Multifractal analysis of extreme fluctuations in hadron multiplicities

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If a limited region of the final state phase space of multiparticle production in high-energy particle collisions is divided into many bins (cells) of equal size, it may happen occasionally that a single bin contains q particles and all the other bins are empty. It is pointed out that the probability $S(q)$ to find such extremely fluctuating events carries useful information which is independent of that carried by the scaled factorial moments F_q. The generalized dimension D_q can be estimated if $S(q)$ obeys a power law. Several models and experimental feasibility are examined in detail. [S0556-2821(97)03721-1]

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I. INTRODUCTION

The observation of seemingly large fluctuations of particle distributions in either cosmic ray [1] or accelerator [2] experiments stimulated further study of nonstatistical fluctuations and the possible multifractal structure behind them [3]. Among others, the Bialas-Peschanski intermittency approach using the scaled factorial moments has been extensively studied for several years [4,5]. The scaled factorial moment $F_q(\Delta)$ is evaluated for the multiplicity distribution in a limited region of the final state phase space of multiparticle production at high energies. Here, Δ is the size of the region called a bin or cell. The main task here is to see if $F_q(\Delta)$ obeys the power law

$$F_q(\Delta)\propto\Delta^{-\phi_q} \tag{1.1}$$

as Δ goes to zero. If such a behavior is observed, one may determine the generalized dimension as

$$D_q=1-\frac{\phi_q}{q-1}. \tag{1.2}$$

The purpose of this paper is to point out that there is another observable which allows an independent estimate of D_q. In Sec. II, we first recall the exclusive analysis of the factorial moments [4] and then derive the main result of the present paper. Several models are examined for illustration in Sec. III. The experimental feasibility of our approach is examined in Sec. IV. Section V is devoted to conclusions.

II. GENERAL FORMALISM

Consider a limited region of the final state phase space (e.g., an interval in the longitudinal rapidity space) and divide it into M bins (cells) of equal size. In the exclusive analysis adopted in Ref. [4], one considers an ensemble of events where just N particles are produced in the region. The fundamental assumption here is that there is an underlying probability distribution $P(p_1,\ldots,p_i,\ldots,p_M)$ which generates the observed joint multiplicity distribution $Q_N(k_1,\ldots,k_i,\ldots,k_M)$ by the relation

$$Q_N(k_1,\ldots,k_i,\ldots,k_M)=\int_0^1 dp_1\cdots\int_0^1 dp_i\cdots\int_0^1 dp_M P(p_1,\ldots,p_i,\ldots,p_M)\times\frac{N!p_1^{k_1}\cdots p_i^{k_i}\cdots p_M^{k_M}}{k_1!\cdots k_i!\cdots k_M!}\quad\text{for } N=1,2,\ldots, \tag{2.1}$$

where p_i is the probability of finding a particle in the ith bin, k_i is the multiplicity of particles in the ith bin,

$$\sum_{i=1}^{M}k_i=N, \tag{2.2}$$

$P(p_1,\ldots,p_i,\ldots,p_M)$ can be nonvanishing only when

$$\sum_{i=1}^{M}p_i=1, \tag{2.3}$$

and $P(p_1,\ldots,p_i,\ldots,p_M)$ is normalized as

$$\int_0^1 dp_1\cdots\int_0^1 dp_i\cdots\int_0^1 dp_M P(p_1,\ldots,p_i,\ldots,p_M)=1. \tag{2.4}$$

It should be noted here that $p_j^{k_j}=1$ for $p_j=k_j=0$, $j=1,2,...,M$ in Eq. (2.1). Bialas *et al.* have shown that [4]

$$\langle p_i^q\rangle=M^{-q}F_q^{(i)}(N)\quad\text{for } q=1,2,3,\ldots, \tag{2.5}$$

where

$$\langle p_i^q\rangle=\int_0^1 dp_1\cdots\int_0^1 dp_i\cdots\int_0^1 dp_M p_i^q P(p_1,\ldots,p_i,\ldots,p_M), \tag{2.6}$$

and

$$F_q^{(i)}(N)=M^q\sum_{k_1=0}^{N}\cdots\sum_{k_i=0}^{N}\cdots\sum_{k_M=0}^{N}k_i(k_i-1)\cdots(k_i-q+1)\times Q_N(k_1,\dots,k_i,\dots,k_M)/\{N(N-1)\cdots(N-q+1)\}. \quad (2.7)$$

Equations (2.1)–(2.7) summarize the general formalism of the scaled factorial moments in the exclusive analysis [4]. The point here is that one can estimate the expectation value of the qth moment of the directly unobservable probability p_i in terms of the observable quantity $F_q^{(i)}(N)$.

Now we would like to make a simple remark that, as is obvious from Eq. (2.1), $\langle p_i^q\rangle$ can also be given in terms of another independent observable:

$$\langle p_i^q\rangle=Q_q(0,\dots,0,k_i=q,0,\dots,0)=S_i(q,M) \quad \text{for } q=1,2,3,\dots . \quad (2.8)$$

It follows from Eqs. (2.5) and (2.8) that

$$M^{-q}F_q^{(i)}(N)=S_i(q,M) \quad \text{for } q=1,2,3,\dots . \quad (2.9)$$

This equality has to be tested experimentally for $q\geqslant 2$ (the $q=1$ case is trivial) in order to prove (or disprove) the validity of the fundamental relation (2.1). In particular, Eq. (2.9) suggests that $F_q^{(i)}(N)$ is independent of N.

According to the theory of multifractals, the generalized dimension D_q may be determined (mathematically) by the following limit [6]:

$$D_q=\frac{1}{1-q}\lim_{M\to\infty}\frac{\ln(\Sigma_{i=1}^{M}\langle p_i^q\rangle)}{\ln M} \quad \text{for } q\neq 1, \quad (2.10)$$

in particular,

$$D_0=\lim_{M\to\infty}\frac{\ln(\Sigma_{i=1}^{M}\langle p_i^0\rangle)}{\ln M}, \quad (2.11)$$

while

$$D_1=-\lim_{M\to\infty}\frac{\Sigma_{i=1}^{M}\langle p_i\ln p_i\rangle}{\ln M}. \quad (2.12)$$

It should be noted here that $\langle p_i^0\rangle$ is not necessarily equal to unity. As the ''empty'' bins have to be excluded when one takes the expectation value of p_i^q (this is particularly crucial when q is negative), $\langle p_i^0\rangle$ represents the probability that p_i is nonzero. It should be noted also that, in Eq. (2.1), $Q_N(k_1,\dots,k_i,\dots,k_M)$ is defined only for $N=1,2,\dots$, and hence $S_i(q,M)$ is defined only for $q=1,2,\dots$, by Eq. (2.8). Then, $\langle p_i^0\rangle$ is formally evaluated by taking the limit

$$\begin{aligned}\langle p_i^0\rangle&=\lim_{q\to 0+}S_i(q,M)\\&=\lim_{q\to 0+}\int_0^1 dp_1\cdots\int_0^1 dp_i p_i^q\cdots\times\int_0^1 dp_M P(p_1,\dots,p_i,\dots,p_M)\\&=\int_0^1 dp_1\cdots\int_{0+}^1 dp_i\cdots\int_0^1 dp_M P(p_1,\dots,p_i,\dots,p_M).\end{aligned} \quad (2.13)$$

On the other hand, the expectation value $\langle p_i\ln p_i\rangle$ is expressed in terms of $S_i(q,M)$ as

$$\langle p_i\ln p_i\rangle=\left.\frac{\partial S_i(q,M)}{\partial q}\right|_{q=1}. \quad (2.14)$$

The proof is as follows. From Eq. (2.1), one has

$$S_i(q,M)=\int_0^1 dp_1\cdots\int_0^1 dp_i\cdots\int_0^1 dp_M p_i^q P(p_1,\dots,p_i,\dots,p_M) \quad (2.15)$$

for $q>0$. Differentiating both sides with respect to q and putting $q=1$ lead to Eq. (2.14) provided that $P(p_1,\dots,p_i,\dots,p_M)$ is independent of q.

Now Eq. (2.10) indicates physically that $S_i(q,M)$ for $q\neq 1$ will show a scaling behavior

$$\sum_{i=1}^{M}S_i(q,M)=M^{(1-q)D_q} \quad (2.16)$$

for increasing M if $P(p_1,\dots,p_i,\dots,p_M)$ possesses a multifractal structure. The experimental test of Eq. (2.16) is independent of the test of Eq. (1.1). Of course, D_q determined by Eq. (1.2) and that by Eq. (2.16) have to coincide with each other if the fundamental formula (2.1) is correct.

Finally, Eq. (2.12) suggests a scaling behavior

$$\sum_{i=1}^{M}\left.\frac{\partial S_i(q,M)}{\partial q}\right|_{q=1}=-D_1\ln M \quad (2.17)$$

for increasing M.

If the interval is so narrow that all the bins are equivalent, i.e., $S_i(q,M)=S(q,M)$, one has

$$S(q,M)=M^{(1-q)D_q-1}, \quad (2.18)$$

and

$$\left.\frac{\partial S(q,M)}{\partial q}\right|_{q=1}=-\frac{D_1\ln M}{M}. \quad (2.19)$$

III. MODELS

For illustration of the results obtained in the preceding section, we consider several models.

A. Statistical model

If all the N particles distribute homogeneously and independently in the interval, the underlying probability distribution is given by

$$P(p_1,\ldots,p_i,\ldots,p_M)=\prod_{j=1}^{M}\delta\left(p_j-\frac{1}{M}\right). \tag{3.1}$$

Substituting Eqs. (3.1) into Eq. (2.15), one obtains

$$S(q,M)=M^{-q}. \tag{3.2}$$

Equations (2.13), (2.14), with Eq. (3.2) give

$$\langle p_i^0\rangle=1, \tag{3.3}$$

$$\langle p_i \ln p_i\rangle=-\ln M/M, \tag{3.4}$$

from which one has

$$D_0=D_1=1. \tag{3.5}$$

Finally, Eq. (2.18) with Eq. (3.2) gives

$$D_q=1 \quad \text{for } q\neq 1. \tag{3.6}$$

To summarize, we have a trivial geometrical dimension $D_q=1$ for any real q.

B. Complete spike model

In order to understand the particular role played by the empty bins, let us consider the distribution

$$P(p_1,\ldots,p_i,\ldots,p_M)=M^{-1}\sum_{j=1}^{M}\left(\prod_{k=1}^{M}\delta(p_k-\delta jk)\right). \tag{3.7}$$

All the bins except for the jth bin are empty (such an event may be called a complete spike event at level M) if particles are produced according to the probability given by the jth term on the right-hand side (RHS). Equation (3.7) yields

$$S(q,M)=1/M \quad \text{for } q>0, \tag{3.8}$$

from which one has

$$\langle p_i^q\rangle=1/M \quad \text{for } q>0 \tag{3.9}$$

and

$$\langle p_i \ln p_i\rangle=0. \tag{3.10}$$

Equation (3.9) gives

$$\langle p_i^0\rangle=\lim_{q\to 0+}S(q,M)=1/M. \tag{3.11}$$

All of these results imply that

$$D_q=0 \quad \text{for } q>0. \tag{3.12}$$

C. α model of intermittency

Both models considered in the preceding subsections are trivial because they give q-independent generalized dimensions. Now we consider a nontrivial model used by Bialas *et al.* [4,5]. In this model, a given interval of the phase space is divided into λ bins of equal size and random numbers are assigned to every bin in the first step. Each bin is divided into λ bins similarly and random numbers are assigned again to every bin in the second step, and the same procedure is repeated ν times. Then the total number of bins is

$$M=\lambda^\nu. \tag{3.13}$$

The probability p_i is then given by

$$p_i=M^{-1}\prod_{a=1}^{\nu}w_a, \tag{3.14}$$

where $\{w_a:\ a=1,2,\ldots,\nu\}$ is a set of non-negative random numbers that are generated along the cascade down to the ith cell at the νth level. The spectral function $\rho(w_a)$ of the random numbers satisfies the conditions

$$\int_0^\infty dw\,\rho(w)=1, \tag{3.15}$$

$$\int_0^\infty dw\,w\rho(w)=1. \tag{3.16}$$

In general, we will use the average of a function $f(w)$ defined as

$$\langle f(w)\rangle=\int_{0+}^\infty dw\,f(w)\rho(w). \tag{3.17}$$

Note that $\langle w^0\rangle$ is less than unity in spite of Eq. (3.15) when $\rho(w)$ has a δ-function-like singularity at $w=0$.

In this model, one has

$$S(q,M)=M^{-q}\langle w^q\rangle^{\ln M/\ln\lambda} \tag{3.18}$$

for a real q, where the relation $\nu=\ln M/\ln\lambda$ has been used. Equations (2.13) and (2.14) then give

$$\langle p_i^0\rangle=\langle w^0\rangle^{\ln M/\ln\lambda}, \tag{3.19}$$

and

$$\langle p_i \ln p_i\rangle=-\ln M/M+M^{-1}\nu\,\langle w\ln w\rangle, \tag{3.20}$$

With Eqs. (2.11) and (2.12), one obtains

$$D_0=1+\frac{\ln\langle w^0\rangle}{\ln\lambda}, \tag{3.21}$$

$$D_1=1-\frac{\langle w\ln w\rangle}{\ln\lambda}. \tag{3.22}$$

In general,

$$D_q=1-\frac{\ln\langle w^q\rangle}{(q-1)\ln\lambda} \quad \text{for } q\neq 1, \tag{3.23}$$

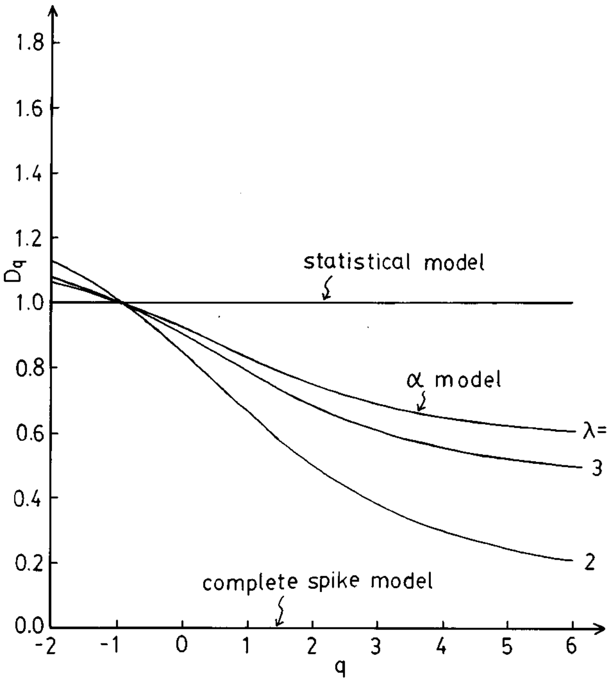

FIG. 1. Generalized dimensions in three models.

which includes Eq. (3.21).

The results of the three models are summarized in Fig. 1. Here, as an example, the spectral function of the α model is chosen as

$$\rho(w)=\alpha\delta(w)+\beta\delta(w-1+a)+(1-\alpha-\beta)\delta(w-1-b), \tag{3.24}$$

with $\alpha=0.1$, $\beta=0.6$, $a=0.3$, and $b=(\alpha+a\beta)/(1-\alpha-\beta)=0.933\cdots$. In this case, the generalized dimension D_q is given by Eq. (3.23) with

$$\langle w^q\rangle=\beta(1-a)^q+(1-\alpha-\beta)^{1-q}\{1-(1-a)\beta\}^q. \tag{3.25}$$

The results for $\lambda=2$, 3, and 4 are shown in Fig. 1.

IV. FEASIBILITY OF SPIKE EVENT ANALYSIS

In this section, we examine the feasibility of determining experimentally the probability $S_i(q,M)$ for detecting the complete spike event. Our approach would turn out to be useless if $S_i(q,M)$'s for interesting ranges of q and M were too small to be determined experimentally even when there are physically interesting fluctuations of nonstatistical origin.

As the first step of the analysis, one has to collect a sufficiently large number (say, 10^4) of events where just q particles are produced in the interval of the phase space under consideration. This may be done rather easily by using an appropriate multiplicity trigger. For simplicity, suppose that

$$S_1(q,M)=S_2(q,M)=\cdots=S_M(q,M)=S(q,M). \tag{4.1}$$

In this case, it is sufficient to consider the total probability for the complete spike event given by

TABLE I. Boundary values of q in two models.

	Statistical model	Random cascade model		
M		$\lambda=2$	$\lambda=3$	$\lambda=4$
2	11	189	25	19
4	6	87	12	10
8	4	53	8	7
16	3	36	6	5
32	3	26	5	4
64	3	19	4	4

$$\sum_{i=1}^{M} S_i(q,M)=MS(q,M). \tag{4.2}$$

If $MS(q,M)$ is of the order of 10^{-3}, one will observe some ten complete spike events among the 10^4 events, obtaining an experimental value of $MS(q,M)$ with some 30% statistical error. Thus we set the following lower bound for practically measurable $MS(q,M)$:

$$MS(q,M)>10^{-3}. \tag{4.3}$$

The most pessimistic estimate of $S(q,M)$ is given by the statistical model defined by Eq. (3.1) because complete spike events take place as a result of statistical fluctuations in this case. We thus put

$$MS(q,M)=M^{1-q}>10^{-3}. \tag{4.4}$$

The boundary values of q as a function of M are shown in Table I. It is obvious that the analysis can be done easily at least up to $q=3$ for a sufficiently large M. An experimental analysis would be difficult for larger q. However, the statistical model gives the result for the most disadvantageous case. The situation is much better if there are nonstatistical fluctuations. In fact, the boundary values of q become much larger if one adopts the random cascade model with the spectral function (3.24). The results for $\lambda=2$, 3, and 4 are shown in Table I. In this case, it will be easy to carry out experimental study for $q=2$ up to several tens.

V. CONCLUSIONS

The standard approach to intermittency or multifractality uses the scaled factorial moments that are supposed to reflect the underlying probability distribution. We have shown that the probability of finding complete spike events carries equivalent information. If $S(q,M)$ is measured experimentally, one should first examine whether or not $\ln\{S(q,M)\}$ behaves as linear functions of $\ln M$ for a sufficiently wide range of M. If the linear-log behavior is observed, one can determine D_q by using Eq. (2.18). Equations (1.1) and (2.18) provide independent tests of the multifractality while Eq. (2.9) provides a consistency check of the existence of the underlying probability distribution. As a byproduct, it is suggested that $F_q^{(i)}(N)$ will be independent of N. There is indeed experimental evidence that the moments have no strong N dependence [7]. The experimental study of the intermittency have been based on the inclusive approach in most cases. An apparent reason is to achieve good statistics. The present work suggests that the exclusive approach with the analysis of complete spike events is complementary to the inclusive approach and is equally important. Furthermore, it has been shown that such an analysis is experimentally most feasible when there are interesting nonstatistical fluctuations.

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