefore, natural to expect that the above results, derived for the SU(2) EYM-Higgs system, continue to hold for a class of EYM systems with higher rank gauge groups and more general bosonic matter fields. Hence, we conjecture that bosonic EYM black holes always have rotating counterparts, whereas bosonic EYM solitons generically

do not admit infinitesimal rotations. The approach presented in this paper offers the possibility for a *systematic* study of these conjectures, which, in case they should turn out to be correct, raise the important question about the physical mechanism preventing bosonic solitons from rotating.

We emphasize that the asymptotic behavior of the perturbations changes qualitatively in the absence of bosonic fields. As a consequence of this, the *pure* EYM system does admit slowly rotating solitons [13].

ACKNOWLEDGMENTS

It is a pleasure to thank Norbert Straumann and Michael Volkov for stimulating discussions. This work was supported by the Swiss National Science Foundation.

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- [1] G. Neugebauer and D. Kramer, *Ann. Phys. (Leipzig)* **24**, 62 (1969).
 - [2] P. Breitenlohner, D. Maison, and G. Gibbons, *Commun. Math. Phys.* **120**, 295 (1988).
 - [3] D. V. Gal'tsov and P. S. Letelier, *Class. Quantum Grav.* **14**, L9 (1997).
 - [4] M. Heusler, *Phys. Rev. D* **56**, 961 (1997).
 - [5] R. Geroch, *J. Math. Phys.* **13**, 394 (1972); W. Kinnersley, *ibid.* **14**, 651 (1973); **18**, 1529 (1977); D. Maison, *ibid.* **20**, 871 (1979).
 - [6] D. V. Gal'tsov, *Phys. Rev. Lett.* **74**, 2863 (1995).
 - [7] B. Carter, *Commun. Math. Phys.* **17**, 233 (1970); *General Relativity: An Einstein Centenary Survey* (Cambridge University Press, Cambridge, England, 1979).
 - [8] M. Heusler, *Helv. Phys. Acta* **69**, 501 (1996); *Class. Quantum Grav.* **12**, 2021 (1995); M. Heusler and N. Straumann, *ibid.* **10**, 1299 (1993).
 - [9] M. Heusler, *Black Hole Uniqueness Theorems* (Cambridge University Press, Cambridge, England, 1996).
 - [10] A. Papapetrou, *Proc. R. Irish Acad.* **51**, 191 (1945), *Ann. Phys. (Leipzig)* **12**, 309 (1953).
 - [11] M. S. Volkov and N. Straumann, *Phys. Rev. Lett.* **79**, 1428 (1997).
 - [12] J. Ehlers, *Les Theories Relativistes de la Gravitation* (CNRS, Paris, 1959).
 - [13] O. Brodbeck, M. Heusler, N. Straumann, and M. Volkov, "Rotating Solitons and Nonrotating, Nonstatic Black Holes," *Phys. Rev. Lett.* (to be published), gr-qc/9707057.
 - [14] O. Brodbeck, *Helv. Phys. Acta* **69**, 321 (1996); O. Brodbeck and N. Straumann, *J. Math. Phys.* **34**, 2412 (1993).
 - [15] P. Boschung, O. Brodbeck, F. Moser, N. Straumann, and M. Volkov, *Phys. Rev. D* **50**, 3842 (1994).
 - [16] O. Brodbeck, M. Heusler, and N. Straumann, *Class. Quantum Grav.* **53**, 754 (1996).