

Order $1/N$ corrections to the time-dependent Hartree approximation for a system of $N + 1$ oscillators

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We solve numerically to order $1/N$ the time evolution of a quantum dynamical system of N oscillators of mass m coupled quadratically to a massless dynamic variable. We use Schwinger's closed time path formalism to derive the equations. We compare two methods which differ by terms of order $1/N^2$. The first method is a direct perturbation theory in $1/N$ using the path integral. The second solves exactly the theory defined by the effective action to order $1/N$. We compare the results of both methods as a function of N . At $N=1$, where we expect the expansion to be quite inaccurate, we compare our results to an exact numerical solution of the Schrödinger equation. In this case we find that when the two methods disagree they also diverge from the exact answer. We also find at $N=1$ that the $1/N$ corrected evolutions track the exact answer for the expectation values much longer than the mean field ($N=\infty$) result. [S0556-2821(97)02921-4]

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I. INTRODUCTION

The large N approximation has a long history in both statistical mechanics and quantum field theory, mostly in determining the phase structure of various theories [1]. It is only recently that this approximation has been used to study the dynamical evolution of various systems, and at present only the lowest order in the large N expansion has been considered. In leading order, the large N expansion is equivalent to using a Gaussian density matrix, and therefore two particle scattering effects are included only indirectly. The leading order in the large N approximation is closely related to a time-dependent Hartree approximation. The exact connection between these methods is discussed in detail in Ref. [2]. Although interesting results in lowest order have been obtained for pair production from strong fields [3,4] as well as the evolution of a chiral phase transition [5], the important effects of the direct two particle elastic scattering, which determines the thermalization time scale of the plasma, is not included in the mean field approximation. In order to compare time scales for rethermalization with, say, the plasma oscillation frequency, as well as the expansion time of an evolving plasma produced in a heavy-ion collision, one needs to go to next order in the $1/N$ expansion. The property of the $1/N$ expansion relevant here is that connected $2n$ -point Green's functions first appear with G_{2n} of order $1/N^{n-1}$. Thus direct scattering of two particles first occurs at order $1/N$. At lowest order in $1/N$, the equations one has to solve are differential equations. At next order one gets integrodifferential equations which depend on the time history of the system. This requires new numerical methods to ensure

that the update conserves energy.

There are two different ways one can determine the $1/N$ corrections. The first method is to iterate the solution of the lowest order calculation in a standard perturbative fashion. However, one might hope that it might be more accurate to first Legendre transform the action obtained to order $1/N$ and to obtain a new effective action, which differs by terms of order $1/N^2$ from the first method. In the second method, one evolves directly the equations of motion obtained from the effective action. By having these two different methods, one has an upper bound on the accuracy of the $1/N$ expansion. When the two methods diverge, this is a signal that $1/N^2$ corrections are important. At $N=1$ this divergence is very close to the place where these two methods diverge from the exact answer. We find that the method based on the effective action is actually less stable, since solutions become unbounded earlier. However, this occurs much after the method is unreliable.

In quantum field theory it is not possible to compare the $1/N$ expansion with an exact calculation of a dynamical evolution because of the large number of degrees of freedom. Thus we thought it appropriate to study a simple quantum-mechanical example which we have studied before in the lowest order mean field approximation [2,6]. The advantage of this simple model is that, at least for $N=1$, comparisons can be made with a direct numerical simulation of the exact problem [7]. Unfortunately, even for the quantum-mechanics case, going beyond $N=1$ is not numerically feasible for the exact problem. So we are testing our large N expansion in a regime where it is not expected to work very well. However, in spite of this short coming, we find that by adding the $1/N$ correction terms, our approximation tracks the exact answer for expectation values a factor of two longer than the lowest order approximation—which is encouraging.

As far as we know, this is the first attempt to use Schwinger's closed time path (CTP) formalism in a calcula-

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tion which is not a perturbation expansion in the coupling constant. The methods of solving the resulting Volterra-like equations, which we present here, can be generalized to the field theory case. Therefore this toy model, which can be compared with a direct solution of the Schrödinger equation, is an ideal problem to test the accuracy of the numerical methods needed for field theory calculations.

II. THE GENERATING FUNCTIONAL

We consider a system of N oscillators of mass m coupled quadratically to a massless oscillator with coupling e . This quantum-mechanical system is a model of a single momentum mode of scalar quantum electrodynamics [8]. The Lagrangian for this system is given by

$$L = \frac{1}{2} \dot{A}^2 + JA + \sum_{a=1}^N \left\{ \frac{1}{2} \dot{\phi}_a^2 - \frac{1}{2} (m^2 + e^2 A^2) \phi_a^2 + j_a \phi_a \right\}. \quad (2.1)$$

We introduce the scaled variables, defined as

$$\begin{aligned} A &\rightarrow A/\sqrt{N}, & \phi_a &\rightarrow \phi_a/\sqrt{N}, & a &= 1, N, \\ J &\rightarrow J/\sqrt{N}, & j_a &\rightarrow j_a/\sqrt{N}, & a &= 1, N \\ e &\rightarrow e\sqrt{N}, & L &\rightarrow L/N. \end{aligned} \quad (2.2)$$

From now on, we use scaled variables. We wish to consider the time evolution of expectation values of observables for an initial value problem $0 < t < \infty$. The way to formulate an initial value problem in quantum mechanics, using a generating functional, was done more than thirty years ago by Schwinger, Bakshi, and Mahanthappa, and later by Keldysh [9]. This formalism, which is in the Heisenberg picture, is related to the fact that in the Schrödinger picture the evolution of the density matrix in quantum mechanics

$$\hat{\rho}(t) = e^{-i\hat{H}t} \hat{\rho}(0) e^{i\hat{H}t}, \quad (2.3)$$

requires both a forward evolution from zero to t and a backward one from t to zero. The average value of observables are given by traces over states of the system

$$\langle \hat{O}(t) \rangle = \text{Tr}\{\hat{\rho}(t)\hat{O}\} = \text{Tr}\{\hat{\rho}(0)\hat{O}(t)\}$$

in the Schrödinger and Heisenberg picture, respectively.

As explained in the Appendix of Ref. [2], this necessitates both positive and negative time ordered operators in the evolution of the observable operators and the introduction of two currents into the path integral for the generating functional. A concise way of writing the needed functions is to define the time integrals along a path in the complex time plane. This closed time path is shown in Fig. 1. The CTP integration contour is given by

$$\int_C F(t) dt = \int_{0:C_+}^{\infty} F_+(t) dt - \int_{0:C_-}^{\infty} F_-(t) dt. \quad (2.4)$$

Using the CTP contour, the generating functional for the causal Green's functions for the theory described by the Lagrangian (2.1) is given by the path integral

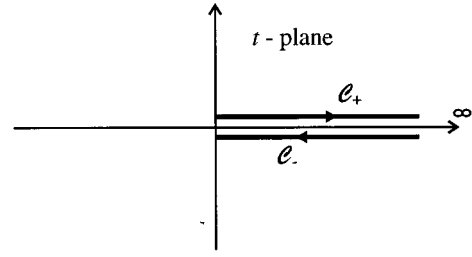


FIG. 1. Complex time contour \mathcal{C} for the closed time path integrals.

$$\begin{aligned} Z[J, j] &= \int d[A] \int d[\phi_a] e^{iNS[A, \phi; J, j]} \\ &\quad \times S[A, \phi; J, j] = \int_C dt L, \end{aligned} \quad (2.5)$$

The full closed time path Green's function for the two point functions is

$$G_{ab}(t, t') = G_{ab}^>(t, t') \Theta_C(t, t') + G_{ab}^<(t, t') \Theta_C(t', t),$$

in terms of the Wightman functions

$$\begin{aligned} G_{ab}^>(t, t') &= i\{\langle \phi_a(t) \phi_b(t') \rangle - \langle \phi_a(t) \rangle \langle \phi_b(t') \rangle\}, \\ G_{ab}^<(t, t') &= i\{\langle \phi_b(t') \phi_a(t) \rangle - \langle \phi_b(t') \rangle \langle \phi_a(t) \rangle\}, \end{aligned}$$

where $\langle \phi_a(t) \phi_b(t') \rangle \equiv \text{Tr}\{\rho(0) \phi_a(t) \phi_b(t')\}$, and where the CTP step function $\Theta_C(t, t')$ is defined by

$$\Theta_C(t, t') = \begin{cases} \Theta(t, t') & \text{for } t \text{ on } C_+ \text{ and } t' \text{ on } C_+, \\ 0 & \text{for } t \text{ on } C_+ \text{ and } t' \text{ on } C_-, \\ 1 & \text{for } t \text{ on } C_- \text{ and } t' \text{ on } C_+, \\ \Theta(t', t) & \text{for } t \text{ on } C_- \text{ and } t' \text{ on } C_-. \end{cases} \quad (2.6)$$

This is equivalent to a 2×2 matrix Green's function on the vector space $\{+, -\}$, often found in the literature.

The large N expansion is obtained by performing the Gaussian integral over the ϕ_a variables to obtain an effective action, and evaluating the remaining integral over A by the method of steepest descent. This gives

$$Z[J, j] = \int d[A] e^{iNS_{\text{eff}}[A; J, j]} \equiv e^{iNW[J, j]},$$

$$W[J, j] = S_{\text{eff}}[A_0; J, j] + \frac{i}{2N} \int_C dt \ln[D^{-1}(t, t)] + \dots, \quad (2.7)$$

where $S_{\text{eff}}[A_0; J, j]$ is given by

$$\begin{aligned}
S_{\text{eff}}[A_0; J, j] = & \int_C dt \left\{ -\frac{1}{2} A_0 \frac{d^2}{dt^2} A_0 + J A_0 \right. \\
& \left. + \frac{i}{2N} \sum_{a=1}^N \ln[G_{0aa}^{-1}(t, t)] \right\} \\
& + \frac{1}{2} \int_C dt \int_C dt' \sum_{a,b=1}^N j_a(t) G_{0ab}(t, t') j_b(t').
\end{aligned} \tag{2.8}$$

The stationary point $A_0(t)$ is determined by

$$\left\{ \frac{d^2}{dt^2} + e^2 \sum_{a=1}^N \left[\phi_{0a}^2(t) + \frac{1}{iN} G_{0aa}(t, t) \right] \right\} A_0(t) = J(t). \tag{2.9}$$

Here ϕ_0 is defined as:

$$\left\{ \frac{d^2}{dt^2} + (m^2 + e^2 A_0^2) \right\} \phi_{0a}(t) = j_a(t), \tag{2.10}$$

and G_0 is given by the solution to

$$\left\{ \frac{d^2}{dt^2} + (m^2 + e^2 A_0^2) \right\} G_{0ab}(t, t') = \delta_{ab} \delta_C(t, t'). \tag{2.11}$$

$\phi_{0a}(t)$ and $A_0(t)$ are to be regarded as functionals of the sources $J(t)$ and $j(t)$. We have defined $\delta_C(t, t') = d\Theta_C(t, t')/dt$.

The inverse propagator $D^{-1}(t, t')$ is

$$\begin{aligned}
D^{-1}(t, t') = & - \left[\frac{\delta^2 S_{\text{eff}}[A; J, j]}{\delta A(t) \delta A(t')} \right]_{A_0} \\
= & D_0^{-1}(t, t') + \Pi_0(t, t'),
\end{aligned} \tag{2.12}$$

where

$$D_0^{-1}(t, t') = \left\{ \frac{d^2}{dt^2} + e^2 \sum_{a=1}^N \left[\phi_a^2(t) + \frac{1}{iN} G_{0aa}(t, t) \right] \right\} \delta_C(t, t') \tag{2.13}$$

and

$$\begin{aligned}
\Pi_0(t, t') = & 2e^4 A_0(t) A_0(t') \sum_{a,b=1}^N \left\{ \frac{i}{N} G_{0ab}(t, t') G_{0ba}(t', t) \right. \\
& \left. - 2\phi_{0a}(t) G_{0ab}(t, t') \phi_{0b}(t') \right\}.
\end{aligned} \tag{2.14}$$

We solve Eq. (2.11) by introducing a complete set of solutions $f(t)$ to the homogeneous equation, satisfying the Wronskian condition

$$f_a^*(t) \dot{f}_a(t) - \dot{f}_a^*(t) f_a(t) = -i. \tag{2.15}$$

The causal Green's functions G_0 can then be written

$$\begin{aligned}
G_{0ab}(t, t') = & i \delta_{ab} \{ f_a(t) f_a^*(t') \Theta_C(t, t') \\
& + f_a(t') f_a^*(t) \Theta_C(t', t) \}.
\end{aligned} \tag{2.16}$$

III. TIME EVOLUTION EQUATIONS

In the Schrödinger picture the Schrödinger equation governs the time evolution of the $N+1$ oscillators

$$i \frac{\partial \Psi(\phi, A, t)}{\partial t} = \left\{ \sum_a \frac{1}{2} \left[-\frac{\partial^2}{\partial \phi_a^2} + (m^2 + e^2 A^2) \phi_a^2 \right] - \frac{1}{2} \frac{\partial^2}{\partial A^2} \right\} \Psi(\phi, A, t). \tag{3.1}$$

It is the solution of this equation that we will compare with our large N equations for the expectation values. Since at leading order in large N an initial Gaussian wave packet stays Gaussian under time evolution, we will start our problem with Gaussian initial data. Also we need to relate the parameters of the wave function at time zero to the values of one and two point functions and their time derivatives, since the Green's functions obey second order differential equations. At $t=0$ we choose our wave function to be a product of Gaussians

$$\Psi(\phi, A, 0) = \Psi_\phi(0) \Psi_A(0), \tag{3.2}$$

where

$$\Psi_\phi(0) = [2\pi G(0)]^{-1/4} \exp\{-x^2[G^{-1}(0)/4 - i\Pi_G(0)]\}$$

and

$$\begin{aligned}
\Psi_A(0) = & [2\pi D(0)]^{-1/4} \exp\{-(A - \sqrt{N}\tilde{A}_0)^2 \\
& \times [D^{-1}(0)/4 - i\Pi_D(0)] \\
& + iP_A(0)(A - \sqrt{N}\tilde{A}_0)\}.
\end{aligned}$$

The variables have the following meaning:

$$\langle \phi_a(0) \phi_b(0) \rangle = \delta_{ab} G(0),$$

$$\Pi_G(0) = \frac{\dot{G}(0)}{4G(0)}.$$

We have chosen $\langle \phi_a \rangle = \langle \dot{\phi}_a \rangle = 0$:

$$D(0) + N\tilde{A}_0^2 = \langle A^2 \rangle_{t=0},$$

$$\Pi_D(0) = \frac{\dot{D}(0)}{4D(0)},$$

$$P_A(0) = \langle -i\partial/\partial A \rangle_{t=0} = \langle \dot{A}(0) \rangle.$$

We will evolve these equations (for $N=1$) using a symplectic integrator as described in Ref. [7] to compare with the results of the large N expansion. The parameter $G(t)$ in the Schrödinger wave function which is the real part of the width of the wave function is related to $G_0(t, t')$ of the Heisenberg approach via $G(t) = G_0(t, t)/i$.

A. Leading order in $1/N$

Let us now consider the equations and the initial conditions for the large N expansion. For simplicity and also because we are considering these equations as the single mode

approximation to scalar electrodynamics we consider the case where $\phi_{0a}=0$ for all t . In terms of the mode functions f_a discussed above we then have

$$G_{0ab}(t,t)/i = \delta_{ab}f_a(t)f_a^*(t). \quad (3.3)$$

The coupled equations which need to be solved are

$$\left\{ \frac{d^2}{dt^2} + \frac{e^2}{N} \sum_{a=1}^N |f_a(t)|^2 \right\} A_0(t) = 0, \quad (3.4)$$

$$\left\{ \frac{d^2}{dt^2} + [m^2 + e^2 A_0^2(t)] \right\} f_a(t) = 0. \quad (3.5)$$

The four initial conditions that have to be specified in lowest order are

$$A(0), \dot{A}(0); G(0), \dot{G}(0).$$

Here we have assumed that the A field can be treated classically, which is the limit where the width of the fluctuations in A , namely D , can be ignored. In general we also have to specify $D(0)$ and $\dot{D}(0)$. In terms of the f_a this translates into the initial conditions

$$f_a(0) = \sqrt{G(0)},$$

$$\dot{f}_a(0) = \left[-\frac{i}{2G(0)} + \frac{\dot{G}(0)}{2G(0)} \right] f_a(0)$$

for all a .

B. Order 1/N corrections

The 1/N corrections to the generating functional Z require the evaluation of the second term in Eq. (2.7). We invert Eq. (2.12) and find

$$D(t,t') = D_0(t,t') - \int_C dt_1 \int_C dt_2 D_0(t,t_1) \Pi_0(t_1,t_2) D(t_2,t'). \quad (3.6)$$

Since we have chosen $\phi_{0a}(t)=0$, we can solve $D_0(t,t')$ as follows. Find the two linearly independent solutions g and g^* to the homogeneous equation,

$$\left\{ \frac{d^2}{dt^2} + \frac{e^2}{N} \sum_{a=1}^N |f_a(t)|^2 \right\} g(t) = 0, \quad (3.7)$$

satisfying the Wronskian condition

$$g^*(t)\dot{g}(t) - \dot{g}^*(t)g(t) = -i. \quad (3.8)$$

In terms of these solutions we have

$$D_0(t,t') = i \{ g(t)g^*(t')\Theta_C(t,t') + g(t')g^*(t)\Theta_C(t',t) \}. \quad (3.9)$$

The initial width of the wave function is then given by

$$D(0) = D_0(0,0)/i = |g(0)|^2. \quad (3.10)$$

Thus we can relate the initial conditions on g to the initial conditions $D(0)$ and $\dot{D}(0)$ of the wave function as

$$g(0) = \sqrt{D(0)},$$

$$\dot{g}(0) = \left[-\frac{i}{2D(0)} + \frac{\dot{D}(0)}{2D(0)} \right] g(0).$$

$\Pi_0(t,t')$ is given by

$$\begin{aligned} \Pi_0(t,t') &= -2ie^4 A_0(t)A_0(t') \\ &\times \left\{ \frac{1}{N} \sum_{a=1}^N [f_a^2(t)f_a^{*2}(t')\Theta_C(t,t') \right. \\ &\left. + f_a^2(t')f_a^{*2}(t)\Theta_C(t',t)] \right\}. \quad (3.11) \end{aligned}$$

Our numerical strategy for solving Eq. (3.6) is discussed in Appendix A.

There are two ways to calculate the order 1/N corrections to the time evolution problem. The first way is by a straight forward perturbation expansion of $W[J,j]$ in powers of 1/N. If we consider the average value of $A(t)$, we have

$$A(t) = \frac{1}{iNZ[j,j]} \left[\frac{\delta Z[J,j]}{\delta J(t)} \right]_j = A_0(t) + \frac{1}{N} A_1(t), \quad (3.12)$$

where

$$A_1(t) = \left[\frac{\delta}{\delta J(t)} \right] \frac{i}{2} \int_C dt' \ln [D^{-1}(t',t')].$$

Computing the derivatives, we obtain

$$A_1(t) = A_1^{(a)}(t) + A_1^{(b)}(t) + A_1^{(c)}(t), \quad (3.13)$$

where

$$A_1^{(a)}(t) = -e^4 \int_C dt' D(t,t') A_0(t') \Sigma_1(t'),$$

$$A_1^{(b)}(t) = -2e^4 \int_C dt' D(t,t') \Omega(t'),$$

$$A_1^{(c)}(t) = 4e^6 \int_C dt' D(t,t') A_0(t') \Sigma_2(t'),$$

where we have introduced

$$\Sigma_1(t) = \frac{1}{N} \sum_{a,b=1}^N \int_C dt' G_{0ab}(t,t') D(t',t') G_{0ba}(t',t),$$

$$\begin{aligned}\Sigma_2(t) &= \frac{1}{N} \sum_{a,b,c=1}^N \int_C dt' G_{0ac}(t,t') A_0(t') \\ &\quad \times \int_C dt'' G_{0cb}(t',t'') D(t',t'') A_0(t'') G_{0ba}(t'',t), \\ \Omega(t) &= \frac{1}{N} \sum_{a,b=1}^N \int_C dt' G_{0ab}(t,t') D(t,t') A_0(t') G_{0ba}(t',t).\end{aligned}$$

Here, all functions are to be evaluated using the first order solutions $A_0(t)$.

C. Effective action approach

A second method of evaluation of the order $1/N$ corrections is to use a Legendre transformation to find the effective action $\Gamma[A, \phi_k]$:

$$\Gamma[A, \phi] = W[J, j] - \int_C dt \left\{ J(t)A(t) + \sum_{a=1}^N j_a(t)\phi_a(t) \right\}. \quad (3.14)$$

Using the fact that $S_{\text{eff}}[A_0; J, j]$ is a stationary point, we find, to order $1/N$,

$$\begin{aligned}\Gamma[A, \phi] &= \int_C dt \left\{ -\frac{1}{2} A \frac{d^2}{dt^2} A + \frac{i}{2N} \sum_{a=1}^N \ln[G_{aa}^{-1}[A](t,t)] \right. \\ &\quad \left. + \frac{i}{2N} \ln[D^{-1}[A, \phi](t,t)] \right\} \\ &\quad - \int_C dt \int_C dt' \frac{1}{2} \sum_{a,b=1}^N \phi_a(t) G_{ab}^{-1}[A](t,t') \phi_b(t'),\end{aligned} \quad (3.15)$$

which is the classical expression plus the trace-log terms. Here, $G_{ab}^{-1}[A](t,t')$ and $D^{-1}[A, \phi](t,t')$ are functionals of the full $A(t)$ and $\phi_a(t)$. Again, in this case we set $\phi_a(t) = 0$. This effective action agrees up to order $1/N$ with the true Legendre transform of the generating functional. One can now ignore where this action came from and directly write nonperturbative equations for A directly from this action which will agree to order $1/N$ with the previous equations. However, at $N=1$ where we will be making a numerical comparison with the exact answer, the results are expected to be quite different. We would like to know how the two results for A differ and which is more accurate at modest N .

The equation of motion for $A(t)$ which follows from varying the effective action Γ is

$$\left\{ \frac{d^2}{dt^2} + \frac{e^2}{iN} \sum_{a=1}^N \mathcal{G}_{aa}(t,t) \right\} A(t) + \frac{1}{N} \int_C dt' K(t,t') A(t') = 0, \quad (3.16)$$

where

$$\begin{aligned}\mathcal{G}_{ab}(t,t') &= G_{ab}(t,t') + \frac{i}{N} \sum_{c,d=1}^N \int_C dt_1 \int_C dt_2 G_{ac}(t,t_1) \\ &\quad \times \Sigma_{cd}(t_1,t_2) G_{db}(t_2,t'),\end{aligned} \quad (3.17)$$

$$\begin{aligned}\Sigma_{cd}(t_1,t_2) &= e^2 \delta_{cd} \delta_C(t_1,t_2) D(t_1,t_2) \\ &\quad - 4e^4 A(t_1) G_{cd}(t_1,t_2) D(t_1,t_2) A(t_2),\end{aligned} \quad (3.18)$$

$$K(t,t') = 2e^4 D(t,t') \frac{1}{N} \sum_{a,b=1}^N \{G_{ab}(t,t') G_{ba}(t',t)\}. \quad (3.19)$$

The equation for A has to be solved simultaneously with the equations for G and D which now depend on the full A . To obtain G we now need to solve for the modes f which satisfy

$$\left\{ \frac{d^2}{dt^2} + m^2 + e^2 A^2(t) \right\} f_a(t) = 0. \quad (3.20)$$

G is again given by

$$\begin{aligned}G_{ab}(t,t') &= i \delta_{ab} \{ f_a(t) f_a^*(t') \Theta_C(t,t') \\ &\quad + f_a(t') f_a^*(t) \Theta_C(t',t) \}.\end{aligned} \quad (3.21)$$

In terms of the full A and these new f we determine D using the integral equation (3.6) where now D_0 is determined using the new f in the equation for g .

These equations of motion agree with the previous ones to order $1/N^2$. We had hoped that these equations which partially resum $1/N$ corrections would be more accurate at late times. However, this turned out not to be the case, and in fact since A becomes unbounded sooner in this second approach, the late time behavior of this approach is worse. At $N=1$ both methods agree with the exact answer for the same amount of time. When they diverge from each other they also diverge from the exact answer.

IV. ENERGY

The expectation value of the energy is given by

$$E/N = \frac{1}{2} \langle \dot{A}^2 \rangle + \frac{1}{2} \sum_{a=1}^N \{ \langle \check{\phi}^2 + m^2 \langle \phi_a^2 \rangle + e^2 \langle A^2 \phi_a^2 \rangle \}. \quad (4.1)$$

The expectation values are related to the full connected Green's functions $\bar{G}, \bar{D}, K_{ab}^3, G_{ab}^4$ by the equations

$$\langle A(t)A(t') \rangle = A(t)A(t') + \frac{1}{iN} \bar{D}(t,t'), \quad (4.2)$$

$$\langle \phi_a(t) \phi_b(t') \rangle = \frac{1}{i} \bar{G}_{ab}(t,t'), \quad (4.3)$$

$$\begin{aligned}
\langle \phi_a(t_1)\phi_b(t_2)A(t_3)A(t_4) \rangle &= -\frac{1}{iN^2}G_{ab}^4(t_3,t_4;t_1,t_2) \\
&+ \frac{1}{i}A(t_3)A(t_4)G_{ab}(t_1,t_2) \\
&- \frac{1}{N}\bar{D}(t_3,t_4)\bar{G}_{ab}(t_1,t_2) \\
&- \frac{1}{N}\{K_{ab}^3(t_1,t_2;t_3) \\
&\times A(t_4) + K_{ab}^3(t_1,t_2;t_4)A(t_3)\}.
\end{aligned} \tag{4.4}$$

We notice that to order 1/N the connected three point function K^3 contributes but not the connected four point function G^4 . If we are using the generating functional approach we can now directly expand the energy in a power series in 1/N by assuming $G^n = G_0^n + 1/N G_1^n + \dots$. We then obtain

$$\langle A^2(t) \rangle = A_0^2(t) + \frac{2}{N}A_0(t)A_1(t) + \frac{1}{iN}D(t,t),$$

$$\langle \dot{A}^2(t) \rangle = \dot{A}_0^2(t) + \frac{2}{N}\dot{A}_0(t)\dot{A}_1(t) + \frac{1}{iN}\left.\frac{\partial^2 D(t,t')}{\partial t \partial t'}\right|_{t=t'},$$

$$\langle \phi_a^2(t) \rangle = \frac{1}{i}\left\{G_{0aa}(t,t) + \frac{1}{N}G_{1aa}(t,t)\right\},$$

$$\langle \dot{\phi}_a^2(t) \rangle = \frac{1}{i}\left\{\left.\frac{\partial^2 G_{0aa}(t,t')}{\partial t \partial t'}\right|_{t=t'} + \frac{1}{N}\left.\frac{\partial^2 G_{1aa}(t,t')}{\partial t \partial t'}\right|_{t=t'}\right\},$$

$$\begin{aligned}
\langle A^2(t)\phi_a^2(t) \rangle &= \frac{1}{i}\left\{A_0^2(t)G_{0aa}(t,t) + \frac{1}{N}A_0^2(t)G_{1aa}(t,t) \right. \\
&+ \left. \frac{2}{N}A_0(t)A_1(t)G_{0aa}(t,t)\right\} \\
&- \frac{1}{N}G_{0aa}(t,t)D(t,t) - \frac{2}{N}A_0(t)K_0^3(t,t;t),
\end{aligned}$$

where to order 1/N,

$$\begin{aligned}
K_0^3(t_1,t_2;t_3) &= -2e^2 \int_C dt' G_0(t_1,t')G_0(t_2,t') \\
&\times D(t_3,t')A_0(t'),
\end{aligned} \tag{4.5}$$

and $G_1(t,t')$ is the sum of three terms

$$G_{1ab}(t,t') = G_{1ab}^{(a)}(t,t') + G_{1ab}^{(b)}(t,t') + G_{1ab}^{(c)}(t,t'), \tag{4.6}$$

with

$$G_{1ab}^{(a)}(t,t') = ie^2 \int_C dt_1 G_{0ab}(t,t_1)D(t_1,t_1)G_{0ba}(t_1,t'),$$

$$\begin{aligned}
G_{1ab}^{(b)}(t,t') &= -4ie^4 \sum_{c,d=1}^N \int_C dt_1 G_{0ac}(t,t_1)A_0(t_1) \int_C dt_2 \\
&\times G_{0cd}(t_1,t_2)D(t_1,t_2)G_{0db}(t_2,t')A_0(t_2),
\end{aligned}$$

$$\begin{aligned}
G_{1ab}^{(c)}(t,t') &= -2e^2 \int_C dt_1 G_{0ab}(t,t_1)A_0(t_1)A_1(t_1) \\
&\times G_{0ba}(t_1,t').
\end{aligned}$$

The above lead to the following expression of the energy:

$$E_Z(t) = E_0(t) + \frac{1}{N}E_1(t). \tag{4.7}$$

For E_0 , we find

$$\begin{aligned}
\frac{E_0(t)}{N} &= \frac{1}{2}\dot{A}_0^2(t) + \frac{1}{2iN} \sum_{a=1}^N \left\{ \left.\frac{\partial^2 G_{0aa}(t,t')}{\partial t \partial t'}\right|_{t=t'} \right. \\
&+ \left. (m^2 + e^2 A_0^2)G_{0aa}(t,t) \right\}.
\end{aligned} \tag{4.8}$$

and where

$$\begin{aligned}
\frac{E_1(t)}{N} &= \dot{A}_0(t)\dot{A}_1(t) + \frac{1}{2iN} \sum_{a=1}^N \left\{ \left.\frac{\partial^2 G_{1aa}(t,t')}{\partial t \partial t'}\right|_{t=t'} \right. \\
&+ (m^2 + e^2 A_0^2)G_{1aa}(t,t) \\
&+ \left. 2e^2 A_0(t)A_1(t)G_{0aa}(t,t) \right\} + \frac{1}{2}\left\{\frac{1}{i}\left.\frac{\partial^2 D(t,t')}{\partial t \partial t'}\right|_{t=t'}\right. \\
&- \left. \frac{e^2}{N} \sum_{a=1}^N G_{0aa}(t,t)D(t,t) \right\} - e^2 A_0(t)K_0^3(t,t;t).
\end{aligned} \tag{4.9}$$

If instead we are using the effective action Γ to determine the equations of motion, then we find the following expressions for the expectation values of the operators:

$$\langle A^2(t) \rangle = A^2(t) + \frac{1}{iN}D(t,t), \tag{4.10}$$

$$\langle \dot{A}^2(t) \rangle = \dot{A}^2(t) + \frac{1}{iN}\left.\frac{\partial^2 D(t,t')}{\partial t \partial t'}\right|_{t=t'}, \tag{4.11}$$

$$\langle \phi_a^2(t) \rangle = \frac{1}{i}\mathcal{G}_{aa}(t,t), \tag{4.12}$$

$$\langle \dot{\phi}_a^2(t) \rangle = \frac{1}{i}\left.\frac{\partial^2 \mathcal{G}_{aa}(t,t')}{\partial t \partial t'}\right|_{t=t'}, \tag{4.13}$$

$$\begin{aligned}
\langle A^2(t)\phi_a^2(t) \rangle &= \frac{1}{i}A^2(t)\mathcal{G}_{aa}(t,t) - \frac{1}{N}G_{aa}(t,t)D(t,t) \\
&- \frac{2}{N}A(t)K^3(t,t;t),
\end{aligned} \tag{4.14}$$

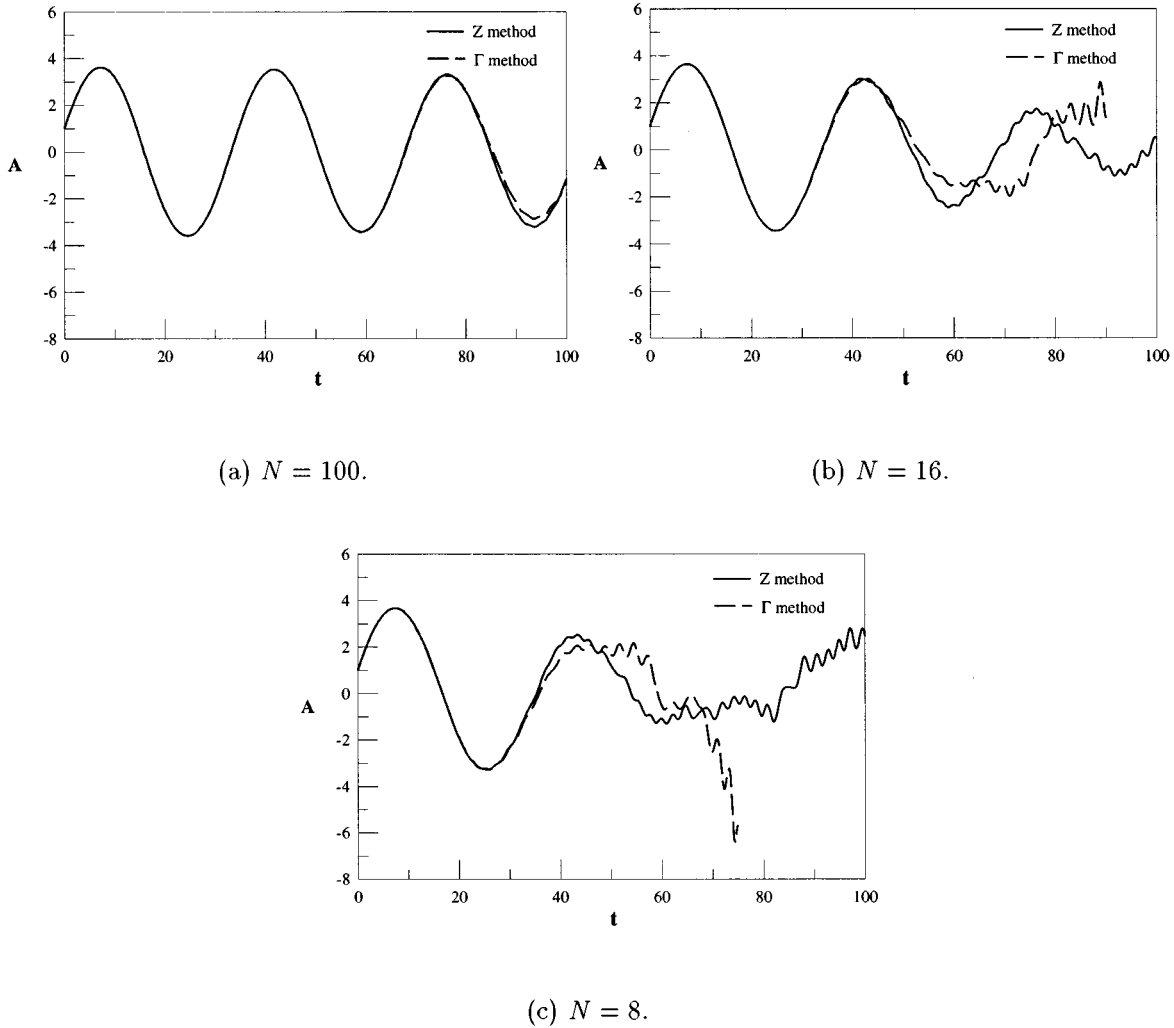


FIG. 2. Time evolution of $A(t)$ for $N=8, 16$, and 100 , using the Z method and the Γ method.

where now we use the full A and not A_0 in determining all the Green's functions in K^3 . The energy for the Γ method can then be calculated as

$$\begin{aligned}
 \frac{E_\Gamma}{N} = & \frac{1}{2} \dot{A}^2(t) + \frac{1}{2iN} \sum_{a=1}^N \left\{ \frac{\partial^2 \mathcal{G}_{aa}(t, t')}{\partial t \partial t'} \right\}_{t=t'} \\
 & + (m^2 + e^2 A^2) \mathcal{G}_{aa}(t, t) \left\} + \frac{1}{2N} \left\{ \frac{1}{i} \frac{\partial^2 D(t, t')}{\partial t \partial t'} \right\}_{t=t'} \\
 & - \frac{e^2}{N} \sum_{a=1}^N \mathcal{G}_{aa}(t, t) D(t, t) \left\} - \frac{e^2}{N} A(t) K^3(t, t; t).
 \end{aligned} \tag{4.15}$$

Using the equations of motion, we can show that all expressions for the energy, Eqs. (4.8), (4.7), (4.15), are time independent, so that the energy is conserved for both the perturbative calculation as well as the effective action method.

V. NUMERICAL RESULTS

In order to find numerical solutions to these equations, we expand all functions in Chebyshev polynomials, and solve

the resulting finite set of equations. In most of our calculations, we used a set of 32 polynomials. Details of this calculation are given in Appendix B.

In order to make contact with our previous lowest order $N=\infty$ results we will use the set of initial conditions ($e=0.3$, $E=1$, $m=1$) given in Ref. [7]. Time is measured in units of $1/m$. We would first like to compare the two methods at large but finite N starting off with Gaussian initial data for the wave function. In a mean field approximation, such as lowest order large N , a wave function which starts out as a Gaussian remains Gaussian. Thus in mean field theory the only physical measurable quantities are $\langle A(t) \rangle$, $\langle [A(t) - \langle A(t) \rangle]^2 \rangle = D(t, t)/i$, and $\langle \phi^2(t) \rangle = \mathcal{G}(t, t)/i$. It is just these three moments that we will plot here to make contact with mean field results. (At infinite N , D is also zero, but it is finite in a Hartree approximation, as discussed in [7].) We will denote the straightforward use of the $1/N$ expansion from the path integral as the Z method and the results coming from the effective action approach, the Γ method.

In Fig. 2, we plot $\langle A(t) \rangle$ for $N=100, 16$, and 8 . We notice that as we reduce N the two different methods start diverging after one period. We also find that the second method which is based on the effective action has corrections which become unbounded at late times which leads us to believe that

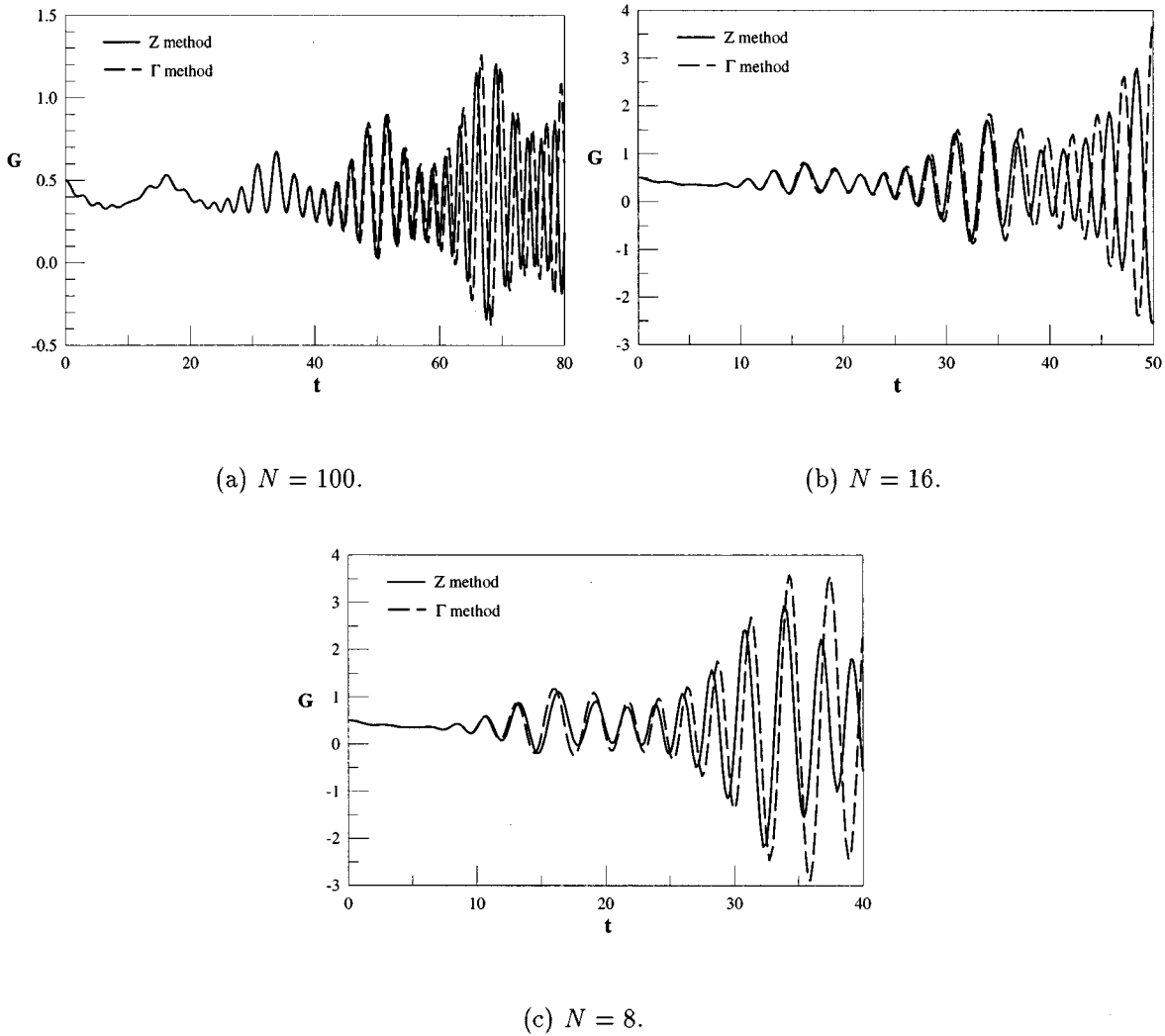


FIG. 3. Time evolution of $\mathcal{G}(t,t)/i$ for $N=8, 16$, and 100 , using the Z method and the Γ method.

the second method is less stable. Both methods seem to have secular behavior (i.e., corrections which grow as a power in time).

In Fig. 3, we show results for $\langle \phi^2(t) \rangle = \mathcal{G}(t,t)/i$ for $N=100, 16$, and 8 . We notice that at late times, this can go negative. This is a result of the fact that the operator ϕ has a $1/N$ expansion $\phi = \phi_0 + (1/N)\phi_1 + \dots$, so that the positive definite width of the wave packet is $\langle (\phi_0 + (1/N)\phi_1)^2 \rangle$ which also includes $1/N^2$ corrections. So once the $1/N$ correction becomes as large as the lowest order term, the expansion breaks down and keeping terms only to order $1/N$ in the expansion of the expectation value can lead to negative results. This negative result occurs after the two methods diverge which is an indication of when the expansion is breaking down.

Now let us look at the effective width of the wave function of the A oscillator. For the Z method, we have that the correlation function $D(t,t')$, defined by Eq. (3.6), is independent of N . However, the width determined in the effective action approach has $1/N$ corrections due to the implicit dependence of $A(t)$ on N . First, in Fig. 4, we compare the two approaches for calculating $D(t,t)/i$ at $N=100$. In Fig. 5, we show the N dependence of $D(t,t)/i$ using the Γ -method

at $N=100, 16$, and 8 . From Fig. 5 we see that when $t \geq 5$ the $1/N^2$ effects are starting to appear.

For the case $N=1$, it is possible to compare the results of both expansions with an exact numerical simulation of the Schrödinger equation which we obtained in Ref. [7]. Figure

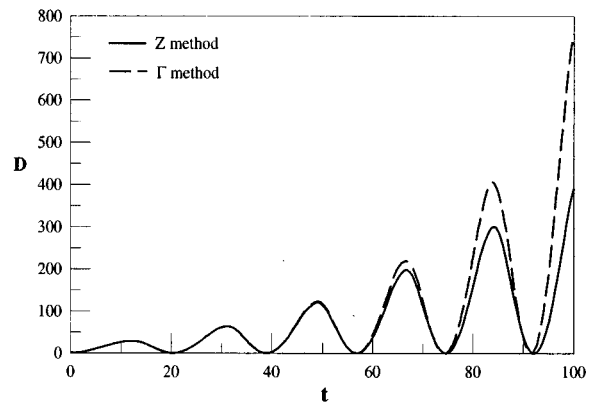


FIG. 4. Time evolution of $D(t,t)/i$ for $N=100$, as computed using the Z method and the Γ method.

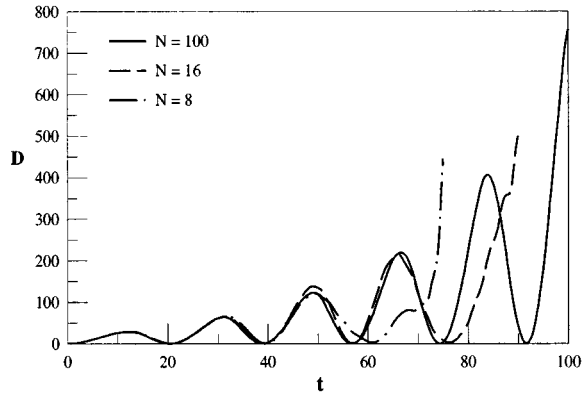


FIG. 5. Time evolution of $D(t,t)/i$ for $N=8,16,100$, as computed using the Γ method.

6 shows the time evolution of $A(t)$ as computed using both the Γ and the Z methods, compared with the exact solution for $0 < t < 25$. For comparison we also include the lowest order in large N result. Both methods agree with each other and with the exact result for $t < 8$. They are accurate for at least twice the time scale as $A_0(t)$, the first order large N result. It is gratifying to see that the two solutions diverge from each other approximately the same time as when they diverge from the exact solution so that the divergence of the two approaches which signals the onset of $1/N^2$ corrections sets the time scale for the accuracy of the result even for $N=1$.

In Fig. 7, we plot the effective width of the ϕ oscillators $\langle [\phi(t) - \langle \phi(t) \rangle]^2 \rangle = \mathcal{G}_{aa}(t,t)/i$ using both methods and compare these results to the exact solution for $N=1$ as well as the lowest order result. Here we notice a marked improvement of the $1/N$ corrected results from the lowest order ones. We find again that the time at which the two methods start diverging from one another they also diverge from the exact answer.

In Fig. 8, we plot the effective width of the A oscillator $D(t,t)/i$ for the Z and Γ methods, as a function of time. In this case the lowest order in $1/N$ is equivalent to a δ function width and so did not give a prediction. Here again we find the two solutions diverge from one another just when the approximation breaks down.

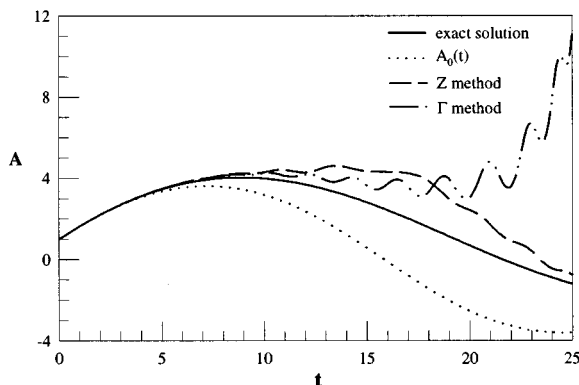


FIG. 6. Time evolution of $A(t)$ for $N=1$, as computed using the Z method and the Γ method, compared with the exact solution.

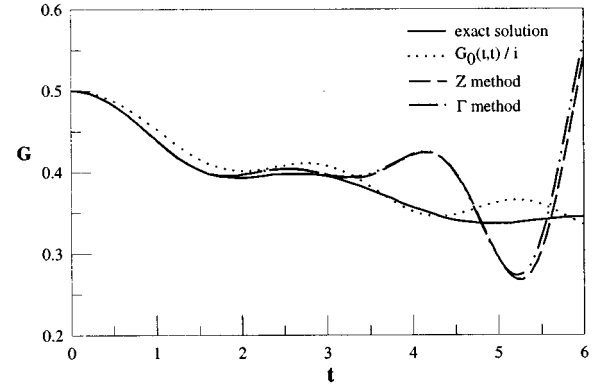


FIG. 7. $\mathcal{G}_{aa}(t,t)/i$ for the first and second order large- N approximations as a function of time.

What we can conclude from the above results are that $1/N^2$ corrections become important rather early in this particular time evolution problem. The period of agreement of our two methods is approximately the period when they give accurate results for the time evolution. We find at $N=1$ that the width of the ϕ oscillator is much better described when we add the $1/N$ corrections. Also the time evolution of the A oscillator is now accurate for about twice the time period found previously at lowest order.

In our calculations, we were able to verify energy conservation to one part in 10^4 . In spite of this, when the $1/N$ expansion breaks down it fails to preserve the positivity of certain expectation values. If we look at Eq. (3.6) for $D(t,t)/i$, which is the positive definite expectation value $\langle [A(t) - \langle A(t) \rangle]^2 \rangle$, we notice that this is an integral equation which sums all the bubbles and thus correctly takes into account the shift in the frequency of the quantum fluctuations of the A oscillator. However in the equation for $\langle A(t) \rangle$ itself, Eq. (3.16), the term which is the time-dependent mass of the oscillator, namely $\langle \phi^2(t) \rangle$, which should also be positive definite, is given by Eq. (3.17). We notice that Eq. (3.17) is not an integral equation but only the leading term in a $1/N$ expanded Green's function. This quantity need not be (and is not) positive definite. We can understand how this can happen in an example. In the vacuum state $\langle \phi^2(t) \rangle = G(t,t)/i$

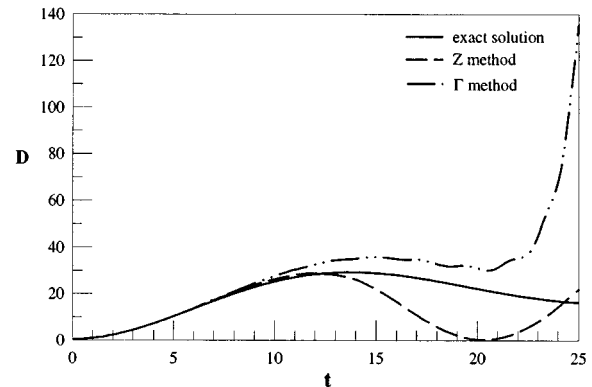


FIG. 8. Comparison of exact numerical simulation of $D(t,t)/i$ with the value computed using the Z and Γ methods.

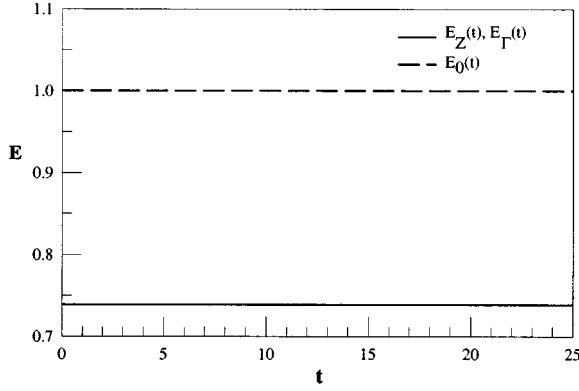


FIG. 9. The energy $E_Z = E_0 + E_1/N$, computed using the Z method, and E_Γ using the Γ method.

time independent and is positive definite. One of the effects of the interactions is to renormalize the mass (change the frequency) of the oscillator. So consider the toy problem

$$G(t,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{[\omega^2 + m_0^2 + (e^2/N)m_1^2]}$$

$$= \frac{1}{2} \left(m_0^2 + \frac{e^2}{N} m_1^2 \right)^{-1/2},$$

where we have kept only the e^2/N correction to the mass shift of the ϕ oscillator. Now m_1^2 is positive since we have assumed the interaction is repulsive. Again reexpanding G we get

$$G(t,t) = \frac{1}{2m_0} \left(1 - \frac{e^2 m_1^2}{2Nm_0^2} \right),$$

so we see that if m_1^2 is large enough, then this quantity which is positive definite can appear negative when reexpanded in $1/N$.

This problem can also be a source of secular terms. For example, consider the fact that the frequency of the classical A oscillator is shifted by $1/N$ corrections so that one has

$$A(t) = A(0) \cos\left(\omega_0 t + \frac{\omega_1}{N} t\right). \quad (5.1)$$

If, however, the actual $1/N$ expansion for the oscillator instead gave the first two terms in the Taylor series we would get

$$A(t) = A(0) \left[\cos(\omega_0 t) - \frac{\omega_1 t}{N} \sin(\omega_0 t) \right]. \quad (5.2)$$

Using this expression would lead to blow up of $A(t)$ in a time scale N/ω_1 . This type of behavior is found in our numerical simulations. One way of solving the secular problem as well as guaranteeing the positivity of expectation values is by appropriately summing the series. That is we replace Eq. (3.17) by

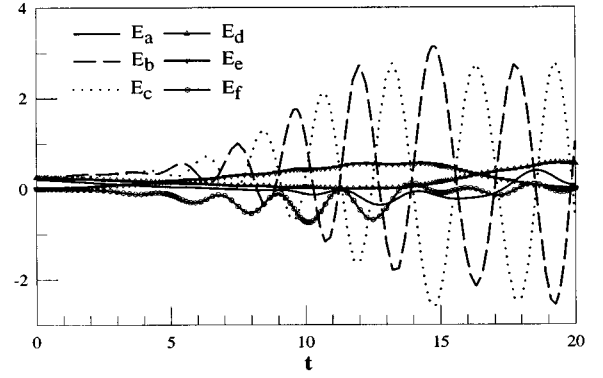


FIG. 10. Various components of the energy, as computed in the Z method, as a function of time. Here we have introduced the notations: $E_a = (1/2)\dot{A}^2(t)$, $E_b = (1/2iN)\sum_{a=1}^N [\partial^2 \mathcal{G}_{aa}(t,t')/\partial t \partial t']|_{t=t'}$, $E_c = (1/2iN)\sum_{a=1}^N (m^2 + e^2 A^2) \mathcal{G}_{aa}(t,t)$, $E_d = (1/2iN)[\partial^2 D(t,t')/\partial t \partial t']|_{t=t'}$, $E_e = -(1/2iN^2)\sum_{a=1}^N e^2 \mathcal{G}_{aa}(t,t) D(t,t)$, $E_f = -(1/N)e^2 A(t) K^3(t,t;t)$.

$$\mathcal{G}_{ab}(t,t') = G_{ab}(t,t') + \frac{i}{N} \sum_{c,d=1}^N \int_c dt_1 \int_c dt_2 G_{ac}(t,t_1) \times \Sigma_{cd}(t_1,t_2) \mathcal{G}_{db}(t_2,t'). \quad (5.3)$$

We then have to modify appropriately Eqs. (2.13), (2.14), (3.18), and (3.19) in order to guarantee energy conservation again.

These modifications only change the results at order $1/N^2$ but are obviously very important because of the early breakdown of the naive methods. However, in this paper we did not want to go beyond what one obtains directly using the two direct approaches to the $1/N$ expansion and will discuss the resummed theory in a separate paper.

This problem with expectation values becoming unbounded is more acute in the Γ method. In Fig. 9 we show that energy is conserved. However, in Figs. 10 and 11 we show the above phenomena that several of the individual contributions to the energy become negative if we keep terms only up to $1/N$.

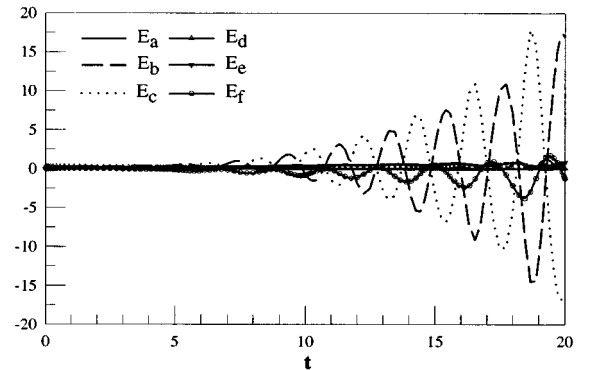


FIG. 11. Various components of the energy, as computed in the Z method, as a function of time. The notations are similar to those of Fig. 10, with the addition that now we have $\mathcal{G}_{aa}(t,t') = G_{0aa}(t,t') + G_{1aa}(t,t')/N$, and $G_{aa}(t,t')$ becomes $G_{0aa}(t,t')$.

The appearance of secular terms in a perturbation expansion is quite well known [10]. As an example for the classical anharmonic oscillator

$$\frac{d^2y}{dt^2} + y + gy^3 = 0,$$

if we assume a solution of the form

$$y = y_0(t) + gy_1(t),$$

and perform a perturbation series in the coupling constant g , secular terms linear in t appear in $y_1(t)$. One can see by doing a large- N expansion in the Schrödinger picture, that one can avoid the secular problem in a simple way. If we solve the Schrödinger equation at large N in terms of the eigenfunctions and eigenvalues we find

$$\Psi(\phi, A, t) = \sum_m c_m \Psi_m(\phi, A) e^{-iE_m t}, \quad (5.4)$$

where the $\Psi_m(\phi, A)$ are the solutions of the time independent Schrödinger equation. We can do a large- N expansion for the eigenfunctions and eigenvalues separately. We then find that $\Psi_m(\phi, A)$ has a Taylor series in $1/\sqrt{N}$ and E_m has a Taylor series in $1/N$. As long as we do not reexpand the exponentials having the time dependence (i.e., keep $\exp\{-i[E_m^0 t + (1/N)E_m^1 t \dots]\}$) then there are no secular terms. However if we reexpand the exponentials and keep only terms of order $1/N$, secular terms proportional to powers of t will appear. We believe that this might very well be occurring in the large N expansion based on the CTP formalism.

VI. CONCLUSIONS

We have presented two methods for calculating the $1/N$ corrections to the time evolution of a system of $N+1$ oscillators using Schwinger's CTP formalism. These two methods differ by terms of order $1/N^2$ so that the divergence of results is an indication of the time during which keeping terms up to order $1/N$ is accurate. This was verified by comparing with direct numerical simulation of the Schrödinger equation for the case $N=1$. We found that for $N=1$, keeping the $1/N$ corrections allows us to extend the range of time for which the expectations values track the exact answer significantly. This was most noticeable for the the quantity which describes the width of the ϕ wave function.

In performing these numerical simulations, we found some shortcomings in the $1/N$ expansion. It does not guarantee the boundedness of various expectation values, even though energy is exactly conserved. As a result of this, we expect that we are seeing secular behavior in our expansion which needs to be cured. One way of curing this problem is to instead look at a $1/N$ expansion for the eigenfunctions and eigenvalues of the Schrödinger equation instead of working with the CTP formalism. Another is to resum the Green's function for the ϕ oscillator and modify the equations for the other variables appropriately so that energy is conserved. These two ideas will be discussed in a future work.

In conclusion, by comparing our two direct approaches to

performing a $1/N$ expansion, one using the generating functional Z and one the effective action Γ , we found that the direct perturbation theory in $1/N$ from Z gave results that were bounded for longer times. However, having both methods, allowed us to determine when $1/N^2$ corrections became important. Thus having both methods will be quite useful in simulating field theories, when there will be no exact solution to compare with. Solving the secular problem we found here will be discussed elsewhere.

ACKNOWLEDGMENTS

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APPENDIX A: EQUATION FOR $D(T, T')$

In this appendix we discuss how we approach the problem of finding the solution of Eq. (3.6):

$$D(t, t') = D_0(t, t') - \int_c dt_1 \int_c dt_2 D_0(t, t_1) \Pi(t_1, t_2) D(t_2, t'). \quad (A1)$$

We first note that the causal Green's functions are symmetric in the sense that $\mathcal{A}_{>}(t, t') = \mathcal{A}_{>}(t', t)$, and obey the additional condition

$$\mathcal{A}_{>,<}(t, t') = -\mathcal{A}_{<,>}^*(t, t') = \mathcal{A}_{<,>}(t', t). \quad (A2)$$

The last equation gives

$$\text{Re}\{\mathcal{A}_{>}(t, t')\} = -\text{Re}\{\mathcal{A}_{<}(t, t')\}, \quad (A3)$$

$$\text{Im}\{\mathcal{A}_{>}(t, t')\} = -\text{Im}\{\mathcal{A}_{<}(t, t')\} \quad (A4)$$

or

$$\mathcal{A}_{>}(t, t') - \mathcal{A}_{<}^*(t, t') = 2\text{Re}\{\mathcal{A}_{>}(t, t')\}, \quad (A5)$$

$$\mathcal{A}_{>}(t, t') + \mathcal{A}_{<}^*(t, t') = 2\text{Im}\{\mathcal{A}_{>}(t, t')\}. \quad (A6)$$

With these equations in mind, the causal Green's function $\mathcal{A}(t, t')$ is fully determined by the component $\mathcal{A}_{>}(t, t') = \text{Re}\{\mathcal{A}_{>}(t, t')\} + i\text{Im}\{\mathcal{A}_{>}(t, t')\}$, and we need to evaluate the function only for $t' \leq t$.

Directing our attention now to the calculation of $D(t, t')$, it is convenient to introduce

$$Q(t, t') = \int_c dt'' D_0(t, t'') \Pi(t'', t'). \quad (A7)$$

Note that even though the functions $D_0(t, t')$ and $\Pi(t, t')$ satisfy the properties of the Green's functions listed above, the new function $Q(t, t')$ satisfies only some: i.e.,

$$Q_{>,<}(t,t') = -Q_{<,>}^*(t,t') \neq Q_{<,>}(t',t). \quad (\text{A8})$$

This does not prevent the Green's function $D(t,t')$ to recover all the desired properties after one more CTP integration. In practice, this translates into the fact that we have to actually calculate the function $Q_{>}(t,t')$ for all moments t and t' , whereas the other causal functions need only some half of the data.

Then, Eq. (A1) becomes

$$D(t,t') = D_0(t,t') - \int_c dt'' Q(t,t'') D(t'',t') \quad (\text{A9})$$

or

$$\begin{aligned} D_{>}(t,t') &= D_{0>}(t,t') - \int_0^t dt'' [Q_{>}(t,t'') \\ &\quad - Q_{<}(t,t'')] D(t'',t') + \int_0^{t'} dt'' Q(t,t'') \\ &\quad \times [D_{>}(t'',t') - D_{<}(t'',t')], \end{aligned} \quad (\text{A10})$$

$$\begin{aligned} D_{<}(t,t') &= D_{0<}(t,t') - \int_0^t dt'' [Q_{>}(t,t'') \\ &\quad - Q_{<}(t,t'')] D_{<}(t'',t') \\ &\quad + \int_0^{t'} dt'' Q_{<}(t,t'') [D_{>}(t'',t') - D_{<}(t'',t')]. \end{aligned} \quad (\text{A11})$$

Using the properties (A2)–(A6), we see that Eq. (A11) is redundant, and we can write Eq. (A10) as

$$\begin{aligned} D_{>}(t,t') &= D_{0>}(t,t') - 2 \int_0^t dt'' \text{Re}\{Q_{>}(t,t'')\} D_{>}(t'',t') \\ &\quad + 2 \int_0^{t'} dt'' Q_{>}(t,t'') \text{Re}\{D_{>}(t'',t')\}. \end{aligned} \quad (\text{A12})$$

We separate now the *real* and the *imaginary* part of Eq. (A12) and obtain the system of integral equations

$$\begin{aligned} \text{Re}\{D_{>}(t,t')\} &= \text{Re}\{D_{0>}(t,t')\} - 2 \int_0^t dt'' \text{Re}\{Q_{>}(t,t'')\} \\ &\quad \times \text{Re}\{D_{>}(t'',t')\} + 2 \int_0^{t'} dt'' \\ &\quad \times \text{Re}\{Q_{>}(t,t'')\} \text{Re}\{D_{>}(t'',t')\}, \end{aligned} \quad (\text{A13})$$

$$\begin{aligned} \text{Im}\{D_{>}(t,t')\} &= \text{Im}\{D_{0>}(t,t')\} - 2 \int_0^t dt'' \text{Re}\{Q_{>}(t,t'')\} \\ &\quad \times \text{Im}\{D_{>}(t'',t')\} + 2 \int_0^{t'} dt'' \text{Im}\{Q_{>}(t,t'')\} \\ &\quad \times \text{Re}\{D_{>}(t'',t')\}. \end{aligned} \quad (\text{A14})$$

The above system of equations has to be solved for $t' \leq t$. Notice that the two equations are independent, which allows

us to solve first Eq. (A13) for the *real* part of $D_{>}(t,t')$, and then use this result to derive the *imaginary* part of $D_{>}(t,t')$ from Eq. (A14).

APPENDIX B: NUMERICAL METHODS

Our numerical technique involves the expansion of all the unknown functions $f(t)$, $g(t)$, $A(t)$, $D(t,t')$ in a Chebyshev polynomial basis. We follow a method developed by El-gendy [11] and have applied it to combined differential and integrals equations of the type we have here. We use the same Chebyshev expansion methods for solving the Green's function equation for $D(t,t')$, in the CTP formalism, as explained in Appendix A. In addition, we divide the time up into small blocks, moving along block by block.

Chebyshev polynomials of the first kind of degree n are defined by

$$T_n(x) = \cos(n \arccos x). \quad (\text{B1})$$

We define the grid in the interval $[-1,1]$ by choosing the positions

$$\tilde{x}_j = \cos \frac{j\pi}{N}, \quad j=0,1,\dots,N. \quad (\text{B2})$$

Then, the Chebyshev polynomials satisfy a discrete orthogonality relation of the form

$$\sum_{k=0}^N {}'' T_i(\tilde{x}_k) T_j(\tilde{x}_k) = \beta_i \delta_{ij}, \quad (\text{B3})$$

where the constants β_i are

$$\beta_i = \begin{cases} \frac{N}{2}, & i \neq 0, N \\ N, & i = 0, N. \end{cases} \quad (\text{B4})$$

An arbitrary function $f(x)$ can be approximated in the interval $[-1,1]$ by the formula

$$f(x) = \sum_{j=0}^N {}'' a_j T_j(x). \quad (\text{B5})$$

We denote $f_k = f(\tilde{x}_k)$. The coefficients a_j are defined by

$$a_j = \frac{2}{N} \sum_{k=0}^N {}'' f(\tilde{x}_k) T_j(\tilde{x}_k), \quad j=0,\dots,N. \quad (\text{B6})$$

and the summation symbol with double primes denotes a sum with first and last terms halved. As shown by El-gendy, we never have to actually compute these expansion coefficients a_j . Instead, we find f_j directly. The advantage of the Chebyshev expansion method is that Eq. (B5), which is an expansion in a *finite* set, is exact for $x = \tilde{x}_j$, $j=0,\dots,N$.

We can approximate the calculation of the indefinite integral $\int_{-1}^x f(t) dt$ by

$$I(x) = \int_{-1}^x f(t) dt = \sum_{j=0}^N {}'' a_j \int_{-1}^x T_j(t) dt, \quad (\text{B7})$$

where the integral $\int_{-1}^x T_j(t)dt$ is given by

$$\begin{aligned} \frac{T_{j+1}(x)}{2(j+1)} - \frac{T_{j-1}(x)}{2(j-1)} + \frac{(-)^{j+1}}{j^2-1} & \text{ if } j \geq 1, \\ \frac{1}{4}[T_2(x)-1] & \text{ if } j=1, \\ T_1(x)+1 & \text{ if } j=0. \end{aligned} \quad (\text{B8})$$

When the variable x takes the values of the grid points (B2), we can write Eq. (B7) in a matrix format, such as

$$I_i = I(\tilde{x}_i) = \sum_{j=0}^N {}'' B_{ij}^{[-1,1]} f_j, \quad (\text{B9})$$

where $B^{[-1,1]}$ is a square matrix of order $(N+1)$, whose elements $B_{ij}^{[-1,1]}$ are equal to

$$\begin{aligned} & \frac{1}{N^2} T_N(\tilde{x}_i) T_{N-1}(\tilde{x}_j) + \frac{1}{2N(N+1)} T_{N+1}(\tilde{x}_i) T_N(\tilde{x}_j) \\ & + \sum_{k=0(k \neq 1)}^N {}'' \frac{2}{N} \frac{(-)^{k+1}}{k^2-1} T_0(\tilde{x}_i) T_k(\tilde{x}_j) - \frac{1}{2N} T_0(\tilde{x}_i) T_1(\tilde{x}_j) \\ & + \frac{1}{N(N-1)} T_{N-1}(\tilde{x}_i) \left[T_{N-2}(\tilde{x}_j) - \frac{1}{2} T_N(\tilde{x}_j) \right] \\ & + \sum_{k=1}^{N-2} \frac{1}{kN} T_k(\tilde{x}_i) [T_{k-1}(\tilde{x}_j) - T_{k+1}(\tilde{x}_j)]. \end{aligned} \quad (\text{B10})$$

In a similar manner, the derivative $df(x)/dx$ is approximated by

$$D(x) = \frac{d}{dx} f(x) = \sum_{j=0}^N {}'' a_j T'_j(x), \quad (\text{B11})$$

where $T'_j(x)$ is the derivative of the Chebyshev polynomial

of the first kind of degree j . By inserting the expression of the a_j coefficients, and evaluating the derivative at $x = \tilde{x}_k$ we obtain a matrix equation:

$$D_i = D(\tilde{x}_i) = \sum_{j=0}^N {}'' \tilde{B}_{ij}^{[-1,1]} f_j. \quad (\text{B12})$$

Here $\tilde{B}^{[-1,1]}$ is given by

$$\tilde{B}_{ij}^{[-1,1]} = \frac{2}{N} \sum_{k=0}^N {}'' T'_k(\tilde{x}_i) T_k(\tilde{x}_j). \quad (\text{B13})$$

Note that the approximations (B9) and (B12) are *exact* for $x = \tilde{x}_j$, $j=0, \dots, N$.

Finally, the approximations (B5), (B9), (B12) can be generalized by allowing the range of the approximation to be between two arbitrary limits a and b , instead of just -1 to 1 . This is done performing the change of variables:

$$x \rightarrow y \equiv \frac{x - (1/2)(b+a)}{(1/2)(b-a)}. \quad (\text{B14})$$

As a consequence, the matrices B and \tilde{B} become

$$B^{[a,b]} = \frac{b-a}{2} B^{[-1,1]}, \quad (\text{B15})$$

$$\tilde{B}^{[a,b]} = \frac{2}{b-a} \tilde{B}^{[-1,1]}, \quad (\text{B16})$$

and the Chebyshev polynomials in Eq. (B5) become now functions of the variable y . The matrices $[f]$, $[\int_{-1}^x f(t)dt]$, and $[df(x)/dx]$ will give then the value of the function $f(x)$, its integral and derivative, at the coordinates $x_k = y_k(b-a)/2 + (b+a)/2$, where y_k are the $(N+1)$ coordinates given by Eq. (B2).

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