Exact results for soft supersymmetry-breaking parameters in supersymmetric gauge theories

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We obtain exact relations (valid to all orders in the coupling constant) for the running gaugino mass in supersymmetric gauge theories, treating the soft supersymmetry-breaking effects in the linear approximation. If a supersymmetry-breaking squark (selectron) mass term is introduced, our relation connects the renormalization group equation for this term with that for the gaugino mass. Exact relations for the threshold effects in the gaugino masses are derived. The key ingredients of our analysis are the use of the Wilsonian action and holomorphy of this action with respect to relevant parameters. [S0556-2821(97)02321-7]

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I. INTRODUCTION

Holomorphy is one of the most powerful tools in explorations of supersymmetric (SUSY) gauge theories. Historically, the first exact result, the so-called Novikov-Shifman-Vainshtein-Zakharov (NSVZ) β function, was obtained [1] by exploiting the holomorphy of the gauge-kinetic term in the Wilsonian action (for a recent review see Ref. [2]). Later on new insights were obtained from similar ideas in a wide range of theories with superpotentials [3]. The role of the holomorphic anomaly was revealed [4].

In this paper we report a new, so far unexplored, application of the method based on holomorphy in SUSY gauge theories with *softly broken* supersymmetry. The SUSY-breaking parameters—the gaugino mass $m_{\tilde{g}}$ and the squark (selectron) mass term $m_{\tilde{q}}$—are considered in the linear approximation (i.e., we disregard effects containing powers of $m_{\tilde{g},\tilde{q}}$ higher than the first), but to all orders in the coupling constant. We obtain the renormalization group (RG) equations governing the running of these parameters, valid *to all orders* in the coupling constant. The simplest example of this type emerges in supersymmetric gluodynamics, i.e., the theory of gluons and gluinos, without matter fields. In this model the combination

$$\frac{\alpha m_{\tilde{g}}}{\beta(\alpha)}=\mathrm{RGI}, \tag{1}$$

where RGI stands for RG invariant, and β is the Gell-Mann–Low function,

$$\beta(\alpha)=-\frac{\alpha^2}{2\pi}\frac{3T(G)}{1-[T(G)\alpha/2\pi]}, \tag{2}$$

where $T(G)$ is (one-half) of the Dynkin index [$T(G)=N$ for SU(N) theories]. In the one-loop approximation the left-hand side of Eq. (1) reduces to $m_{\tilde{g}}/\alpha$. The fact that this ratio is RG invariant in the leading approximation is well known. Equation (1) generalizes this result to all orders. Formula (1) is general. It holds also in supersymmetric gauge theories with matter [with the corresponding β function: see Eq. (41)] provided there are no super-Yukawa (trilinear) couplings in the superpotential. If trilinear couplings are introduced, Eq. (1) is replaced by a more general relation: see Eq. (42) below.

A particular but very interesting issue belonging to the given range of questions is the impact of the mass thresholds. We show how the threshold effects can be exactly incorporated in the gluino mass.

Section II introduces our notation and conventions. We begin our analysis (Sec. III) with a simple case of supersymmetric electrodynamics (SQED). In this problem nontrivial dynamics arises only from loops with the matter fields. One can consider the soft supersymmetry breaking due to the photino mass and due to the selectron mass term. The latter will be chosen in a special form. We then derive an exact RG relation between these parameters.

In Sec. IV softly broken non-Abelian gauge theories are considered. A subtle issue is to which particular action the holomorphy-based arguments apply. Two distinct constructions go under the name "effective action": The first, $\Gamma(\mu)$, is the generator of one-particle irreducible vertices, and the second, $S(\mu)$, is the Wilsonian effective action, where all infrared contributions are excluded, by definition. As was shown in Ref. [5] the holomorphic dependence refers to the Wilsonian action. At the same time, such parameters as the gauge coupling constants and the gluino mass are introduced through Γ. Exact results for the renormalization of the gluino mass can be obtained due to the fact that the relation between the parameters in Γ and S is known. All parameters appearing in the Wilsonian action will be marked by the subscript W.

We also consider in Sec. V a toy model of grand unified theory (GUT) and derive prototype "GUT relations" valid to all orders. In Sec. VI we confront all-order predictions with explicit two-loop calculations of the gluino mass known in the literature, and find perfect agreement. Finally, Sec. VII summarizes our results.

II. PRELIMINARIES

In this section we will briefly review our notation and conventions and discuss a mechanism through which the soft SUSY-breaking parameters will be introduced.

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Supersymmetric generalization of pure gluodynamics, the theory of gluons and gluinos, is described by the component Lagrangian [6]

$$\mathcal{L}_{\rm SYM} = -\frac{1}{4g_0^2} G^a_{\mu\nu} G^a_{\mu\nu} + \frac{\vartheta}{32\pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu} + \frac{1}{g_0^2}[i\lambda^{a\alpha} D_{\alpha\dot\beta} \bar{\lambda}^{a\dot\beta}], \tag{3}$$

where the spinorial notation is used. In the superfield language the Lagrangian can be written as

$$\mathcal{L}_{\rm SYM} = \int d^2\theta \frac{1}{4g^2} \,\mathrm{Tr}\, W^2 + \mathrm{H.c.}, \tag{4}$$

where

$$\frac{1}{g^2} = \frac{1}{g_0^2} - \frac{i\vartheta}{8\pi^2}.$$

In what follows the vacuum angle ϑ will play no special role. It is important, however, that g^{-2} can be treated as a complex parameter. Our conventions regarding the superfield formalism are summarized, e.g., in a recent review [7].[1] We will limit ourselves to the SU(N) gauge group [the generators of the group T^a are in the fundamental representation, so that $\mathrm{Tr}(T^a T^b) = (1/2)\delta^{ab}$].

The matter fields are assumed, for simplicity, to belong to the fundamental representation of SU(N). (Our final results are independent of this assumption.) In this case each flavor consists of two subflavors Q and $\tilde{Q}$. These superfields are in the representation N and $\bar{N}$ of the gauge group, respectively. The Lagrangian of the matter sector has the form

$$\mathcal{L}_{\rm M} = \frac{1}{4}\int d^2\theta d^2\bar{\theta}^2 (\bar{Q} e^V Q + \bar{\tilde{Q}} e^{-V} \tilde{Q}) + \left(\int d^2\theta \frac{m_0}{2} Q^\alpha \tilde{Q}_\alpha + \mathrm{H.c.}\right), \tag{5}$$

where α is the color index, $\alpha = 1,2,\dots,N$. The subscript 0 of the matter mass term m_0 indicates that it is the bare mass that enters the original Lagrangian; this parameter is complex. It is assumed that the matter mass matrix is diagonal in flavor. Such a diagonalization can always be achieved.

In SQED the gauge part of the Lagrangian takes the form

$$\mathcal{L}_{\rm SQED} = \int d^2\theta \frac{1}{8g^2} W^2 + \mathrm{H.c.}, \tag{6}$$

while the matter part is the same as in Eq. (5) with the omission of the color indices.

[1] These conventions are essentially those of Bagger and Wess [8]. The distinctions are that we use the metric $(+---)$ and the Grassmannian differentials are normalized as $\theta^2 d^2\theta = 2$.

Now we must discuss how the soft supersymmetry breaking is introduced. To this end $1/g^2$ in Eqs. (4) or (6) is substituted by a chiral superfield $\mathcal{S}$, so that the expectation value of the lowest component

$$\langle \mathcal{S} \rangle = \frac{1}{g^2}.$$

The expectation value of the F component generates the gluino (photino) mass $m_{\tilde{g}}$: namely,

$$\mathcal{S} \to \frac{1 - 2m_{\tilde{g}}\theta^2}{g^2}. \tag{7}$$

By the same token we substitute the parameter m in Eq. (5) by a chiral superfield $\mathcal{M}$. The expectation value of the lowest component,

$$\langle \mathcal{M} \rangle = m,$$

yields the supersymmetric matter mass term. The expectation value of the F component generates the squark (selectron) mass. If

$$\mathcal{M} = m(1 - b\theta^2), \tag{8}$$

where b is a parameter of dimension of mass, the nonsupersymmetric squark (selectron) mass term takes the form

$$\Delta\mathcal{L}_m = -mb\phi\tilde{\phi} + \mathrm{H.c.}, \tag{9}$$

where ϕ and $\tilde{\phi}$ are the lowest components of the superfields Q and $\tilde{Q}$.

In order to use the holomorphic nature of the gauge-kinetic term and the superpotential we introduce an infrared cutoff parameter μ and the ultraviolet cutoff parameter Λ. It is assumed that $\mu \gg m, b, m_{\tilde{g}}$, while Λ is much larger than any of the physical parameters of the dimension of the mass. In principle, the ultraviolet cutoff parameter Λ can be regarded as a chiral superfield, too. The theory is regularized in the ultraviolet by introducing the Pauli-Villars fields (within the background field technique) and higher derivatives. We do not need to know the precise form of the regulator sector. All we need to know is that such a regularization exists and that it preserves supersymmetry. The ultraviolet parameter Λ is the mass of the Pauli-Villars fields or a dumping factor in the covariant derivative term. If Λ is treated as a chiral superfield, we assume that only its lowest component develops an expectation value.

In evolving the Lagrangian (5) from the ultraviolet point Λ down to μ a Z factor appears in front of the kinetic term of the matter fields:

$$(\bar{Q} e^V Q + \bar{\tilde{Q}} e^{-V} \tilde{Q}) \to Z(\bar{Q} e^V Q + \bar{\tilde{Q}} e^{-V} \tilde{Q}).$$

In the theory where the gauge coupling is substituted by $\mathcal{S}$ the Z factor becomes a superfield, too. We will denote this superfield by $\mathcal{Z}$; its decomposition takes the form

$$\mathcal{Z} = Z\left(1 + \frac{1}{2}\zeta\theta^2 + \frac{1}{2}\zeta^\dagger \bar{\theta}^2 + \cdots\right). \tag{10}$$

Other components than those indicated above are irrelevant for our consideration; ζ and $\zeta^\dagger$ must be (and actually are) treated in the linear approximation. Then, it is convenient to introduce

$$\mathcal{Z}_L = Z(1+\zeta\theta^2) \tag{11}$$

and

$$\mathcal{Z}_R = Z(1+\zeta^\dagger \overline{\theta^2}). \tag{12}$$

III. SUPERSYMMETRIC ELECTRODYNAMICS

We begin our derivations from SQED since the relation between the gauge couplings in Γ and S are especially simple in this case. Let us first assume that in the bare Lagrangian b_0 is put to zero, so that the only source of SUSY breaking is the photino mass. We will see that in evolving the theory from Λ down to μ we do generate the selectron mass b, necessarily.

Let us briefly recall how the exact β function is obtained in SQED without SUSY breaking [9,5]. The relation between the Wilsonian gauge coupling and that in Γ is

$$\left(\frac{8\pi^2}{g^2}\right)_W = \frac{8\pi^2}{g^2} + 2\ \ln Z, \tag{13}$$

where Z stands for the Z factor of the matter fields, and the renormalization of the Wilsonian gauge coupling is exactly one loop,

$$\left(\frac{8\pi^2}{g^2}\right)_W = \left(\frac{8\pi^2}{g_0^2}\right)_W + 2\ln\frac{\Lambda}{\mu}. \tag{14}$$

Thus, the Gell-Mann–Low function for the Wilsonian couplings is one loop. The conventional definition of the coupling constants refers, however, to Γ, not to the Wilsonian action. Then combining Eqs. (13) and (14) we arrive at [9]

$$\beta(\alpha) = \frac{\alpha^2}{\pi}[1-\gamma(\alpha)], \tag{15}$$

where $\gamma(\alpha)$ is the anomalous dimension of the electron (selectron) field,

$$\gamma = -\mu\frac{d\ \ln Z}{d\mu}. \tag{16}$$

In the leading (one-loop) order

$$Z = 1 - \frac{\alpha}{\pi}\ln\frac{\Lambda}{\mu} \ \text{ and } \ \gamma = -\frac{\alpha}{\pi}.$$

What is to be changed in this derivation if $1/g^2$ is substituted by a superfield $\mathcal{S}$? It is clear that the one-loop nature of the renormalization of the gauge kinetic term in the Wilsonian action, Eq. (14), remains intact. The only difference is the fact that $Z \to \mathcal{Z}$, and $\mathcal{Z}$ depends now on $\mathcal{S}$. An additional term in $\mathcal{Z}$ arises, linear in $F_{\mathcal{S}}$. This term is obviously involved in the renormalization of the photino mass. An analogue of Eq. (13), describing the transition from the Wilsonian gauge coupling to that in Γ, now takes the form[2]

$$(8\pi^2\mathcal{S})_W = (8\pi^2\mathcal{S}) + 2\ \ln\ \mathcal{Z}_L \tag{17}$$

and

$$[8\pi^2\mathcal{S}(\mu)]_W = (8\pi^2\mathcal{S}_0)_W + 2\ \ln\frac{\Lambda}{\mu}. \tag{18}$$

Since there is no F component in the logarithm of Λ/μ and $\mathcal{Z}_0$ is put to unity, by definition, we conclude that

$$[(4\pi^2 F_{\mathcal{S}}) + \zeta]_\mu = [(4\pi^2 F_{\mathcal{S}})]_\Lambda . \tag{19}$$

Let us discuss the dependence of the $\mathcal{Z}$ factor on the superfield $\mathcal{S}$. As a warm-up exercise consider the leading-logarithmic approximation for the Z factor. The corresponding analysis will determine the running of $m_{\tilde{g}}$ up to two loops. In the leading-logarithmic approximation,[3]

$$Z = \frac{\alpha(\mu)}{\alpha_0} \to \mathcal{Z}_L = \frac{\alpha(\mu)}{\alpha_0}\,\frac{1-2m_{\tilde{g}0}\theta^2}{1-2m_{\tilde{g}}\theta^2}. \tag{20}$$

In other words,

$$\zeta = 2(m_{\tilde{g}} - m_{\tilde{g}0}). \tag{21}$$

Using the definition of $\mathcal{S}$ [see Eq. (7)] and Eqs. (19) and (21) we immediately conclude that

$$\frac{m_{\tilde{g}}}{\alpha}\left(1-\frac{\alpha}{\pi}\right) = \text{RGI}. \tag{22}$$

This relation is nothing but the two-loop truncation of the general expression (1).

Generically,

$$Z = \exp\left(\int_\alpha^{\alpha_0}\frac{\gamma(\alpha)}{\beta(\alpha)}d\alpha\right). \tag{23}$$

Inclusion of the F component of the superfield $\mathcal{S}$ reduces to the following changes in Eq. (23):

$$\alpha_0 \to \alpha_0 + 2m_{\tilde{g}0}\alpha_0\theta^2, \quad \alpha \to \alpha + 2m_{\tilde{g}}\alpha\theta^2, \quad Z \to \mathcal{Z}_L .$$

Therefore, the all-order result for ζ is

$$\zeta = 2\left[-\frac{\alpha\gamma(\alpha)}{\beta(\alpha)}m_{\tilde{g}} + \frac{\alpha_0\gamma(\alpha_0)}{\beta(\alpha_0)}m_{\tilde{g}0}\right]. \tag{24}$$

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[2]When writing the action in terms of the renormalized fields Q_r and $\tilde{Q}_r$,

$$Q_r(\tilde{Q}_r) = \mathcal{Z}_L^{1/2}Q(\mathcal{Z}_L^{1/2}\tilde{Q}),$$
$$\overline{Q}(\bar{\tilde{Q}}_r) = \mathcal{Z}_R^{1/2}\overline{Q}(\mathcal{Z}_R^{1/2}\bar{\tilde{Q}}),$$

the Konishi anomaly [10] generates terms

$$\frac{1}{32\pi^2}\int d^2\theta(\ln\ \mathcal{Z}_L)W^2 + \text{H.c.}$$

as a Jacobian of the measure [11]. Thus, Eq. (17) is justified.

[3]Here it is necessary to note that α in the Z factor means $1/[2\pi(\mathcal{S}+\mathcal{S}^\dagger)]$ after $1/g^2$ is substituted by $\mathcal{S}$.

Using this result in Eq. (19) we obtain the all-order prediction for the running of the photino mass presented in the general relation (1). This relation is valid as long as there are no trilinear couplings in the superpotential.

Note that even though at the ultraviolet cutoff the mass parameter was assumed above to be supersymmetric, i.e., $b_0=0$, the evolution from Λ down to μ does produce a nonsupersymmetric selectron mass. Since $\mathcal{M}=m_0/\mathcal{Z}_L$, it is not difficult to see that

$$b=\zeta=2\pi\left\{-\frac{m_{\tilde{g}}}{\alpha}\frac{\gamma(\alpha)}{1-\gamma(\alpha)}+\frac{m_{\tilde{g}0}}{\alpha_0}\frac{\gamma(\alpha_0)}{1-\gamma(\alpha_0)}\right\}. \quad (25)$$

Therefore, Eq. (19) can be obviously rewritten as

$$\frac{m_{\tilde{g}}}{\alpha}-\frac{b}{2\pi}=\mathrm{RGI}. \quad (26)$$

In this form the prediction is valid even if a nonvanishing selectron mass $b_0\neq 0$ is introduced in the original Lagrangian, as long as $\mu\gg m,b$. Also, even if there is a trilinear coupling in the superpotential, this is valid. This assertion follows from the fact that Eq. (26) can be proved directly from Eq. (19), being combined with the relation between b and b_0:

$$b=b_0+\zeta. \quad (27)$$

The photino mass, the selectron mass, and the gauge coupling constant run in such a way that the combination (26) stays μ independent.

So far, the normalization point μ was assumed to lie above the mass thresholds. In conclusion of this section we turn to the question what happens if the evolution of the gauge coupling is complete, i.e., if the normalization point μ becomes lower than m—we run all the way down until the point where the gauge coupling becomes frozen. Likewise, the evolution of $m_{\tilde{g}}$ freezes at $\mu/m\to 0$. (It is assumed that $m_{\tilde{g}}\ll m$.) As was noted in Ref. [2], some curious relations between the frozen low-energy values of the parameters and those in the original Lagrangian emerge in this formulation. If we dive below the threshold of the matter fields, the exact expression for the gauge coupling constant looks as if it were exactly one loop, but with the fake value of the threshold:

$$\alpha_{\mathrm{LE}}=\alpha_0\left\{1+\frac{\alpha_0}{\pi}\ln\frac{\Lambda}{m_0}\right\}^{-1}. \quad (28)$$

Here the subscript ''LE'' marks the low-energy (frozen) quantities. This result has been known for a long time [9]. We can get a similar expression for the photino mass. Passing through the matter threshold modifies Eq. (26). Say, if we descend down to the domain of freezing,

$$\left(\frac{m_{\tilde{g}}}{\alpha}\right)_{\mathrm{LE}}=\frac{m_{\tilde{g}0}}{\alpha_0}-\frac{b_0}{2\pi}. \quad (29)$$

The second term on the right-hand side of Eq. (29) corresponds to a finite correction to the gaugino mass, Fig. 1. There is no explicit γ factor here; all nontriviality associated with γ is hidden completely.

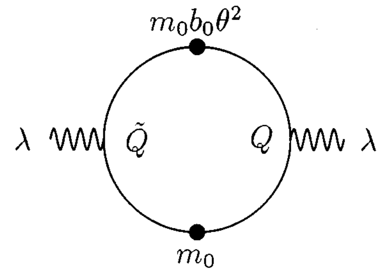

FIG. 1. The contribution in the gaugino mass arising from SUSY-violating selectron mass.

It is worth emphasizing that Eqs. (28) and (29) take into account the threshold at $\mu\sim m$ in full and exactly.

IV. SUPERSYMMETRIC GLUODYNAMICS

Our task now is to extend the method to non-Abelian theories. Although technically this case is somewhat more complicated, conceptually we will encounter no new elements. Let us first treat the case when there are no matter fields.

The main distinction is that even in the absence of the matter fields the Wilsonian couplings are now different from those in Γ. According to Eq. (7), the Wilsonian coupling

$$\left(\frac{1}{g_0^2}\right)_W\to\left(\frac{1-2m_{\tilde{g}}\theta^2}{g^2}\right)_W, \quad (30)$$

where $m_{\tilde{g}W}$ is a Wilsonian gluino mass. Below we will need to exploit the fact that $m_{\tilde{g}W}$ is *not* what is usually called the gluino mass. Indeed, the mass term is usually defined as a parameter in front of $(1/g^2)\lambda\lambda$ in Γ. It is not difficult to establish a relation between these two parameters. If in the Wilsonian Lagrangian the mass perturbation is

$$\left(\frac{m_{\tilde{g}}}{g^2}\right)_W\lambda\lambda, \quad (31)$$

in passing to Γ, we get

$$\left(\frac{m_{\tilde{g}}}{g^2}\right)_W\frac{1}{1-[T(G)g^2]/(8\pi^2)}\lambda\lambda\bigg|_{\mathrm{ext}}, \quad (32)$$

to be identified with

$$\frac{m_{\tilde{g}}}{g^2}.$$

From here we conclude that

$$\frac{m_{\tilde{g}}}{g^2}\left(1-\frac{T(G)g^2}{8\pi^2}\right)=\left(\frac{m_{\tilde{g}}}{g^2}\right)_W. \quad (33)$$

Note that in obtaining Eq. (32) we used the fact [5] that the matrix element of the operator W^2 is

$$\frac{1}{1-[T(G)g^2]/(8\pi^2)}W^2\bigg|_{\mathrm{ext}}.$$

Once the relation between the Wilsonian and conventional mass parameters is established, we can forget for a while about the conventional parameter, and focus on what happens with the Wilsonian action under renormalizations. As known from Ref. [4], in the Wilsonian action, where the infrared contributions are not included by definition, the holomorphy is preserved—F terms should depend only on the chiral superfields, not antichiral, and so on. In the Wilsonian action the coupling constant is renormalized only at one loop:

$$\left(\frac{1}{g^2}\right)_W = \left(\frac{1}{g_0^2}\right)_W - \frac{3T(G)}{8\pi^2}\ln\frac{\Lambda}{\mu}. \tag{34}$$

Now, in this relation it is perfectly legitimate to substitute each bracket by the same bracket with $(1-2m_{\tilde g}\theta^2)$ in the numerator. Both, the left- and right-hand sides appear as coefficients of the F terms in the Wilsonian action, and we multiply by the chiral superfield. From Eq. (34), inspecting the coefficient in front of θ^2, we immediately deduce that

$$\left(\frac{m_{\tilde g}}{g^2}\right)_W = \mathrm{RGI}. \tag{35}$$

Invoking now Eq. (33) we see that

$$\frac{m_{\tilde g}}{g^2}\left(1-\frac{T(G)g^2}{8\pi^2}\right) = \mathrm{RGI}. \tag{36}$$

Taking into account the explicit form of the NSVZ β function in the case at hand [see Eq. (2)], we conclude that Eq. (36) is in full accordance with the general expression (1).

The same result can be obtained in a slightly different way. The vacuum expectation value of the *operator* W^2 is a physical quantity; it is RG invariant. This vacuum expectation value can be written as [12,4]

$$\langle W^2\rangle = (\text{numer. const.})\times\mu^3\ \exp(-4\pi^2\mathcal{S}_W). \tag{37}$$

In both the left- and right-hand sides we have only chiral superfields, as it should be. If $\mathcal{S}$ develops an F term, we expand in it. The expectation value of the F term of the operator W^2 is proportional to the vacuum energy density.[4] The F term on the right-hand side is proportional to $(m_{\tilde g}/g^2)_W$. In this way we arrive at Eq. (35).

If the matter fields are switched on, the argument becomes somewhat more subtle in the part referring to the matter field Z factors. Let us sketch here the basic points. The main idea is to continue using the Wilsonian action.

Under the renormalization the Wilsonian action

$$\left(\frac{1-2m_{\tilde g 0}\theta^2}{g_0^2}\right)_W W^2$$

goes into

[4]This is another reason for the absence of renormalization.

$$\left\{\left(\frac{1-2m_{\tilde g 0}\theta^2}{g_0^2}\right)_W + \left[\frac{\Sigma_i T(R_i)-3T(G)}{8\pi^2}\ln\frac{\Lambda}{\mu} - \sum_i \frac{T(R_i)}{8\pi^2}\ln\mathcal{Z}_{Li}\right]\right\}W^2, \tag{38}$$

where $T(R_i)$ are the Dynkin indices for the matter fields. In the fundamental representation $T=1/2$ for each subflavor; $T=1$ for one flavor. The sum runs over all matter fields.

We must now derive an analogue of Eq. (24). A straightforward calculation yields

$$\zeta_i = 4\pi\left\{\left(\frac{m_{\tilde g}}{\alpha}\right)_W \frac{\gamma_i}{3T(G)-\Sigma_i T(R_i)(1-\gamma_i)} - \left(\frac{m_{\tilde g 0}}{\alpha_0}\right)_W \frac{\gamma_{0i}}{3T(G)-\Sigma_i T(R_i)(1-\gamma_{i0})}\right\}. \tag{39}$$

Here $\gamma_i=\gamma_i(\alpha)$ and $\gamma_{i0}=\gamma_i(\alpha_0)$. From where we immediately conclude that

$$\left(\frac{m_g}{g^2}\right)_W\left(1-\frac{T(R_i)\gamma_i}{3T(G)-T(R_i)(1-\gamma_i)}\right) = \mathrm{RGI}. \tag{40}$$

Invoking again Eq. (33) and using the fact that the NSVZ β function in the case at hand has the form

$$\beta(\alpha) = -\frac{\alpha^2}{2\pi}\frac{3T(G)-T(R_i)(1-\gamma_i)}{1-T(G)\alpha(2\pi)^{-1}}, \tag{41}$$

we reproduce[5] Eq. (1).

An alternative form of the RGI relation is obtained if the squark mass terms are introduced. As in SQED, $b_i=b_{i0}+\zeta_i$, which implies, in turn, that

$$\frac{m_{\tilde g}}{\alpha}\left(1-\frac{T(G)\alpha}{2\pi}\right) - \sum_i \frac{T(R_i)b_i}{4\pi} = \mathrm{RGI}, \tag{42}$$

to be compared with Eq. (26) in SQED.

V. EXACT GUT RELATION OF THE GAUGINO MASSES

So far, we assumed that the chiral multiplets Q and $\tilde Q$ do not have vacuum expectation values, that is, that the spontaneous breaking of the gauge symmetry does not occur. It is interesting to discuss the issue of the gaugino mass renormalization in the presence of the spontaneous breaking of the gauge symmetry, keeping in mind possible applications in theories of grand unification (GUT).

Let us consider an SU(3) gauge model as a prototypical example; we will introduce a chiral multiplet Σ^a in the adjoint representation $(a=1,2,\ldots,8)$. The vacuum expectation value of Σ induces the gauge symmetry breaking SU(3)→SU(2)×U(1) (see Ref. [2] for a discussion of the

[5]The anomalous dimensions of the matter fields *per se* cannot be determined to all orders since the holomorphy arguments do not apply in this case. At one loop $\gamma_i=-C(R_i)\alpha/\pi$ where $C(R_i)=T(R_i)\,\dim(\mathrm{adj})/\dim(R_i)$.

running gauge coupling in this model).

The original superpotential of the adjoint chiral multiplet is

$$P=\frac{1}{4g_0^2}(m_0\Sigma^a\Sigma^a+y_0 d_{abc}\Sigma^a\Sigma^b\Sigma^c), \tag{43}$$

where d_{abc} are the d symbols of SU(3). Here again, we will substitute the mass m and the Yukawa coupling y by the chiral superfields $\mathcal{M}$ and $\mathcal{Y}$, respectively, and assume nonvanishing vacuum expectation values of the F components, in order to introduce soft SUSY breaking:

$$\begin{aligned}\langle\mathcal{M}_0\rangle&=m_0(1-b_0\theta^2),\\ \langle\mathcal{Y}_0\rangle&=y_0(1-a_0\theta^2).\end{aligned} \tag{44}$$

Here a and b are parameters of dimension of mass. We assume that a, b, and $m_{\tilde g}\ll m$. By inspecting the superpotential (43) we observe that Σ gets the vacuum expectation value

$$\langle\Sigma\rangle=\frac{1}{2\sqrt{3}}\begin{pmatrix}1&&\\&1&\\&&-2\end{pmatrix}\mathcal{V}_0, \tag{45}$$

breaking SU(3) down to SU(2)$\times$U(1); here,[6]

$$\mathcal{V}_0=\frac{2\mathcal{M}_0}{\sqrt{3}\mathcal{Y}_0}[1+O(b/m,a/m)]=\frac{2m_0}{\sqrt{3}y_0}[1+(a_0-b_0)\theta^2]. \tag{46}$$

As a result of the vacuum expectation value $\mathcal{V}_0$, four out of eight gauge multiplets get a mass and form, together with two SU(2) doublets from Σ absorbed in the super-Higgs mechanism, massive vector multiplets. One SU(2) triplet and one singlet from Σ survive. The mass of the SU(2) triplet is

$$M_\Sigma=\mathcal{M}_0. \tag{47}$$

Below the masses of the heavy gauge bosons (the ''elephants'') and the masses of the surviving fields from Σ (i.e., below the unification thresholds), the SU(2) and U(1) gauge couplings evolve separately and so do the corresponding gaugino masses. The gauge couplings diverge. Our task is to express the low-energy values of the gauge couplings and gaugino masses (far below the thresholds) in terms of the high-energy (i.e., above-threshold) parameters.

In other words, we choose the normalization point μ far below the lowest threshold and the ultraviolet cutoff Λ far above the highest one. The Wilsonian couplings of the SU(2) and U(1) gauge multiplets are given by

$$\begin{aligned}(\mathcal{S}_{\mathrm{SU}(2)})_W=(\mathcal{S}_0)_W&-\frac{6}{8\pi^2}\ln\frac{\Lambda}{\mu}-\frac{2}{8\pi^2}\ln\frac{\Lambda}{\sqrt{3}\mathcal{V}_0/2}\\&+\frac{2}{8\pi^2}\ln\frac{\Lambda}{\mathcal{M}_0},\end{aligned} \tag{48}$$

$$(\mathcal{S}_{\mathrm{U}(1)})_W=(\mathcal{S}_0)_W-\frac{6}{8\pi^2}\ln\frac{\Lambda}{\sqrt{3}\mathcal{V}_0/2}. \tag{49}$$

The second and fourth terms in Eq. (48) are the contributions from the SU(2) gauge multiplet and SU(2) triplet in Σ, respectively, and the others come from the massive vector multiplets. By taking the F components of Eqs. (48) and (49) and using Eq. (33) we arrive at

$$\begin{aligned}\frac{m_{\tilde g_2}}{\alpha_2}\left(1-\frac{T(\mathrm{SU}(2))\alpha_2}{2\pi}\right)&=\frac{m_{\tilde g0}}{\alpha_0}\left(1-\frac{T(\mathrm{SU}(3))\alpha_0}{2\pi}\right)\\&\quad-\frac{1}{2\pi}(a_0-b_0)-\frac{1}{2\pi}b_0,\end{aligned} \tag{50}$$

$$\frac{m_{\tilde g_1}}{\alpha_1}=\frac{m_{\tilde g0}}{\alpha_0}\left(1-\frac{T(\mathrm{SU}(3))\alpha_0}{2\pi}\right)-\frac{3}{2\pi}(a_0-b_0). \tag{51}$$

Here $m_{\tilde g_2}$ $[m_{\tilde g_1}]$ and α_2 $[\alpha_1]$ are the SU(2) [U(1)] gaugino mass and gauge coupling constant at[7] μ. Then, we can get an exact relation for the gaugino mass at low energy:

$$\frac{m_{\tilde g_2}}{\alpha_2}\left(1-\frac{T(\mathrm{SU}(2))\alpha_2}{2\pi}\right)-\frac{m_{\tilde g_1}}{\alpha_1}=\frac{1}{\pi}(a_0-b_0)-\frac{1}{2\pi}b_0. \tag{52}$$

Equation (52) is usually referred to as the *GUT relation*. It is worth emphasizing that the relation we derived is valid to all orders in the coupling constants and exactly takes into account the threshold effects.

In the diagrammatic calculation, the first term on the right-hand side in Eq. (52) comes from the mass of the fermionic partners of the Goldstone bosons, and the second term comes from a diagram similar to that of Fig. 1, in which the surviving SU(2) triplet from Σ propagates [13].

Let us consider a particular case $a_0=b_0=0$; i.e., all SUSY-breaking parameters, except the gaugino mass, are zero at Λ. It is remarkable that in this case the ratio of the low-energy gaugino masses is completely determined by *only* the low-energy gauge couplings, with no dependence on details of the model at the gauge symmetry-breaking scale:

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[6]In the diagrammatic calculation of the radiative correction to the gaugino masses in this model, we have to keep the shift of the scalar component of $\mathcal{V}$ from its supersymmetric value due to introduction of a and b. This is because this shift leads to a nonvanishing mass of the fermionic partners of the Nambu-Goldstone bosons [13]. However, the result for the gaugino mass is not changed compared to ours. This follows from the fact that the shift of the scalar component of $\mathcal{V}_0$ has a correlation with the value of the F component of $\mathcal{V}_0$.

[7]It can be proved by using the relation

$$\mu\frac{da}{d\mu}=\frac{3}{2}\mu\frac{db}{d\mu}$$

that above the threshold these equations give the same RG expressions.

$$\frac{m_{\widetilde{g}_2}}{m_{\widetilde{g}_1}}=\frac{\alpha_2}{\alpha_1}\left(1-\frac{\alpha_2}{\pi}\right)^{-1}.$$

VI. CONFRONTING OUR RESULTS WITH EXPLICIT CALCULATIONS AT THE TWO-LOOP LEVEL

Equation (1) is our basic all-order prediction for the running gaugino mass. It is instructive to check it ''empirically.'' The Gell-Mann–Low function is scheme independent up to two loops. The γ factors are scheme independent only in the leading (one-loop) order, but, as we have seen, the one-loop γ factors will affect the renormalization of $m_{\widetilde{g}}$ at the two-loop level only. Therefore, the running of $m_{\widetilde{g}}$ up to two loops is unambiguously given by Eq. (1) if there are no trilinear couplings in the superpotential. Here we compare our result with the direct two-loop calculation of the gluino mass reported in Ref. [14] where the dimensional reduction with modified minimal subtraction ($\overline{\mathrm{DR}}$) scheme is adopted.

The two-loop RG equations of the gauge coupling constant and the gaugino mass in the $\overline{\mathrm{DR}}$ scheme are given as follows:

$$\mu\frac{d}{d\mu}\alpha=\frac{\alpha^2}{2\pi}\left\{\sum_i T(R_i)-3T(G)\right\}+\frac{\alpha^2}{2\pi}\left\{\left(\sum_i T(R_i)-3T(G)\right)T(G)\frac{\alpha}{2\pi}-\sum_i T(R_i)\gamma_i\right\}, \quad (53)$$

$$\mu\frac{d}{d\mu}\left(\frac{m_{\widetilde{g}}}{\alpha}\right)=\frac{1}{2\pi}\left\{\left(\sum_i T(R_i)-3T(G)\right)T(G)\frac{\alpha}{2\pi}-\sum_i T(R_i)\widetilde{\gamma}_i\right\}. \quad (54)$$

Here γ_i and $\widetilde{\gamma}_i$ are defined as

$$\gamma_i=-\mu\frac{d\ \ln Z_i}{d\mu},$$

$$\widetilde{\gamma}_i=-\frac{1}{2}\mu\frac{d\zeta_i}{d\mu},$$

and are given, at the one-loop level, by the following expressions:

$$\gamma_i=-\frac{\alpha}{\pi}C(R_i),$$

$$\widetilde{\gamma}_i=-\frac{\alpha}{\pi}C(R_i)m_{\widetilde{g}}. \quad (55)$$

[Note that C is defined as $C\delta_j^i=(\Sigma_a T^aT^a)_i^j$, and $C=(N^2-1)/2N$ for the fundamental representation of SU(N).] From Eqs. (53) and (54) we can get

$$\mu\frac{d}{d\mu}\left(\frac{\alpha m_{\widetilde{g}}}{\beta(\alpha)}\right)=\frac{\alpha^3}{[\beta(\alpha)]^2}\frac{1}{2\pi}\sum_i T(R_i)\left\{\beta(\gamma_i)m_{\widetilde{g}}-\frac{\alpha}{2\pi}\left(\sum_i T(R_i)-3T(G)\right)\widetilde{\gamma}_i\right\}, \quad (56)$$

where $\beta(\gamma_i)$ is the β function for γ_i,

$$\beta(\gamma_i)=\mu\frac{d\gamma_i(\alpha)}{d\mu}.$$

It is easy to prove, by using the explicit forms of γ_i and $\widetilde{\gamma}_i$ at the one-loop level, Eqs. (55), that the right-hand side of Eq. (56) vanishes. Thus, Eq. (1) is confirmed at two-loop level.

Equations (26) and (42) can be derived directly by using the RG equation for the SUSY-breaking parameter b_i:

$$\mu\frac{db_i}{d\mu}=-2\widetilde{\gamma}_i. \quad (57)$$

Here explicit forms of γ and $\widetilde{\gamma}$ are not needed, as is expected from the fact that Eqs. (26) and (42) are valid even in the presence of the trilinear couplings in the superpotential.

VII. CONCLUSIONS AND DISCUSSION

In this paper, in SUSY gauge theories with the soft SUSY breaking, we studied the running of the gaugino and/or squark (selectron) masses. Several exact (i.e., all-order) predictions are obtained by exploiting the relation between the gaugino masses in the generator of the 1PI vertices and the Wilsonian action and by using the holomorphic nature of the F term in the Wilsonian action. Our main findings can be summarized as follows.

In the absence of the trilinear couplings in the superpotential we derived a general *exact* formula relating the running gluino mass to the running gauge coupling constant,

$$\frac{\alpha m_{\widetilde{g}}}{\beta(\alpha)}=\mathrm{RGI}. \quad (58)$$

This formula is valid even if the matter sector is chiral; no mass terms are possible [e.g., SU(5) theory with an equal number of quintets and antidecuplets].

If the matter sector is nonchiral—i.e., supersymmetric mass terms are possible—and the squark SUSY-breaking masses are introduced through Eq. (9), then

$$\frac{m_{\widetilde{g}}}{\alpha}\left(1-\frac{T(G)\alpha}{2\pi}\right)-\sum_i\frac{T(R_i)b_i}{4\pi}=\mathrm{RGI}. \quad (59)$$

This formula is valid even if there are trilinear couplings in the superpotential.

In certain instances the mass threshold effects in the gaugino mass can be taken into account exactly. In SQED this assertion is illustrated by Eq. (29), in non-Abelian GUT's by Eq. (52).

As was just mentioned, if the Yukawa (trilinear) couplings in the superpotential are present (let us call them generically h), the exact result that we managed to get refers to a linear combination of the gluino and (SUSY-breaking) squark mass terms, rather than to the gluino mass *per se*. Technically the reason is evident: Unlike the case of pure gauge interactions, now the derivative of $Z(\alpha,h)$ with respect to α is unrelated with $\gamma(\alpha,h)$. It is still possible to obtain the exact expressions of the type (1), in the closed form, for a specific choice of the points on the $\{\alpha,h\}$ plane. We mean fixed points of the gauge coupling constant and the Yukawa coupling constants. If the initial set of parameters is such that the condition

$$\gamma(\alpha,h)=\frac{1}{3}\,\frac{\beta(\alpha)}{\alpha} \tag{60}$$

is met, the ratio of the gauge coupling constant to the Yukawa coupling constant in the superpotential is RG invariant. In Ref. [15] it was shown, by examining explicit formulas of the RG equations, that Eq. (60) does have a solution at least up two loops (the so-called ''$P=Q/3$'' rule). More interestingly, the authors of Ref. [15] argue that if Eq. (60) is satisfied, the gaugino mass and the SUSY-breaking trilinear scalar coupling constant a, associated with the Yukawa coupling, have a fixed point at

$$a=-m_{\tilde{g}}\,. \tag{61}$$

Now we are able to solve the question of the fixed-point behavior of the SUSY-breaking parameters beyond two loops. Equation (61) follows from the holomorphic nature of the F terms. In fact, it can be proved that Eq. (61) is valid to all orders provided Eq. (60) is satisfied to all orders. If that is the case, Eq. (1) remains valid even in the presence of the Yukawa couplings.

What remains to be done? So far, we have not considered models with the chiral matter sector (no mass term possible), with the Yukawa (trilinear) couplings of the chiral superfields in the superpotential. It is possible to show that a RGI relation for the gaugino mass *at any order* has the form

$$\frac{m_{\tilde{g}}}{\alpha}\left(1-\frac{T(G)\alpha}{2\pi}\right)-\sum_i\frac{T(R_i)\zeta_i}{4\pi}=\text{RGI}. \tag{62}$$

The task is to evaluate ζ_i from supersymmetric Z factors of the chiral multiplets.

Another problem is quite obvious too. In Ref. [16] a perturbative renormalization $\overline{\text{DR}}$-related scheme was identified, yielding the NSVZ β function up to three loops. The anomalous dimensions of the matter fields are known in this scheme in two loops. If the running of the gluino and squark masses were known in three loops in this scheme, one could have verified our all-order predictions by comparing them with the explicit calculations up to three loops.

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