Static solitons with nonzero Hopf number

Jens Gladikowski* and Meik Hellmund†
Department of Mathematical Sciences, South Road, Durham DH1 3LE, England
(Received 3 September 1996)

We investigate a generalized nonlinear O(3) σ model in three space dimensions where the fields are maps from $\mathbb{R}^3 \cup \{\infty\}$ to S^2. Such maps are classified by a homotopy invariant called the Hopf number which takes integer values. The model exhibits soliton solutions of closed vortex type which have a lower topological bound on their energies. We numerically compute the fields for topological charge 1 and 2 and discuss their shapes and binding energies. The effect of an additional potential term is considered and an approximation is given for the spectrum of slowly rotating solitons. [S0556-2821(97)00520-1]

PACS number(s): 11.27.+d, 11.10.Lm

I. INTRODUCTION

The nonlinear O(3) σ model in (3+1)-dimensional spacetime is a scalar field theory whose target space is S^2. The static fields are maps $\mathbb{R}^3 \cup \{\infty\} \mapsto S^2$ and can be classified by a homotopy invariant which is called the Hopf number. Such a model in three space dimensions must include higher order terms in the field gradient in order to allow non-singular, topologically nontrivial, static solutions. The corresponding ''σ model with a Skyrme term'' was proposed long ago by Faddeev [1]. For this model the Hopf number provides a lower topological bound on the energy [2].

Early studies on ''Hopfions'' (soliton solutions of Hopf number unity) in classical field theory, including estimates for their size and mass, were carried out by de Vega [3]. Subsequently it was suggested to employ them in an effective chiral theory describing low-energy hadron dynamics; in that respect they are similar to Skyrmions [4]. It was later shown by Kundu and Rybakov [5] that Hopfions in the O(3) σ model are of closed vortex type.

Models with nonzero Hopf number have also been investigated in condensed matter physics for the description of three-dimensional ferromagnets and superfluid ^{3}He [6,7]. These are effective theories of Ginzburg-Landau type where the fields are interpreted as physical order parameters. However, a field configuration which is a solution of the full equations of motion has not been found for any of the mentioned theories.

In this paper we mainly study classical static Hopfions. Our model is defined in Sec. II where also an ansatz of azimuthal symmetry is introduced which is later used for numerical computations. In Sec. III we present our numerical results which are minima of the energy functional for Hopf number one and two. We discuss their shapes and binding energies as well as their relation to (2+1)-dimensional solitons. Our model has a self-interaction coupling parameter and we study the dependence of the energy on this coupling. In addition, the effect of a symmetry breaking potential term is described. In Sec. IV we give a simple approximation for the excitation spectrum of a Hopfion slowly rotating around its axis of symmetry. We conclude with Sec. V where we also remark on possible further investigations.

II. HOPF MAPS AND TOROIDAL ANSATZ

We are almost exclusively interested in static solutions and therefore define our model by the following energy functional on $\mathbb{R}^3$:

$$E_{\mathrm{stat}}[\boldsymbol{\phi}] = \Lambda \int_{\mathbb{R}^3} d\mathbf{x} \frac{1}{2} (\partial_i \boldsymbol{\phi})^2 + \frac{g_1}{8} (\partial_i \boldsymbol{\phi} \times \partial_j \boldsymbol{\phi})^2 + \frac{g_2}{8} (\partial_i \boldsymbol{\phi})^2 (\partial_j \boldsymbol{\phi})^2. \tag{1}$$

For $g_2 = 0$ this is equivalent to the static energy of the Faddeev-Skyrme model [1,2]. The field $\boldsymbol{\phi}$ is a three-component vector in isospace, subject to the constraint $\boldsymbol{\phi}^2 = 1$. The cross product is taken in internal space and the coordinate indices i,j run from 1 to 3.

For $g_1 = g_2 = 0$ minima of E_{stat} [Eq. (1)] are harmonic maps from $\mathbb{R}^3$ to S^2. As shown in [8], all nonconstant harmonic maps are orthogonal projections $\mathbb{R}^3 \mapsto \mathbb{R}^2$, followed by a harmonic map $\mathbb{R}^2 \mapsto S^2$ and therefore have infinite energy.

Consistently, simple scaling arguments along the line of the Hobart-Derrick theorem [9] show that the fourth-order terms in the energy functional are required to stabilize the soliton against shrinkage. We include here the most general combination of global O(3)-invariant fourth-order terms.

The parameter Λ is a constant of dimension energy/length and determines the models energy unit. The couplings g_1 and g_2 are of dimension (length)^2. The ratio g_1/g_2 is the only physically relevant coupling since an overall scaling of g_1 and g_2 can be absorbed by a rescaling of length and energy units. Using $(\partial_i \boldsymbol{\phi} \times \partial_j \boldsymbol{\phi})^2 = (\partial_i \boldsymbol{\phi})^2 (\partial_j \boldsymbol{\phi})^2 - (\partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi})^2$ and the inequality

$$2 \sum_{ij} (\partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi})^2 \geq \sum_{ij} (\partial_i \boldsymbol{\phi})^2 (\partial_j \boldsymbol{\phi})^2 \geq \sum_{ij} (\partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi})^2, \tag{2}$$

*Electronic address: Jens.Gladikowski@durham.ac.uk
†Permanent address: Institut für Theoretische Physik, Augustusplatz, D-04109 Leipzig, Germany. Electronic address: hellmund@tph100.physik.uni-Leipzig.de

one sees that the allowed ranges for the coupling constants are $g_2 \geqslant 0$ and $g_1 > -2g_2$.

For finite energy solutions one requires $\boldsymbol{\phi} \to \mathbf{n}$ as $|\mathbf{r}| \to \infty$, where $\mathbf{n}$ is a constant unit vector. Thus $\mathbb{R}^3$ can be one-point compactified to S^3 and the fields $\boldsymbol{\phi}$ are maps

$$\boldsymbol{\phi}: S^3 \mapsto S^2. \tag{3}$$

Because $\pi_3(S^2) = \mathbb{Z}$, every $\boldsymbol{\phi}$ falls into a class of topologically equivalent maps, where each class is characterized by an integer: the Hopf number H.

Although it is not a simple ''winding number,'' H has an elementary geometric interpretation. The preimage of every point of the target space S^2 is isomorphic to a circle. All those circles are interlinked with each other in the sense that any circle intersects the disc spanned by any other one. The Hopf number just equals the multiplicity by which two arbitrary circles are linked.

H also has a differential geometric representation [10]: If f is a generator of the de Rham cohomology $H^2_{dR}(S^2)$, its pullback F under $\boldsymbol{\phi}$ is necessarily exact since $H^2_{dR}(S^3) = 0$. Hence a one-form A with $F = dA$ exist and $H \sim \int A \wedge F$.

In coordinate language, the dual of F is $B_i = \varepsilon_{ijk} \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} \times \partial_k \boldsymbol{\phi}$ and

$$H = -\frac{1}{(8\pi)^2} \int_{\mathbb{R}^3} d\mathbf{x} \mathbf{B} \cdot \mathbf{A}. \tag{4}$$

It was proved in [2] that the energy, Eq. (1), has a lower topological bound in terms of H. For $g_1 \geqslant 0$ it is given by

$$E_{\text{stat}} \geqslant \Lambda k H^{3/4}, \tag{5}$$

where $k = \sqrt{2g_1}(2\pi)^2 3^{3/8}$ [5].

The variational equations resulting from Eq. (1) are coupled nonlinear partial differential equations. It would be useful to find a parametrization of $\boldsymbol{\phi}$ which carries nonzero Hopf charge and allows the equations to be reduced to ordinary differential equations. There have been two proposals for such fields in the literature. One of them uses spherical coordinates and is a composition of the standard Hopf map and a map $S^2 \mapsto S^2$ for which a hedgehog ansatz is employed [7,11]. Alternatively, a closed vortex ansatz in toroidal coordinates was suggested [3,4,12,13]. However, as shown in [14], even for $g_2 = 0$ none of these proposals allows a consistent separation of variables in the variational equations derived from Eq. (1).

At this point it is instructive to look at the symmetries of the field. It was shown in [5] that the maximal subgroup of $\mathrm{O}(3)_X \otimes \mathrm{O}(3)_I$ under which fields with nonvanishing Hopf number can be invariant is

$$G = \mathrm{diag}[\mathrm{O}(2)_X \otimes \mathrm{O}(2)_I]. \tag{6}$$

Here $\mathrm{O}(2)_X$ and $\mathrm{O}(2)_I$ denote rotations about a fixed axis in space and isospace, respectively. We choose the z and ϕ_3 axes as the axes of symmetry. According to the Coleman-Palais theorem we expect to find the minimal energy solution in the class of G-invariant configurations [17]. Therefore we use the most general G-invariant ansatz, written in terms of two functions $w(\xi_1, \xi_2)$ and $v(\xi_1, \xi_2)$. They depend on coordinates ξ_1 and ξ_2 which form an orthogonal coordinate system together with α, the angle around the z axis:

$$\phi_1 + i\phi_2 = \sqrt{1 - w^2(\xi_1, \xi_2)}\, e^{i[N\alpha + v(\xi_1, \xi_2)]},$$
$$\phi_3 = w(\xi_1, \xi_2). \tag{7}$$

We have checked the consistency of this ansatz with the variational equations derived from Eq. (1). The components ϕ_1 and ϕ_2 have to vanish along the z axis for the field to be well defined. This is realized by setting $\boldsymbol{\phi}(0,0,z) = \mathbf{n} = (0, 0, 1)$, which also defines the vacuum state of the theory. In order to describe a nontrivial map, $\boldsymbol{\phi}$ has to be surjective. Hence there is at least one point $\mathbf{r}_0$ with $\boldsymbol{\phi}(\mathbf{r}_0) = -\mathbf{n}$. Under the action of G, $\mathbf{r}_0$ represents a full circle around the z axis. We fix our coordinate system such that this circle lies in the xy plane and define $a \equiv |\mathbf{r}_0|$. On every trajectory from the circle to the z axis or infinity, $w(\xi_1, \xi_2)$ runs at least once from -1 to 1. Therefore the surfaces of constant w are homeomorphic to tori.

This structure prompts us to choose toroidal coordinates (η, β, α), related to cylindrical coordinates (r, z, α) as

$$r = \frac{a \sinh \eta}{\tau}, \quad z = \frac{a \sin \beta}{\tau}, \tag{8}$$

where $\tau = \cosh \eta - \cos \beta$. Surfaces of constant η describe tori about the z axis, while each of these tori is parametrized by the two angles (β, α). The two cases $\eta = 0$ and $\eta = \infty$ correspond to degenerated tori, $\eta = 0$ being the z axis and $\eta = \infty$ the circle of radius a in the xy-plane.

The function $w(\eta, \beta)$ is subject to the boundary conditions $w(0, \beta) = 1, w(\infty, \beta) = -1$ and is periodic in β. $v(\eta, \beta)$ is an angle around ϕ_3 and can include windings around β. Therefore we set $v(\eta, \beta) = M\beta + v_0(\eta, \beta)$ where $v_0(., \beta): S^1 \mapsto S^1$ is homotopic to the constant map. Since v is ill defined for $w = \pm 1$, it is not restricted by any boundary condition at $\eta = 0, \infty$.

The ''potential'' $\mathbf{A}$ and the ''field strength'' $\mathbf{B}$ for this ansatz are given by

$$A_\alpha = 2\frac{\tau}{a \sinh \eta} N(w-1), \quad A_\beta = 2\frac{\tau}{a}(M + \dot{v}_0)(w+1),$$
$$A_\eta = 2\frac{\tau}{a} v_0'(w+1), \quad B_\alpha = 2\frac{\tau^2}{a^2}[w'(M + \dot{v}_0) - v_0' \dot{w}],$$
$$B_\beta = -2\frac{\tau^2}{a^2 \sinh \eta} N w', \quad B_\eta = 2\frac{\tau^2}{a^2 \sinh \eta} N \dot{w}, \tag{9}$$

where the dot and prime denote derivatives with respect to β and η, respectively. Note that the field $\mathbf{A}$ is well defined on all of $\mathbb{R}^3$. The gauge has been chosen such that A_α vanishes for $\eta = 0$ (where the coordinate α is undefined) and analogously A_β vanishes for $\eta = \infty$.

Equation (4) then gives $H = NM$ in accordance with the linking number definition given above. The energy Eq. (1) of ansatz equation (7) is given by

$$E[w(\eta,\beta),v(\eta,\beta),a]=\pi\Lambda\int d\eta d\beta\frac{a^3\sinh\eta}{\tau^3}\left\{\frac{(\nabla w)^2}{1-w^2}+(1-w^2)\left((\nabla v)^2+\frac{N^2\tau^2}{a^2\sinh^2\eta}\right)\right.$$
$$\left.+\frac{g_1}{2}\left(\frac{N^2\tau^2}{a^2\sinh^2\eta}(\nabla w)^2+(\nabla w\times\nabla v)^2\right)+\frac{g_2}{4}\left[\frac{(\nabla w)^2}{1-w^2}+(1-w^2)\left((\nabla v)^2+\frac{N^2\tau^2}{a^2\sinh^2\eta}\right)\right]^2\right\}. \tag{10}$$

In toroidal coordinates the gradient includes a factor a^{-1}. Hence the term quadratic in the gradients is proportional to a while the quartic terms are inverse proportional to it. For soliton solutions, the energy functional has to be varied with respect to w, v, and a.

III. NUMERICAL RESULTS

The variational equations for Eq. (10) are highly nonlinear coupled partial differential equations and numerically hard to tackle. Therefore we solved the problem by a minimization of the energy functional which was discretized on an (η,β) grid. The search for the minimum in a high-dimensional space is feasible using the NETLIB routine $ve08$ with an algorithm described in [15]. This method is applicable if the objective function is a sum $f(\mathbf{x})=\Sigma f_i(\mathbf{x})$ of simpler functions f_i, each of which is nonconstant only for a few components of the (multidimensional) vector $\mathbf{x}$. Thus the Hessian matrix is very sparse and can be updated locally. This saves a considerable amount of memory and time compared to a more naive implementation of a conjugate gradient search.

We obtain field configurations as displayed in Fig. 1(a) where the Hopf number equals 1. In this plot the field $\boldsymbol{\phi}$ is viewed from above the north pole of target S^2. Isovectors in the northern hemisphere terminate in a cross, those in the southern hemisphere in a dot. The toroidal structure of the fields is clearly visible. Also note that the fields in the southern hemisphere span a torus indeed.

There is an interesting interpretation of such configurations in terms of the O(3) σ model in (2+1) dimensions, the solutions of which we call (anti-) baby Skyrmions. The fields in the positive and negative x half-plane of Fig. 1 are baby Skyrmions and antibaby Skyrmions, respectively. This can be understood in the following way. Wilczek and Zee [16] show that a (2+1)-dimensional configuration of Hopf number one can be produced by creating a baby-Skyrmion–antibaby-Skyrmion pair from the vacuum, rotating the (anti-) Skyrmion adiabatically by 2π and then annihilating the pair. In our model time corresponds to the third space dimension, hence Fig. 1(a) displays a ''snapshot'' at the time when the antibaby Skyrmion is rotated by π. Baby Skyrmions are classified by a homotopy invariant $Q\in\mathbb{Z}$ due to $\pi_2(S^2)=\mathbb{Z}$. The analytic expression for Q is given by

$$Q=\frac{1}{4\pi}\int_{\mathbb{R}^2}d\mathbf{x}\boldsymbol{\phi}\cdot\partial_1\boldsymbol{\phi}\times\partial_2\boldsymbol{\phi}, \tag{11}$$

where 1 and 2 denote Cartesian coordinates in $\mathbb{R}^2$. The topological charge density is half the α component of $\mathbf{B}$. The integral over the whole plane vanishes because the contributions for negative and for positive x exactly cancel. However, if integrated over the positive half-plane only, Eq. (11) yields the baby Skyrmion number for ansatz (7):

$$Q=\frac{1}{8\pi}\int_0^{2\pi}d\beta\int_0^{\infty}d\eta\frac{a^2}{\tau^2}B_\alpha=M, \tag{12}$$

where we use B_α of Eq. (9).

Next we turn to Hopfions of topological charge two. For parametrization, Eq. (7), there are two ways of creating a Hopfion with $H=2$, namely by setting either N or M to 2. Both cases correspond to two Hopfions sitting on top of each other. In order to determine which configuration represents the true ground state we computed their energies and found that the configuration with $N=2$, $M=1$ yields the lower energy for all couplings. The interpretation of the $H=2$ solutions in terms of a (2+1)-dimensional soliton-antisoliton pair is equivalent to the one given above for the one-Hopfion. Because the multiplicity of the azimuthal rotation is $N=2$ for the two-Hopfion, the antibaby Skyrmion in the negative x half-plane [see Fig. 1(b)] has a relative angle of π compared to the antibaby Skyrmion of Fig. 1(a).

It is instructive to investigate how the inclusion of a potential term $V[\boldsymbol{\phi}]$ alters the configuration. Its energy can be lowered by rescaling $\mathbf{x}\rightarrow\lambda\mathbf{x}$ $(\lambda\rightarrow 0)$ under which $V\rightarrow\lambda^3V$. This means that the potential term induces a ''shrinkage'' of the configuration in the sense that the favored position of the fields is closer to their vacuum value. This effect is counterbalanced by the higher order derivatives in the energy functional Eq. (1).

Any potential explicitly breaks the model's global O(3) symmetry because O(3) acts transitively on the target space. We chose $V=m^2\int d\mathbf{x}(1-\mathbf{n}\cdot\boldsymbol{\phi})$, where the parameter m is of dimension $(\text{length})^{-1}$ and, in a quantum version of the theory, becomes the mass of the elementary excitations. The minimum energy solution for $m=4$ can be seen in Fig. 1(c). The tubelike region where the field is in the southern hemisphere has clearly shrunk. Adding a linear potential term also means that the fields fall off exponentially at large distances. The reason is that the equations of motion become in the asymptotic limit those of the massive Klein-Gordon equation.

The fields of minimal energy correspond, via Eq. (1), to energy distributions which are displayed in Fig. 2. Despite the toroidal structure of the fields, we find that the energy for the Hopfion of $H=1$ is lump-shaped, see Fig. 2(a). Although unexpected, this is not entirely unlikely, because the field changes far more rapidly within the disc $|\mathbf{r}|\leq a$ than outside it. Hence the gradient energy can be concentrated in the vicinity of the origin.

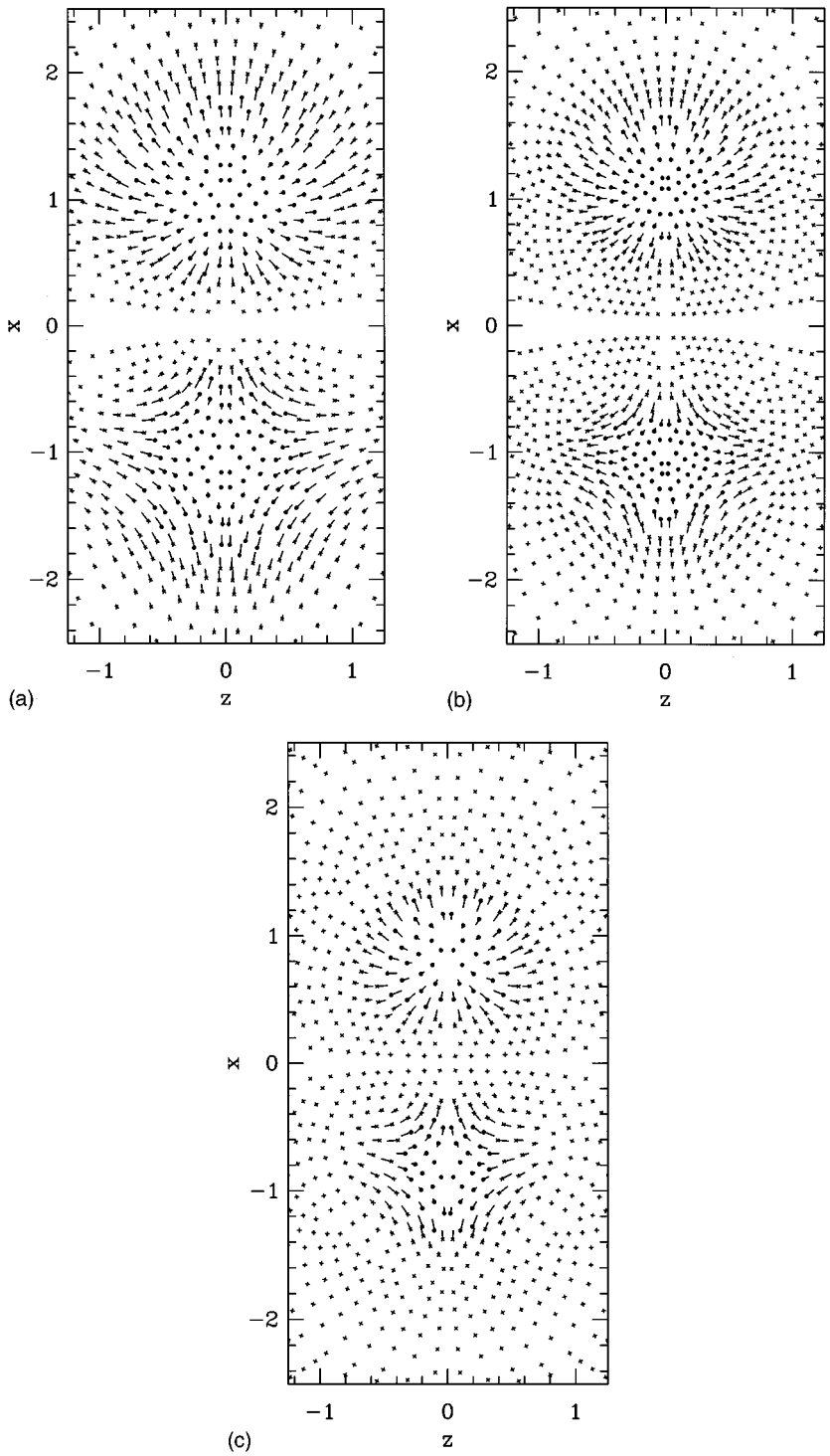

FIG. 1. (a) Field configuration in the xz plane for $H=1, g_1=0.4, g_2=0.4$. (b) Field configuration of $H=2, g_1=0.4, g_2=0.4$. (c) Field configuration with potential term, $H=1, g_1=1, g_2=0, m=4$. The field is projected into the $\phi_1\phi_2$ plane. A cross indicates $\phi_3>0$, a dot $\phi_3<0$. Therefore the vacuum state is denoted by a cross only.

If the potential term becomes very large compared to the gradient terms one expects the energy to become more localized around the filament where the fields are far away from the vacuum. We observe this transition to a toroidal energy distribution at $m\approx 4$ for $g_1=1, g_2=0$.

The energy distribution of the two-Hopfion is of toroidal shape (for all m), as shown in Fig. 2(b). It is a common feature in many soliton theories that solutions of topological charge two are tori, notably Skyrmions, baby Skyrmions, and magnetic monopoles. It is interesting to ask whether the two-Hopfion is in a stable state or likely to decay into two Hopfions of charge one. As an estimate for short range interactions one can compare the energy per Hopfion for the solution of $H=1$ and $H=2$ and conclude from the sign of the energy gap whether there is a repulsive or attractive channel. Our results are plotted in Fig. 3(a), which also

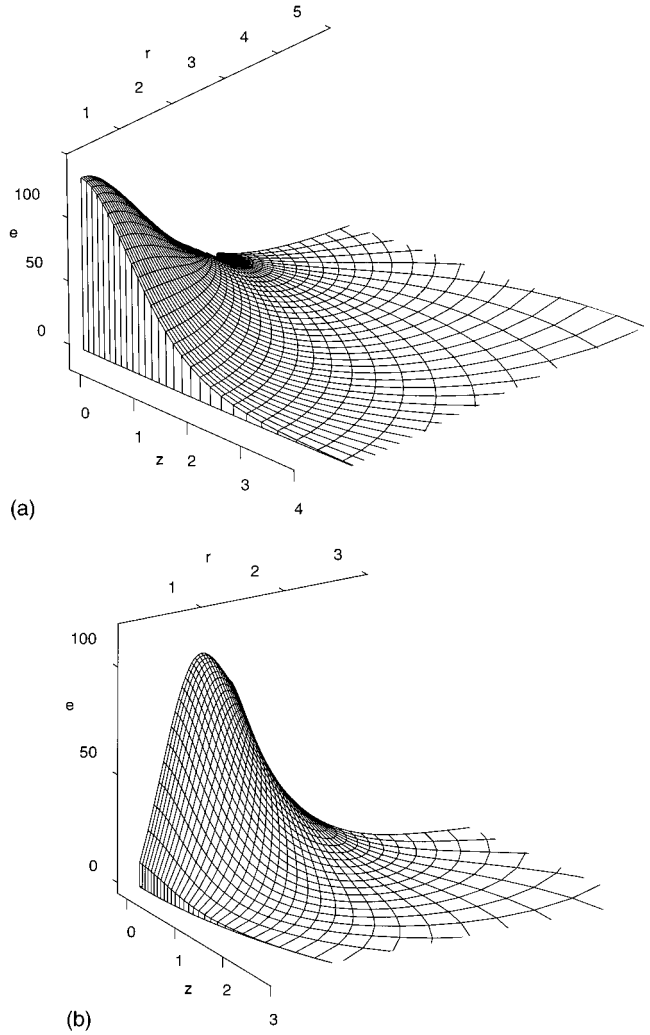

FIG. 2. (a) Energy density e (arbitrary units) for $H=1, g_1=0.4, g_2=0$ in cylindrical coordinates r,z. (b) Energy density e for $H=2, g_1=0.4, g_2=0.8$ over r,z.

shows the topological bound Eq. (5). For a pure Skyrme coupling we obtain energies of 197Λ and $2*158\Lambda$ for the one-Hopfion and two-Hopfion, respectively. Moreover, it turns out that for all couplings the two-Hopfion has a lower energy per topological unit than the one-Hopfion. This indicates that there is a range where the forces are attractive and that the two-Hopfion might be stable at least under small perturbations. Of course, there can be a range in which the forces are repulsive, however, an investigation of such interactions would require a full (3+1)-dimensional simulation which is beyond our present means. Also note that the gap between the energies per Hopfion is largest when the fourth-order terms are purely the Skyrme term. On the other hand, for $g_1 \to -2g_2$ (i.e., $g \to 1$) the energy of the quartic terms tends to zero. Hence the energy of the soliton vanishes as a consequence of the above-mentioned Hobart-Derrick theorem.

IV. SPINNING HOPFIONS

Finally, we study the effect of a slow rotation around the axis of symmetry. For this we use a Lorentz-invariant extension of our model into (3+1) dimensional space-time. The energy of the rotating Hopfion $E=E_{\rm rot}+E_{\rm stat}$, where $E_{\rm stat}$ is the static energy given by Eq. (1) and $E_{\rm rot}$ is the rotation energy functional:

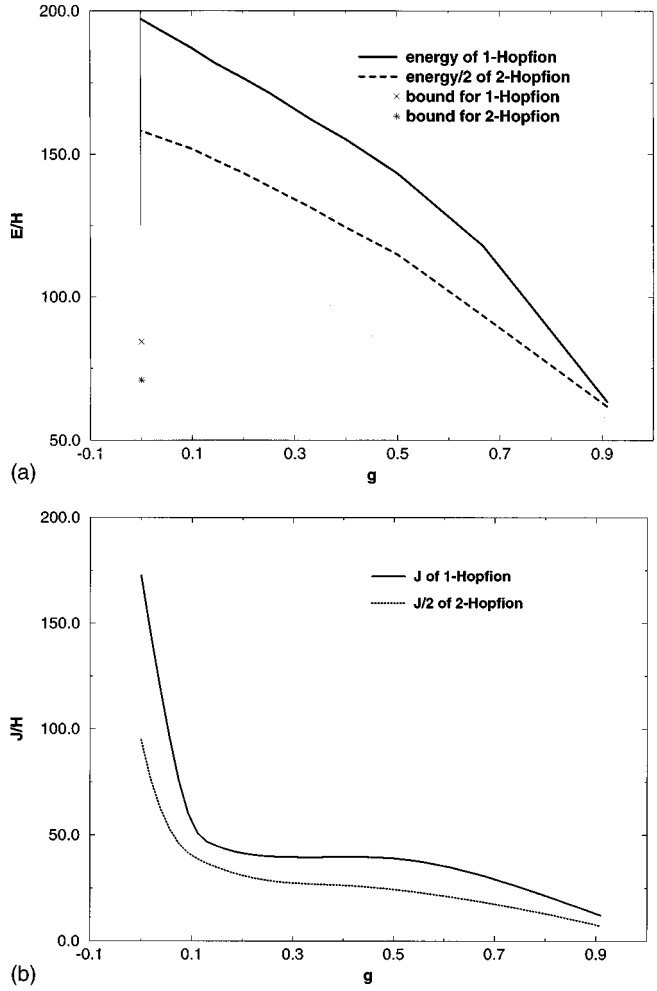

FIG. 3. (a) Dependence of the energy $E_{\rm stat}$ per Hopfion on the quartic couplings. They are parametrized as $g_1=1-3g$, $g_2=g$. Hence $g=0$ corresponds to pure $(\partial_i \boldsymbol{\phi}\times\partial_j \boldsymbol{\phi})^2$ coupling, $g=1/3$ to pure $(\partial_i \boldsymbol{\phi})^2(\partial_j \boldsymbol{\phi})^2$ coupling and $g=1$ to the case $(\partial_i \boldsymbol{\phi})^2(\partial_j \boldsymbol{\phi})^2-2(\partial_i \boldsymbol{\phi}\times\partial_j \boldsymbol{\phi})^2$. The energy is given in units of Λ. The topological bounds for pure Skyrme coupling are also displayed. (b) Dependence of the moment of inertia J per Hopfion on the coupling g.

$$E_{\rm rot}[\boldsymbol{\phi}]=\Lambda\int_{\mathbb{R}^3} d\mathbf{x}\frac{1}{2}(\partial_t \boldsymbol{\phi})^2+\frac{g_1}{8}(\partial_t \boldsymbol{\phi}\times\partial_i \boldsymbol{\phi})^2 + \frac{g_2}{8}(\partial_t \boldsymbol{\phi})^2(\partial_i \boldsymbol{\phi})^2+O[(\partial_t \boldsymbol{\phi})^4]. \quad (13)$$

In the spirit of a moduli space approximation we assume that the configuration does not alter its shape due to the rotation (''rigid rotor''), i.e., it is given at any time by a static solution (see [17] for a review on similar treatment of the Skyrmion). We impose time dependence on the azimuthal angle by $\alpha \to \alpha+(\omega/N)t$ with constant velocity ω. $E_{\rm rot}$ leads to a term in the energy that is proportional to ω^2:

$$E = E_{\rm stat} + \frac{J}{2}\omega^2, \tag{14}$$

where terms $O(\omega^4)$ are neglected. J is the moment of inertia and, using Eq. (7), given by

$$J = 2\pi\Lambda \int d\eta d\beta \left\{ 1 + \frac{g_1}{2}\frac{(\nabla w)^2}{1-w^2} + \frac{g_2}{2} \times \left[\frac{(\nabla w)^2}{1-w^2} + \left((\nabla v)^2 + \frac{N^2\tau^2}{a^2 \sinh^2\eta} \right)(1-w^2) \right] \right\} (1-w^2). \tag{15}$$

J can be measured explicitly on the individual solution. We plotted the values for $H=1$ and $H=2$ in Fig. 3(b). The moment of inertia per Hopfion is always larger for the $H=1$ solution, with an increasing gap for decreasing g. This should be compared with the dependence of $E_{\rm stat}$ on g.

The functional $E_{\rm stat}$ [Eq. (1)] is invariant under α-rotations while the fields of ansatz (7) are clearly not. Therefore, upon quantization, the coordinate α describes a zero mode and requires treatment as a collective coordinate. This is similar to the problem of the rotating radially symmetric Skyrmion. In analogy to the Skyrme model we therefore use, as a first approximation, the spectrum obtained by a straightforward quantization. The canonical momentum is $l = i(d/d\alpha)(\hbar = 1)$ and the rotational energy $E_{\rm rot} = -l^2/2J$. It is then trivial to solve the eigenvalue problem $E_{\rm rot}\psi = \lambda\psi$, which gives $\lambda_n = n^2/2J$.

V. CONCLUSIONS

We have studied topological solitons in a generalized nonlinear O(3) σ model in three space dimensions. Physically one may think of them as a model for hadronic matter or topological defects in a condensed matter system. By using a general ansatz for the fields we obtained explicit numerical solutions for soliton number one and two. Unexpectedly, the energy of the one-Hopfion is distributed as a lump. We also observed that two solitons sitting on top of each other attract, thus indicating a stable configuration.

There are several interesting questions which remain unanswered. In particular, the stability of Hopfions of higher topological charge deserves some scrutiny. It is worthwhile asking how multisolitons which sit on top of each other, or at least are very close, behave under induced perturbations. In analogy to planar O(3) σ models there might be several decay channels into less symmetric configurations [18].

At the opposite end of the scale, it would be instructive to look in greater detail at the interaction potential of two or more well-separated Hopfions. This is also interesting in comparison to the well-studied dynamics of Skyrmions and monopoles. Clearly, a first step in such an investigation would be to determine the asymptotic fields of the Hopf soliton. It seems obvious that intersoliton forces will depend on the orientation of the Hopfions.

The complete description of Hopfion dynamics would require a huge numerical effort which can, however, possibly be reduced by an appropriate approximation scheme. For Bogomol'nyi solitons, the low-energy behavior can be approximated via the truncation of the dynamics to the moduli space. Although our numerical results show that Hopfions are not of Bogomol'nyi type, given that the static forces between them are weak, there is a chance that their dynamics can be described by some kind of moduli space approximation, in analogy to Skyrmions (which are also not of Bogomol'nyi type).

Finally, it seems worthwhile to study spinning Hopfions in a more sophisticated way. This should include an assessment of the back reaction of the rotation on the matter fields. From this one expects a nontrivial shift of the energy levels in the rotation spectrum and possibly radiation of excessive energy.

ACKNOWLEDGMENTS

It is a pleasure to thank Wojtek Zakrzewski, Jacek Dziarmaga, and Rob deJeu for helpful discussions. We also wish to thank Bernd Schroers for making [2] available to us. J.G. acknowledges an EPSRC Grant No. 94002269. M.H. was supported by Deutscher Akademischer Austauschdienst.

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