Structure of resonance in preheating after inflation

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We consider preheating in the theory $\frac{1}{4}\lambda\phi^4+\frac{1}{2}g^2\phi^2\chi^2$, where the classical oscillating inflaton field $\phi(t)$ decays into χ particles and ϕ particles. The parametric resonance which leads to particle production in this conformally invariant theory is described by the Lamé equation. It significantly differs from the resonance in the theory with a quadratic potential. The structure of the resonance depends in a rather nontrivial way on the parameter g^2/λ. We find an ''unnatural selection'' rule: the most efficient creation of particles occurs not for particles which have the strongest coupling to the inflaton field, but for those which have the greatest characteristic exponent μ. We construct the stability-instability chart in this theory for arbitrary g^2/λ. We give simple analytic solutions describing the resonance in the limiting cases $g^2/\lambda\ll 1$ and $g^2/\lambda\gg 1$, and in the theory with $g^2=3\lambda$, and with $g^2=\lambda$. From the point of view of parametric resonance for χ, the theories with $g^2=3\lambda$ and with $g^2=\lambda$ have the same structure, respectively, as the theory $\frac{1}{4}\lambda\phi^4$, and the theory $(\lambda/4N)(\phi_i^2)^2$ of an N-component scalar field ϕ_i in the limit $N\to\infty$. We show that in some of the conformally invariant theories such as the simplest model $\frac{1}{4}\lambda\phi^4$, the resonance can be terminated by the back reaction of produced particles long before $\langle\chi^2\rangle$ or $\langle\phi^2\rangle$ become of the order ϕ^2. We analyze the changes in the theory of reheating in this model which appear if the inflaton field has a mass m. In this case the conformal invariance is broken, and the resonance may acquire the features of stochasticity and intermittancy even if the mass is very small, so that $(m^2/2)\phi^2\ll(\lambda/4)\phi^4$. We give a classification of different resonance regimes for various relations between the coupling constants, masses, and the amplitude of the oscillating inflaton field ϕ in a general class of theories $\pm(m^2/2)\phi^2+(\lambda/4)\phi^4+(g^2/2)\phi^2\chi^2$. [S0556-2821(97)05122-9]

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I. INTRODUCTION

The theory of reheating of the universe is one of the most important and least developed parts of inflationary cosmology. Recently it was found that in many realistic versions of chaotic inflation reheating begins with a stage of parametric resonance [1]. At this stage the energy is rapidly transferred from the inflaton field to other scalar and vector fields interacting with it. This process occurs far away from thermal equilibrium, and therefore we called it *preheating*. The theory of preheating is rather complicated. In [1] we gave only a brief summary of its basic features. A detailed investigation of preheating in the simplest chaotic inflation model describing a massive inflaton field ϕ interacting with a massless scalar field χ was contained in our recent paper [2]. It was found, in particular, that the resonance in such theories can be efficient only if it is extremely broad. In such a situation preheating in an expanding universe looks like a stochastic process.

In this paper we will concentrate on the theory of preheating in a class of conformally invariant theories such as $(\lambda/4)\phi^4+(g^2/2)\phi^2\chi^2$. Different aspects of preheating in such theories have been studied in Refs. [1,3–11]. A specific feature of these models is that by a conformal transformation one can reduce the investigation of preheating in these theories in an expanding universe to a much simpler theory of preheating in Minkowski space-time [1]. As a result, the parametric resonance does not exhibit the stochasticity found in [2]. However, stochastic resonance may appear again at the late stages of preheating if the fields ϕ and χ have bare masses which break conformal invariance.

We will investigate preheating in the theories of the type of $(\lambda/4)\phi^4+(g^2/2)\phi^2\chi^2$ for various relations between the coupling constants g^2 and λ. During this investigation (see specifically Secs. V and XII), we will discuss how the results of the previous papers on this subject are related to the picture which emerges from the current study. We will show that the development of the resonance in the various conformally invariant theories can be very different, depending on the particular values of parameters and the structure of the theory. For example, the model $(\lambda/4)\phi^4+(g^2/2)\phi^2\chi^2$ with $g^2=\lambda$ or $g^2=3\lambda$ has only one instability band, but the structure of the bands and the characteristic exponents μ_k are completely different. It is enough to change the ratio g^2/λ only slightly, and the number of the instability bands immediately becomes infinitely large. For this reason, it is dangerous to extrapolate the results obtained for a theory with one choice of parameters to a theory with another choice of parameters. As we will see, not only is the structure of the resonances different in different models, but the self-consistent dynamical evolution with an account taken of the back reaction of produced particles can also be qualitatively different.

The main purpose of the present paper is to study the structure of the parametric resonance in the conformally invariant theories. These theories may describe many bose fields χ interacting with the inflaton field ϕ with different coupling constants:

$$\mathcal{L}=-\frac{M_p^2}{16\pi}R+\frac{1}{2}\phi_{,i}\phi^{,i}-\frac{\lambda}{4}\phi^4 + \frac{1}{2}\sum_m \left(\chi_{m,i}\chi_m{}^{,i}-g_m^2\phi^2\chi_m^2-\xi_m R\chi_m^2\right). \quad (1)$$

Here χ_m stands for the mth scalar field interacting with the inflaton field with the coupling constant g_m, and interacting with curvature R with the coupling constant ξ_m.

Strictly speaking, this model is conformally invariant only for a specific choice of the parameters ξ_m: $\xi_m=\frac{1}{6}$. Nevertheless, in this paper we will consider the simplest models with $\xi_m=0$. As we will see shortly, this difference is not going to be very important because the average value of R vanishes when $|\phi|\ll M_p$.

We will see that for the conformally invariant theories the only parameter actually responsible for the structure of the resonance for the field χ_m is the ratio g_m^2/λ. Furthermore, we will find that the strength of the resonance and the number and widths of the instability bands for the field χ_m in the theory (1) depends on g_m^2/λ nonmonotonically. To get a general picture, we will construct the stability-instability chart for the equation for fluctuations on the two-dimensional plane $(k^2,g^2/\lambda)$, see Fig. 4. The stability-instability chart gives us insight into the structure of the resonances in the conformally invariant theories. From this it will immediately be clear which of the fields χ_m of Eq. (1) will be most amplified during preheating. The stability-instability chart unifies our knowledge of the resonance for the various conformal models thus far considered in the literature.

Note that the class of theories we are going to investigate include in particular the theory $(\lambda/4)(\sum_{i=1}^N\phi_i^2)^2$ of an N-component scalar field ϕ_i. This theory has O(N) symmetry. One can identify the inflaton field ϕ in this theory with the field ϕ_1. Then the quantum fluctuations of this field, just like the quantum fluctuations in the theory of a one-component field $(\lambda/4)\phi^4$, will have effective mass squared $3\lambda\phi^2$, whereas the fluctuations of all other components will have effective mass squared $\lambda\phi^2$. Therefore, the equation for the growth of the fluctuations of the field $\phi=\phi_1$ (neglecting backreaction) will coincide with the equation for the growth of fluctuations of the field χ coupled to the field ϕ with the coupling constant $g^2=3\lambda$. Meanwhile, the equation for the growth of the fluctuations of the fields ϕ_i, $i\neq 1$, will coincide with the equation for the growth of fluctuations of the field χ with the coupling constant $g^2=\lambda$. This regime is especially important in the limit $N\to\infty$, where the main contribution to particle production is given by the modes with $i\neq 1$. Thus, the cases $g^2=\lambda$ and $g^2=3\lambda$ are especially interesting and deserve careful investigation.

This paper is organized as follows. In Sec. II we will describe the evolution of the background inflaton field $\phi(t)$ after inflation in the theory with the effective potential $V(\phi)=\frac{1}{4}\lambda\phi^4$. We will give an analytic solution for the motion of the field $\phi(t)$ in the regime of oscillations, when $|\phi|\ll M_p$. Then, in Sec. III, we derive the equations for fluctuations of the fields χ and ϕ in the conformally invariant theory, and reduce these to equations in Minkowski spacetime. We show that these equations can ultimately be reduced to a single Lamé equation with just one parameter, g^2/λ. In Sec. IV we solve the Lamé equation numerically for an arbitrary g^2/λ and arbitrary momentum, k, of fluctuations. This allows us to produce the main result of our paper; we construct the stability/instability chart for fluctuations in the conformally invariant theories. In Sec. V we discuss the particular ranges and values of the parameter g^2/λ where the analytic methods for the description of the resonance can be developed. In Secs. VI–IX we perform an analytic investigation of the resonance for some particular values of g^2/λ. For different values of g^2/λ different analytic approaches will be developed. We report a new method to treat the resonance when $g^2/\lambda=n(n+1)/2$, where n is an integer. We show that the solutions for $g^2/\lambda=n(n+1)/2$ can be found in closed form, in terms of integrals of algebraic functions, instead of complicated theta functions. This is done explicitly for the most interesting cases, $n=1$ and $n=2$ (i.e., for $g^2=\lambda$ and $g^2=3\lambda$), in Secs. VI, VII, and the Appendix. We also consider the two opposite limits $g^2/\lambda\ll 1$ and $g^2/\lambda\gg 1$ in Secs. VIII and IX, respectively. Section X contains a discussion of the self-consistent dynamics of the system including backreaction of the created particles. In Sec. XI we describe the restructuring of the resonance which occurs when the back reaction is incorporated into the equations for fluctuations. We show that this is the leading effect which terminates the resonance in the theory $(\lambda/4)\phi^4$. In Sec. XII we discuss the modifications of the theory of preheating which appear when the inflaton field ϕ is massive. This allows us to unify the results obtained in this paper with the results of our preceding investigation of preheating in the theory of a massive inflaton field [2]. We find out that even a very small mass, $m\ll\sqrt{\lambda}\Phi$, where Φ is the amplitude of oscillations of the field ϕ, may change the nature of the resonance, making it stochastic as in the theory of a massive inflaton field considered in Ref. [2]. This is a rather surprising result, which was not anticipated in the earlier studies of this issue. Indeed, one could expect that the presence of a term $(m^2/2)\phi^2$ cannot influence the nature of the resonance if this term is much smaller than $(\lambda/4)\phi^4$. We give a classification of different resonance regimes for various relations between the coupling constants, masses, and the amplitude of the oscillating inflaton field ϕ in a general class of theories with the effective potential $\pm(m^2/2)\phi^2+(\lambda/4)\phi^4+(g^2/2)\phi^2\chi^2$. In Sec. XIII, we give a summary of our results and discuss their possible implications.

II. EVOLUTION OF THE INFLATON FIELD

We consider chaotic inflation with the potential $V(\phi)=\frac{1}{4}\lambda\phi^4$. During inflation the leading contribution to the energy-momentum tensor is given by the inflaton scalar field ϕ. The evolution of the (flat) Friedmann-Robertson-Walker

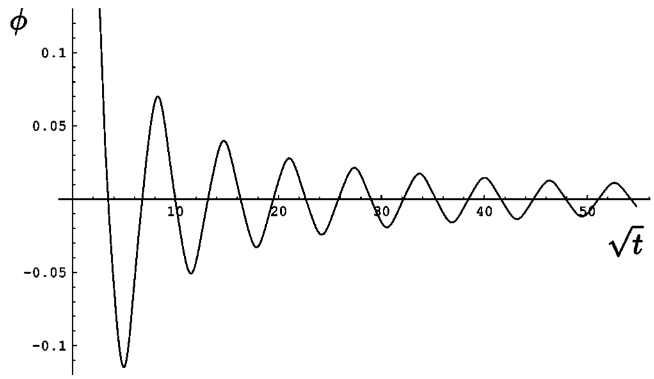

FIG. 1. Oscillations of the field ϕ after inflation in the theory $\lambda\phi^4/4$. The value of the scalar field here and in all other figures in this paper is measured in units of M_p, time is measured in units of $(\sqrt{\lambda}M_p)^{-1}$.

(FRW) universe is described by the Friedmann equation

$$H^2=\frac{8\pi}{3M_p^2}\left(\frac{1}{2}\dot{\phi}^2+\frac{\lambda\phi^4}{4}\right), \tag{2}$$

where $H=\dot{a}/a$. Let us note one more useful relationship between $H(t)$ and $\phi(t)$ which follows from the Einstein equations

$$\dot{H}=-\frac{4\pi\dot{\phi}^2}{M_p^2}. \tag{3}$$

The equation for the classical field $\phi(t)$ is

$$\ddot{\phi}+3H\dot{\phi}+\lambda\phi^3=0. \tag{4}$$

For sufficiently large initial values of $\phi > M_p$, the friction term, $3H\dot{\phi}$, in Eq. (4) dominates over $\ddot{\phi}$ and the potential term in Eq. (2) dominates over the kinetic term. This is the inflationary stage, where the universe expands quasiexponentially, $a(t)=a_0\exp[\int dtH(t)]$, and the field ϕ slowly decays, $\phi\sim M_p\exp[-(\sqrt{\lambda}/\sqrt{6\pi})M_Pt]$. With a decrease of the field ϕ below M_p, the "drag" term $3H\dot{\phi}$ gradually becomes less important and inflation terminates at $\phi\sim M_p/2$. After a short stage of fast rolling down, the inflaton field rapidly oscillates around the minimum of $V(\phi)$ with the initial amplitude $\Phi_0 \sim 0.1M_p$. Although this value is below the magnitude needed for inflation, it is still very large.

The character of the classical oscillations of the homogeneous scalar field depends on the shape of its potential $V(\phi)$. In Ref. [2] we considered the theory with the quadratic potential $V(\phi)=\frac{1}{2}m\phi^2$. In that theory the fluctuations are harmonic, $\phi(t)=\Phi(t)\sin mt$, with the amplitude decreasing like $\Phi(t)\approx(M_p/\sqrt{3\pi}mt)\propto a^{-3/2}$. The scale factor at the stage of oscillations is $a(t)\approx a_0t^{2/3}$, and the energy density of the inflaton field decreases in the same way as the energy density of nonrelativistic matter $\propto a^{-3}$.

In the theory with the potential $V(\phi)=\frac{1}{4}\lambda\phi^4$, which we consider in this paper, the inflaton oscillations are not sinusoidal. The amplitude Φ of the oscillations of the scalar field ϕ in the limit $t\to\infty$ approaches the asymptotic regime

$$\Phi(t)\approx\frac{1}{\sqrt{t}}\left(\frac{3M_p^2}{8\pi\lambda}\right)^{1/4}\sim\frac{M_p}{10N}, \tag{5}$$

where N is the number of oscillations after the end of inflation, see Fig. 1.

To make calculations in this theory, and in particular, to find the form of the oscillations, it is convenient to make a conformal transformation of the space-time metric and the fields. For this we use the conformal time

$$\eta=\int\frac{dt}{a(t)}, \tag{6}$$

and the conformal field

$$\varphi=a\phi. \tag{7}$$

In the coordinates $(\eta,\mathbf{x})$ the Klein-Gordon equation (4) for φ is

$$\varphi''+\lambda\varphi^3-\frac{a''}{a}\varphi=0, \tag{8}$$

where $'$ stands for the derivative with respect to the conformal time, $d/d\eta$. The Friedmann equation (2) in these variables is

$$a'^2=\frac{8\pi}{3M_p^2}\left[\frac{1}{2}\left(\varphi'-\varphi\frac{a'}{a}\right)^2+\frac{\lambda\varphi^4}{4}\right]. \tag{9}$$

As one can see from Eq. (8), the equation of motion for the field φ in the time variable η does not look exactly as the equation for the theory $(\lambda/4)\varphi^4$ in Minkowski space. In order to achieve it one would need to add the term $(\phi^2/12)R$ to the Lagrangian. However, this subtlety is not very important. First of all, soon after the end of inflation one has $(\lambda/4)\phi^4 \gg (\phi^2/12)|R|$, and $\lambda\varphi^3\gg(a''/a)\varphi$. Moreover, it is known that the energy-momentum tensor of the field ϕ in the theory $(\lambda/4)\phi^4$, when averaged over several oscillations, is traceless ($p=\rho/3$) [12]. In this case one has $R=0$, $a(\eta)\sim\eta$, and $a''=0$, so that the last term in Eq. (8) vanishes:

$$\varphi''+\lambda\varphi^3=0. \tag{10}$$

The Friedmann equation (9) averaged over several oscillations of the field ϕ in the regime $\phi\ll M_p$ also takes a very simple form:

$$a'^2=\frac{8\pi}{3M_p^2}\left(\frac{1}{2}\varphi'^2+\frac{\lambda\varphi^4}{4}\right)\equiv\frac{8\pi\rho_\varphi}{3M_p^2}, \tag{11}$$

where we have introduced the conformal energy density, $\rho_\varphi=\frac{1}{2}\varphi'^2+(\lambda/4)\varphi^4$.

It is convenient to express ρ_φ in terms of the amplitude $\tilde{\varphi}$ of the oscillations of the field φ: $\rho_\varphi=(\lambda/4)\tilde{\varphi}^4$. Equation (10) has an oscillatory solution with a constant amplitude and the conformal energy ρ_φ. Then from Eq. (11) we find

$$a(\eta)=\sqrt{\frac{2\pi\lambda}{3}}\frac{\tilde{\varphi}^2}{M_p}\eta,\quad t=\sqrt{\frac{\pi\lambda}{6}}\frac{\tilde{\varphi}^2}{M_p}\eta^2. \tag{12}$$

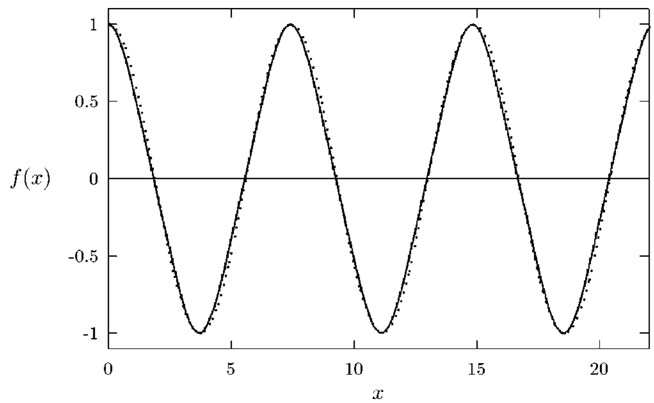

FIG. 2. The exact solution (14) for the oscillations of the inflaton field after inflation in the conformally invariant theory $\frac{1}{4}\lambda\phi^4$. We show the field in rescaled conformal field and time variables, $f(x)=\varphi/\tilde{\varphi}$ (solid curve) and the first term, $\cos(0.8472x)$, in its harmonic expansion (15) (dotted curve).

As we expected, in this regime the last term $(a''/a)\varphi$ in Eq. (10) vanishes.

Equation (10) can be reduced to the canonical equation for an elliptic function. Indeed, let us use a dimensionless conformal time variable

$$x\equiv\sqrt{\lambda}\tilde{\varphi}\eta=\left(\frac{6\lambda M_p^2}{\pi}\right)^{1/4}\sqrt{t}. \qquad (13)$$

Then we can rescale the function $\varphi\equiv a\phi=\tilde{\varphi}f(x)$. The function $f(x)$ has an amplitude equal to unity and obeys the canonical equation for the elliptic function. The integral of this equation, $f'^2=\frac{1}{2}(1-f^4)$, has the solution in terms of an elliptic cosine

$$f(x)=cn\left(x-x_0,\frac{1}{\sqrt{2}}\right). \qquad (14)$$

As claimed, oscillations in this theory are not sinusoidal but are given by an elliptic function. The energy density of the field ϕ decreases in the same way as the density of radiation, i.e., as a^{-4}.

The solution (14) has some interesting properties which are not usually elucidated in the literature. It matches the solution describing the slow rolling of the field φ at the end of inflation if one takes $x_0\approx 2.44$. The period of the oscillations (in units of x) is $T=4K(1/\sqrt{2})=\Gamma^2(1/4)/\sqrt{\pi}\approx 7.416$, K stands for the complete elliptic integral of the first kind. The effective frequency of oscillations is $2\pi/T\approx 0.8472$ [1]. The value of f^4 averaged over a period is $\frac{1}{3}$. The potential energy density $\frac{1}{4}\lambda\varphi^4$ averaged over a period of oscillation is equal to $\frac{1}{3}\rho_\varphi$, and the average kinetic energy $\frac{1}{2}\varphi'^2$ is given by $\frac{2}{3}\rho_\varphi$.

The elliptic cosine can be represented as follows:

$$f(x)=\frac{8\pi\sqrt{2}}{T}\sum_{n=1}^{\infty}\frac{e^{-\pi(n-1/2)}}{1+e^{-\pi(n-1/2)}}\cos\frac{2\pi(2n-1)x}{T}. \qquad (15)$$

The amplitude of the first term in this sum is 0.9550; the amplitude of the second term is much smaller, 0.043 05. The full solution (14) is plotted in Fig. 2 (solid curve), alongside the leading harmonic term in the series (15), $\cos 0.8472x$ (dotted curve). Although the first harmonic term is very close to the actual form of oscillations, it will be important for the investigation of the general structure of stability-instability bands in this theory that $f(x)$ is not exactly equal to $\cos(2\pi x/T)$.

III. EQUATIONS FOR QUANTUM FLUCTUATIONS OF THE FIELDS ϕ AND χ

We will consider here the interaction between the *classical* inflaton field, ϕ, and the massless, *quantum* scalar field, $\hat{\chi}$, with the Lagrangian (1). The Heisenberg representation of the quantum scalar field $\hat{\chi}$ is

$$\hat{\chi}(t,\mathbf{x})=\frac{1}{(2\pi)^{3/2}}\int d^3k[\hat{a}_k\chi_k(t)e^{-i\mathbf{k}\cdot\mathbf{x}}+\hat{a}_k^+\chi_k^*(t)e^{i\mathbf{k}\cdot\mathbf{x}}],$$

where $\hat{a}_k$ and $\hat{a}_k^+$ are the annihilation and creation operators. For a flat Friedmann background with scale factor $a(t)$, the temporal part of the eigenfunction with comoving momentum $\mathbf{k}$ obeys the following equation:

$$\ddot{\chi}_k+3\frac{\dot{a}}{a}\dot{\chi}_k+\left(\frac{k^2}{a^2}+g^2\phi^2\right)\chi_k=0. \qquad (16)$$

As we mentioned in the previous section, at the stage of oscillations when $\phi\ll M_p$ the average value of the curvature R vanishes, so one can neglect the term $\sim\xi\phi^2R$.

The self-interaction $\frac{1}{4}\lambda\phi^4$ also leads to the generation of fluctuations of the field ϕ. The equation for the eigenmodes $\phi_{\mathbf{k}}(t)$ is

$$\ddot{\phi}_k+3\frac{\dot{a}}{a}\dot{\phi}_k+\left(\frac{k^2}{a^2}+3\lambda\phi^2\right)\phi_k=0. \qquad (17)$$

Note that this equation is identical to Eq. (16) with $g^2=3\lambda$. Therefore, the study of the fluctuations ϕ_k in the $\frac{1}{4}\lambda\phi^4$ model is a particular case of the general equation for fluctuations (16).

The physical momentum, $\mathbf{p}=\mathbf{k}/\mathbf{a(t)}$, in Eq. (16) is redshifted in the same manner as the background field amplitude, $\phi(t)=\varphi/a(t)$. Therefore, the redshifting of momenta can be eliminated from the evolution of χ_k. Indeed, let us use the conformal transformation of the mode function $X_k(t)=a(t)\chi_k(t)$ and rewrite the mode equation for $X_k(t)$ with the dimensionless conformal time x [see Eq. (13)]:

$$X_k''+\left[\kappa^2+\frac{g^2}{\lambda}cn^2\left(x,\frac{1}{\sqrt{2}}\right)\right]X_k=0, \qquad (18)$$

where for simplicity we drop the initial value of $x_0=2.44$. In this form the equation for fluctuations does not depend on the expansion of the universe and is completely reduced to the similar problem in Minkowski space-time. This is a special feature of the conformally invariant theory $\frac{1}{4}\lambda\phi^4+\frac{1}{2}g^2\phi^2\chi^2$.

For the fluctuations of the field $\varphi=a\phi$ one has

$$\varphi_k'' + \left[\kappa^2 + 3cn^2\left(x, \frac{1}{\sqrt{2}} \right) \right] \varphi_k = 0. \tag{19}$$

Equation (18) will be the master equation for our investigation of the resonance in the conformally invariant theory. The comoving momentum k enters the equation in the combination

$$\kappa^2 = \frac{k^2}{\lambda \tilde{\varphi}^2}. \tag{20}$$

Therefore the natural units of the momenta k is $\sqrt{\lambda}\tilde{\varphi}$. Equation (18) describes oscillators, X_k, with a variable frequency

$$\omega_k^2 = \kappa^2 + \frac{g^2}{\lambda} cn^2\left(x, \frac{1}{\sqrt{2}} \right), \tag{21}$$

which periodically depends on time, x. It is well known that in this case the solutions X_k are exponentially unstable: $X_k(x) \propto e^{\mu_k x}$. If we choose the vacuum positive-frequency initial condition, $X_k(x) \simeq e^{-i\kappa x}/\sqrt{2\kappa}$, we then expect the exponentially fast creation of χ particles ($n_k \propto e^{2\mu_k x}$) as the inflaton field oscillates. The strength of interaction with the periodic oscillations $cn^2(x, 1/\sqrt{2})$ is given by the dimensionless coupling parameter g^2/λ. This means that the condition of a broad parametric resonance does not require a large initial amplitude of the inflaton field, Φ_0, as in the case of the quadratic potential [1]. As we will see, the combination of parameters g^2/λ ultimately defines the structure of the parametric resonance in the theory. It turns out that the strength of the resonance depends rather nontrivially (nonmonotonically) on this parameter.

From a mathematical point of view, the mode equation (18) belongs to the class of Lamé equations [13]. In the context of preheating this was first noticed in [1], and then thoroughly studied for the O(N) symmetric theory in the limit $N \to \infty$ (i.e., for $g^2 = \lambda$) in [7] and for the $\frac{1}{4}\lambda\phi^4$ theory ($g^2 = 3\lambda$) in [8]. In this paper we perform a numerical and analytical investigation of the parametric amplification of fluctuations in the conformally invariant theory $\frac{1}{4}\lambda\phi^4 + \frac{1}{2}g^2\phi^2\chi^2$ for an arbitrary parameter g^2/λ. In the next section, we present the two-dimensional chart of the stability-instability bands for the Lamé equation (18) in terms of variables κ^2 and g^2/λ. In subsequent sections, we give a new analytic treatment of the Lamé equation in the case $g^2/\lambda = n(n+1)/2$ with integer n. We will also perform an analytical investigation of the resonance for $g^2/\lambda \ll 1$ and for $g^2/\lambda \gg 1$.

IV. STABILITY-INSTABILITY CHART IN THE CONFORMAL THEORY

As was shown in the previous section, the equation for vacuum fluctuations interacting with the inflaton oscillations in the conformal theories can be reduced to the similar problem in the Minkowski space. The equation for fluctuations (18) in this case contains only two parameters. The first parameter is g^2/λ, which characterizes the strength of the interaction. The second parameter is the momentum of vacuum fluctuations κ in units of the frequency of the inflaton oscillations. As is well known, the solutions X_k of this equation may be stable or unstable depending on the particular values for κ and g^2/λ considered. At the stage of the free resonance when we do not take into account the back reaction of the unstable fluctuations, Eq. (18) is an equation with periodic coefficients, which belongs to the class of the Lamé equations. The stability-instability chart of another equation with periodic coefficients, the Mathieu equation, is well known and can be found in many textbooks, see, e.g., [14]. We are unaware of the stability-instability charts for the Lamé equation, which describes preheating in the conformally invariant theories. Therefore in this section we present the stability-instability chart Eq. (18) in variables ($\kappa^2, g^2/\lambda$), which we obtained by solving this equation numerically.

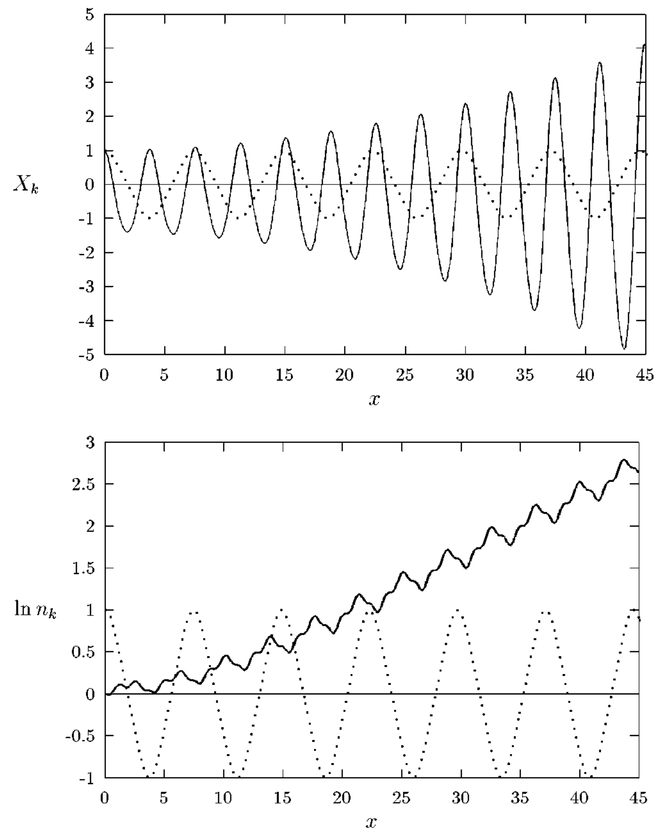

FIG. 3. The typical resonant production of particles at the particular choice of rescaled comoving momentum $\kappa^2 = 1.6$, and the parameter $g^2/\lambda = 3$. The upper plot shows the amplification of the real part of the eigenmode $X_k(x)$. The lower plot shows the logarithm of the comoving particle number density, n_k, calculated with formula (22). The number of particles grows exponentially, $\ln n_k \approx 2\mu_k x$. In this case, $\mu_k \approx 0.035$.

Figure 3 shows a typical resonant solution of Eq. (18). Though we have plotted the particular case $\kappa^2 = 1.6$, $g^2/\lambda = 3$, the form of the resonant solution is generic. The upper plot demonstrates the amplification of the real part of the eigenmode $X_k(x)$ (solid curve) in the oscillating inflaton background (dotted curve).

In addition to the investigation of the rapidly oscillating functions $X_k(x)$, it is convenient for analytical and numerical work to consider the evolution of the comoving number density of created χ particles, n_k, with comoving momentum k. This can be defined from the comoving energy density and the energy per particle, ω_k:

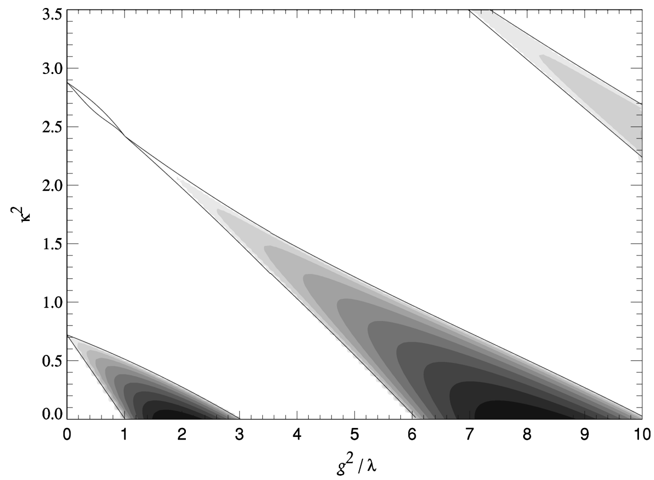

FIG. 4. The stability-instability chart for the Lamé equation for fluctuations $X_k(x)$ in the variables $(\kappa^2, g^2/\lambda)$, obtained from the numerical solution of Eq. (18). Shaded (unshaded) areas are regions of instability (stability). For instability bands, the darker shade implies a larger characteristic exponent μ_k. Altogether, there are 10 color steps. One color step corresponds to the increment $\Delta\mu_k = 0.0237$, so the darkest shade corresponds to maximal $\mu_k = 0.237$, the least dark shade in the instability bands corresponds to $\mu_k = 0.009$. For positive κ^2, there is only one instability band for the particular values of the parameter $g^2/\lambda = 1$ and 3. This occurs because the higher bands shrink into nodes as g^2/λ approaches 1 and 3.

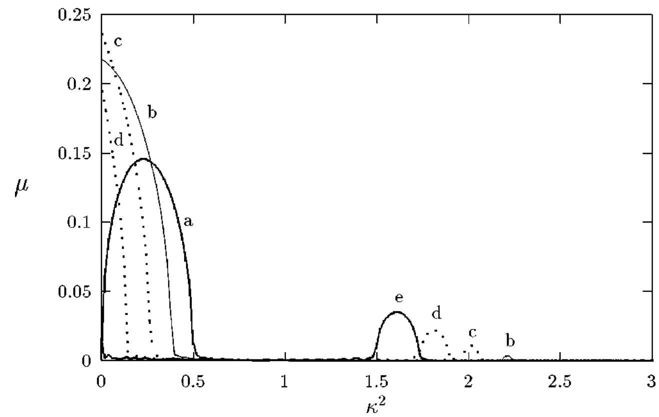

FIG. 5. Slices of the stability-instability chart, Fig. 4, reveal the dependence of the characteristic exponent, μ_k, on κ^2 for several particular values of g^2/λ. For the top panel $g^2/\lambda = 1.0$, 1.5, 2.0, 2.5, and 3.0, labeled a through e, respectively. The numerical curves a and e for $g^2/\lambda = 1$ and $g^2/\lambda = 3$ are identical to the analytic predictions (34) of Sec. VI and (40) of Sec. VII. For the lower panel $g^2/\lambda = 6.0$, 7.0, 8.0, 9.0, and 10, labeled a through e, respectively.

$$n_k = \frac{\omega_k}{2}\left(|X_k|^2 + \frac{|\dot{X}_k|^2}{\omega_k^2}\right) - \frac{1}{2}. \tag{22}$$

The lower plot of Fig. 3 shows the evolution of the logarithm of n_k (solid curve) and the inflaton field (dotted curve). For the growing solutions after an initial transitional period the number of particles increases exponentially, $\ln n_k \approx 2\mu_k x$, where μ_k is the characteristic exponent of the unstable solution. In the particular case shown, $\mu_k \approx 0.035$.

For arbitrary values of κ and g^2/λ, we can obtain a numerical solution of Eq. (18) and exploit the simple relation $\ln n_k \approx 2\mu_k x$ to extract the characteristic exponent for the growing modes. For the regions of stability the characteristic exponent formally is imaginary. In this way, the stability-instability chart for the Lamé equation, Fig. 4, is constructed. Shaded (unshaded) regions of the chart indicate values of κ^2 and g^2/λ for which the solutions are unstable (stable). For the instability bands, a darker shade indicates a larger characteristic exponent. An immediate result is that, for a given range of g^2/λ, the largest characteristic exponent will occur for $\kappa^2 = 0$ between the integer values $g^2/\lambda = n(n+1)/2$ with n integer.

This is demonstrated in Fig. 5, where slices of the stability-instability chart show the characteristic exponent as a function of κ^2 for various values of g^2/λ. The top panel of Fig. 5 plots the cases $g^2/\lambda = 1.0, 1.5, 2.0, 2.5, 3.0$, labeled a through e, respectively. $g^2/\lambda = 1$ corresponds to $n = 1$, $g^2/\lambda = 3$ corresponds to $n = 2$. As claimed, we see that the largest value of the characteristic exponent occurs for $\kappa^2 = 0$ at a value of g^2/λ between the limits 1 and 3 (curve c). Similarly, the lower panel of Fig. 5 plots the cases $g^2/\lambda = 6.0, 7.0, 8.0, 9.0, 10.0$, labeled a through e, respectively. The values $g^2/\lambda = 6$ and 10 correspond to $n = 3$ and 4. Again we see that the largest value of the characteristic exponent occurs for $\kappa^2 = 0$ at a value of g^2/λ between the limits 6 and 10 (curve c).

This stability-instability chart is very similar to the stability-instability chart of the Mathieu equation, but there are important differences as well. For the Mathieu equation there are infinitely many instability bands corresponding to each value of the parameter q, which is analogous to our parameter g^2/λ. Meanwhile for the Lamé equation some of the instability bands may occasionally shrink to a point. As a result, for $g^2/\lambda = 1$ and for $g^2/\lambda = 3$ (Fig. 5, curves a and e, respectively) there is only one instability band. This will be shown analytically in Secs. VI and VII. From the stability-instability chart for the Lamé equation, Fig. 4, we see that, topologically, this occurs because all the higher instability bands shrink to nodes as g^2/λ approaches 1 and 3.

Similarly, there are a finite number of instability bands for positive κ^2 whenever $g^2/\lambda = n(n+1)/2$. Again, it occurs as other higher instability bands shrink to nodes. However, as for the Mathieu equation, all other values of g^2/λ have an infinite number of instability bands. This is true in particular for $g^2/\lambda \ll 1$ where, as we will show in Sec. VIII, the Lamé equation (18) may be formally transformed into the Mathieu equation (43) with the parameters $A \approx 1.3932\kappa^2$ and $q \approx 0.3464 g^2/\lambda \ll 1$. Thus, with this change of variables and in the limit $g^2/\lambda \ll 1$, the stability-instability chart for the Lamé equation is found to coincide exactly with that for the Mathieu equation [1,2].

As we have shown in [1,2], the maximum value of the characteristic exponent $\mu_{\max}$ for the Mathieu equation (43) (for $A \geqslant 2q$) is $\mu_{\max} \approx 0.28$. As we will see below, similarly, the maximum value of the characteristic exponent for the Lamé equation (18) is $\mu_{\max} \approx 0.2377$. This limit can be easily related to that of the Mathieu equation. Indeed, $0.2377 = 0.28(T_M/T_L)$, where T_M and T_L are the periods of the harmonic oscillations $\cos x$ and the oscillations $f(x)$ given by Eq. (14), correspondingly; $T_M = 2\pi$ and $T_L \approx 7.416$, see Sec. II.

As we noticed in Sec. II, oscillations $f(x)$ only insignificantly differ from the leading harmonic term cos0.8472x of the series (15). Meanwhile, in general the solutions of the Lamé equation cannot be obtained by perturbative corrections to the solutions of the Mathieu equation. Overall, the stability-instability chart of the Lamé equation, Fig. 4, is quite different from that of the Mathieu equation. This is one of the manifestations of the nonperturbative nature of the parametric resonance.

We now proceed to develop the new analytic results for preheating in the physically interesting theories with $g^2/\lambda = n(n+1)/2$, which as we have seen, are hinted at by the stability-instability chart for the Lamé equation, Fig. 4.

V. ANALYSIS OF THE EQUATION FOR FLUCTUATIONS

In this section we begin the analytic investigation of the Lamé equation (18) for the fluctuations $X_k(x)$. In particular, in the next two sections we will try to find the values of the parameter g^2/λ for which analytical solutions can be obtain in closed form, and construct these solutions.

We will also investigate the resonance in two limiting cases: $g^2/\lambda \ll 1$ and $g^2/\lambda \gg 1$. In the first case one can use perturbation theory in the small parameter $g^2/\lambda \ll 1$, see Sec. VIII [16]. In the opposite limit, $g^2/\lambda \gg 1$, we can implement the method of successive parabolic scattering [2], see Sec. IX.

It is known that the Lamé equation can be solved in terms of the transcendental Jacobi functions, which in turn are given by series expansions. Earlier we reported the result for the characteristic exponent $\mu = 0.0359$ for $\lambda\phi^4$ theory [15,4]. Analytic investigation of the resonance using these transcendental functions gives the width of the unstable zone and the maximum of the characteristic exponent, μ_k, in the physically interesting cases of the O(N) symmetric theory in the limit $N \to \infty$ (which is equivalent to $g^2/\lambda = 1$ in our convention) and the $\frac{1}{4}\lambda\phi^4$ self-interacting theory ($g^2/\lambda = 3$) [7,8].

However, calculations involving these transcendental functions are extremely tedious. Fortunately, it turns out that for

$$\frac{g^2}{\lambda} = \frac{n(n+1)}{2}, \tag{23}$$

with n an integer, one can obtain simple, closed-form solutions to the master equation (18). This includes in particular the most interesting cases $g^2 = \lambda$ and $g^2 = 3\lambda$.

To find the solutions of the fluctuation equation (18) for $g^2/\lambda = n(n+1)/2$, we will rewrite Eq. (18) in the so-called algebraic form. We will use the ''time'' variable z instead of x:

$$z(x) = cn^2\left(x, \frac{1}{\sqrt{2}}\right), \quad \frac{d}{dx} = -\sqrt{2z(1-z^2)}\frac{d}{dz}, \tag{24}$$

Equation (18) for fluctuations becomes

$$2z(1-z^2)\frac{d^2X_k}{dz^2} + (1-3z^2)\frac{dX_k}{dz} + \left(\kappa^2 + \frac{g^2}{\lambda}z\right)X_k = 0. \tag{25}$$

Omitting the lower index k for simplicity, let $X_1(z)$ and $X_2(z)$ be two linearly independent solutions of Eq. (25). One of them exponentially grows, another exponentially decreases during the resonance. Let us also introduce the bilinear combinations X_1^2, X_2^2, and X_1X_2. From Eq. (25) it follows that these bilinear combinations obey a third-order equation

$$2z(z^2-1)\frac{d^3M}{dz^3} + (9z^2-3)\frac{d^2M}{dz^2} - 2\left[\left(2\frac{g^2}{\lambda} - 3\right)z + 2\kappa^2\right]\frac{dM}{dz} - 2\frac{g^2}{\lambda}M = 0. \tag{26}$$

The three solutions, $M(z)$, of this equation correspond to the three bilinear combinations of X_1 and X_2. The crucial observation is that for $g^2/\lambda = n(n+1)/2$ Eq. (26) admits a polynomial solution of degree n. In the particular cases $n=1$ and $n=2$, we have

$$n=1: \quad M_1(z) = z - 2\kappa^2,$$

$$n=2: \quad M_2(z) = z^2 - \frac{2}{3}\kappa^2 z - 1 + \frac{4}{9}\kappa^4. \tag{27}$$

Obviously, the polynomial function $M(z)$ must be the product of an exponentially growing solution and an exponentially decreasing one, i.e., $M(z) = X_1(z)X_2(z)$ in the resonance zone. From this, as we will show in the next two sections, one can construct the closed-form solutions $X(z)$.[1]

Therefore, in the physically interesting cases $n=1$ and $n=2$ we will obtain simple closed-form solutions instead of the complicated transcendental functions. This significantly simplifies the study of preheating in these cases. In particular, we will find the form of the characteristic exponent μ_k as a function of κ^2 in each case.

VI. CLOSED FORM SOLUTION FOR $g^2/\lambda = 1$

In the case $g^2 = \lambda$ Eq. (26) in the resonance band gives

$$X_1(z)X_2(z) = M_1(z), \tag{28}$$

where

$$M_1(z) = z - 2\kappa^2. \tag{29}$$

The Wronskian of Eq. (25) for $X(z)$ is

$$X_1\frac{dX_2}{dz} - X_2\frac{dX_1}{dz} = \frac{C}{\sqrt{z(1-z^2)}}, \tag{30}$$

[1]The solutions $X(z)$ involve a normalization factor N to be defined by the physical initial conditions. The auxiliary functions $M(z)$ are also defined up to a normalization factor N^2. For sake of simplicity we set $N=1$ in Eq. (27) and in the rest of the paper. This does not affect the calculation of the characteristic exponent, which is our primary interest in this paper. Determination of the factor N is a straightforward operation, see Ref. [20], where it was shown that for the vacuum initial condition $N = |M(1)|^{-1/2}$.

where C is some constant, $C=C_1$, to be defined. From Eqs. (28) and (30) we immediately obtain the closed-form solutions

$$X_{1,2}(z)=\sqrt{|M_1(z)|}\exp\left(\pm\frac{C_1}{2}\int\frac{dz}{\sqrt{z(1-z^2)}M_1(z)}\right). \tag{31}$$

Now, substituting this solution back into Eq. (25) for $X(z)$, we find the constant C_1:

$$C_1=\sqrt{2\kappa^2(1-4\kappa^4)}. \tag{32}$$

For exponentially growing solutions, C_1 must be real; therefore the exponentially growing solutions for fluctuations with $\kappa^2>0$ take place in a single instability band for which

$$0<\kappa^2<\frac{1}{2}, \tag{33}$$

in agreement with the result of [7]. The growing solution of Eq. (18) has the form $X(x)=e^{\mu_k x}P[z(x)]$, where $P[z(x)]$ is a periodic function of the conformal time x. Using Eq. (31), we can now find the characteristic exponent μ_k as a function of κ. The technical details can be found in the Appendix.

The final answer is

$$\mu_k(\kappa)=\frac{2}{T}\sqrt{2\kappa^2(1-4\kappa^4)}I(\kappa), \tag{34}$$

where an auxiliary function $I(\kappa)$ is

$$I(\kappa)=\int_0^{\pi/2}d\theta\frac{\sin^{1/2}\theta}{1+2\kappa^2\sin\theta}. \tag{35}$$

Recall that $T\approx 7.416$. Equation (34) is one of the most important analytic results of our paper. Some numerical values of μ_k as function of κ^2 for $g^2/\lambda=1$ calculated with Eq. (34) are listed in the upper half of Table I below.

The analytic form (34) is in excellent agreement with the numerical results for this case plotted in the top panel of Fig. 5 as curve a. The maximum value of the characteristic exponent for $g^2/\lambda=1$ is $\mu_{\max}\approx 0.1470$ at $\kappa^2\approx 0.228$, in agreement with the numerical value for $\mu_{\max}$ of Fig. 6.

VII. CLOSED FORM SOLUTION FOR $g^2/\lambda=3$

The method of obtaining a closed-form analytic solution, $X_k(z)$, in the case $g^2=3\lambda$ is similar to that of the previous section. In the resonance zone with $g^2=3\lambda$, Eq. (26) gives

$$X_1(z)X_2(z)=M_2(z), \tag{36}$$

where now

$$M_2(z)=z^2-\frac{2}{3}\kappa^2 z-1+\frac{4}{9}\kappa^2. \tag{37}$$

The Wronskian of equation (25) is the same as in Eq. (30), but with a new constant, $C=C_2$. Therefore, the closed form solutions are the same as in Eq. (31), but with $M_2(z)$ in place of $M_1(z)$. Substituting this solution into Eq. (25), we find the constant C_2 in this case:

TABLE I. Numerical values of μ_k for various g^2/λ and κ^2.

g^2/λ	κ^2	μ_k
1	0.0	0.000
1	0.1	0.1238
1	0.2	0.1460
1	0.21	0.1466
1	0.22	0.1469
1	0.228	0.1470
1	0.23	0.1470
1	0.24	0.1468
1	0.25	0.1465
1	0.3	0.1411
1	0.4	0.1117
1	0.5	0.000
3	1.5	0.000
3	1.55	0.02981
3	1.60	0.03570
3	1.61	0.03595
3	1.615	0.03598
3	1.62	0.03594
3	1.625	0.03583
3	1.65	0.03427
3	1.70	0.02460
3	1.732	0.00

$$C_2=\sqrt{\frac{32}{81}\kappa^2\left(\kappa^4-\frac{9}{4}\right)(3-\kappa^4)}. \tag{38}$$

Therefore, in the case $g^2/\lambda=3$ for $\kappa^2>0$, there is also only a single instability band corresponding to

$$\frac{3}{2}<\kappa^2<\sqrt{3}, \tag{39}$$

in agreement with [8].

For illustration, we plot the resonant solution $X_k(x)$ in the top panel of Fig. 3. Notice that $X_k(x)$ oscillates twice within one inflaton oscillation. Using solution (31) with $M_2(z)$ and C_2, we can find μ_k in this case; see the Appendix for details.

The resulting characteristic exponent for $g^2/\lambda=3$ is

$$\mu_k=\frac{8\sqrt{2}}{9T}\sqrt{\kappa^2\left(\kappa^4-\frac{9}{4}\right)(3-\kappa^4)}J(\kappa), \tag{40}$$

where the auxiliary function $J(\kappa)$ is

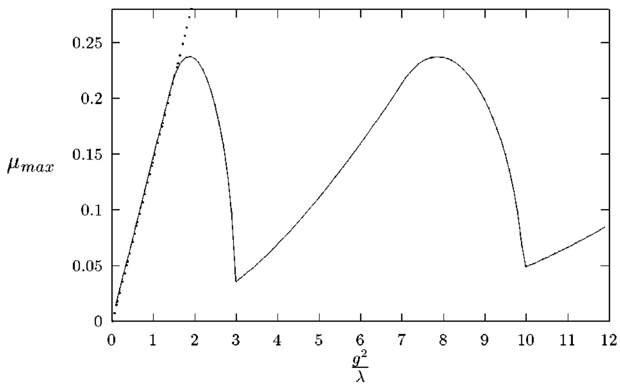

FIG. 6. The maximum value of the characteristic exponent $\mu_{\max}$ extracted from the stability-instability chart, Fig. 4, is plotted as a function of g^2/λ (solid curve). The function $\mu_{\max}(g^2/\lambda)$ is nonmonotonic. The universal upper limit of $\mu_{\max}$ is 0.2377. The local minima of the function are gradually increasing with g^2/λ, and asymptotically approach 0.2377. The dotted line is the prediction $\mu_{\max}\approx 0.1467(g^2/\lambda)$ for $g^2/\lambda \ll 1$, when the mode equation (18) is effectively reduced to the Mathieu equation (43).

$$J(\kappa)=\int_0^{\pi/2} d\theta \frac{\sin^{3/2}\theta}{1+(2/3)\kappa^2\sin\theta+[(4/9)\kappa^4-1]\sin^2\theta} \tag{41}$$

in this case. Formula (40) is another important result of our paper. Some numerical values of μ_k as a function of κ^2 for $g^2/\lambda=3$ calculated with Eq. (40) are listed in the lower half of Table I. The analytic form (40) is in agreement with the numerical results for this case plotted in the top panel of Fig. 5 as curve e. The maximum value of the characteristic exponent for $g^2/\lambda=3$ is $\mu_{\max}\approx 0.035\,98$ at $\kappa^2\approx 1.615$, in agreement with the numerical value for $\mu_{\max}$ of Fig. 6.

VIII. SOLUTION FOR $g^2/\lambda \ll 1$

In this section we investigate the equation for fluctuations (18) in the limiting case $g^2/\lambda \ll 1$. Let us recall that $f(x)$ is given by the series (15), and hence, $f^2(x)$ in Eq. (18) can be decomposed as

$$f^2(x)=F_0+F_1\cos\left(\frac{4\pi x}{T}\right)+F_2\cos\left(\frac{8\pi x}{T}\right)+\cdots, \tag{42}$$

where $F_0=0.4570$, $F_1=0.4973$, $F_2=0.042\,90$ and so on, but $\Sigma_{k=0}^{\infty}F_k=1$. One can seek $X_k(x)$ in the form of a harmonic series of terms $\cos(2n\pi x/T)$ with slowly varying coefficients. If g^2/λ is a small parameter, one can develop an iterative solution with respect to g^2/λ. It is easy to show that the leading contribution to $X_k(x)$ comes from the lower harmonic: $\cos(4\pi x/T)$. Keeping only this term, the equation for $X_k(x)$ can be reduced to the Mathieu equation

$$\frac{d^2X_k}{d\tau^2}+(A+2q\cos 2\tau)X_k=0, \tag{43}$$

where $\tau=2\pi x/T$, $A=(T\kappa/2\pi)^2$, and $q=(g^2/2\lambda)(T/2\pi)^2F_1$. Thus, our theory is effectively reduced to the Mathieu equation only in the limit $q\ll 1$, where it has instabilities in very narrow resonant bands around $\kappa^2=2\pi m/T$, $m=1,2,\ldots$. The results of the numerical investigation of the instability zones plotted in Fig. 4 indeed show that for $g^2/\lambda \ll 1$ the parametric resonance corresponds to that of the Mathieu equation.

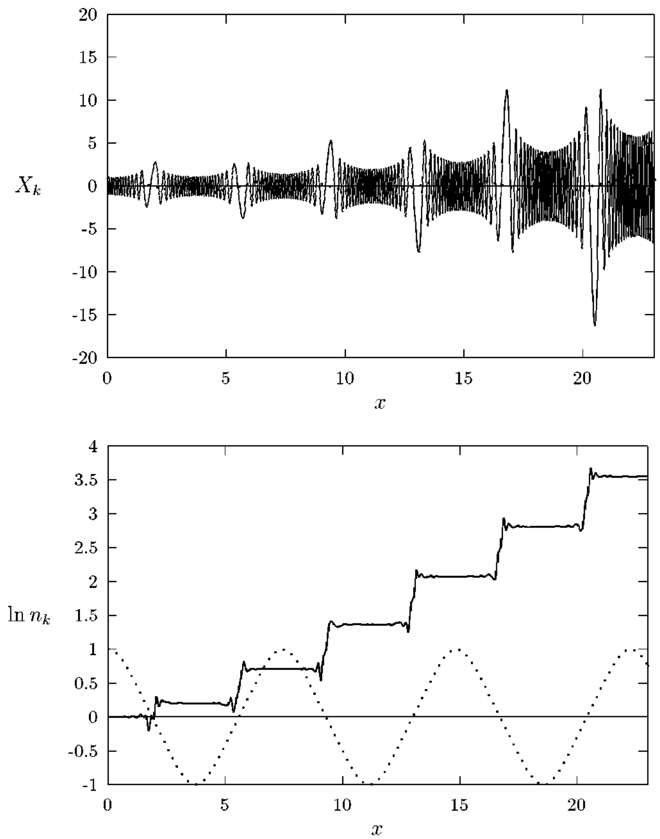

FIG. 7. The same as in Fig. 3 but for a large value $g^2/\lambda \gg 1$, here for the particular choice $g^2/\lambda=5050$ and $\kappa^2=29.0$. The upper plot shows the time dependence of the real part of the eigenmode $X_k(x)$, which demonstrates the adiabatic (semiclassical) behavior between zeros of the inflaton oscillations (dotted line), where the comoving occupation number n_k of created particles is constant (lower plot). The lower plot shows $\ln n_k$ as a function of time x. Particle creation occurs in a steplike manner only in the vicinity of the zeros of the inflaton field, where the adiabaticity is broken. The envelope of $\ln n_k$ is approximated by $2\mu_k x$. The characteristic exponent for this example is $\mu_k\approx 0.1$.

The exponentially growing solution of the Mathieu equation, $X_k(x)\propto e^{\mu_k x}$, has a maximum characteristic exponent (in the first zone)

$$\mu_{\max}=\frac{g^2}{4\lambda}\left(\frac{T}{2\pi}\right)^2 F_1\approx 0.1467\frac{g^2}{\lambda}. \tag{44}$$

In Fig. 6 we plot the maximum value of the characteristic exponent as a function of g^2/λ together with the prediction (44) for $\mu_{\max}$ from the Mathieu equation. As one can see from Fig. 6, Eq. (44) works extremely well even up to $g^2/\lambda\simeq 1$.

IX. ANALYTIC SOLUTION FOR $g^2/\lambda \gg 1$

In this section we consider the limiting case when the parameter g^2/λ is very large. In the upper panel of Fig. 7 we plot the time evolution of fluctuations $X_k(x)$ in this case. In the lower panel of Fig. 7 we plot the number of particles $n_k(x)$ in a given mode as a function of time x calculated from $X_k(x)$ with Eq. (22).

The basic observation is that, for $g^2/\lambda \gg 1$, the evolution of the modes $X_k(x)$ is adiabatic and the number of particles $n_k(x)$ is constant between the zeros of the background field. Changes in the number density of particles occur only near times $x=x_j$ when the amplitude of the inflaton field crosses zero, i.e., $\varphi(x_j)=0$. To describe the effect of a single kick at $x=x_j$, it is enough to consider the evolution of $X_k(x)$ in the interval when $\varphi^2(x)$ is small and can therefore be represented by its quadratic part $\propto (x-x_j)^2$. This process looks like wave propagation in a parabolic potential. Outside of these time intervals, $X_k(x)$ has a simple, semiclassical (adiabatic) form. We can combine the action of the subsequent parabolic potentials to find the net effect of particle creation. This method of successive parabolic scattering was formulated and applied to the broad parametric resonance for the quadratic inflaton potential in [2]. This method, as we see, can also be applied to the conformally invariant theory for $g^2/\lambda \gg 1$.

We expect that the semiclassical solution is valid everywhere but around x_j. Thus, prior to scattering at x_j, the mode function $X_k(x)$ has the adiabatic form

$$X_k^j(x) = \frac{\alpha_k^j}{\sqrt{2\omega_k}} \exp\left(-i\int_0^x \omega_k dx\right) + \frac{\beta_k^j}{\sqrt{2\omega_k}} \exp\left(+i\int_0^x \omega_k dx\right), \tag{45}$$

where the coefficients α_k^j and β_k^j are constant for $x_{j-1}<x<x_j$, $\alpha_k^0=1$, $\beta_k^0=0$, and normalization yields $|\alpha_k|^2-|\beta_k|^2=1$. After scattering when $x=x_j$, $X_k(x)$ in the interval $x_j<x<x_{j+1}$ again has the adiabatic form of Eq. (45) but with new constant coefficients, α_k^{j+1} and β_k^{j+1}.

The form is essentially the asymptotic expression of the incoming waves (for $x<x_j$) and similarly for the outgoing waves (for $x>x_j$) scattered from a parabolic potential $(x-x_j)^2$ at the moment x_j. Therefore, the outgoing amplitudes, α_k^{j+1} and β_k^{j+1}, can be expressed in terms of the incoming amplitudes, α_k^j and β_k^j, with the help of the reflection and transmission amplitudes for scattering at a parabolic potential [2]. For this we need the mode equation around a single parabolic potential at $x=x_j$. In the vicinity of x_j, $cn(x,1/\sqrt{2}) \approx (1/\sqrt{2})(x-x_j)$. Then Eq. (18) around x_j is reduced to the simple equation

$$\frac{d^2X_k}{dx^2} + \left(\kappa^2 + \frac{g^2}{2\lambda}(x-x_j)^2\right) X_k = 0. \tag{46}$$

The mapping of α_k^j, β_k^j into α_k^{j+1}, β_k^{j+1} in terms of parameters in Eq. (46) reads

$$\begin{pmatrix} \alpha_k^{j+1} \\ \beta_k^{j+1} \end{pmatrix} = \begin{pmatrix} \sqrt{1+e^{-\pi\epsilon^2}}\, e^{i\zeta_k} & i e^{-(\pi/2)\,\epsilon^2 + 2i\theta_k^j} \\ -i e^{-(\pi/2)\,\epsilon^2 - 2i\theta_k^j} & \sqrt{1+e^{-\pi\epsilon^2}}\, e^{-i\zeta_k} \end{pmatrix} \begin{pmatrix} \alpha_k^j \\ \beta_k^j \end{pmatrix}, \tag{47}$$

where $\zeta_k = \arg\Gamma[(1+i\epsilon^2)/2] + (\epsilon^2/2)[1+\ln(2/\epsilon^2)]$, and $\epsilon^2 = \sqrt{2\lambda/g^2}\,\kappa^2 = k^2/\sqrt{\lambda/2}\,\tilde{\varphi}^2 g$. The phase accumulated by the moment x_j is $\theta_k^j = \int_0^{x_j} dx\, \omega_k(x) = j\,\theta_k$, where $\theta_k = 2\int_0^{T/4} dx \sqrt{\kappa^2 + (g^2/\lambda) f^2(x)}$ is the phase accumulating within half of a period of the inflaton oscillation.

In the regime when a large number of particles have been created, $n_k^j = |\beta_k^j|^2 \gg 1$, we have $|\alpha_k^j| \approx |\beta_k^j|$, so α_k^j and β_k^j are distinguished by their phases only. In this case there is a simple solution of the matrix equation (47):

$$\alpha_k^j = \frac{1}{\sqrt{2}} \exp\left[\left(\mu_k \frac{T}{2} + i\,\theta_k\right)\cdot j\right], \quad \beta_k^j = \frac{1}{\sqrt{2}} e^{i\vartheta} \exp\left[\left(\mu_k \frac{T}{2} - i\,\theta_k\right)\cdot j\right], \tag{48}$$

where ϑ is a constant phase and μ_k is the characteristic exponent. $T \approx 7.416$ is the period of oscillations of the inflaton field in the variable x, so the number of particles grows as $e^{2\mu_k x}$. Another solution is similar to Eq. (48) but with the substitution $\theta_k \to \theta_k + \pi$.

Substituting solution (48) into Eq. (47), we get an equation for the parameters μ_k and θ_k:

$$e^{\mu_k (T/2)} = |\cos(\theta_k - \zeta_k)| \sqrt{1+e^{-\pi\epsilon^2}} + \sqrt{(1+e^{-\pi\epsilon^2})\cos^2(\theta_k - \zeta_k) - 1}. \tag{49}$$

In the instability zones, the parameter μ_k of Eq. (49) should be real. From this we obtain the condition

$$|\tan(\theta_k - \zeta_k)| \leq e^{-(\pi/2)\epsilon^2}. \tag{50}$$

for the momentum k to be in a resonance band.

To further analyze the conditions for the strength (50) and widths (49) of the resonance, one should calculate the phase $\theta_k - \zeta_k$. For $g^2/\lambda \gg 1$ we have

$$\theta_k - \zeta_k = 2\int_0^{T/4} dx \sqrt{\kappa^2 + \frac{g^2}{\lambda} f^2(x)} - \arg\Gamma\left(\frac{1+i\epsilon^2}{2}\right) - \frac{\epsilon^2}{2}\left(1+\ln\frac{2}{\epsilon^2}\right) \approx \pi\sqrt{\frac{g^2}{2\lambda}} + \kappa^2 \sqrt{\frac{\lambda}{8g^2}} \ln\frac{g^2}{\lambda}. \tag{51}$$

Using Eqs. (51), (50), and (49), we find the characteristics of the resonance in the regime $g^2/\lambda \gg 1$. From Eq. (50) it follows that the resonance is efficient for $\epsilon^2 \leqslant \pi^{-1}$, i.e., for

$$\kappa^2 \leqslant \sqrt{\frac{g^2}{2\pi^2\lambda}}. \tag{52}$$

Equation (50) transparently shows that, for a given g^2/λ, there will be a sequence of stability-instability bands as a function of κ. The width of an instability band, where the resonance occurs, is $\Delta\kappa^2 \simeq \sqrt{g^2/2\lambda}$. Let the integer part of the large number $\sqrt{g^2/2\lambda}$ be l. From Eq. (51) it follows that if we vary κ^2 within the range $2\pi^2\sqrt{g^2/2\pi^2\lambda}[\ln(g^2/\lambda)]^{-1}$, then within this interval of κ^2 the phase $\theta_k - \zeta_k$ reaches either $l\pi$ or $(l+1)\pi$. Then within this resonance band we get the maximum value $\mu_{\max}$ defined by Eq. (49) with $|\cos(\theta_k-\zeta_k)|=1$:

$$e^{(T/2)\mu_{\max}} = \sqrt{1+e^{-\pi\epsilon^2}} + e^{-\pi\epsilon^2/2}. \tag{53}$$

The characteristic exponent $\mu_{\max}$ is a nonmonotonic function of g^2/λ. If the value of the parameter g^2/λ is exactly equal to $2l^2$ where l is an integer, then the strongest resonance occurs at $\kappa^2=0$,[2] and from Eq. (53) we get

$$\mu_{\max} = \frac{2}{T}\ln(1+\sqrt{2}) \approx 0.2377. \tag{54}$$

This is actually a general result for the upper limit of $\mu_{\max}$ for an arbitrary g^2/λ, see Fig. 6. One may compare it with a similar result for $\mu_{\max}$ for the harmonic oscillations in the case of the Mathieu equation, where $T=2\pi$ and $\mu_{\max} = (1/\pi)\ln(1+\sqrt{2}) \approx 0.28$ [1]. If g^2/λ is not exactly equal to $2l^2$, then $\mu_{\max}$ occurs at a nonzero κ^2 and is smaller than 0.2377. It is interesting that in the formal limit $g^2/\lambda \to \infty$ the function $\mu_{\max}(g^2/\lambda)$ asymptotically approaches the value 0.2377 for arbitrary g^2/λ. To see this, we have to check that a variation of $\kappa^2 \sim 2\pi^2\sqrt{g^2/2\pi^2\lambda}[\ln(g^2/\lambda)]^{-1}$ is compatible with the condition for an efficient resonance, $\epsilon^2 \leqslant \pi^{-1}$. In Fig. 6 we see that the minimal value of μ as a function of g^2/λ very slowly increases towards 0.2377. Therefore, although $\mu_{\max}$ is not a monotonic function of g^2/λ, for $g^2/\lambda \gg 1$ the resonance is stronger both in terms of the characteristic exponent $\mu_{\max}$ and the width κ^2.

X. BACK REACTION OF CREATED PARTICLES

Thus far we have considered the parametric resonance in the conformally invariant theory (1) in an expanding universe neglecting the back reaction of the amplified fluctuations of the fields ϕ and χ. In the next two sections we will study the effects related to the back reaction.

In the theory $\frac{1}{4}\lambda\phi^4 + \frac{1}{2}g^2\phi^2\chi^2$, the equation of motion for the inflaton field $\phi(t)$ looks as follows:

$$\ddot{\phi} + 3H\dot{\phi} + \lambda\phi^3 + 3\lambda\langle\phi^2\rangle\phi + g^2\langle\chi^2\rangle\phi = 0. \tag{55}$$

The two additional terms are due to the one-loop Hartree diagrams; $\langle\phi^2\rangle$ and $\langle\chi^2\rangle$ stand for the quantum fluctuations of the fields ϕ and χ, respectively. There are also higher-loop corrections such as $2g^2\langle\phi\chi^2\rangle$ and $6\lambda\langle\phi^3\rangle$, which are not necessarily negligible at the end of preheating when we may expect $n_\chi \sim 1/g^2$ and $n_\phi \sim 1/\lambda$ (or even higher, if one takes into account rescattering). Here we will work in the one-loop approximation.

If again we use the conformal transformation $\eta = \int(dt/a)$, $\chi_k = X_k/a$ and similarly $\phi_k = \varphi_k/a$, then the equation for the background field $\varphi(\eta)$ is

$$\varphi'' + \lambda\varphi^3 + 3\lambda\langle\varphi^2\rangle\varphi + g^2\langle X^2\rangle\varphi = 0, \tag{56}$$

with the comoving vacuum expectation values for X and φ correspondingly

$$\langle X^2\rangle = \frac{1}{(2\pi)^3}\int d^3k|X_k|^2, \quad \langle\varphi^2\rangle = \frac{1}{(2\pi)^3}\int d^3k|\varphi_k|^2. \tag{57}$$

The integral of Eq. (56) coincides with the energy density

$$\rho_{\text{tot}} = \frac{1}{2}\varphi'^2 + \frac{\lambda}{4}\varphi^4 + \rho_\varphi + \rho_X. \tag{58}$$

The first two terms describe the energy of the classical field φ, ρ_φ, and ρ_X correspond to the energy density of φ particles and X particles, respectively:

$$\rho_\varphi = \frac{1}{(2\pi)^3}\int d^3k\sqrt{k^2+3\lambda\varphi^2}\,n_k^\varphi, \tag{59}$$

$$\rho_X = \frac{1}{(2\pi)^3}\int d^3k\sqrt{k^2+g^2\varphi^2}\,n_k^X. \tag{60}$$

Here n_k^φ and n_k^X correspond to the occupation numbers of the φ particles and X particles. It is easy to show that $\rho'_\chi = g^2\langle\chi^2\rangle\varphi\varphi'$ and $\rho'_\varphi = 3\lambda\langle\varphi^2\rangle\varphi\varphi'$, and therefore Eq. (58) is an integral of Eq. (56).

To close the set of self-consistent equations we need the equations for the modes $\varphi_k(x)$ and X_k:

$$\varphi''_k(\eta) + [k^2 + \Pi_\varphi + 3\lambda\varphi^2(\eta)]\varphi_k = 0, \tag{61}$$

$$X''_k(\eta) + [k^2 + \Pi_X + g^2\varphi^2(\eta)]X_k = 0. \tag{62}$$

The polarization operator Π_φ consists of $\Pi^1_\varphi = 3\lambda\langle\varphi^2\rangle + g^2\langle X^2\rangle$ and the nonlocal term Π^2_φ which emerges in the one-loop approximation beyond the Hartree diagram, see Fig. 8.

The calculation of the polarization operator Π^2_φ in the regime of parametric resonance is rather complicated. Estimates of Π^2_φ performed in [2] indicate that it can be of the same order of magnitude as the standard Hartree polarization operator Π^1_φ. The polarization operator Π^2_φ was not taken into account in the previous treatment of the self-consistent equations for the eigenmodes in the $1/N$ approximation [6,7], but in fact it may survive in the limit $N \to \infty$ [2]. This may imply that in the context of the theory of preheating the standard $1/N$ approximation breaks down.

[2]It is easy to see from Eq. (51) that the mode $\kappa^2=0$ is within the resonance band if $2l^2 - l < g^2/\lambda < 2l^2 + l$.

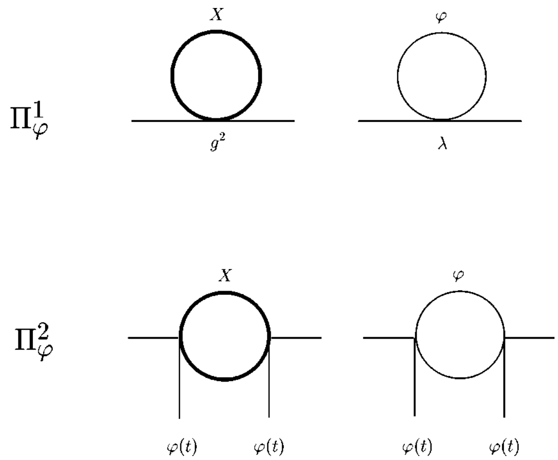

FIG. 8. The diagrams for the polarization operator of the field φ_k. Thin and thick lines represent the fields ϕ and χ, respectively. Vertical lines correspond to the oscillating background field $\varphi(t)$. Π^1_φ corresponds to the Hartree approximation which takes into account the contribution of $\langle X^2\rangle$ and $\langle\varphi^2\rangle$. The contributions of Π^1_φ and Π^2_φ to the effective mass of φ particles can be comparable to each other.

Similarly, the polarization operator Π_X is equal to $g^2\langle\varphi^2\rangle$, plus an additional nonlocal term Π^2_X. We expect that $\Pi_\varphi \geqslant 0$, $\Pi_X \geqslant 0$, as suggested by the Hartree approximation.

A complete calculation of the polarization operators Π_φ and Π_X is outside the scope of this paper. Fortunately, as we will see in the next section, one need not really know exact expressions for Π_φ and Π_X in order to make an estimate of the density of produced particles at the time when the feedback of the amplified fluctuations terminates the parametric resonance.

XI. DYNAMICAL RESTRUCTURING OF THE RESONANCE

In this paper we found that the structure of the parametric resonance in terms of its strength and width strongly depends on the parameters of the model. For example, the parametric resonance in the simplest conformally invariant theory $\frac{1}{4}\lambda\phi^4+\frac{1}{2}g^2\phi^2\chi^2$ is very different from that in the theory $\frac{1}{2}m_\phi\phi^2+\frac{1}{2}g^2\phi^2\chi^2$ [2]. In the simplest conformally invariant theories which we consider in this paper the structure of the resonance is determined by the combination g^2/λ.

How does the resonance develop if the back reaction of the accumulating fluctuations is taken into account? The answer to this question also strongly depends on the parameter g^2/λ.

For illustration we consider the model of the self-interacting inflaton field $\frac{1}{4}\lambda\phi^4$, no χ field is involved. In this case we shall take $g^2=0$ in all the equations (56), (61), (62). As we already mentioned, if one neglects the back reaction, the equations describing the resonance for the modes φ_k in this theory coincide with the equations for the modes χ_k in the theory with $g^2=3\lambda$. Thus, we can use the results of the investigation of the theory with $g^2=3\lambda$ obtained in Sec. VII for our analysis.

Historically, the model $\frac{1}{4}\lambda\phi^4$ was one of the first models illustrating the general idea of preheating. The investigation of the stability-instability chart for the Lamé equation has shown that this model in a certain sense is the least favorable for the development of the resonance: it has only one resonance band, and the characteristic exponent μ for this theory is anomalously small, see Figs. 4 and 6. Originally it was expected that preheating in this model would rapidly transfer about half of the energy of the oscillating scalar field to the ϕ particles, after which the decay of the field ϕ would continue at a much slower pace. However, the results of computer simulations of preheating in this theory indicated that the stage of efficient preheating ends as soon as the fluctuations of produced particles $\langle\varphi^2\rangle$ grow to $0.05\tilde{\phi}^2$ [5]. The interpretation of this result, however, was not quite clear. It was conjectured that the resonance terminates because of rescattering of the ϕ particles. It was not clear also whether the decay of the field ϕ continues at a slower pace until this field completely decays, or its decay eventually shuts down.

A complete investigation of this issue is rather difficult. First of all, the theory of rescattering is not fully developed: various approximations often break down near the end of preheating when the occupation numbers of particles are anomalously large ($n_k\sim\lambda^{-1}$) [1,2]. Even in the Hartree approximation (or in the $1/N$-approximation) an investigation is very complicated [7,8] because it is very difficult to work with the solutions of equations for the growing modes in terms of the transcendental Jacobi functions. It may be easier to work with the solutions obtained in Secs. VI and VII. We will not perform a full investigation of this issue here because, as we argued in the previous section, one may need to calculate the polarization operator beyond the Hartree approximation, see [2]. Instead, we will make some simple estimates which will allow us to elucidate the mechanism which terminates the resonance in the theory $\frac{1}{4}\lambda\phi^4$.

As we will see, the main reason for the termination of the resonance in the theory $\frac{1}{4}\lambda\phi^4$ is the restructuring of the resonance band due to the back reaction of created particles. This process occurs at $\langle\varphi^2\rangle\ll\varphi^2$ because the resonance band is very narrow. In the beginning of preheating in the theory $\frac{1}{4}\lambda\phi^4$ the instability band is given by the condition $1.5\lambda\tilde{\varphi}_0^2<k^2<1.73\lambda\tilde{\varphi}_0^2$, where $\tilde{\varphi}_0$ is the initial amplitude of the oscillations of the field φ (39). It is sufficient to shift the position of the resonance band in momentum space by few percent, and the leading resonant modes χ_k which have been growing since the start of the parametric resonance will not grow anymore. This will effectively shut down the resonance.

There are two different effects which lead to a restructuring of the resonance band, and these effects act in opposite directions. First of all, particle production reduces the energy of the scalar field, and therefore reduces the amplitude of its oscillations. This effect tends to reduce the frequency of the oscillations and to move the resonance band towards smaller k. On the other hand, the effective mass of the field φ grows due to its interaction with the ϕ particles. This effect increases the frequency of oscillations and tends to shift the resonance band towards larger k. We will consider here both of these effects.

To investigate the decrease of the amplitude of the oscillations due to particle production, one should compare the total energy of the system before and after the appearance of $\langle\phi^2\rangle$:

$$\frac{\lambda}{4}\tilde{\varphi}_0^4 \approx \frac{\lambda}{4}\tilde{\varphi}^4(\eta) + \frac{1}{(2\pi)^3}\int d^3k\sqrt{k^2+3\lambda\tilde{\varphi}^2}n_k. \quad (63)$$

Here we calculate the energy density at the moment when $\varphi'=0$, and the oscillating field is equal to its amplitude $\tilde{\varphi}(\eta)$. This amplitude is smaller than $\tilde{\varphi}_0$ due to the transfer of energy to the created particles.

The resonance is most efficient in a small vicinity of $k^2 \approx 1.6\lambda\tilde{\varphi}^2$. Therefore, the leading contribution to ρ_ϕ is given by integration near $k^2=1.6\lambda\tilde{\varphi}^2$:

$$\rho_\phi \approx \frac{1}{(2\pi)^3}\int d^3k\sqrt{4.6\lambda\tilde{\varphi}^2}n_k = \sqrt{4.6\lambda}\tilde{\varphi}n_\phi. \quad (64)$$

Equations (63) and (64) give

$$\tilde{\varphi}(\eta) \approx \tilde{\varphi}_0 - \frac{\sqrt{4.6\lambda}n_\phi}{\tilde{\varphi}^2}. \quad (65)$$

Thus, the creation of φ particles diminishes the frequency of oscillations, because the frequency of oscillations of the field φ in the theory $\lambda\phi^4$ is proportional to its amplitude. To evaluate the significance of this effect one may express it in terms of $\langle\phi^2\rangle$ calculated at $\varphi(\eta)=\tilde{\varphi}$:

$$\langle\phi^2\rangle \approx \frac{1}{(2\pi)^3}\int \frac{d^3k n_k}{\sqrt{k^2+3\lambda\tilde{\varphi}^2}} \approx \frac{n_\phi}{\sqrt{4.6\lambda}\tilde{\varphi}}. \quad (66)$$

From the last two equations one obtains

$$\frac{\tilde{\varphi}^2(\eta)}{\tilde{\varphi}_0^2} \approx 1 - 9.2\frac{\langle\phi^2\rangle}{\tilde{\varphi}_0^2}, \quad (67)$$

which leads to a proportional shift of the resonance band towards smaller k^2. This indicates that even a very small amount of fluctuations $\langle\phi^2\rangle \sim 10^{-2}\tilde{\varphi}_0^2$ may shift the resonance band away from its original position, which may terminate the resonance for the leading modes φ_k.

This effect is partially compensated by the growth of the effective mass of the field φ. We will analyze this effect in the Hartree approximation, in which the field φ acquires the effective mass squared $\Pi_\phi = 3\lambda\langle\varphi^2\rangle$. One may relate $\Pi_\phi = 3\lambda\langle\varphi^2\rangle$ to the number density of ϕ-particles in the following way:

$$\Pi_\phi \approx \frac{3\lambda}{(2\pi)^3}\int \frac{d^3k n_k(\eta)}{\sqrt{k^2+3\lambda\varphi^2(\eta)}} \approx \frac{3\lambda}{(2\pi)^3}\int \frac{d^3k n_k}{\sqrt{1.6\tilde{\varphi}^2+3\lambda\varphi^2}} = \frac{3\lambda n_\phi(\eta)}{\sqrt{1.6\tilde{\varphi}^2+3\lambda\varphi^2(\eta)}}. \quad (68)$$

Note, that this quantity is time dependent. It oscillates; its magnitude changes considerably several times within a single oscillation of the inflaton field, and it also grows exponentially during the resonance. The number density of φ particles also oscillates and grows exponentially, but typically its oscillations are less wild than the oscillations of $\langle\varphi^2\rangle$. In the first approximation, we will neglect the oscillations of $n_\phi(\eta)$. Also, we are trying to find the time when the resonance terminates, and at that time the average number density of particles n_ϕ becomes nearly constant. It is still difficult to find an analytic solution for φ_k with the time-dependent polarization operator (68), but one can easily find the solution numerically.

The result of the combined investigation of the two effects discussed above shows that the resonance on the leading modes φ_k effectively terminates as soon as $\langle\phi^2\rangle$ grows up to

$$\langle\phi^2\rangle \approx 0.05\tilde{\phi}^2. \quad (69)$$

Note that even after this moment the resonance may continue for a while for the new modes which can be amplified in the restructured resonance band. However, this process is much less efficient. Thus, in the pure $\lambda\phi^4$ theory the rapid development of the resonance ends when the dispersion of amplified fluctuations is about 20% of the amplitude of the inflaton field, which corresponds to only 0.2% of the total energy. This result is based on rather rough estimates neglecting rescattering. It is interesting, however, that it is in complete agreement with the result of the lattice simulation of the parametric resonance in the theory $\lambda\phi^4$ [5].

We should emphasize that there are several specific reasons why the resonance in the particular case of the theory $\lambda\phi^4$ is relatively inefficient. First of all, the resonance band in this theory is narrow and the characteristic exponent μ is very small. This is no longer the case when one considers, for example, the theory describing a χ field with $g^2=\lambda$ or with $g^2=2\lambda$. In these theories the characteristic exponent is much greater, the resonance band is rather broad, and it begins at $k=0$. As a result, it is much more difficult to shut down the resonance in such theories.

In the theories with a massive inflaton field there is an additional effect which makes the resonance more stable. Broad parametric resonance in such theories is stochastic, which makes it more difficult to shut down [2]. Now we are going to study what happens to the resonance in the conformally invariant theories if this invariance is broken by a small mass term. As we will see, stochastic resonance may appear in such theories as well.

XII. PREHEATING IN THE THEORY OF A MASSIVE SELF-INTERACTING INFLATON FIELD

In our previous paper [2] we investigated parametric resonance in the theory $(m^2/2)\phi^2+(g^2/2)\phi^2\chi^2$. We have found that reheating can be efficient in this theory only if $g\Phi \gg m$, where Φ is the amplitude of oscillations of the inflaton field. This amplitude is extremely large immediately after inflation, $\phi \sim 10^{-1}M_p$, and later it decreases as

$$\Phi \sim \frac{M_p}{3mt}. \quad (70)$$

Due to this decrease, the ratio $g\phi/m$ rapidly changes. As a result, the broad parametric resonance regime in this theory is a stochastic process, which we called *stochastic resonance*.

Here we studied the theory $(\lambda/4)\phi^4+(g^2/2)\phi^2\chi^2$ for various relations between the coupling constants g^2 and λ. In this theory the amplitude of the field ϕ also decreases in an expanding universe, but it does not make the resonance stochastic because all parameters of the resonance scale in the same way as Φ due to the conformal invariance. One may wonder, what is the relation between these two theories? Indeed, neither of these two theories is completely general. In the theory of the massive scalar field one may expect terms $\sim(\lambda/4)\phi^4$ to appear because of radiative corrections. On the other hand, in many realistic theories the effective potential is quadratic with respect to ϕ near the minimum of the effective potential.

To address this question, let us study the theory $(m^2/2)\phi^2+(\lambda/4)\phi^4+(g^2/2)\phi^2\chi^2$. One may expect that for $\phi\gg m/\sqrt{\lambda}$ parametric resonance in this theory occurs in the same way as in the model $(\lambda/4)\phi^4+(g^2/2)\phi^2\chi^2$, whereas for $\phi\ll m/\sqrt{\lambda}$ the resonance develops as in the theory $(m^2/2)\phi^2+(g^2/2)\phi^2\chi^2$. Let us check whether this is really the case, ignoring for simplicity the effects of back reaction of created particles, which is always possible in the beginning of the resonance regime.

First of all, one should remember that at the beginning of the stage of oscillations in this theory one has $\Phi\sim10^{-1}M_p$. Therefore there are two basic possibilities. If $m/\sqrt{\lambda}\gg10^{-1}M_p$, then the term $(\lambda/4)\phi^4$ never plays any role in determining the frequency of oscillations of the field ϕ. Also, in this regime the particles ϕ are not produced by parametric resonance, because the condition $\sqrt{\lambda}\phi>m$ (analogous to the condition $g\phi>m$ for the production of χ particles [2]) is violated. In such a case χ particles can be produced if $10^{-1}gM_p\gg m$. The theory of this process is described in [2]; we do not have anything new to add here.

Another possibility, which we are going to study here in more detail, is that $m/\sqrt{\lambda}\ll10^{-1}M_p$. Then in the beginning the mass term $(m^2/2)\phi^2$ does not affect the frequency of the oscillating scalar field ϕ. Therefore, one could expect that as the amplitude Φ decreases from $10^{-1}M_p$ to $m/\sqrt{\lambda}$, the theory of parametric resonance coincides with the one described in this paper.

However, for large g^2/λ the situation is more complicated. Even though the mass term for $10^{-1}M_p>\Phi\gg m/\sqrt{\lambda}$ does not affect the frequency of oscillations, it may affect the nature of the broad parametric resonance by inducing an additional rotation of the phase θ of the modes χ_k (see Sec. IX).

The reason why the broad resonance in the theory $(m^2/2)\phi^2+(g^2/2)\phi^2\chi^2$ was stochastic can be explained as follows. The χ particles are produced when the field $\phi(t)$ comes close to the point $\phi=0$, which happens once during each time period $\Delta t=\pi/m$. During this time the phase of each mode χ_k grows approximately by $g\Phi(t)\pi m^{-1}$. During the next half of a period of an oscillation it changes by $g\Phi(t+\pi/m)\pi m^{-1}\approx g\Phi(t)\pi m^{-1}+g\dot{\Phi}(t)\pi^2m^{-2}$. This destroys the phase coherence required for the ordinary resonance and makes the resonance stochastic if $|g\dot{\Phi}(t)\pi^2m^{-2}|\gtrsim1$.

The condition for the stochastic resonance in the theory $(m^2/2)\phi^2+(g^2/2)\phi^2\chi^2$ can be obtained from Eq. (70):

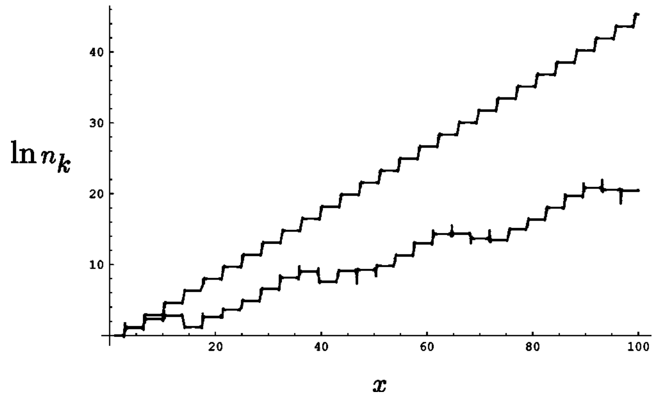

FIG. 9. Development of the resonance in the theory $(m^2/2)\phi^2+(\lambda/4)\phi^4+(g^2/2)\phi^2\chi^2$ for $g^2/\lambda=5200$. The upper curve corresponds to the massless theory, the lower curve describes stochastic resonance with a theory with a mass m which is chosen to be much smaller than $\sqrt{\lambda}\phi$ during the whole period of calculations. Nevertheless, the presence of a small mass term completely changes the development of the resonance.

$$\Phi\gtrsim\sqrt{\frac{mM_p}{g}}. \tag{71}$$

In particular, for $\Phi=m/\sqrt{\lambda}$ it gives $g/\sqrt{\lambda}\gtrsim\sqrt{\lambda}M_p/m$. Note that by our assumption $\sqrt{\lambda}M_p/m\gg1$.

The generalization of this result for the theory $(m^2/2)\phi^2+(\lambda/4)\phi^4+(g^2/2)\phi^2\chi^2$ is straightforward, but the result is somewhat unexpected. As a rough estimate of the time Δt one can take $\pi(2\lambda\Phi^2+m^2)^{-1/2}=\pi[2\lambda\varphi^2a^{-2}(t)+m^2]^{-1/2}$, where $\varphi\equiv\Phi a^{-1}(t)$ is the time-independent amplitude. The phase shift during this time is given by $g\varphi\pi[2\lambda\varphi^2+m^2a^2(t)]^{-1/2}$. Thus, for $m=0$ this quantity is time independent, and one can have a regular stable resonance. In the limit $\Phi\gg m/\sqrt{\lambda}$ one can represent the phase shift as $(g\pi/\sqrt{2\lambda})[1-m^2a^2(t)/4\lambda\varphi^2]$. The change in this shift during one oscillation is $g\pi^2m^2H/4\lambda^2\Phi^3$, where $H=\dot{a}/a=\sqrt{2\pi\lambda}\Phi^2/\sqrt{3}M_p$. This gives the following condition for stochastic resonance:

$$\Phi\lesssim\frac{g}{\sqrt{\lambda}}\frac{\pi^2m^2}{3\lambda M_p}. \tag{72}$$

Again, for $\Phi=m/\sqrt{\lambda}$ it gives $g/\sqrt{\lambda}\gtrsim\sqrt{\lambda}M_p/m$.

This conclusion is illustrated by Fig. 9, where we show the development of the resonance both for the massless theory with $g^2/\lambda\sim5200$, and for the theory with a small mass m. As we see, in the purely massless theory the logarithm of the number density n_k for the leading growing mode increases linearly in time x, whereas in the presence of a mass m, which we took to be much smaller than $\sqrt{\lambda}\phi$ during the whole process, the resonance becomes stochastic.

In fact, the development of the resonance is rather complicated even for smaller g^2/λ. The resonance for a massive field with $m\ll\sqrt{\lambda}\phi$ in this case is not stochastic, but has a feature of intermittancy: it may consist of stages of regular resonance separated by the stages without any resonance, see Fig. 10.

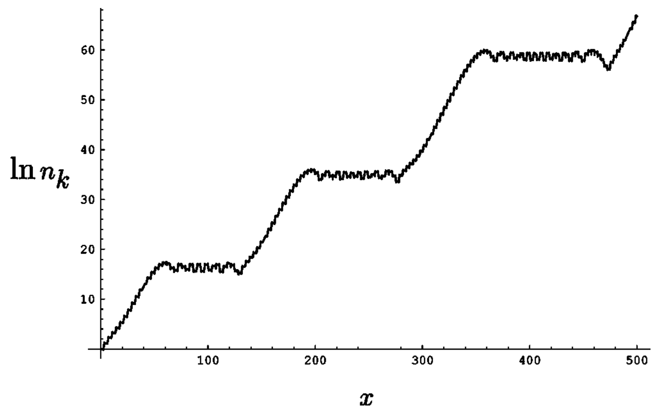

FIG. 10. Development of the resonance in the theory $(m^2/2)\phi^2+(\lambda/4)\phi^4+(g^2/2)\phi^2\chi^2$ with $m^2 \ll \lambda\phi^2$ for $g^2/\lambda=240$. In this particular case the resonance is not stochastic. As time x grows, the relative contribution of the mass term to the equation describing the resonance also grows. This shifts the mode from one instability band to another.

Thus we see that the presence of the mass term $(m^2/2)\phi^2$ can modify the nature of the resonance even if this term is much smaller than $(\lambda/4)\phi^4$. This is a rather unexpected conclusion, which is an additional manifestation of the nonperturbative nature of preheating. This subject deserves separate investigation.

Different regimes of parametric resonance in the theory $(m^2/2)\phi^2+(\lambda/4)\phi^4+(g^2/2)\phi^2\chi^2$ are shown in Fig. 11. We suppose that immediately after inflation the amplitude Φ of the oscillating inflaton field is greater than $m/\sqrt{\lambda}$. If $g/\sqrt{\lambda} \lesssim \sqrt{\lambda}M_p/m$, the χ particles are produced in the regular stable resonance regime until the amplitude $\Phi(t)$ decreases to $m/\sqrt{\lambda}$, after which the resonance occurs as in the theory $(m^2/2)\phi^2+(g^2/2)\phi^2\chi^2$ [2]. The resonance never becomes stochastic.

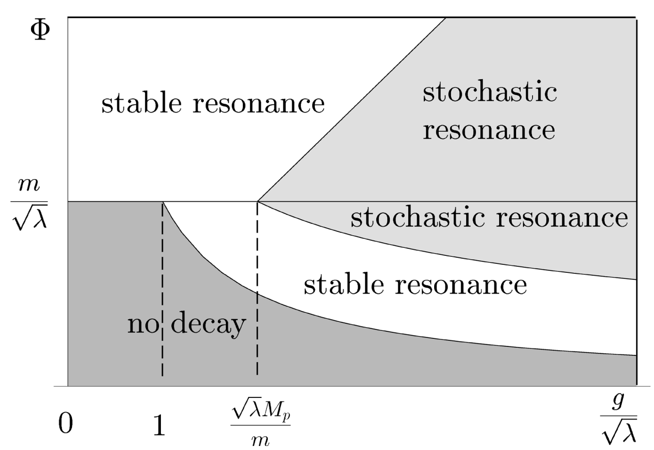

FIG. 11. Schematic representation of different regimes which are possible in the theory $(m^2/2)\phi^2+(\lambda/4)\phi^4+(g^2/2)\phi^2\chi^2$ for $m/\sqrt{\lambda} \ll 10^{-1}M_p$ and for various relations between g^2 and λ in an expanding universe. The theory developed in this paper describes the resonance in the white area above the line $\Phi=m/\sqrt{\lambda}$. The theory of preheating for $\Phi<m/\sqrt{\lambda}$ is given in [2]. A complete decay of the inflaton is possible only if additional interactions are present in the theory which allow one inflaton particle to decay to several other particles, for example, an interaction with fermions $\bar{\psi}\psi\phi$.

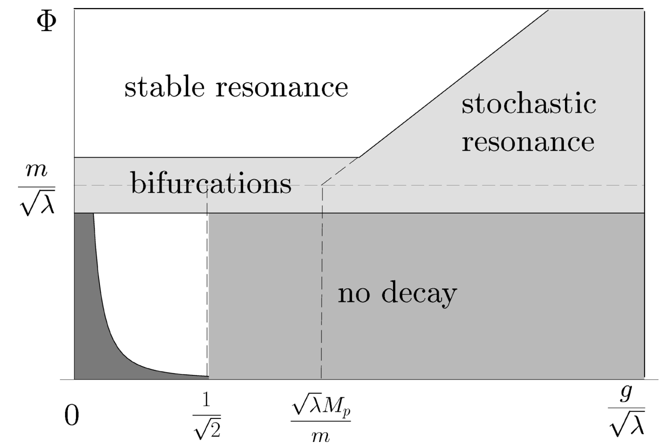

FIG. 12. Schematic representation of different regimes which are possible in the theory $-(m^2/2)\phi^2+(\lambda/4)\phi^4+(g^2/2)\phi^2\chi^2$. White regions correspond to the regime of a regular stable resonance, a small dark region in the left corner near the origin corresponds to the perturbative decay $\phi\to\chi\chi$. Unless additional interactions are included (see the previous figure), a complete decay of the inflaton field is possible only in this small area.

If $g/\sqrt{\lambda} \gtrsim \sqrt{\lambda}M_p/m$, the resonance originally develops as in the conformally invariant theory $(\lambda/4)\phi^4+(g^2/2)\phi^2\chi^2$, but with a decrease of $\Phi(t)$ the resonance becomes stochastic. Again, for $\Phi(t) \lesssim m/\sqrt{\lambda}$ the resonance occurs as in the theory $(m^2/2)\phi^2+(g^2/2)\phi^2\chi^2$. In all cases the resonance eventually disappears when the field $\Phi(t)$ becomes sufficiently small. As we already mentioned in [1,2], reheating in this class of models can be complete only if there is a symmetry breaking in the theory, i.e., $m^2<0$, or if one adds interaction of the field ϕ with fermions. In both cases the last stages of reheating are described by perturbation theory [17,18].

Adding fermions does not alter the description of the stage of parametric resonance. Meanwhile the change of sign of m^2 does lead to substantial changes in the theory of preheating, see Fig. 12. We will investigate preheating in the theory $-(m^2/2)\phi^2+(\lambda/4)\phi^4+(g^2/2)\phi^2\chi^2$ in a separate publication [19]. Here we will briefly describe the structure of the resonance for various g^2 and λ neglecting effects of back reaction. This will give us a more general perspective on the theory of reheating.

First of all, at $\Phi \gg m/\sqrt{\lambda}$ the field ϕ oscillates in the same way as in the massless theory $(\lambda/4)\phi^4+(g^2/2)\phi^2\chi^2$. Moreover, the condition for the resonance to be stochastic remains the same as before: $\Phi \lesssim (g/\sqrt{\lambda})(\pi^2m^2/3\lambda M_p)$, see Eq. (72).

However, as soon as the amplitude Φ drops down to $m/\sqrt{\lambda}$, the situation changes dramatically. First of all, depending on the values of parameters the field rolls to one of the minima of its effective potential at $\phi=\pm m/\sqrt{\lambda}$. The description of this process is rather complicated. Depending on the values of parameters and on the relation between $\sqrt{\langle\phi^2\rangle}$, $\sqrt{\langle\chi^2\rangle}$, and $\sigma\equiv m/\sqrt{\lambda}$, the universe may become divided into domains with $\phi=\pm\sigma$, or it may end up in a single state with a definite sign of ϕ. We will describe this bifurcation period in [19]. After this transitional period the field ϕ oscillates near the minimum of the effective potential at $\phi=\pm m/\sqrt{\lambda}$ with an amplitude $\Phi \ll \sigma = m/\sqrt{\lambda}$. These oscillations lead to parametric resonance with χ-particle production which can be (approximately) described as a narrow resonance in the first instability band of the Mathieu equation with $A_k = 4(k^2+g^2\sigma^2)/m^2$, $q=4g^2\sigma\Phi/m^2$. For definiteness we will consider here the regime $\lambda^{3/2}M_p < m \ll \lambda^{1/2}M_p$. The resonance in this instability band is possible only if $g^2/\lambda < \frac{1}{2}$; the resonance in higher instability bands is very inefficient and rapidly shuts down due to the expansion of the universe. Using the results of [2] one can show that the resonance in the first band also terminates at $\Phi < \lambda m^2/g^4 M_p$. By taking the upper limit of this inequality at $\Phi \sim m/\sqrt{\lambda}$ one concludes that this resonance is possible only for $g/\sqrt{\lambda} \gtrsim (m/\sqrt{\lambda}M_p)^{1/4}$. (The resonance may terminate somewhat earlier if the particles produced by the parametric resonance give a considerable contribution to the energy density of the universe.) However, this is not the end of reheating, because the perturbative decay of the inflaton field remains possible. It occurs with the decay rate $\Gamma(\phi\rightarrow\chi\chi) = g^4 m/8\pi\lambda$. This is the process which is responsible for the last stages of the decay of the inflaton field. It occurs only if one ϕ particle can decay into two χ particles, which implies that $g^2/\lambda < \frac{1}{2}$.

XIII. DISCUSSION

In this paper we investigated the development of parametric resonance in the conformally invariant theories of the type of $(\lambda/4)\phi^4 + (g^2/2)\phi^2\chi^2$. We have found that the development of parametric resonance in these theories does not depend on the expansion of the universe, and can be classified in terms of the ratio g^2/λ. This ratio determines the structure of the stability-instability bands for the equations describing the resonance.

We have found that the behavior of the resonance with respect to χ-particle production is a nonmonotonic function of g^2/λ. For example, for $g^2=\lambda$ and for $g^2=3\lambda$ equation for the perturbations of the field χ has only one instability band, for $g^2/\lambda = n(n+1)/2$ there is only a finite number of instability bands, whereas for all other values of g^2/λ the number of instability bands is infinite.

It is interesting that χ-particle production is least efficient for $g^2 \ll \lambda$ and for $g^2=3\lambda$. For example, the characteristic exponent $\mu_{\max}$ for $g^2=2\lambda$ and for $g^2=8\lambda$ is almost 7 times greater than $\mu_{\max}$ for $g^2=3\lambda$, see Fig. 6. Meanwhile the characteristic exponent for the production of ϕ particles in the theory $(\lambda/4)\phi^4$ coincides with that of the field χ for $g^2=3\lambda$. Therefore χ-particle production is typically more efficient than the production of ϕ particles (unless $g^2 \ll \lambda$). The nonmonotonic dependence of μ on the ratio g^2/λ suggests that there exists an ''unnatural selection'' rule: The particles which are especially intensively produced during preheating are not the ones which have the strongest coupling to the inflaton field, but those for which the characteristic exponent μ is the greatest.

In the conformally invariant theories the expansion of the universe does not hamper the resonance, so it ends only due to the back reaction of the produced particles. There are several different mechanisms which may terminate the parametric resonance. First of all, creation of particles leads to a decrease in the amplitude of oscillations of the field $\varphi = a\phi$, which otherwise would remain constant. This leads to a proportional decrease in the frequency of oscillations in terms of the conformal time η, which may shift the position of the instability band towards smaller momenta. There is also an opposing effect which increases the frequency of oscillations due to the interaction of the homogeneous inflaton field with the produced particles. Finally, quantum fluctuations of the fields ϕ and χ acquire contributions to their masses, which changes their spectra. A combination of all these effects leads to restructuring of the instability bands. This terminates the amplification of the leading modes which have been growing from the very beginning of preheating. Additionally, one may envisage effects related to rescattering of produced particles, which may terminate the resonance even somewhat earlier. In this respect it is interesting that our estimates ignoring the process of rescattering give results which are in a very good numerical agreement with the results of computer simulations of reheating in the theory $\lambda\phi^4$ performed in [5] where all of these effects including rescattering have been taken into account.

Rescattering may be more important for $g^2 \gg \lambda$ [9–11]. However, in this regime one may need to take into account possible small mass terms which should be present in realistic versions of the theory. As we have found, for $g^2 \gg \lambda$ these mass terms lead to a radical change in the structure of the resonance not at $\Phi \lesssim m/\sqrt{\lambda}$, as one could naively expect, but much earlier, at $\Phi \lesssim (g/\sqrt{\lambda})(\pi^2 m^2/3\lambda M_p)$. In this regime the resonance becomes stochastic, the effective width of the resonance band increases, making it much more stable with respect to various back reaction effects including rescattering [2].

We should emphasize again that preheating is but the first stage of reheating, which does not lead to a complete decay of the inflaton field in any models which we studied so far. The last stages of preheating are always described by the perturbation theory [17], which will be developed further in our subsequent publication [18]. To illustrate this point, we described the development of the parametric resonance in the general class of models with the effective potential $V(\phi,\chi) = \pm(m^2/2)\phi^2 + (\lambda/4)\phi^4 + (g^2/2)\phi^2\chi^2$. We have found that in these theories (without any other fields being added) the inflaton field can completely decay only if the sign of the term $(m^2/2)\phi^2$ is negative, which corresponds to spontaneous symmetry breaking. Moreover, this process is completed only for $g^2/\lambda < \frac{1}{2}$, see Figs. 11 and 12.

A complete inflaton decay is possible for $g^2/\lambda > \frac{1}{2}$ as well, even without spontaneous symmetry breaking, but only if the inflaton field has some other interactions, such as an interaction with fermions $\overline{\psi}\psi\phi$ with mass $m_\psi < m/\sqrt{2}$ [18]. This conclusion implies that the decay of the inflaton field is by no means automatic even if it is heavy and strongly interacts with other fields. Generically, the inflaton field accumulates an enormously large energy density, which can be completely released only if it interacts with other particles in a very specific way [1].

To understand how these results may change our point of view on the thermal history of the universe, let us suppose for a moment that the inflaton field does not have any interactions with light fermions, and that it has an effective potential $-(m^2/2)\phi^2 + (\lambda/4)\phi^4 + (g^2/2)\phi^2\chi^2$ with $\lambda \sim g^2 \sim 10^{-13}$ and with a small mass $m \sim 10^2$ GeV protected by supersymmetry. Then the final stage of reheating of the universe will begin only after the symmetry breaking in this theory, and the reheating temperature estimated in accordance with [2] will be smaller than 10^2-10^3 GeV. In such a theory the electroweak phase transition may never happen, or it may occur in an entirely different way. From the end of inflation until the symmetry breaking and the final stage of reheating, the universe will remain far away from thermal equilibrium, and various nonthermal phase transitions and explosive processes of particle production may occur. In such a model one should reconsider all issues related to the primordial gravitino problem, moduli field problem, baryogenesis, etc.

The main conclusion of our investigation can be formulated as follows. The first stages of the process of the inflaton decay may occur much more efficiently than was previously thought, due to the effect of parametric resonance. The last stage of this process may be completely inefficient even if the coupling of the inflaton field to matter is very strong, or it may be efficient only in a very narrow range of parameters, see Figs. 11 and 12. As a result, the complete thermal (and nonthermal) history of the universe in the context of the inflationary universe scenario may be dramatically different from the standard lore of the hot Big Bang cosmology.

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APPENDIX

Here we show how one can derive Eq. (34) or (40) for the characteristic exponent μ_k from the analytic solution (31). We will first consider here the case $g^2=\lambda$. Equation (31) describes both solutions, $X_1(z)$ and $X_2(z)$. The resonant solution $X(z)$ consists of four monotonic parts within a single period of the inflaton oscillation, see Fig. 3. It turns out that at different quarters of the period either $X_1(z)$ or $X_2(z)$ correspond to the exponentially growing solution. Indeed, the square of the resonant solution within the first quarter of a period is

$$X^2(z)=X_0^2\exp\left[\int_0^z\frac{dz}{M_1(z)}\left(1-\frac{C_1}{\sqrt{z(1-z^2)}}\right)\right], \quad \text{(A1)}$$

where $M_1(z)$ is given by Eq. (29), C_1 is given by Eq. (32), and X_0^2 is the square of the resonant solution in the beginning of the period when $z=0$.

Within the second quarter of the period one has

$$X^2(z)=X_{1/4}^2\exp\left[\int_1^z\frac{dz}{M_1(z)}\left(1+\frac{C_1}{\sqrt{z(1-z^2)}}\right)\right], \quad \text{(A2)}$$

where $X_{1/4}$ is the value of X_z after the first quarter of the period $X_{1/4}\equiv X(z=1)$.

Then the value of X^2 after half of a period is

$$X_{1/2}^2=X_0^2\exp\left(-2C_1\int_0^1\frac{dz}{M_1(z)\sqrt{z(1-z^2)}}\right), \quad \text{(A3)}$$

where the integral is understood as its principal value. The resonant solution has the generic form $X[z(x)]=P[z(x)]e^{\mu x}$, where $P(z)$ is a periodic function. Since P has a period equal to half of the period of the inflaton oscillation, Eq. (A3) is sufficient to find μ:

$$\frac{\mu T}{2}=-C_1\int_0^1\frac{dz}{M_1(z)\sqrt{z(1-z^2)}}>0. \quad \text{(A4)}$$

The integral in this equation can be reduced to $I(\kappa^2)$ given by Eq. (35):

$$-\int_0^1\frac{dz}{M_1(z)\sqrt{z(1-z^2)}}=\int_0^{\pi/2}\frac{d\theta\sin^{1/2}\theta}{1+2\kappa^2\sin\theta}\equiv I(\kappa^2). \quad \text{(A5)}$$

Similar calculations can be repeated for the case $g^2=3\lambda$ considered in Sec. VII.

The square of the resonant solution within the first quarter of a period is

$$X^2(z)=X_0^2\exp\left[\int_0^z\frac{dz}{M_2(z)}\left(2x-\frac{2}{3}\kappa^2+\frac{C_2}{\sqrt{z(1-z^2)}}\right)\right], \quad \text{(A6)}$$

where $M_2(z)$ is given by Eq. (37), C_2 is given by Eq. (38), and X_0^2 is the square of the resonant solution in the beginning of the period when $z=0$.

Within the second quarter one has

$$X^2(z)=X_{1/4}^2\exp\left[\int_1^z\frac{dz}{M_2(z)}\left(2x-\frac{2}{3}\kappa^2-\frac{C_2}{\sqrt{z(1-z^2)}}\right)\right]. \quad \text{(A7)}$$

Then the value of X^2 after half of a period is

$$X_{1/2}^2=X_0^2\exp\left(2C_2\int_0^1\frac{dz}{M_2(z)\sqrt{z(1-z^2)}}\right), \quad \text{(A8)}$$

where the integral is understood as its principal value.

The equation for the characteristic exponent in this case follows from Eq. (A8):

$$\frac{\mu T}{2}=C_2\int_0^1\frac{dz}{M_2(z)\sqrt{z(1-z^2)}}>0. \quad \text{(A9)}$$

The integral in Eq. (A9) can be reduced to $J(\kappa^2)$ given by Eq. (41):

$$\int_0^1\frac{dz}{M_2(z)\sqrt{z(1-z^2)}} =\int_0^{\pi/2}d\theta\frac{\sin^{3/2}\theta}{1+(2/3)\kappa^2\sin\theta+[(4/9)\kappa^4-1]\sin^2\theta} \equiv J(\kappa^2). \quad \text{(A10)}$$

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