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Perturbative $O(\alpha_s^2)$ corrections to the hadronic cross section near heavy quark-antiquark thresholds in e^+e^- annihilation

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It is demonstrated how perturbative $O(\alpha_s^2)$ relativistic corrections to the nonrelativistic stable heavy quark-antiquark production cross section in e^+e^- annihilation based on a Coulombic QCD potential can be systematically calculated using the concept of effective field theories. The $O(\alpha_s^2)$ corrections from the relativistic energy-momentum relation, the relativistic phase space corrections and the $O(\underline{\alpha_s^2})$ corrections involving the group theoretical factors C_F^2 and C_FT are determined explicitly. For the case of $t\bar{t}$ production the sum of these corrections amounts to 3%–7% over the whole threshold regime and is insensitive to variations in the top width. Perturbative $O(\alpha_s^2)$ corrections to the leptonic decay width of heavy quark-antiquark ${}^3S_1$ vector resonances are extracted. [S0556-2821(97)00421-9]

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In the kinematic regime where the c.m. energy is much larger than the quark masses, the total hadronic cross section in e^+e^- collisions $R_{\rm had}=\sigma(e^+e^-\to\text{hadrons})/\sigma_{pt}$, where σ_{pt} is the point cross section, belongs to the theoretically best known and understood quantities in electroweak physics. Inspired by the astonishing precision of the experiments at the CERN e^+e^- collider LEP $R_{\rm had}$ has been calculated up to $O(\alpha_s^3)$ accuracy for LEP energies based on sophisticated multiloop techniques [1]. In the kinematic regime near quark-antiquark thresholds, however, which is characterized by the relation

$$|\beta|\lesssim\alpha_s\,,\quad \beta\equiv\sqrt{1-\frac{4M_Q^2}{s+i\epsilon}}\,,\tag{1}$$

where M_Q is the quark mass[1] and $\sqrt{s}$ the c.m. energy, the theoretical understanding of $R_{\rm had}$ is much poorer. Although perturbative methods to describe the hadronic cross section in the threshold regime in general seem to be more than questionable due to nonperturbative effects and due to the large size of the strong coupling, it is well accepted that a perturbation theory based description of the hadronic cross section near a quark-antiquark threshold should be possible if the quark mass is sufficiently larger than the hadronic scale $\Lambda_{\rm QCD}$ [2]. The prototype application of perturbative methods in the threshold regime is therefore the case of $t\bar{t}$ production[2] since nonperturbative effects are suppressed due to the large top quark mass and width [3,4] and since the strong coupling is sufficiently small, $\alpha_s(C_FM_t\alpha_s)\approx0.15$. The most common approach found in literature is based on numerical solutions of a nonrelativistic Schrödinger equation with a phenomenological QCD potential where the short-distance part originates from loop calculations and the long-distance part is obtained from fits to charmonium and bottomonium spectra [7]. Although this approach leads to applicable results, the use of a phenomenological potential makes the systematic implementation of dynamic as well as kinematic relativistic corrections difficult, if not impossible. As a consequence, all potential model predictions for the cross section in the threshold region contain intrinsic uncertainties of relative order $|\beta^2|\approx\alpha_s^2$. The actual size of these $O(\alpha_s^2)$ corrections[3] and their dependence on the c.m. energy is unknown because a consistent and systematic determination of these corrections has never been achieved. It is the purpose of this paper to demonstrate how relativistic perturbative $O(\alpha_s^2)$ QCD corrections to the nonrelativistic stable heavy quark-antiquark cross section predicted by the nonrelativistic Schrödinger equation with a Coulombic QCD potential $V_{\rm QCD}=-C_F\alpha_s/r$ and a fixed α_s can be calculated. We want to emphasize that this work is not meant to present all calculational details, but to show the main steps and results of our calculations. A more detailed and explicit work will be published later. We also would like to mention that at no point in this work are nonperturbative effects and electroweak corrections taken into account.

The formula for the single photon mediated (i.e., vector-current induced) nonrelativistic cross section valid also for complex energies reads [8]

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[1]Throughout this paper the quark mass M_Q is understood as the pole mass and β is called "velocity."

[2]An application of perturbative methods might also be possible for the $b\bar{b}$ production cross section for low radial excitations of the Υ family because in that case nonperturbative effects seem to be well under control [5,6] and the strong coupling is still small enough that a perturbative calculation might be justified.

[3]Relativistic corrections to the nonrelativistic cross section actually result in a series in powers of α_s and β. Because we consider the kinematic regime $|\beta|\lesssim\alpha_s$, we count powers of β as powers of α_s. For simplicity we will call the $O(\alpha_s^2)$, $O(\alpha_s\beta)$ and $O(\beta^2)$ relativistic corrections briefly "$O(\alpha_s^2)$ corrections."

$$R^{Q\overline{Q},\mathrm{NR}}=\left[\frac{\sigma(e^+e^-\stackrel{\gamma^*}{\rightarrow}Q\overline{Q})}{\sigma_{pt}}\right]_{\mathrm{NR}}=\frac{3}{2}N_c e_Q^2\ \mathrm{Im}\bigg\{i\beta-C_F\alpha_s \times\left[\gamma+\ln(-i\beta)+\Psi\left(1-i\frac{C_F\alpha_s}{2\beta}\right)\right]\bigg\}, \tag{2}$$

where Ψ is the digamma function, $\Psi(z)\equiv d/dz\ \ln\Gamma(z)$, $\sigma_{pt}=4\pi\alpha^2/(3s)$, and $N_c=3$ is the number of colors. e_Q is the electric heavy quark charge and α denotes the fine structure constant. In the nonrelativistic limit and for unstable quarks, where the relation between the energy $E\equiv\sqrt{s}-2M_Q$ relative to the threshold point and the velocity can be approximated as $\beta=[(E+i\Gamma_Q)/M_Q]^{1/2}$, Γ_Q being the width of the heavy quark, formula (2) coincides with an expression given in [3,4]. For stable quarks and above threshold expression (2) leads to the famous Sommerfeld expression

$$R^{Q\overline{Q},\mathrm{NR}}_{\Gamma=0,\beta>0}=\frac{3}{2}N_c e_Q^2\ \frac{C_F\alpha_s\pi}{1-\exp\left(-\frac{C_F\alpha_s\pi}{\beta}\right)}, \tag{3}$$

whereas for stable quarks and below threshold Eq. (2) develops narrow resonances at the well-known Coulomb energy levels [9,8]. So far only $O(\alpha_s)$ QCD corrections to the nonrelativistic cross section from the running of the strong coupling [7,10,5] and from short distances [11] have been successfully calculated. The knowledge of the $O(\alpha_s^2)$ corrections to expressions (2) and (3) is important in order to understand the structure and importance of relativistic corrections and might even lead to hints on the form of nonperturbative effects. Because it is quite unclear how the $O(\alpha_s^2)$ relativistic corrections to the nonrelativistic cross section obtained from a Coulombic potential have to be implemented consistently into a potential model approach, they will serve as an order of magnitude estimate for the uncertainties inherent in potential model predictions.

In this paper we explicitly determine the $O(\alpha_s^2)$ corrections to the total single photon mediated heavy quark pair production cross section from the relativistic energy-momentum relation, the relativistic phase space corrections and those $O(\alpha_s^2)$ corrections which are multiplied by the SU(3) group theoretical factors C_F^2 and C_FT ($C_F=4/3$, $T=1/2$), where for the latter contribution only the effects from the heavy quark itself are taken into account. We would like to stress that these corrections represent a gauge invariant subset of all $O(\alpha_s^2)$ QCD corrections and that no model-like assumptions are imposed for our calculation. The $O(\alpha_s^2)$ QCD corrections involving the SU(3) group theoretical factors C_AC_F and C_FTn_l, where $C_A=3$ and n_l is the number of massless quark flavors, including the $O(\alpha_s^2)$ effects from the running of the strong coupling are not treated in this work, but can be determined along the lines presented here. For the case of a non-negligible width of the heavy quark (as in $t\overline{t}$ production) further types of corrections have to be taken into account. Apart from the perturbative (multiloop) corrections to the width of the free heavy quark, also corrections from the off-shellness of the decaying heavy quark, from the interaction among the decay products and the other heavy quark (if it is not decayed yet) and from time dilatation effects need to be considered. Although the size and the interplay of all these effects have been examined at various places in literature (see e.g., [12,13,14]) they are still not completely understood yet as far as $O(\alpha_s^2)$ corrections to the cross section are concerned. However, it is agreed in [12,13,14] that the bulk of these effects can be accounted for by using a momentum dependent width instead of a constant one. This difficulty is completely ignored in this work. Therefore, all formulas presented here are, strictly speaking, only valid for stable quarks. However, during our calculations we never assume that the squared velocity β^2 is a real number. Therefore, our results could be easily implemented in a more complete approach which treats all finite width effects properly. For now we will use the naive replacement $E\rightarrow E+i\Gamma_Q$ in the spirit of [3,4], where Γ_Q represents an appropriately chosen constant heavy quark width, which is not necessarily equal to the decay width of a free heavy quark. We think that this procedure is justified in order to demonstrate the size of the $O(\alpha_s^2)$ corrections calculated in this work in the presence of a large quark width. We finally would like to emphasize that relativistic corrections from the exchange of noninstantaneous gluons (responsible for Lamb shift type corrections and real radiation effects for $s>4M_Q^2$) are of order α_s^3 [15,16] and beyond the level of accuracy intended in this work.

Let us start by reminding the reader that by means of the optical theorem the nonrelativistic cross section $R^{Q\overline{Q},\mathrm{NR}}$, Eq. (2), can be written as [3,4]

$$R^{Q\overline{Q},\mathrm{NR}}=\mathrm{Im}\left[N_c e_Q^2\frac{24\pi}{s}G_c(0,0)\right], \tag{4}$$

where G_c is the Coulomb Green function satisfying the equation of motion

$$\left[-\frac{1}{M_Q}\vec{\nabla}_{\vec{x}}^2-\frac{a}{|\vec{x}|}-\frac{\vec{p}^{\,2}}{M_Q}\right]G_c(\vec{x},\vec{y})=\delta^{(3)}(\vec{x}-\vec{y}),\quad a\equiv C_F\alpha_s. \tag{5}$$

$\vec{p}$ is called the ''external three momentum'' for the rest of this work. It should be noted that $\vec{p}^{\,2}/M_Q$ is equal to the energy $E=\sqrt{s}-2M_Q$ only up to higher orders in E. In quantum mechanics textbooks usually the convention is employed where these higher order terms are set to zero. As explained later, we set $\vec{p}^{\,2}$ equal to the squared (relativistic) three momentum of heavy quarks in the c.m. frame. An explicit analytic expression for G_c in coordinate space representation has to our knowledge been calculated the first time in [17]. Our strategy is to calculate relativistic corrections to the Coulomb Green function G_c by using (textbook quantum mechanics) time-independent perturbation theory (TIPT). This approach is justified because only the exchange of instantaneous gluons needs to be taken into account. Using the concept of effective field theories the arising UV divergences are removed by matching to recent two-loop results for the cross section in the threshold region in the framework of QED [15]. The cross section is then obtained via the optical theorem, Eq. (4). Conceptually we follow the lines presented in [8].

The relativistic corrections to the Coulomb Green function arise from three different sources: (a) the relativistic energy-momentum relation, (b) $1/M_Q^2$ corrections to the Coulomb potential $V_{\rm QCD}$, and (c) $1/M_Q^2$ corrections from the electromagnetic current which produces and annihilates the quark-antiquark pair. The corrections from the relativistic energy-momentum relation can be easily determined by taking into account that G_c can be written in the form [4]

$$G_c(\vec{x},0)=\int \frac{d^3\vec{p}_0}{(2\pi)^3}\, e^{i\vec{p}_0\vec{x}}\,\frac{M_Q}{\vec{p}_0^{\,2}-\vec{p}^{\,2}-i\epsilon}\times\sum_{m=0}^{\infty}\prod_{n=1}^{m}\int \frac{d^3\vec{p}_n}{(2\pi)^3}\,\frac{4\pi a}{(\vec{p}_{n-1}-\vec{p}_n)^2}\,\frac{M_Q}{\vec{p}_n^{\,2}-\vec{p}^{\,2}-i\epsilon}. \tag{6}$$

The relativistic expression for the particle-antiparticle propagation $\int d\vec{p}_n^{\,3}/(2\pi)^3 M_Q/(\vec{p}_n^{\,2}-\vec{p}^{\,2}-i\epsilon)$ reads

$$-i\int \frac{d^4p_n}{(2\pi)^4}\,S[p_n+(\sqrt{M_Q^2+\vec{p}^{\,2}},\vec{0})]\times S[p_n-(\sqrt{M_Q^2+\vec{p}^{\,2}},\vec{0})],\quad p_n=(p_n^0,\vec{p}_n), \tag{7}$$

where S is the common Dirac propagator. Because the Coulomb interaction is instantaneous, the p_n^0 integration can be carried out. An expansion in $1/M_Q^2$ then yields that the corrections from the relativistic energy-momentum relation can be implemented into expression (6) by the replacement

$$\frac{M_Q}{\vec{p}_n^{\,2}-\vec{p}^{\,2}-i\epsilon}\to\frac{M_Q}{\vec{p}_n^{\,2}-\vec{p}^{\,2}-i\epsilon}\left[1+\frac{\vec{p}_n^{\,2}+\vec{p}^{\,2}}{4M_Q^2}\right] \tag{8}$$

for each particle-antiparticle propagator. As mentioned earlier, the form of the correction factor on the right-hand side (RHS) of Eq. (8) differs from the usual kinetic energy correction used in quantum mechanics textbooks because the relation between the external momentum $\vec{p}$ and the c.m. energy reads

$$\frac{\vec{p}^{\,2}}{M_Q^2}\equiv\frac{\beta^2}{1-\beta^2}. \tag{9}$$

For stable quarks, definition (9) leads to the relation $\vec{p}^{\,2}/M_Q=E+E^2/(4M_Q)+O(E^3)$ between the external momentum $\vec{p}$ and the energy E. Any other definition of $\vec{p}^{\,2}$ would lead to a different form of the RHS of Eq. (8). The final result for the cross section, of course, is independent of this choice. We have chosen definition (9) to facilitate our calculations. The $1/M_Q^2$ corrections to the interaction potential are well known and read

$$\widetilde{V}(\vec{Q})=-\frac{4\pi a}{\vec{Q}^2}+\frac{\pi a}{M_Q^2}+\frac{4\pi a}{M_Q^2}\left[\vec{S}_1\vec{S}_2-\frac{(\vec{Q}\vec{S}_1)(\vec{Q}\vec{S}_2)}{\vec{Q}^2}\right]-\frac{4\pi a}{M_Q^2}\left[\frac{\vec{p}^{\,2}}{\vec{Q}^2}-\frac{(\vec{Q}\vec{p})^2}{\vec{Q}^4}\right]-i\,\frac{6\pi a}{M_Q^2}\left[(\vec{S}_1+\vec{S}_2)\,\frac{\vec{Q}\times\vec{p}}{\vec{Q}^2}\right] \tag{10}$$

in momentum space representation, where $\vec{Q}$ is the (three) momentum flowing through the gluons and the $\vec{S}_{1/2}$ represent the quark/antiquark spin operators. In Eq. (10) the Coulomb interaction is also displayed. The $1/M_Q^2$ corrections from the electromagnetic current which produces and annihilates the heavy quark-antiquark pair lead to the insertion of the factor $\{1-\vec{p}_0^{\,2}/(3M_Q^2)[\frac{3}{4}+\vec{S}_1\vec{S}_2]\}$ into expression (6).

Taking into account that the $Q\overline{Q}$ pair is produced in a $(J^{PC}=1^{--})$ $^3S_1$ state,[4] the relativistic corrections to the Coulomb Green function can be rewritten in terms of corrections induced by an effective interaction potential which in coordinate space representation takes the simple form

$$V_{3S1}(\vec{x})=-\frac{a}{|\vec{x}|}\left[1+\frac{3\vec{p}^{\,2}}{2M_Q^2}\right]+\frac{11}{3}\,\frac{\pi a}{M_Q^2}\,\delta^{(3)}(\vec{x})-\frac{5}{4M_Q}\,\frac{a^2}{|\vec{x}|^2}, \tag{11}$$

where the Coulomb potential is also displayed. In addition, there remains a correction to the Coulomb Green function, which cannot be expressed in terms of an interaction potential. This correction takes the form

$$\delta G_c(0,0)=-\lim_{|\vec{x}|\to 0}\left[\frac{1}{M_Q^2}\left(\frac{7}{6}\,\vec{\nabla}^2+\vec{p}^{\,2}\right)G_c(\vec{x},0)\right] \tag{12}$$

and essentially represents relativistic phase space corrections.

It is now straightforward to determine all $O(\alpha_s^2)$ corrections to the Coulomb Green function. The corrections from the first term on the RHS of Eq. (11), called kinetic energy corrections later in this paper, can be trivially implemented by the replacement $a\to a(1+3\vec{p}^{\,2}/2M_Q^2)$ in G_c. The corrections from the second and third term, later called dynamical corrections, can be calculated via coordinate space TIPT. The phase space corrections of Eq. (12) can be evaluated by employing the equation of motion (5). The arising UV (short-distance) divergences can be regularized by considering corrections to $G_c(\vec{x},0)$ and taking the limit $|\vec{x}|\to 0$ afterwards.[5] Taking into account that the relation between the cross section $R^{Q\overline{Q}}$ and the Green function, Eq. (4), leads to another $1/M_Q^2$ phase space correction and using relation (9), the result for the cross section reads

$$R^{Q\overline{Q}}=\frac{3}{2}N_c e_Q^2 a\ \mathrm{Im}[H_a(a,\beta)]\left\{1+a[\mathrm{div}]+a^2[\mathrm{div}]+\frac{2}{3}a^2\ \mathrm{Re}[H_a(a,\beta)]\right\}, \tag{13}$$

where

[4]The production of a $^3D_1$ state is proportional to the modulus squared of the second derivative of the heavy quark-antiquark wave function at the origin and therefore suppressed by $|\beta|^4\approx\alpha_s^4$. This is beyond the intended accuracy.

[5]We emphasize that this method is not claimed to be a consistent way of UV regularization in coordinate space. However, in our case a quite sloppy treatment of UV divergences is allowed because we later match our result directly to the two-loop expression for the cross section in the threshold region. (See also [8].)

$$H_a(a,\beta)\equiv\left(1-\frac{1}{3}\beta^2\right)\left\{i\frac{\beta}{a}-(1+\beta^2)\left[\gamma+\ln(-i\beta)+\Psi\left(1-ia\frac{1+\beta^2}{2\beta}\right)\right]\right\}. \tag{14}$$

It should be noted that the term $\ln(-i\beta)$ in the function H_a does not lead to a singular behavior of H_a in the limit $\beta\to 0$ because this logarithm is cancelled by a corresponding logarithmic term generated by the digamma function in the same limit. It is a remarkable fact that in the framework of an expansion in Feynman diagrams the combination $\gamma+\Psi[1-ia(1+\beta^2)/2\beta]$ is generated entirely by diagrams of higher order than the diagrams which produce the explicit term $\ln(-i\beta)$. (See also [8].) The terms in Eq. (13) symbolized by [div] represent divergent and β-independent contributions. The divergences originate from the integration region $|\vec{x}|\to 0$ in TIPT and have to be considered as UV divergences which indicate that the electromagnetic current which produces and annihilates the heavy quark pair has to be renormalized. This renormalization is usually achieved by the determination of the corresponding counterterm via matching to amplitudes calculated in covariant (multiloop) perturbation theory in the framework of QCD [18,19]. Fortunately this lengthy procedure is not necessary in our case because expression (13) can be matched directly to a recent two-loop calculation of the fermion-antifermion cross section in the threshold region in the framework of QED [15] which is sufficient to determine the $O(\alpha_s^2)$ corrections in which we are interested. This ''direct matching'' procedure [8] is carried out in the formal limit $a\ll\beta\ll 1$ for stable quarks, where predictions in the nonrelativistic effective theory and conventional multiloop perturbation theory in QCD have to coincide. Expanding up to next-to-next-to-leading order in β [and including only the $O(\alpha_s^2)$ contributions with the color factors C_F^2 and C_FT in which we are interested] the two-loop expression (i.e., including Born, one-loop, and two-loop contributions) for the cross section reads [15]

$$R_{2\ \text{Loop}}^{Q\overline{Q}}=N_ce_Q^2\left\{\left(\frac{3}{2}\beta-\frac{1}{2}\beta^3\right)+\frac{C_F\alpha_s}{\pi}\left(\frac{3\pi^2}{4}-6\beta+\frac{\pi^2}{2}\beta^2\right)+\alpha_s^2\left[\frac{C_F^2\pi^2}{8\beta}-3C_F^2+\left(\frac{5C_F^2\pi^2}{24}+\frac{3}{2}C_2-C_F^2\ln\beta\right)\beta\right]\right\}, \tag{15}$$

where

$$C_2\equiv C_F^2\left[\frac{1}{\pi^2}\left(\frac{39}{4}-\zeta_3\right)+\frac{4}{3}\ln 2-\frac{35}{18}\right]+C_FT\left[\frac{4}{9}\left(\frac{11}{\pi^2}-1\right)\right]. \tag{16}$$

Expanding Eq. (13) in the same way and demanding equality to expression (15) the divergent contributions in Eq. (13) are unambiguously removed and replaced by constant (i.e., β-independent) terms. The final result for the cross section then reads

$$R^{Q\overline{Q}}=\frac{3}{2}N_ce_Q^2C_F\alpha_s\text{Im}[H_a(C_F\alpha_s,\beta)]\left\{1-4C_F\frac{\alpha_s}{\pi}+\alpha_s^2C_2+\frac{2}{3}C_F^2\alpha_s^2\text{Re}[H_a(C_F\alpha_s,\beta)]\right\}. \tag{17}$$

In Eq. (17) the well-known $O(C_F\alpha_s)$ short-distance correction $-4C_F\alpha_s/\pi$ [11] is successfully recovered. It is an interesting fact that the size of the contribution from $C_2=-0.24$ is an order of magnitude smaller than the one of the function H_a. For convenience, C_2 will be called $O(\alpha_s^2)$ short-distance correction in the following discussion. However, we would like to stress that a unique identification of short-distance and long-distance contributions in the combination $\alpha_s^2C_2+\frac{2}{3}C_F^2\alpha_s^2\text{Re}[H_a]$ is impossible because such a procedure is cutoff-dependent. In the language of conventional perturbation theory (in the number of loops), expression (17) resums all contributions of order $\beta(C_F\alpha_s/\beta)^n[1,C_F\alpha_s,\beta^2,C_F\alpha_s\beta,C_F^2\alpha_s^2,C_FT\alpha_s^2]$, $n=0,1,2,\ldots,\infty$, in an expansion for small β. Because expression (17) is also valid for complex energies it is applicable for $t\bar{t}$ production, where the large top width has to be taken into account.[6]

Although the $O(C_AC_F\alpha_s^2)$ and $O(C_FTn_l\alpha_s^2)$ corrections to the heavy quark-antiquark cross section in the threshold region are still unknown, it is instructive to examine the $O(\alpha_s^2)$ corrections contained in Eq. (17) for the case of $t\bar{t}$ production. In Fig. 1 the sum of the relative $O(\alpha_s^2)$ kinetic energy and the phase space corrections, $\Delta^1\equiv(\frac{2}{3}N_ce_Q^2C_F\alpha_s\text{Im}[H_a]-R^{Q\overline{Q},\text{NR}})/R^{Q\overline{Q},\text{NR}}$ (dashed lines), the $O(\alpha_s^2)$ dynamical corrections including the $O(\alpha_s^2)$ short-distance contribution, $\Delta^2\equiv\alpha_s^2C_2+\frac{2}{3}C_F^2\alpha_s^2\text{Re}[H_a]$ (dashed-dotted lines), and their sum, $\Delta^1+\Delta^2$ (solid lines), are plotted for $\alpha_s=0.13$ (thin lines) and $\alpha_s=0.16$ (thick lined) in the energy range $-10\text{ GeV}<E<10\text{ GeV}$, where $E=\sqrt{s}-2M_t$ and $M_t=175\text{ GeV}$. As mentioned earlier, the top decay width is implemented by the naive replacement $E\to E+i\Gamma_t$, which leads to the relation $\beta=[1-4M_t^2/(E+i\Gamma_t+2M_t)^2]^{1/2}$ between the velocity β and the energy E. In Fig. 1(a) $\Gamma_t=1.55\text{ GeV}$, whereas in Fig. 1(b) we have chosen $\Gamma_t=0.80\text{ GeV}$. It is striking that the strong energy dependence of Δ^1 and Δ^2 around the $1S$ peak is cancelled in their sum leaving a fairly stable correction between 3% and 7% over the whole threshold region.[7] This shows that the $1S$ peak, which is the most important characteristic of the total

[6]For complex energies there is an ambiguity in the definition of the function H_a of order β^2 which cannot be removed by the matching to the two-loop result calculated for stable quarks. For the case of unstable quarks this ambiguity amounts to $\text{Im}[a\beta^2]\sim C_F\alpha_s\Gamma_Q/M_Q$ in the total cross section, where Γ_Q is the quark width. For the case of $t\bar{t}$ production ($\Gamma_t\approx 1.5$ GeV) this ambiguity is of order 0.1% which is beyond the intended accuracy. For bottom and charm quarks this ambiguity can be ignored.

[7]We would like to mention that Δ^2 contains contributions of order $\alpha_s^2\beta^2$ coming from the β^2 terms in the function H_a. These terms represent contributions beyond the intended accuracy and are not included in our analysis. The size of these contributions to Δ^2 does not exceed 0.5% in the considered energy range.

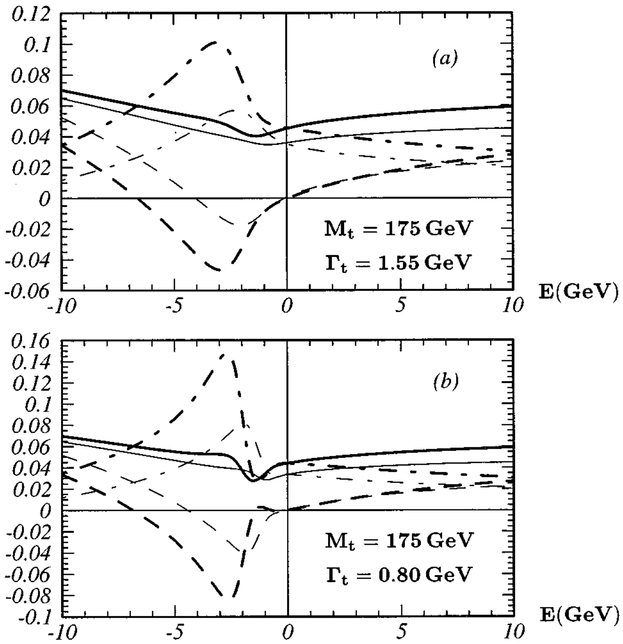

FIG. 1. The relative $O(\alpha_s^2)$ corrections to the total nonrelativistic $t\bar{t}$ production cross section Δ^1 (dashed lines), Δ^2 (dashed-dotted lines) and $\Delta^1+\Delta^2$ (solid lines) for $\alpha_s=0.13$ (thin lines) and $\alpha_s=0.16$ (thick lined) as described in the text.

$t\bar{t}$ production cross section in the threshold region [7], is barely shifted by the $O(\alpha_s^2)$ corrections determined in this work. [See also the near cancellation of the $O(\alpha_s^4)$ contributions to the energy levels E_n in Eq. (20) for $n=1$.] In particular, the sum $\Delta^1+\Delta^2$ is fairly insensitive to variations in the choice of the value of Γ_t indicating that the size of the $O(\alpha_s^2)$ corrections calculated in this work is not affected by our ignorance of a consistent treatment of all finite width effects. The variation of the size of $\Delta^1+\Delta^2$ for different choices of α_s further shows that once all $O(\alpha_s^2)$ corrections are calculated, the remaining relative theoretical uncertainty for the total cross section can be expected at the level of 1%. Taking the size of $\Delta^1+\Delta^2$ as an order of magnitude estimate for the sum of all $O(\alpha_s^2)$ corrections and because a consistent $O(\alpha_s^2)$ analysis has never been accomplished in the framework of potential models for $t\bar{t}$ production, we come to the conclusion that an uncertainty of the order 5% is contained in all predictions for the total $t\bar{t}$ production cross section based on phenomenological potentials.

Evaluating formula (17) for stable quarks above threshold we obtain

$$R^{Q\bar{Q}}_{\Gamma=0,\beta>0}=\frac{3}{2}N_c e_Q^2\beta\left(1-\frac{1}{3}\beta^2\right)\frac{\left(\dfrac{2C_F\alpha_s\pi}{v_{\rm rel}}\right)}{1-\exp\left(-\dfrac{2C_F\alpha_s\pi}{v_{\rm rel}}\right)}\times\left\{1-4C_F\frac{\alpha_s}{\pi}+\alpha_s^2\left[C_2-\frac{2}{3}C_F^2\left(\gamma+\ln\beta+\mathrm{Re}\Psi\left(1-i\frac{C_F\alpha_s}{2\beta}\right)\right)\right]\right\}, \tag{18}$$

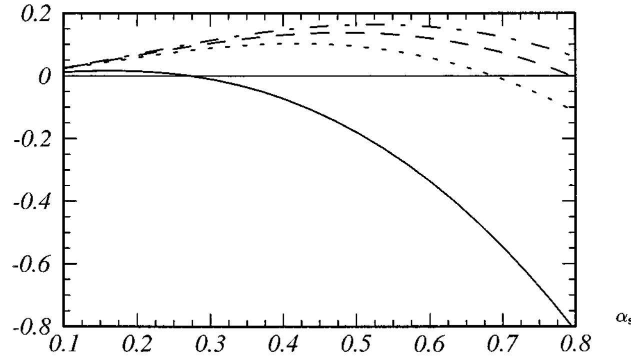

FIG. 2. The relative $O(\alpha_s^2)$ corrections δ_n for $n=1$ (solid line), 2 (dashed line), 3 (dashed-dotted line) and ∞ (dotted line) for values of α_s in the range $0.1<\alpha_s<0.8$ as described in the text.

where $v_{\rm rel}=2\beta/(1+\beta^2)$ is the relativistic relative velocity of the produced quark pair. This verifies a suggestion for the form of the relativistic extension of the Sommerfeld expression made in [15]. For $\beta\to0$ this leads to the finite expression

$$R^{Q\bar{Q}}_{\Gamma=0,\beta=0}=\frac{3}{2}N_c e_Q^2 C_F\alpha_s\pi\left\{1-4C_F\frac{\alpha_s}{\pi}+\alpha_s^2\left[C_2-\frac{2}{3}C_F^2\left(\ln\left(\frac{C_F\alpha_s}{2}\right)+\gamma\right)\right]\right\}. \tag{19}$$

For stable quarks and below threshold formula (17) develops narrow resonances at the spin triplet (n^3S_1) energy levels[8] ($n=1,2,\ldots,\infty$)

$$E_n=-\frac{M_Q C_F^2\alpha_s^2}{4n^2}+\frac{M_Q C_F^4\alpha_s^4}{4n^4}\left[\frac{11}{16}-\frac{2}{3}n\right]. \tag{20}$$

Parameterizing the resonances in the form ($\alpha=1/137$)

$$R^{Q\bar{Q}}_{\Gamma=0,-i\beta>0}=\frac{9\pi}{\alpha^2}\sum_{n=1}^{\infty}\Gamma(n\to e^+e^-)M_n\delta(s-M_n^2), \tag{21}$$

where the $M_n\equiv 2M_Q+E_n$ are the vector resonance masses, we can extract the corrections to the leptonic widths ($n=1,2,\ldots,\infty$),

$$\Gamma(n\to e^+e^-)=N_c e_Q^2\,\frac{16\pi\alpha^2}{3}\,\frac{|\Psi_n^c(0)|^2}{M_n^2}\left\{1-4C_F\frac{\alpha_s}{\pi}+\delta_n\right\}, \tag{22}$$

[8]The derivation of the energy levels in Eq. (20) from expression (17) is carried out according to Sec. IV in [8]. We also would like to note that the $O(\alpha_s^4)$ contributions to the n^3S_1 energy levels in Eq. (20) are consistent with the effective potential V_{3S1} given in Eq. (11).

$$\delta_n = \alpha_s^2 \left\{ C_2 - \frac{2}{3} C_F^2 \left[\frac{5}{2n^2} + \ln\left(\frac{C_F \alpha_s}{2n} \right) - \frac{1}{n} + \Psi(n) + \gamma \right] \right\}. \tag{23}$$

$|\Psi_n^c(0)|^2 = M_Q^3 C_F^3 \alpha_s^3 / 8\pi n^3$ is the modulus squared of the (unperturbed) Coulomb wave function at the origin with the radial quantum number n. It should be noted that the relative $O(\alpha_s^2)$ correction δ_n remains finite in the limit $n \to \infty$ (called ''small binding limit'' in [20]),

$$\delta_\infty \equiv \lim_{n\to\infty} \delta_n = \alpha_s^2 \left\{ C_2 - \frac{2}{3} C_F^2 \left[\ln\left(\frac{C_F \alpha_s}{2} \right) + \gamma \right] \right\}. \tag{24}$$

In Fig. 2 the relative $O(\alpha_s^2)$ correction δ_n is plotted for $n = 1$ (solid line), 2 (dashed line), 3 (dashed-dotted line) and ∞ (dotted line) for values of α_s in the range $0.1 < \alpha_s < 0.8$.

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