Leptonic decay rates of charmonium *S* and *D* states

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We calculate the leptonic decay rates of the $^3S_1$ and the $^3D_1$ states of charmonium including their relativistic and single gluonic radiative corrections within the framework of a nonsingular potential model proposed by Gupta, Johnson, Repko, and Suchyta. We find that the relativistic corrections and the single gluonic radiative corrections are both significant. But single gluonic radiative corrections are significantly smaller than the well-known static limit results. Since we work in a formalism where the quarks are assumed to be on mass shell, there is some ambiguity about treating the infrared divergent part of the radiative correction terms. [S0556-2821(97)06621-6]

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I. INTRODUCTION

An important feature of the $^3S_1$ and $^3D_1$ states of charmonium is the decay of these states into lepton pairs such as e^+e^-. Since these states have the same J^{PC} quantum numbers as the photon, they can directly annihilate into a virtual photon which can then produce the lepton pair. For $^3S_1$ states the width has been found to be [1]

$$\Gamma(n^3S_1\rightarrow e^+e^-)=\frac{16\pi\alpha^2e_Q^2}{M^2}|\psi(0)|^2\left(1-\frac{16\alpha_s}{3\pi}\right) \quad (1)$$

if one includes a single gluonic radiative correction at the annihilation vertex of the quarks. The wave function ψ is obtained by solving the eigenvalue problem, perhaps through a variational method or some other approach, for a potential which yields good results for the spectra. The result of Eq. (1) above is valid only to lowest order in $\langle p\rangle/m$ where $\langle p\rangle$ is the average magnitude of the quark (or antiquark) momentum in the charmonium rest frame.

In Eq. (1) the term involving $16\alpha_s/3\pi$ is the static limit (zero quark momentum limit) of the contribution due to the exchange of a transverse gluon between the quark and antiquark, and except for an additional factor of 4/3 arising from color it is exactly the same term obtained many years ago in positronium when studying, in electrodynamics, lowest order radiative corrections to the annihilation graph. In that work (see, e.g., Schwinger [2]) one must remove from the one photon vertex correction ''the Coulomb piece'' since it is already accounted for in the wave function ψ and it is crucial to avoid double counting. Likewise, in the present problem we must also remove from the vertex correction all terms which are already accounted for in $\psi(0)$. We will return to the specifics of this later. A second important point concerning this radiative correction is that it has been calculated in an approximate way [2] by assuming the bound quarks to be on-shell particles of respective three momenta $\vec{p}$ and $-\vec{p}$, and subsequently integrating over the momentum distribution. A more rigorous approach would utilize the Bethe-Salpeter formalism.

Charmonium is at best only an approximately nonrelativistic system with $(v^2)/c^2$ roughly 0.2 or 0.25. From this we conclude that relativistic corrections to Eq. (1) could be important both for the leading term and for the radiative correction. In fact, a naive extension of the assumptions leading to Eq. (1), when applied to $^3D_1$ states, leads to zero width for the leptonic annihilation decay. On the other hand, the $\psi''(3770)$, which is supposed to be predominantly the 1^3D_1 state, has a measured leptonic decay rate given by width 0.26 keV. Thus relativistic corrections are clearly necessary for D state decays and they may also be important for S states. Moreover, since for S states the suppression factor is very substantial, any deviations from Eq. (1) are likely to be important in the consideration of QCD subprocesses in which such radiative corrections are relevant.

Another issue, briefly mentioned earlier, is that $\psi(0)$ already contains a good part of the $c\overline{c}$ interaction and consequently one must avoid overcounting. For example, one of the more successful recent potential models, namely, the nonsingular potential model of Gupta, Johnson, Repko, and Suchyta (GJRS) [3] includes in its potential the instantaneous one gluon exchange in the Coulomb gauge (Coulomb and transverse term). Thus $\psi(0)$ is obtained for this potential and consequently the radiative correction should arise only as a vertex correction reflecting only the difference between one gluon exchange and the instantaneous Coulomb gauge one gluon exchange. It is not obvious that this will produce the factor $16\alpha_s/3\pi$ since that factor normally emerges from the difference between one gluon and Coulomb pieces. We will later show that in the static limit there is, in fact, no double counting, but we stress that this is true only in this limit.

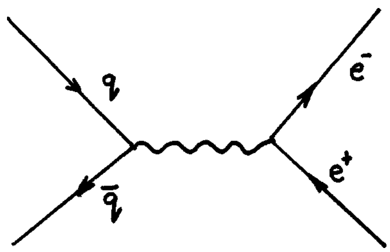

FIG. 1. Simplest graph for annihilation of quark and antiquark into an electron positron pair.

In this paper we calculate the decay rate of charmonium into lepton pairs, including the residual single gluon radiative correction, without making any static approximations. However, to the extent that this is possible, we treat the quarks as on shell particles in order to simplify the calculation of vertex functions and avoid the use of the Bethe-Salpeter formalism. We find that relativistic corrections and radiative corrections are significant. To proceed we first calculate the probability amplitude for the on-mass-shell quark and antiquark with three momenta $\vec{p}$ and $-\vec{p}$ to annihilate into a virtual photon which then creates an electron positron pair (see Fig. 1). We use wave functions obtained from the potential model of Ref. [3], and thus the amplitude is obtained by integrating over the momentum space wave function. It should be mentioned that relativistic effects are included in this model and the wave functions are determined variationally. The only term dropped and treated as a perturbation to the energy is the tensor interaction. The inclusion of instantaneous Coulomb and Breit interactions in $\psi(0)$ implies their removal from the vertex correction. Thus, as illustrated in Fig. 2, we must calculate the difference shown. Although the present calculation contains relativistic effects we show that the static limit agrees with previous results. An important feature of the nonstatic calculation is the appearance of an infrared difficulty caused by placing the quarks on shell. We use several different methods to cope with this singularity and compare the results.

The format for the rest of the paper is as follows. In Sec. II, following the work of Ref. [4], we set up the problem and show that the leptonic decay rates of the $^3S_1$ and the $^3D_1$ states can be expressed entirely in terms of the decay constants f_{V,n^3S_1} and f_{V,n^3D_1} which define the matrix element of the electromagnetic current between the vacuum and the vector meson state. We then derive expressions for these decay constants in terms of integrals involving form factors G_1 and G_2 and appropriate wave functions. In Sec. III we calculate G_1 and G_2. In Sec. IV we carry out numerical integration over wave functions, presenting results for 1^3S_1, 2^3S_1, and 1^3D_1 states using several treatments to deal with the infrared problem previously noted. In Sec. V we discuss our results and make some concluding remarks.

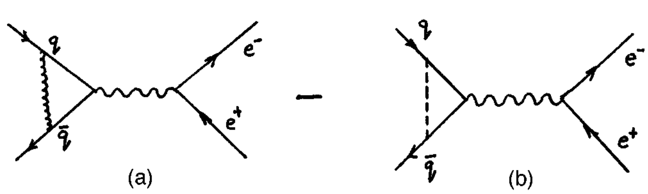

FIG. 2. (a) Gluonic radiative correction to simplest graph. (b) Subtraction of the Coulomb gauge instantaneous part of the gluonic correction.

II. FORMULATION OF THE PROBLEM

We consider the quantum states of charmonium which have the same quantum numbers as the photon. In spectroscopic notation this could be the $^3S_1$ or the $^3D_1$ state of charmonium. In these states the charmonium acts as a vector meson, with $J^{PC}=1^{--}$, the same J^{PC} of the photon. We assume that the vector meson decays into a virtual photon with $q^2=M^2$ which then decays into a lepton-antilepton pair. M is the mass of the vector meson of the charmonium state $^3S_1$ or $^3D_1$. Quite generally, the S matrix element for the above process, when the vector meson is at rest, can be written as

$$S_{fi}=(2\pi)^4\delta^4(k-p_1-p_2)\frac{1}{\sqrt{2MV}}\sqrt{\frac{m}{E_1V}}\sqrt{\frac{m}{E_2V}}\mathcal{M}, \tag{2}$$

where $\mathcal{M}$, Lorentz invariant Feynman amplitude, can be written as

$$\mathcal{M}=\langle 0|J^{\mu}_{em}(0)|V;\lambda\rangle \times\langle e^-(p_1)\ell^+(p_2)|J^{\nu}_{em}(0)|0\rangle\sqrt{2MV}\sqrt{\frac{E_1V}{m}}\sqrt{\frac{E_2V}{m}} \times\frac{(-i)g_{\mu\nu}}{q^2+i\varepsilon}(e^2), \tag{3}$$

where m is the lepton mass. We assume box normalization in a box of volume V with periodic boundary conditions. In general, the covariant matrix element of the electromagnetic current between the vacuum state and the vector meson at rest can be written as

$$\sqrt{2MV}\langle 0|J^{\mu}_{em}(0)|V;\lambda\rangle=f_VM^2\varepsilon^{\mu}(\lambda), \tag{4}$$

where $\varepsilon^{\mu}(\lambda)$ is the polarization four vector of the vector meson. The constant f_V is called the decay constant. Also,

$$\sqrt{\frac{E_1V}{m}}\sqrt{\frac{E_2V}{m}}\langle\ell^-(r_1p_1)\ell^+(r_2p_2)|J^{\nu}_{em}(0)|0\rangle =\overline{U}_{r_1}(p_1)\gamma^{\nu}V_{r_2}(p_2). \tag{5}$$

Substituting Eqs. (2), (3), and (4) into Eq. (1) and doing the usual calculations, we get for the leptonic decay rate of the vector meson the following result:

$$\Gamma_{V\to \ell^+\ell^-}=\frac{4}{3}\pi\alpha^2 f_V^2 M\left(1+\frac{2m_\ell^2}{M^2}\right)\sqrt{1-\frac{4m_\ell^2}{M^2}}. \tag{6}$$

When the lepton is the electron we can write the above formula to an excellent approximation as

$$\Gamma_{V\to e^+e^-}=\frac{4}{3}\pi\alpha^2 f_V^2 M. \tag{7}$$

The problem in all theoretical calculations is to calculate f_V according to some model. Here we assume that the vector meson state in question is the bound state of a c quark and its antiquark bound by some potential in the n^3S_1 or the n^3D_1 state. In the quark model these states are represented by the state vectors

$$|n^3S_1,\lambda\rangle=\frac{V}{(2\pi)^3}\int d^3p\frac{(2\pi)^{3/2}}{\sqrt{V}}\phi_{nS}(p)Y_{00}(\theta,\phi)\sum_{rs}\left\langle\frac{1}{2}r;\frac{1}{2}s\middle|1\lambda\right\rangle|\vec{p}r;-\vec{p}s\rangle, \tag{8}$$

$$|n^3D_1,\lambda\rangle=\frac{V}{(2\pi)^3}\int d^3p\sum_{m\nu}\frac{(2\pi)^{3/2}}{\sqrt{V}}\phi_{nD}(p)Y_{2m}(\theta,\phi)\langle 1\lambda|2m;1\nu\rangle\sum_{rs}\left\langle\frac{1}{2}r;\frac{1}{2}s\middle|1\nu\right\rangle|\vec{p}r;-\vec{p}s\rangle. \tag{9}$$

In Eqs. (8) and (9), λ is the polarization index of the vector meson and $\phi_{nS}(p)$ and $\phi_{nD}(p)$ are the momentum space radial wave functions of the n^3S_1 and the n^3D_1 states. Also the state vector $|\vec{p}r;-\vec{p}s\rangle$ represents an on mass shell quark-antiquark state with three momenta $+\vec{p}$ and $-\vec{p}$ and the spin indices r and s, respectively.

In Eqs. (8) and (9) if the state vectors are normalized to 1, the radial wave functions are also normalized to 1. That is,

$$\int_0^\infty|\phi_{ns}(p)|^2p^2dp=1$$

and

$$\int_0^\infty|\phi_{nD}(p)|^2p^2dp=1. \tag{10}$$

In this model, the problem of calculating f_V reduces to the problem of calculating the matrix element $\langle 0|J_\mu^{em}(0)|\vec{p}r;-\vec{p}s\rangle$. In general this matrix element can be written as

$$\langle 0|J_\mu^{em}(0)|\vec{p}r;-\vec{p}s\rangle=\sqrt{\frac{m}{E_pV}}\sqrt{\frac{m}{E_pV}}e_V V_s(-\vec{p})\Lambda_\mu U_r(\vec{p}). \tag{11}$$

In Eq. (11) the constant e_V is a color factor which for charmonium with $e_c=+2/3e$ takes the value

$$e_V=\frac{2}{\sqrt{3}}. \tag{12}$$

We will later calculate Λ_μ in Eq. (11) using the diagrams of Figs. 1 and 2. In Fig. 2 the quark and the antiquark exchange a gluon before they combine to annihilate into a photon. Now we parameterize the matrix element of Eq. (11) the way it enters into the calculation of f_V as

$$\sum_{rs}\left\langle\frac{1}{2}r;\frac{1}{2}s\middle|1\nu\right\rangle\overline{V_s}(-\vec{p})\Lambda_\mu Ur(\vec{p})=[G_1(p)\varepsilon_\mu(\nu)+G_2(p)(\varepsilon\cdot p)p_\mu], \tag{13}$$

where G_1 and G_2 are two form factors introduced by Bergström, Snellman, and Tengstrand [4]. One can invert Eq. (13) and express the form factors G_1 and G_2 as

$$G_1=\frac{1}{2}\sum_\nu\sum_{\mu s}\left\langle\frac{1}{2}r;\frac{1}{2}s\middle|1\nu\right\rangle\overline{V_s}(-\vec{p})\left[\vec{\varepsilon}^*(\nu)\cdot\vec{\Lambda}-\frac{1}{|\vec{p}|^2}(\vec{p}\cdot\vec{\Lambda})[\vec{\varepsilon}^*(\nu)\cdot\vec{p}]\right]U_r(\vec{p}), \tag{14}$$

$$G_2=\frac{1}{2|\vec{p}|^2}\sum_\nu\sum_{rs}\langle rs|1\nu\rangle\overline{V_s}(-\vec{p})\left[\vec{\varepsilon}^*(\nu)\cdot\vec{\Lambda}-\frac{3}{|\vec{p}|^2}(\vec{p}\cdot\vec{\Lambda})[\vec{\varepsilon}^*(\nu)\cdot\vec{p}]\right]U_r(\vec{p}). \tag{15}$$

We will use Eqs. (14) and (15) to calculate G_1 and G_2 once the $\vec{\Lambda}$'s are determined from Feynman diagrams of Figs. 1 and 2.

We will now express the decay constants $f_{V,nS}$ and $f_{V,nD}$ in terms of the form factors G_1 and G_2 and the momentum space radial wave function for the n^3S_1 and the n^3D_1 states. First of all we notice from Eq. (4) that

$$f_V M^2 = \frac{1}{3}\sqrt{2MV}\sum_{\lambda} \langle 0|\vec{\varepsilon}^*(\lambda)\cdot \vec{J}_{em}(0)|V,\lambda\rangle. \quad (16)$$

Using Eqs. (8), (9), (11), and (13) and using explicit expressions for $\vec{\varepsilon}^*(\lambda)$ and $\vec{\varepsilon}(\nu)$ for $\lambda,\nu=+1,0,-1$ in Eq. (15) we obtain

$$f_{V,n^3S_1} = \frac{m}{\pi} M^{-3/2} e_V \int_0^\infty \frac{p^2 dp}{E_p}\,\phi_{nS}(p)\left[G_1 - \frac{1}{3}p^2 G_2\right] = \frac{m}{\pi} M^{-3/2} e_V I_{nS}, \quad (17)$$

$$f_{V,n^3D_1} = \frac{\sqrt{2}}{3}\frac{m}{\pi} M^{-3/2} e_V \int_0^\infty \frac{p^2 dp}{E_p}\,\phi_{nD}(p) G_2(p) p^2 = \frac{\sqrt{2}}{3}\frac{m}{M} M^{-2/3} e_V I_{nD}. \quad (18)$$

III. CALCULATION OF THE FORM FACTORS G_1 AND G_2 TO FIRST ORDER IN α_s

We now proceed to calculate the form factors with wave functions which are eigenfunctions of the Hamiltonian which includes the Coulomb, as well as the instantaneous part of the transverse one-gluon exchange potentials.

Once we have wave functions for the $^3S_1$ and the $^3D_1$ states, the calculation of the leptonic decay constant of the vector mesons f_V in Eqs. (17) and (18) reduces to a calculation of the form factors G_1 and G_2. We can calculate these form factors G_1 and G_2 from Eqs. (14) and (15) once we know the vertex functions Λ_μ of Eq. (11). We will calculate the vertex function Λ_μ to the first order in α_s. If we assume that the quark and the antiquark are on mass shell, the only Feynman diagrams we have to consider are those in Figs. 1 and 2. The virtual one gluon insertions on the external lines lead to the usual wave function renormalization of the charge. The wave functions of the GJRS model [3] already contain the contribution of the instantaneous part of the one-gluon exchange. In the Coulomb gauge, this means that the potential includes the Coulomb part of the one-gluon exchange and the instantaneous part of the transverse one-gluon exchange. So we have to subtract the part of Λ_μ which corresponds to these exchanges. Let us call the part of Λ_μ to be subtracted as $\Lambda^{(0)}_{\mu,1g}$.

Let us call the form factors obtained from Eqs. (14) and (15) with Λ^μ calculated from Figs. 1 and 2(a) as G_1' and G_2'. In other words, G_1' and G_2' includes the contributions from the basic vertex part of Fig. 1(a) and the full contribution from the one-gluon exchange diagram of Fig. 2(a). The form factors obtained from Eqs. (14) and (15) where Λ_μ is replaced by $\Lambda^{(0)}_{\mu,1g}$ of Fig. 2(b) are called $G^{(0)}_{1,1g}$ and $G^{(0)}_{2,1g}$. Then the form factors G_1 and G_2 to be substituted in Eqs. (17) and (18) for the calculation of the decay constants f_{V,n^3S_1} and f_{V,n^3D_1} are given by

$$G_1(p) = G_1'(p) - G^{(0)}_{1,1g}(p),$$

$$G_2(p) = G_2'(p) - G^{(0)}_{2,1g}(p). \quad (19)$$

Next we turn to the calculation of G_i' and $G^{(0)}_{i,1g}(i=1,2)$. From Lorentz covariance and current conservation alone, the vertex function originating from the diagrams of Figs. 1 and 2(a) can be written as

$$\overline{V_s}(p_2)\Lambda_\mu U_r(p_1) = \overline{V_s}(p_2)\left[F_1(q^2)\gamma^\mu - \frac{-i}{2m}\frac{(4/3)\alpha_s}{2\pi}F_2(q^2)\sigma^{\mu\nu}q_\nu\right]U_r(p_1), \quad (20a)$$

where

$$q = p_1 + p_2, \quad (20b)$$

$$p_1 = (E_p, +\vec{p}) \text{ and } p_2 = (E_p, -\vec{p})$$

$$\text{and } q^2 = (p_1+p_2)^2 = 4M^2. \quad (20c)$$

Equation (20a) is the counterpart of the corresponding equation in QED with $\alpha = e^2/4\pi$ replaced by $4/3\alpha_s$ in QCD to take into account the color factor 4/3. The form factor is multiplied by $4/3\alpha_s$ for convenience since F_2 is nonzero only because of the Feynman diagram in Fig. 2(a). Notice the minus sign in front of F_2. This was introduced deliberately to conform with Schwinger's convention. After the charge renormalization, the vertex function originating from Fig. 2(b) is finite and Schwinger [2] has given expressions for the form factors F_1 and F_2 in QED. We can use the same expressions in QCD provided we replace the fine-structure constant, α by $4/3\alpha_s$. In contrast to the static result of Schwinger in which the infrared piece drops out, we cannot ignore the infrared divergent part in the expressions since these remain in the nonstatic limit and are cut off by the ''true'' off-shell nature of the quarks. Later we will present calculations using several procedures to deal with this infrared ambiguity which would not be present in a fully rigorous bound state formalism, based on the Bethe-Salpeter equation. Schwinger's expressions [2] are given in terms of the relativistic velocity $v = p/E_p$. They are

$$F_1(v)=1-4/3\frac{\alpha_s}{2\pi}\left\{\frac{(1+v^2)}{2v}\left[-\pi^2+\frac{1}{2}\left(\ln\frac{(1-v)}{2}\right)^2-\frac{1}{2}\left(\ln\frac{(1+v)}{2}\right)^2-2\ln 2\ln\frac{(1+v)}{(1-v)}+4\ell(v)-\ell(v^2)-\ell\left(\frac{1+v}{2}\right)+\ell\left(\frac{1-v}{2}\right)\right]+2\ln 2+\left[2\ln\frac{2m}{\mu}-2\right]\left[\frac{(1+v^2)}{2v}\ln\frac{(1+v)}{(1-v)}-1\right]+\frac{1}{2v}\ln\frac{(1+v)}{(1-v)}\right\}, \tag{21a}$$

where

$$\ell(x)=\int_0^x\frac{dt}{t}\ln\left(\frac{1}{(1-t)}\right)=\sum_{n=1}^{\infty}\frac{x^n}{n^2};\quad 0<x<1, \tag{21b}$$

$$F_2(v)=-\frac{(1-v^2)}{2v}\ln\frac{(1+v)}{(1-v)}. \tag{22}$$

Substituting Eq. (20a) in Eqs. (14) and (15) and doing the ν, r, and s summation by the well known trace method [5]

$$\sum_{rs}\left\langle\frac{1}{2}r;\frac{1}{2}s\middle|1\nu\right\rangle\overline{V_s}(p_2)OU_r(p_1) =\frac{1}{4\sqrt{2}m(E_p+m)}\,\mathrm{Tr}\{(\not p_2-m)\times O(\not p_1+m)(1+\gamma^0)\not\varepsilon(\nu)\} \tag{23}$$

we obtain

$$G_1'(p)=-\frac{\sqrt{2}E_p}{m}\left(F_1(v)+\frac{4/3\alpha_s}{2\pi}F_2(v)\right), \tag{24}$$

$$G_2'(p)=-\frac{\sqrt{2}}{m(E_p+m)}\left[F_1(v)-\frac{(4/3\alpha_s)}{2\pi}F_2(v)\frac{E_p}{m}\right]. \tag{25}$$

Next we turn to the more difficult part of calculating $G_{1,1g}^{(0)}$ and $G_{2,1g}^{(0)}$. The contributions of the Feynman diagrams in Figs. 1 and 2(a) can be written as

$$\Gamma^\mu(p_1,p_2)=ie\overline{V(p_2)}\Lambda^\mu U(p_1), \tag{26}$$

where

$$\Lambda^\mu(p_1,p_2)=\gamma^\mu+(4\pi)\frac{4}{3}\alpha_s\Lambda_{1g}^\mu(p_1,p_2), \tag{27}$$

where Λ_{1g}^μ is the regularized one-gluon vertex correction. The unregularized vertex correction is called $\Lambda_{1g}'^\mu$ and it is given by

$$\Lambda_{1g}'^\mu(p_1,p_2)=-\frac{i}{(2\pi)^4}\int\frac{d^4k}{k^2+i\varepsilon}\gamma^\alpha\frac{1}{(-\not p_2-\not k-m+i\varepsilon)}\gamma^\mu\frac{1}{(\not p_1-\not k-m+i\varepsilon)}\gamma_\alpha. \tag{28}$$

The integral in Eq. (28) is both ultraviolet and infrared divergent. The finite part of it was given by Eqs. (21) and (22). The instantaneous part of $\Lambda_{1g}'^\mu(p_1,p_2)$, given by Fig. 2(b), which is already contained in the QCD potential of the GJRS model, is called $\Lambda_{(0)1g}'^\mu$ and is given by

$$\Lambda_{(0),1g}'^\mu(p_1,p_2)=\frac{-1}{(2\pi)^4}\left[\int\frac{d^4k}{(-|\vec{k}|^2+i\varepsilon)}\gamma^\alpha\frac{1}{(-\not p_2-\not k-m+i\varepsilon)}\gamma^\mu\frac{1}{(\not p_1-\not k-m+i\epsilon)}\gamma_\alpha+\int\frac{d^4k}{(-|\vec{k}|^2+i\varepsilon)}(\vec{\gamma}\cdot\hat{k})\frac{1}{(-\not p_2-\not k-m+i\varepsilon)}\gamma^\mu\frac{1}{(\not p_1-\not k-m+i\varepsilon)}(\vec{\gamma}\cdot\hat{k})\right]. \tag{29}$$

To obtain the instantaneous part of Eq. (28) we replaced $k^2=k_0^2-|\vec{k}|^2$ in the photon propagator by $-|\vec{k}|^2$. Equation (29) is the instantaneous part in the Coulomb gauge. By subtracting the contribution of Eq. (29), we are trying to include only the retarded part of the transverse one-gluon exchange. In order to calculate what is included in the potential we also have to consider the appropriate parts of the fermion propagators. We write

$$S_F(p)=\frac{i}{(\not p-m+i\varepsilon)}=\left\{\frac{\Lambda_+(\vec{p})}{p^0-E_p+i\varepsilon}+\frac{\Lambda_-(\vec{p})}{p^0+E_p-i\varepsilon}\right\}i\gamma^0, \tag{30}$$

where $\Lambda_+(\vec{p})$ and $\Lambda_-(\vec{p})$ are the positive and the negative energy projection operators:

$$\Lambda_+(\vec{p})=\frac{E_p+(\vec{\alpha}\cdot\vec{p}+\beta m)}{2E_p}=\frac{(\not p+m)}{2E_p}\gamma^0, \tag{31}$$

$$\Lambda_-(\vec{p})=\frac{E_p-(\vec{\alpha}\cdot\vec{p}+\beta m)}{2E_p}=\gamma^0\frac{(\not p-m)}{2E_p}. \tag{32}$$

In the quark propagator involving p_1 we only include the Λ_+ part and in the antiquark propagator involving p_2 we only include the Λ_- part. We thus obtain

$$\Lambda'^{\mu}_{(0),1g}(p_1,p_2)=+\frac{i}{(2\pi)^4}\int\frac{d^3k dk_0}{(|\vec{k}|^2-i\varepsilon)}\Bigg[\frac{\gamma^\alpha\Lambda_-(-\vec{p}_2-\vec{k})}{(-p_2^0-k^0+E_{-\vec{p}_2-\vec{k}}-i\varepsilon)}\frac{\gamma^0\gamma^\mu\Lambda_+(\vec{p}_1-\vec{k})}{(p_1^0-k^0-E_{\vec{p}_1-\vec{k}}+i\varepsilon)}\gamma^0\gamma_\alpha$$
$$+\frac{\vec{\gamma}\cdot\hat{k}\Lambda_-(-\vec{p}_2-\vec{k})}{(-p_2^0-k^0+E_{-\vec{p}_2-\vec{k}}-i\epsilon)}\frac{\gamma^0\gamma^\mu\Lambda_+(\vec{p}_1-\vec{k})}{(p_1^0-k^0-E_{\vec{p}_1-\vec{k}}+i\varepsilon)}\gamma^0\overline{\gamma\cdot}\hat{k}\Bigg]. \tag{33}$$

Integration over k^0 is rather trivial now. Integrating over k^0 from $-\infty$ to $+\infty$ and closing the contour in the lower half plane of the complex k^0 plane, we will only pick up the pole at $k^0=E_{-\vec{p}_1-\vec{k}}-p_2^0-i\varepsilon$. By Cauchy's theorem this integral is $(-2\pi i)$ times the residue at that pole. We thus get, after changing the integration variable from $\vec{k}$ to $\vec{p}\,'=\vec{p}-\vec{k}$,

$$\Lambda'^{\mu}_{(0),1g}(p_1,p_2)=\frac{-1}{(2\pi)^3}\int\frac{d^3p'}{|\vec{p}-\vec{p}\,'|^2}[\gamma^\alpha\Lambda_-(\vec{p}_1')\gamma^0\gamma^\mu\Lambda_+(\vec{p}_1')\gamma^0\gamma_\alpha+\vec{\gamma}\cdot\hat{k}\Lambda_-(\vec{p}_1')\gamma^0\gamma^\mu\Lambda_+(\vec{p}_1')\gamma^0\vec{\gamma}\cdot\hat{k}]/2(E_p-E_p'+i\varepsilon). \tag{34}$$

The integral in Eq. (34) is ultraviolet divergent. We make it ultraviolet convergent by subtracting the value of the same vertex function at the unphysical value $q^2=0$ where $q=p_1+p_2$. One can easily see that

$$\Lambda'^{\mu}_{(0),1g}(p_1,p_2)|_{q^2=0}=\frac{+1}{(2\pi)^3}\int\frac{d^3p'}{|\vec{p}-\vec{p}\,'|^2}\left[\frac{\gamma^\alpha\Lambda_-(\vec{p}\,')\gamma^0\gamma^\mu\Lambda_+(\vec{p}\,')\gamma_0\gamma_\alpha+\vec{\gamma}\cdot\hat{k}\Lambda_-(\vec{p}\,')\gamma^0\gamma^\mu\Lambda_+(\vec{p}\,')\gamma^0(\vec{\gamma}\cdot\hat{k})}{2E_p'}\right]. \tag{35}$$

In Eqs. (33)–(35),

$$\hat{k}=\frac{(\vec{p}-\vec{p}\,')}{|\vec{p}-\vec{p}\,'|}. \tag{36}$$

The subtraction at $q^2=0$ is equivalent to the usual renormalization of the electric charge. After subtraction, we get the physically relevant part which is finite:

$$\Lambda^{\mu}_{(0)1g}=\Lambda'^{\mu}_{(0)1g}-\Lambda'^{\mu}_{(0)1g}|_{q^2=0}$$
$$=-\frac{1}{(2\pi)^3}\int\frac{d^3p'}{|\vec{p}-\vec{p}\,'|^2}[\gamma^\alpha\Lambda_-(\vec{p}\,')\gamma^0\gamma^\mu$$
$$\times\Lambda_+(\vec{p}\,')\gamma^0\gamma_\alpha+\vec{\gamma}\cdot\hat{k}\Lambda_-(\vec{p}\,')\gamma^0\gamma^\mu$$
$$\times\Lambda_+(\vec{p}\,')\gamma^0\vec{\gamma}\cdot\hat{k}]E_p/2E_{p'}(E_p-E_{p'}+i\varepsilon). \tag{37}$$

We can write the right-hand side of Eq. (37) as the sum of two parts, one corresponding to the $\gamma^\alpha\ldots\gamma_\alpha$ piece in Eq. (37), which we will call $\Lambda^{\mu,\mathrm{covar}}_{(0),1g}$ or the covariant piece, and the other corresponding to the $\vec{\gamma}\cdot\hat{k}\ldots\vec{\gamma}\cdot\hat{k}$ piece in Eq. (37) which we will call $\Lambda^{\mu L}_{(0),1g}$ or the longitudinal piece. That is,

$$\Lambda^{\mu}_{(0),1g}=\Lambda^{\mu,\mathrm{covar}}_{(0),1g}-\Lambda^{\mu L}_{(0),1g}, \tag{38}$$

where

$$\Lambda^{\mu,\mathrm{covar}}_{(0),1g}=-\frac{1}{(2\pi)^3}\int\frac{d^3p'}{|\vec{p}-\vec{p}\,'|^2}\times\frac{\gamma^\alpha\Lambda_-(\vec{p}\,')\gamma^0\gamma^\mu\Lambda_+(\vec{p}\,')\gamma^0\gamma_\alpha E_p}{2E_p'(E_p-E_p'+i\varepsilon)}, \tag{39}$$

$$\Lambda^{\mu L}_{(0),1g}=-\frac{1}{(2\pi)^3}\int\frac{d^3p'}{|\vec{p}-\vec{p}\,'|^2}\times\frac{\vec{\gamma}\cdot\hat{k}\Lambda_-(\vec{p}\,')\gamma^0\gamma^\mu\Lambda_+(\vec{p}\,')\gamma^0\vec{\gamma}\cdot\hat{k}E_p}{2E_p'(E_p-E_p'+i\varepsilon)}. \tag{40}$$

Substituting the spatial part of the four vectors in Eqs. (39) and (40) into Eqs. (14) and (15), we get $G^{(0)\mathrm{covar}}_{i,1g}$ and $G^{(0)L}_{i,1g}(i=1,2)$. The true G_1 and G_2 to be substituted into Eqs. (17) and (18) to get the decay constants $f_{V,S}$ and $f_{V,D}$ for the S and D states

$$G_i(p)=G_j'-G^{(0)\mathrm{covar}}_{i,1g}+G^{(0)L}_{i,1g}\quad(i=1,2). \tag{41}$$

The sum over the spin indices r and s in Eqs. (14) and (15) can be carried out by the trace theorem of Eq. (23). We thus get,

$$G_{1,1g}^{(0)\mathrm{covar},L}=-\frac{4\pi}{3}\alpha_s E_p\frac{1}{16\sqrt{2}m(E_p+m)}\int\frac{d^3p'}{(2\pi)^3}\frac{1}{E_{p'}^3(E_p'-E_p-i\varepsilon)}\frac{1}{|\vec{p}-\vec{p}\,'|^2}\sum_\nu\left[T^{\mathrm{covar},L}(\vec{p},\vec{p}\,',r=\epsilon^*(\nu))-\frac{1}{p^2}\cdot\vec{\varepsilon}^*(\nu)\cdot\vec{p}T^{\mathrm{covar}}(\vec{p},\vec{p}\,',r=p)\right] \quad (42)$$

$$G_{2,1g}^{(0)\mathrm{covar},L}=-\frac{1}{p^2}\left(\frac{4\pi}{3}\alpha_s\right)E_p\frac{1}{16\sqrt{2}m(E_p+m)}\int\frac{d^3p'}{(2\pi)^3}\frac{1}{E_{p'}^3(E_{p'}-E_p-i\varepsilon)}\frac{1}{|\vec{p}-\vec{p}\,'|^2}\sum_\nu\left[T^{\mathrm{covar},L}(\vec{p},\vec{p}\,',r=\varepsilon^*(\nu))-\frac{3}{p^2}\vec{\varepsilon}^*(\nu)\cdot\vec{p}T^{\mathrm{covar},L}(\vec{p},\vec{p}\,',r=p)\right], \quad (43)$$

where in Eqs. (42) and (43) $G_{i,1g}^{(0),\mathrm{covar},L}$ can be $G_{i,1g}^{(0)\mathrm{covar}}$ or $G_{i,1g}^{(0)L}$ depending on whether the trace T is T^{covar} or T^L. The traces T^{covar} and T^L are given by the following expressions:

$$T^{\mathrm{covar}}(\vec{p}.\vec{p}\,',r)=\mathrm{Tr}\{(\not{p}_2-m)[-2\not{p}'\not{r}\not{p}''+2m^2\not{r}+8m\vec{p}\,'\cdot\vec{r}]\times(\not{p}_1+m)(1+\gamma^0)\not{\varepsilon}(\nu)\}, \quad (44)$$

$$T^L(\vec{p},\vec{p}\,',r)=-\mathrm{Tr}\{(\not{p}_2-m)\not{s}(\not{p}''-m)\not{r}(\not{p}'+m)\not{s}(\not{p}_1+m)\times(1+\gamma^0)\not{\varepsilon}(\nu)\}. \quad (45)$$

In Eqs. (44) and (45) the different four vectors are defined as follows:

$$\begin{aligned}p_1&=E_p,+\vec{p},\\ p_2&=E_p,-\vec{p},\\ p'&=E_{p'},+\vec{p}\,',\\ p''&=E_{p'},-\vec{p}\,',\\ r&=(0,\vec{\varepsilon}^*(\nu))\quad\text{or}\quad(0,\vec{p}),\\ s&=(0,\hat{k}),\end{aligned} \quad (46)$$

where $\hat{k}$ is the unit three vector,

$$\hat{k}=\frac{(\vec{p}-\vec{p}\,')}{|\vec{p}-\vec{p}\,'|}. \quad (47)$$

After computing the traces and summing over the polarization index ν and using the result

$$\sum_\nu\varepsilon_\alpha^*(Q,\nu)\varepsilon_\beta(Q,\nu)=-g_{\alpha\beta}+\frac{Q_\alpha Q_\beta}{M^2}, \quad (48)$$

where Q and M are the four momentum vectors and the mass of the vector meson, we obtain

$$\sum_\nu\left[T^{\mathrm{covar}}(\vec{p},\vec{p}\,',r=\varepsilon^*(\nu))-\frac{1}{p^2}\vec{\varepsilon}^*(\nu)\cdot\vec{p}T^{\mathrm{covar}}(\vec{p},\vec{p}\,',r=p)\right]=-32E_p(E_p+m)(E_{p'}^2+m^2)-64E_{p'}(E_p+m)\vec{p}\cdot\vec{p}\,'-\frac{32}{p^2}E_p(E_p+m)(\vec{p}\cdot\vec{p}\,')^2, \quad (49)$$

$$\sum_\nu\left[T^{\mathrm{covar}}(\vec{p},\vec{p}\,',r=\varepsilon^*(\nu))-\frac{3}{p^2}\vec{\varepsilon}^*(\nu)\cdot\vec{p}T^{\mathrm{covar}}(\vec{p},\vec{p}\,',r=p)\right]=-\frac{32}{p^2}(E_p+m)(E_p+2m)(\vec{p}\cdot\vec{p}\,')^2-64(E_p+m)(E_{p'}-2m)\vec{p}\cdot\vec{p}\,'-32(E_p+m)[(E_p-2m)E_{p'}^2+E_pm^2], \quad (50)$$

$$\sum_{\nu}\left[T^L(\vec{p},\vec{p}\,',r=\varepsilon^*(\nu))-\frac{1}{p^2}\vec{\varepsilon}^*(\nu)\cdot\vec{p}T^L(\vec{p},\vec{p}\,',r=p)\right]$$

$$=-16E_p(E_p+m)p'^2+16\frac{E_p(E_p+m)}{p^2}(\vec{p}\cdot\vec{p}\,')^2+32\frac{E_p{}'(E_p+m)}{p^2}(E_{p'}E_p-m^2)(\hat{k}\cdot\vec{p})^2$$

$$+32E_p(E_p+m)(\hat{k}\cdot\vec{p}\,')^2+32E_{p'}(E_p+m)(\hat{k}\cdot\vec{p})(\hat{k}\cdot\vec{p}\,')-32\frac{E_p(E_p+m)}{p^2}(\hat{k}\cdot\vec{p})(\hat{k}\cdot\vec{p}\,')(\vec{p}\cdot\vec{p}\,'), \tag{51}$$

$$\sum_{\nu}\left[T^L(\vec{p},\vec{p}\,',r=\varepsilon^*(\nu))-\frac{3}{p^2}\vec{\varepsilon}^*(\nu)\cdot\vec{p}T^L(\vec{p},\vec{p}\,',r=p)\right]$$

$$=16\frac{(E_p+2m)(E_p+m)}{p^2}(\vec{p}\cdot\vec{p}\,')^2-32m(E_p+m)(\vec{p}\cdot\vec{p}\,')-32\frac{(E_p+2m)(E_p+m)}{p^2}(\vec{p}\cdot\vec{p}\,')(\hat{k}\cdot\vec{p})(\hat{k}\cdot\vec{p}\,')$$

$$+8p^2(E_{p'}+m)^2+8(E_p+m)^2(m^2+2E_{p'}m-3E_{p'}^2)+32E_p(E_p+m)(\hat{k}\cdot\vec{p}\,')^2$$

$$+32E_{p'}(E_p+m)(\hat{k}\cdot\vec{p})(\hat{k}\cdot\vec{p}\,')+32\frac{E_p'(E_p+m)}{p^2}[E_pE_{p'}-m^2+2m(E_{p'}-E_p)](\hat{k}\cdot\vec{p})^2 \tag{52}$$

In Eqs. (51) and (52), p^2 and p'^2 are the squares of the magnitudes of the three momenta. After substituting Eqs. (49)–(52) into Eqs. (42) and (43), the angular integrations (the integration over the directions of the three vector $\vec{p}\,'$) can be performed easily. After substituting Eqs. (49)–(52) into Eqs. (42) and (43) and then doing the angular integrations, we find that in general any of the form factor $G_{i,1g}^{(0)\text{covar}}$ or $G_{i,1g}^{(0)L(i=1,2)}$ takes the following form:

$$G_{i,1g}^{(0)\text{covar},L}=\left(\frac{\alpha_s}{3\pi}\right)\left(\frac{\sqrt{2}E_p}{m}\right)\int_0^\infty\frac{p'^2dp'}{(p'^2-p^2-i\varepsilon)}F_i^{\text{covar},L}(p,p'), \tag{53}$$

where $F_i^{\text{covar},L}(p,p')$ is a real function of the variables p and p'. Writing

$$\frac{1}{p'^2-p^2-i\varepsilon}=P\left(\frac{1}{p'^2-p^2}\right)+i\pi\delta(p'^2-p^2) \tag{54}$$

we notice that the imaginary part of $G_{i,1g}^{(0)\text{covar},L}$ does not contribute to the leptonic decay rate to first order in α_s. So we only have to concern ourselves with the principal part of the integral in Eq. (53) since we intend to calculate the leptonic decay rate to first order in α_s. After evaluating the well defined principal part and changing the integration variable to x such that $(p'/p)=x$ for $p'<p$ and $(p/p')=x$ for $p'>p$, we finally find the following integral expressions for $G_{i,1g}^{(0)\text{covar}}$ and $G_{i,1g}^{(0)L}(i=1,2)$:

$$G_{1,1g}^{(0)\text{covar}}=-\frac{\alpha_s}{3\pi}\left(\frac{\sqrt{2}E_p}{m}\right)\Bigg\{\frac{1}{v}\int_0^1\frac{dx}{(1-x^2)}\ln\frac{(1+x)}{(1-x)}\Bigg[(\sqrt{1+u^2/x^2}+\sqrt{1+u^2})\frac{(2+u^2/x^2)}{x(1+u^2/x^2)^{3/2}}-\frac{x(\sqrt{1+u^2x^2}+\sqrt{1+u^2})}{(1+u^2x^2)^{3/2}}$$
$$\times(2+u^2x^2)\Bigg]+2u\int_0^1\frac{dx}{(1-x^2)}\left[\frac{(1+x^2)}{2x}\ln\frac{(1+x)}{(1-x)}-1\right]\left[\frac{(\sqrt{1+u^2}+\sqrt{1+u^2/x^2})}{x^2(1+u^2/x^2)}-x^2\frac{(\sqrt{1+u^2}+\sqrt{1+u^2x^2})}{(1+u^2x^2)}\right]$$
$$+\left(\frac{E_p}{2m}\right)u\int_0^1\frac{dx(1+x^2)}{(1-x^2)}\left[\frac{(1+x^2)}{2x}\ln\frac{(1+x)}{(1-x)}-1\right]\left[\frac{1}{x^4}\frac{(\sqrt{1+u^2}+\sqrt{1+u^2/x^2})}{(1+u^2/x^2)^{3/2}}-\frac{x^2(\sqrt{1+u^2}+\sqrt{1+u^2x^2})}{(1+u^2x^2)^{3/2}}\right]\Bigg\}, \tag{55}$$

where v is the relativistic velocity and u is the nonrelativistic velocity.

$$v=\frac{p}{E_p}, \tag{56}$$

$$u=\frac{p}{m}, \tag{57}$$

$$G_{2,g}^{(0)\text{covar}}=-\frac{1}{p^2}\left(\frac{\alpha_s}{3\pi}\right)\left(\frac{\sqrt{2}E_p}{m}\right)\Bigg[\Bigg[u\int_0^1\frac{dx}{(1-x^2)}\ln\frac{(1+x)}{(1-x)}\Bigg[\frac{1}{x}\frac{(\sqrt{1+u^2/x^2}+\sqrt{1+u^2})}{(1+u^2/x^2)^{3/2}}\Bigg\{\frac{2}{(1+\sqrt{1+u^2})}+\frac{(\sqrt{1+u^2}-2)}{x^2}\Bigg\}$$
$$-x\frac{(\sqrt{1+u^2x^2}+\sqrt{1+u^2})}{(1+u^2x^2)^{3/2}}\Bigg\{\frac{2}{(1+\sqrt{1+u^2})}+(\sqrt{1+u^2}-2)x^2\Bigg\}\Bigg]+2u\int_0^1\frac{dx}{(1-x^2)}\Bigg[\frac{(1+x^2)}{2x}\ln\frac{(1+x)}{(1-x)}-1\Bigg]$$
$$\times\Bigg[\frac{(\sqrt{1+u^2}+\sqrt{1+u^2/x^2})(\sqrt{1+u^2/x^2}-2)}{x^2(1+u^2/x^2)^{3/2}}-x^2\frac{(\sqrt{1+u^2}+\sqrt{1+u^2x^2})(\sqrt{1+u^2x^2}-2)}{(1+u^2x^2)^{3/2}}\Bigg]$$
$$+\frac{(E_p+2m)}{2m}u\int_0^1 dx\frac{(1+x^2)}{(1-x^2)}\Bigg[\frac{(1+x^2)}{2x}\ln\frac{(1+x)}{(1-x)}-1\Bigg]$$
$$\times\Bigg[\frac{1}{x^4}\frac{(\sqrt{1+u^2}+\sqrt{1+u^2/x^2})}{(1+u^2/x^2)^{3/2}}-x^2\frac{(\sqrt{1+u^2}+\sqrt{1+u^2x^2})}{(1+u^2x^2)^{3/2}}\Bigg]\Bigg],\tag{58}$$

$$G_{1,1g}^{(0)L}=-\left(\frac{\alpha_s}{6\pi}\right)\left(\frac{\sqrt{2}E_p}{m}\right)\Bigg[\Bigg[\frac{E_p}{m}u\int_0^1\frac{dx}{(1-x^2)}\ln\frac{(1-x)}{(1+x)}\Bigg[\frac{x^3(\sqrt{1+u^2}+\sqrt{1+u^2x^2})}{(1+u^2x^2)^{3/2}}-\frac{1}{x^3}\frac{(\sqrt{1+u^2}+\sqrt{1+u^2/x^2})}{(1+u^2/x^2)^{3/2}}\Bigg]$$
$$+\frac{1}{2}\frac{E_p}{m}u\int_0^1 dx\frac{(1+x^2)}{(1-x^2)}\Bigg[\frac{(1+x^2)}{2x}\ln\frac{(1+x)}{(1-x)}-1\Bigg]\Bigg[x^2\frac{(\sqrt{1+u^2}+\sqrt{1+u^2x^2})}{(1+u^2x^2)^{3/2}}-\frac{1}{x^4}\frac{1}{(1+u^2/x^2)^{3/2}}(\sqrt{1+u^2}$$
$$+\sqrt{1+u^2/x^2})\Bigg]+2\frac{1}{u}\int_0^1\frac{dx}{(1-x^2)}\Bigg[x^2\frac{(\sqrt{1+u^2}+\sqrt{1+u^2x^2})(\sqrt{1+u^2x^2}\sqrt{1+u^2}-1)}{(1+u^2x^2)}$$
$$-\frac{1}{x^2}\frac{(\sqrt{1+u^2}+\sqrt{1+u^2/x^2})(\sqrt{1+u^2/x^2}\sqrt{1+u^2}-1)}{(1+u^2/x^2)}\Bigg]$$
$$+2\frac{1}{u}\int_0^1\frac{dx}{2x}\ln\frac{(1+x)}{(1-x)}\Bigg[\frac{x^2(\sqrt{1+u^2}+\sqrt{1+u^2x^2})(\sqrt{1+u^2x^2}\sqrt{1+u^2}-1)}{(1+u^2x^2)}$$
$$+\frac{1}{x^2}\frac{(\sqrt{1+u^2}+\sqrt{1+u^2/x^2})(\sqrt{1+u^2/x^2}\sqrt{1+u^2}-1)}{(1+u^2/x^2)}\Bigg]+2\left(\frac{E_p}{m}\right)u\Bigg\{\int_0^1\frac{dx}{(1-x^2)}\Bigg[\frac{x^2(\sqrt{1+u^2}+\sqrt{1+u^2x^2})}{(1+u^2x^2)^{3/2}}$$
$$-\frac{1}{x^2}\frac{(\sqrt{1+u^2}+\sqrt{1+u^2/x^2})}{(1+u^2/x^2)^{3/2}}\Bigg]+\int_0^1\frac{dx}{2x}\ln\frac{(1-x)}{(1+x)}\Bigg[\frac{x^2(\sqrt{1+u^2}+\sqrt{1+u^2x^2})}{(1+u^2x^2)^{3/2}}+\frac{1}{x^2}\frac{(\sqrt{1+u^2}+\sqrt{1+u^2/x^2})}{(1+u^2/x^2)^{3/2}}\Bigg]$$
$$-\frac{1}{2}\left(\frac{E_p}{m}\right)u\int_0^1\frac{dx}{(1-x^2)}\Bigg[(1+x^2)+\frac{(1-x^2)^2}{2x}\ln\frac{(1-x)}{(1+x)}\Bigg]\Bigg[x^2\frac{(\sqrt{1+u^2}+\sqrt{1+u^2x^2})}{(1+u^2x^2)^{3/2}}-\frac{1}{x^4}\frac{(\sqrt{1+u^2}+\sqrt{1+u^2/x^2})}{(1+u^2/x^2)^{3/2}}\Bigg]\Bigg],\tag{59}$$

and

$$G_{2,1g}^{(0)L}=-\left(\frac{\alpha_s}{6\pi}\right)\left(\frac{\sqrt{2}E_p}{m}\right)\frac{1}{p^2}\Bigg[\Bigg[\frac{1}{u}\int_0^1\frac{dx}{(1-x^2)}\ln\frac{(1+x)}{(1-x)}\Bigg[x\frac{(\sqrt{1+u^2}+\sqrt{1+u^2x^2})}{(1+u^2x^2)^{3/2}}\{\sqrt{1+u^2}(\sqrt{1+u^2x^2}+1)^2-2(1+u^2x^2)$$
$$\times(\sqrt{1+u^2}+1)\}-\frac{1}{x}\frac{(\sqrt{1+u^2}+\sqrt{1+u^2/x^2})}{(1+u^2/x^2)^{3/2}}\{\sqrt{1+u^2}(\sqrt{1+u^2/x^2}+1)^2-2(1+u^2/x^2)(\sqrt{1+u^2}+1)\}\Bigg]$$
$$+2u\int_0^1\frac{dx}{(1-x^2)}\Bigg[1+\frac{(1+x^2)}{2x}\ln\frac{(1-x)}{(1+x)}\Bigg]\Bigg\{x^2\frac{(\sqrt{1+u^2}+\sqrt{1+u^2x^2})}{(1+u^2x^2)^{3/2}}-\frac{1}{x^2}\frac{(\sqrt{1+u^2}+\sqrt{1+u^2/x^2})}{(1+u^2/x^2)^{3/2}}\Bigg\}$$
$$+\frac{(E_p+2m)}{2m}u\int_0^1 dx\frac{(1+x^2)}{(1-x^2)}\Bigg[\frac{(1+x^2)}{2x}\ln\frac{(1+x)}{(1-x)}-1\Bigg]\Bigg[\frac{x^2}{(1+u^2x^2)^{3/2}}\Big(\sqrt{1+u^2}+\sqrt{1+u^2x^2}\Big)$$

$$-\frac{1}{x^4}\frac{1}{(1+u^2/x^2)^{3/2}}\Big(\sqrt{1+u^2}+\sqrt{1+u^2/x^2}\Big)\Bigg]-\frac{(E_p+2m)}{2m}u\int_0^1\frac{dx}{(1-x^2)}\Bigg[\Big(1+x^2\Big)+\frac{(1-x^2)^2}{2x}\ln\frac{(1-x)}{(1+x)}\Bigg]$$
$$\times\Bigg[\frac{x^2(\sqrt{1+u^2}+\sqrt{1+u^2x^2})}{(1+u^2x^2)^{3/2}}-\frac{1}{x^4}\frac{(\sqrt{1+u^2}+\sqrt{1+u^2/x^2})}{(1+u^2/x^2)^{3/2}}+2\left(\frac{E_p}{m}\right)u\Bigg\{\int_0^1\frac{dx}{(1-x^2)}\Bigg[\frac{x^2(\sqrt{1+u^2}+\sqrt{1+u^2x^2})}{(1+u^2x^2)^{3/2}}$$
$$-\frac{1}{x^2}\frac{(\sqrt{1+u^2}+\sqrt{1+u^2/x^2})}{(1+u^2/x^2)^{3/2}}\Bigg]+\int_0^1\frac{dx}{2x}\ln\frac{(1-x)}{(1+x)}\Bigg[\frac{x^2(\sqrt{1+u^2}+\sqrt{1+u^2x^2})}{(1+u^2x^2)^{3/2}}+\frac{1}{x^2}\frac{(\sqrt{1+u^2}+\sqrt{1+u^2/x^2})}{(1+u^2/x^2)^{3/2}}\Bigg]\Bigg\}$$
$$+2\frac{1}{u}\Bigg\{\Bigg\{\int_0^1\frac{dx}{(1-x^2)}\Bigg[\frac{x^2(\sqrt{1+u^2}+\sqrt{1+u^2x^2})}{(1+u^2x^2)}\Big\{\sqrt{1+u^2}\sqrt{1+u^2x^2}-1+2\Big(\sqrt{1+u^2x^2}-\sqrt{1+u^2}\Big)\Big\}$$
$$-\frac{1}{x^2}\frac{(\sqrt{1+u^2}+\sqrt{1+u^2/x^2})}{(1+u^2/x^2)}\Big\{\sqrt{1+u^2}\sqrt{1+u^2/x^2}-1+2\Big(\sqrt{1+u^2/x^2}-\sqrt{1+u^2}\Big)\Big\}\Bigg]$$
$$+\int_0^1\frac{dx}{2x}\ln\frac{(1+x)}{(1-x)}\Bigg[\frac{x^2(\sqrt{1+u^2}+\sqrt{1+u^2x^2})}{(1+u^2x^2)}\Big\{\sqrt{1+u^2}\sqrt{1+u^2x^2}-1+2\Big(\sqrt{1+u^2x^2}-\sqrt{1+u^2}\Big)\Big\}$$
$$+\frac{1}{x^2}\frac{(\sqrt{1+u^2}+\sqrt{1+u^2/x^2})}{(1+u^2/x^2)}\Big\{\sqrt{1+u^2}\sqrt{1+u^2/x^2}-1+2\Big(\sqrt{1+u^2/x^2}-\sqrt{1+u^2}\Big)\Big\}\Bigg]\Bigg\}\Bigg\}\Bigg]. \qquad (60)$$

Substituting Eqs. (55)–(60) into Eq. (41), we get the form factors G_1 and G_2 to be substituted into Eqs. (17) and (18) which give the decay constants f_{V,n^3S_1} and f_{V,n^3D_1}. Once we have the decay constants f_V, the leptonic decay rates can be obtained from Eqs. (6) or (7). It is interesting to see how our results lead to the leptonic decay rate in the static approximation that is in the limit when u or v go to zero. Using Eqs. (21) and (22) in Eqs. (24) and (25) and taking the limit $v\to 0$, we find that

$$\lim_{v\to 0}G_1'(p)=-\sqrt{2}\frac{E_p}{m}\left[1+\frac{\pi\alpha_s}{3v}-\frac{8\alpha_s}{3\pi}\right], \qquad (61)$$

$$\lim_{v\to 0}p^2G_2'(p)=0, \qquad (62)$$

$$\lim_{v,u\to 0}G_{1,1g}^{(0)\text{covar}}$$
$$=-\frac{\alpha_s}{3\pi}\left(\frac{\sqrt{2}E_p}{m}\right)\frac{1}{v}\int_0^1\frac{dx}{(1-x^2)}\ln\frac{(1+x)}{(1-x)}4\left(\frac{1}{x}-x\right)$$
$$=-\frac{\alpha_s}{3\pi}\left(\frac{\sqrt{2}E_p}{m}\right)\frac{4}{v}\int_0^1\frac{dx}{x}\ln\frac{(1+x)}{(1-x)}$$
$$=-\frac{\alpha_s}{3\pi}\left(\frac{\sqrt{2}E_p}{m}\right)\frac{4}{v}\frac{\pi^2}{4}-\left(\frac{\sqrt{2}E_p}{m}\right)\frac{\pi\alpha_s}{3v}, \qquad (63)$$

$$\lim_{v,u\to 0}p^2G_{2,1g}^{(0)\text{covar}}=0, \qquad (64)$$

$$\lim_{u\to 0}G_{1,1g}^{(0)L}=0, \qquad (65)$$

$$\lim_{u\to 0}p^2G_{2,1g}^{(0)L}=0. \qquad (66)$$

It is worth pointing out that the terms multiplying $-1/u$ in the expressions for $G_{1,1g}^{(0)L}$ and $G_{2,1g}^{(0)L}$ vanish since the integrand is proportional to u^2 for small u.

Substituting Eqs. (61)–(66) we find

$$\lim_{v\to 0}\left[G_1(p)-\frac{1}{3}p^2G_2(p)\right]=-\frac{\sqrt{2}E_p}{m}\left[1-\frac{8\alpha_s}{3\pi}\right]. \qquad (67)$$

Substituting Eq. (67) into Eq. (17) we obtain the following result for the decay constant f_{V,n^3S_1} in the static limit:

$$f_{V,n^3S_1}=\frac{m}{\pi}M^{-3/2}\frac{2}{\sqrt{3}}\int_0^\infty\frac{p^2dp}{E_p}\phi_{ns}(p)\left(\frac{-\sqrt{2}E_p}{m}\right)\left(1-\frac{8\alpha_s}{3\pi}\right)$$
$$=-\frac{m}{\pi}M^{-3/2}\frac{2}{\sqrt{3}}\frac{\sqrt{2}}{m}\left(1-\frac{8\alpha_s}{3\pi}\right)\int_0^\infty p^2dp\,\phi_{ns}(p) \qquad (68)$$
$$=-\frac{2\sqrt{2}}{\pi\sqrt{3}}M^{-3/2}\left(1-\frac{8\alpha_s}{3\pi}\right)\sqrt{2}\pi\psi(\vec{r}=0)$$
$$=-\frac{4}{\sqrt{3}}M^{-3/2}\left(1-\frac{8\alpha_s}{3\pi}\right)\psi(0). \qquad (69)$$

Substituting Eq. (68) into Eq. (6) we reproduce the decay rate to first order in α_s.

As we pointed out earlier the static approximation can lead to significant errors in charmonium since (v^2/c^2) is of the order of 0.2–0.3 in this system. Also in the static approximation the decay rate for the D state is exactly zero whereas experimentally it is quite significant, about 0.26 keV. So we intend to use the full G_1 and G_2 given by our equations to calculate the leptonic decay constant f_V for the 1^3S_1, 2^3S_1, and $^3D_1$ states of charmonium. We will do that in Sec. IV using the wave functions of the GJRS model [3].

IV. NUMERICAL RESULTS FOR $1\,^3S_1$, $2\,^3S_1$, and $1\,^3D_1$ STATES

Equation (69) gives the decay constant for the S states in the static limit. As shown earlier, this can be derived by letting $v \to 0$ in the expressions for the form factors. This leads to the suppression factor $1-8\alpha_s/3\pi$ in the decay constant when lowest order radiative corrections in α_s are calculated and also leads to Eq. (1) for the width.

One purpose of the present work was to investigate the relaxation of the above approximation by integrating over the bound state wave functions. A rigorous calculation requires a systematic bound state treatment with off-shell quarks and is very difficult. Instead of pursuing this approach, we elected to use on-shell vertex functions, which unfortunately have an infrared divergence. This divergence disappears in the $v \to 0$ limit, as can be seen in Eq. (21a) above, since ln $2m/\mu$ multiplies a function which goes to zero in this limit. However, for finite v this term does not vanish and we need some rationale for its evaluation.

We have carried out calculations using the wave functions obtained from the model of GJRS. These wave functions are provided in the Appendix. To do the calculations we need to chose an IR cutoff μ. We have done this in two ways to see how sensitive the result is to the choice of μ. The first choice utilizes $\mu=\mu_p$, where $\mu_p=\int_0^\infty p^2 dp \cdot p|\phi(p)|^2$ where $\phi(p)$ is the normalized radial wave function for the state. In the second choice we use $\mu=\mu_0$ where $\mu_0=[(-(M/2)^2+m^2)/M]$ is an invariant measure of the extent to which the quarks are off their mass shell. The results of our numerical integrations for the integrals I_{nS} and I_{nD} appearing in Eqs. 17 and 18 are as follows:

Values of integrals I_{nS} and I_{nD}

State	Infrared mass (GeV)	Static limit	Current result
$1S$	$\mu_p=0.855$	$-1.08\left(1-\frac{8\alpha_s}{3\pi}\right)$	$-1.007+0.459\alpha_s$
	$\mu_0=0.800$	$-1.08\left(1-\frac{8\alpha_s}{3\pi}\right)$	$-1.007+0.474\alpha_s$
$2S$	$\mu_p=0.878$	$-0.88\left(1-\frac{8\alpha_s}{3\pi}\right)$	$-0.803+0.408\alpha_s$
	$\mu_0=0.402$	$-0.88\left(1-\frac{8\alpha_s}{3\pi}\right)$	$-0.803+0.621\alpha_s$
$1D$	$\mu_p=1.024$	0	$0.178-0.0148\alpha_s$
	$\mu_0=0.351$	0	$0.178-0.075\alpha_s$

Numerical estimate of the leptonic decay rates and their comparison with experiment

$\alpha_s=0.313$, $m_c=2.208$ GeV, $M_{1\,^3S_1}=3.097$ GeV, $M_{2\,^3S_1}=3.685$ GeV, $M_{1\,^3D_1}=3.77$ GeV.

We used the GJRS model [3]:

State	Value of the infrared parameter (μ)	Predicted decay rate (keV)	Experiment: (keV)
$1\,^3S_1$	$\mu=0.855(\mu_p)$	11.42	(5.26 ± 0.37)
	$\mu=0.800$	11.30	
$2\,^3S_1$	$\mu=0.878$	4.93	(2.14 ± 0.21)
	$\mu=0.402$	4.00	
$1\,^3D_1$	$\mu=1.024$	0.069	(0.26 ± 0.04)
	$\mu=0.351$	0.055	

V. CONCLUDING REMARKS

The results quoted in the previous section have several problems. First, they do not agree with experiment. Second, they also seem to depend on the infrared cutoff parameter μ. We are forced to introduce the cutoff parameter μ because of the infrared divergent term in the Schwinger's form factor F_1. The origin of this divergence is due to the fact that we took the quark and the antiquark on mass shell. If we were dealing with the leptonic decay of a free quark-antiquark pair, this infrared divergence in the decay rate to order α_s will be canceled by the infrared divergence due to the emission or absorption of soft gluons by the quark and the antiquark. But here we have a color singlet quark-antiquark bound state where the quark and the antiquark are truly off mass shell. If we work within the framework of a truly bound state formalism such as Bethe-Salpeter, the quark and the antiquark will be off mass shell and we will not encounter any infrared divergences. We believe that the result of a Bethe-Salpeter calculation will be similar to our result except that in the infrared divergent logarithmic term, instead of the fictitious gluon mass μ we will have some parameter which has the dimensions of mass, and which indicates how far the quark or the antiquark is off mass shell in the bound state. In the absence of a more complete calculation involving the bound state, we can only make an educated guess what this parameter will be. It is not surprising that the result is somewhat sensitive to this parameter. But we do not think the qualitative nature of our results will change in a more complete calculation.

The essential conclusions which we draw from our results are the following. For the $1S$ and the $2S$ states the relativistic and the one-gluon radiative corrections lead to a suppression of the width significantly less than that predicted in the static limit. In the original work of Gupta, Johnson, Repko, and Suchyta (GJRS) [3], they were able to get reasonable agreement with experiment only because the radiative correction in the static limit was quite large, of the order of 60%. But as we have argued the static limit is not a reasonable approximation for charmonium. When we take into account the relativistic corrections, the suppression factor is reduced to ten to fifteen percent and we have disagreement with experiment for the leptonic decay rates in the GJRS model [3]. We do not believe the ambiguity in the infrared cutoff parameter will change this important conclusion and consequently for this reason we have not yet attempted to carry out the very difficult calculation which would lead to a result independent of an IR cutoff. For the $1D$ state we find that the inclusion of the relativistic and the radiative corrections produce a result different from the static nonrelativistic result of zero. A small mixing of $2\ ^3S_1$ state with the $1\ ^3D_1$ state would give good agreement with experiment for $\Gamma[\psi''(3770)\to e^+e^-]$, but as yet a realistic mechanism to produce sufficient mixing has not been found.

We should also mention that there are other approaches that have been applied to calculating the leptonic decay rates of quarkonium bound states, including their radiative corrections. For example, Durand and Durand [6], Dentamaro and Goldstein [7], and Duke and Kimel [8] have used the so-called ''duality relation'' and time-reversal invariance to relate $\Gamma((q\overline{q})_{\rm bound}\to e^+e^-)$ to $\sigma(e^+e^-\to(q\overline{q})_{\rm free})$. The cross-section $\sigma(e^+e^-\to(q\overline{q})_{\rm free})$ has been calculated to order α_s^2 before [9]. Infrared divergences in this calculation are canceled by the infrared divergences in $\sigma(e^+e^-\to(q\overline{q})_{\rm free}$ + real soft gluon). The observed $\sigma(e^+e^-\to(q\overline{q})_{\rm free})$ is the sum of $\sigma(e^+e^-\to(q\overline{q})_{\rm free})$ and $\sigma(e^+e^-\to(q\overline{q})_{\rm free}$ + real soft gluon). Such a procedure would not apply in our formalism. The duality relation [6] has only been derived in the JWKB approximation [6] for the cross sections without radiative corrections. It is not clear how it will apply when gluonic radiative corrections are also included.

The above-mentioned authors [6–8] have used other potential models [10]. The main reason they are able to get better agreement with experiment is that their radiative corrections of order α_s are much larger (of the order of forty percent) and their potential models give slightly smaller values for the leptonic decay rates without the radiative corrections. The radiative correction of order α_s^2 obtained by Dentamaro and Goldstein [7] has the opposite sign to that of the order α_s correction. So inclusion of such higher-order corrections is not likely to improve our results either.

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APPENDIX

We have calculated the momentum space radial wave functions for the GJRS [3] model using a variational approach. This model, which has a rather large charmed quark mass of 2.208 GeV produces excellent spectra in close agreement with experiment, as well as good results for radiative decays. The wave functions are

$$\phi_{^3S_1}(p)=\sum_{n=1}^{10} a_n\sqrt{\frac{2}{\pi}}R^2\frac{\Gamma(n+1)}{p}(\cos\,\theta)^{n+1}\,\sin[(n+1)\theta]$$

for $1\ ^3S_1$ and $2\ ^3S_1$ states

$$\phi_{1\,^3D_1}(p)=\sum_{n=1}^{8} a_n\sqrt{\frac{2}{\pi}}R^2\frac{\Gamma(n+3)}{p}(\cos\,\theta)^{n+3}\times\sin[(n+3)\theta]\left\{1+\frac{3}{n+1}+\frac{3}{pR(n+1)(n+2)}\times\left[\frac{(n+3)\cos[(n+3)\theta]}{\sin[(n+3)\theta]}-\frac{1}{Rp}\right]\right\}\quad\text{for}\quad 1\ ^3D_1.$$

In this expression the a_n are coefficients given below and $\Gamma(n+1)=n!$, $\tan\,\theta=pR$. The values of R are $R_{1\,^3S_1}=0.43$ GeV^{-1}, $R_{2\,^3S_1}=0.82$ GeV^{-1}, $R_{1\,^3D_1}=0.46$ GeV^{-1}.

The a_n are:

	$1\ ^3S_1$	$2\ ^3S_1$	$1\ ^3D_1$
a_1	1.34684183060	1.10370094192	0.03104816149
a_2	1.47543941745	1.30702688329	0.09780895082
a_3	0.30738386571	−0.72789877235	0.00001099680
a_4	0.07174239176	−0.07715517807	0.00452835801
a_5	0.01038137911	−0.00525212711	−0.00008205698
a_6	0.00118180693	0.00031138962	0.00016043162
a_7	0.00007979753	0.00021966952	−0.00000626100
a_8	−0.00001003665	0.00004228280	0.00000028000
a_9	0.00000060216	−0.00000789835	
a_{10}	0.00000008718	0.00000028197	

[1] R. Barbieri, R. Gatto, R. Kögenler, and Z. Kunszt, Phys. Lett. **57B**, 455 (1975); W. Celmaster, Phys. Rev. D **19**, 1517 (1979); E. C. Poggio and H. J. Schnitzer, *ibid.* **20**, 1175 (1979).

[2] J. Schwinger, *Particles*, *Sources and Fields* (Addison Wesley, Reading, MA, 1970), Vol. II.

[3] S. N. Gupta, J. M. Johnson, W. W. Repko, and C. J. Suchyta III, Phys. Rev. D **49**, 1551 (1994).

[4] L. Bergström, H. Snellman, and G. Tengstrand, Part. Fields **4**, 215 (1980).

[5] J. H. Kuhn, J. Kaplan, and E. G. O. Safiani, Nucl. Phys. **B157**, 125 (1979); L. Bergström, H. Grotch, and R. W. Robinett, Phys. Rev. D **43**, 2157 (1991).

[6] B. Durand and L. Durand, Phys. Lett. **113B**, 338 (1982); Phys. Rev. D **25**, 2312 (1982).

[7] A. V. Dentamaro and G. R. Goldstein, Can. J. Phys. **63**, 1481 (1985); **66**, 112 (1988).

[8] D. W. Duke and J. D. Kimel, Phys. Rev. D **25**, 2960 (1982).

[9] T. W. Appelquist and H. D. Politzer, Phys. Rev. D **12**, 1404 (1975); M. Dine and J. Sapirstein, Phys. Rev. Lett. **43**, 668 (1979).

[10] W. Buchmuller and S.-H. H. Tye, Phys. Rev. D **24**, 132 (1981); **22**, 3003 (1981); E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, and T. M. Yan, *ibid.* **17**, 3090 (1978); **21**, 203 (1980); A. Martin, Phys. Lett. **93B**, 338 (1980).