PHYSICAL REVIEW D VOLUME 56, NUMBER 8 15 OCTOBER 1997

Thermal and curvature effects to dynamical symmetry breaking

Tomohiro Inagaki
Institute for Cosmic Ray Research, University of Tokyo, Tanashi, Tokyo 188, Japan

Ken-ichi Ishikawa
Department of Physics, Hiroshima University, Higashi-Hiroshima, Hiroshima 739, Japan

(Received 8 November 1996)

We investigate the thermal and curvature effects to dynamical symmetry breaking in a four-fermion model. The effective potential is evaluated in the leading order of the $1/N$ expansion at finite temperature on positive curvature space $R\otimes S^{D-1}$ or on negative curvature space $R\otimes H^{D-1}$ with varying temperature and curvature. We argue the combined effects of the temperature and curvature to the phase structure. The broken chiral symmetry is restored for a sufficiently high temperature and/or large positive curvature. The negative curvature enhances the chiral symmetry breaking. It is found that the thermal effect restores the symmetry at high temperature even in negative curvature spacetime. [S0556-2821(97)05020-0]

PACS number(s): 11.10.Wx, 04.62.+v, 11.30.Qc

I. INTRODUCTION

A phase transition at the early stage of the universe has various influences on the evolution of the universe. The symmetry breaking of the grand unified theory (GUT) may cause the inflationary expansion of the universe. There is a possibility of investigating the mechanism of the symmetry breaking at the GUT era in astrophysical observations [1].

Much interest has been paid especially to the phase structure of symmetry breaking to classify the models of the GUT. One of the interesting mechanisms to break symmetry is dynamical symmetry breaking (DSB) which is caused by the nonvanishing vacuum expectation value of the composite operator constructed by a fermion and an antifermion without introducing any elementary scalar field [2]. In the present paper we focus on the DSB at GUT era where we cannot neglect the thermal and curvature effects. Since there is much uncertainty in the model of the fundamental theory at the GUT era, the four-fermion models are often used as a prototype model to study the phase structure of the DSB.

Many physicists pay attention to thermal and curvature effects of DSB. Thermal restoration of the broken symmetry is discussed in the literature [3,4]. In the four-fermion models the minimum of the effective potential is shifted by varying the temperature. Evaluating the effective potential of the model it has been found that the broken chiral symmetry is restored for a sufficiently high temperature through the second-order phase transition. The analytical expression for the critical point was known in arbitrary dimensions within a simple model. On the other hand, a curvature-induced phase transition is discussed in the literature [5–7]. Using the four-fermion model it is found that the broken chiral symmetry is restored for a sufficiently large positive curvature and the chiral symmetry is always broken down in a negative curvature spacetime. In some compact spaces (S^D and $R\otimes S^{D-1}$) the critical point was known analytically by exact calculations without making any approximation in the spacetime curvature. The phase transition is of second order in these compact spaces. However there is a little work for combined effects of the temperature and curvature. The vacuum energy density for free fermion is calculated at finite temperature in $R\otimes S^{D-1}$ [8]. The thermal and curvature effects for dynamical symmetry breaking are studied by using the four-fermion model in the positive weak curvature limit [9].

In the present paper we use the four-fermion interaction model with N-component fermions. As a nonperturbative approach is necessary to study the phase transition, the model is treated nonperturbatively at the large N limit by using the $1/N$ expansion. We suppose that the system is in equilibrium and introduce the temperature. This assumption is not accepted in a general curved spacetime. In the spacetime which has no time evolution the equilibrium state can be defined. We then restrict ourselves in the positive curvature spacetime $R\otimes S^{D-1}$ and the negative curvature spacetime $R\otimes H^{D-1}$. To find the ground state at finite temperature and curvature we calculate the effective potential and analyze its stationary condition by the gap equation in the leading order of the $1/N$ expansion.

The main purpose of this paper is to show the importance of the combined effects of the temperature and curvature to DSB. One of the important problems is whether the thermal effect restores the broken symmetry in a negative curvature spacetime. It may give some effect to the evolution of the universe. The four-fermion model may be too simple to discuss DSB at the GUT era, but we expect that the model has some fundamental properties.

The paper is organized in the following way. In Sec. II we briefly review the dynamical symmetry breaking in a four-fermion interaction model. The effective potential described by the spinor two-point function in the leading order of the $1/N$ expansion, keeping only the effects of fermion loops. In Minkowski space the shape of the effective potential is of the double well for a sufficiently large four-fermion coupling and the chiral symmetry is broken down dynamically. In Sec. III we investigate the thermal and curvature effects to DSB. Starting from the theory with broken chiral symmetry we study the phase transition induced by the thermal and curvature effects. Temperature and curvature are introduced to the

 © 1997 The American Physical Society

effective potential through the spinor two-point function at finite temperature and curvature. We derive the explicit form of the effective potential at finite temperature on $R\otimes S^{D-1}$ and $R\otimes H^{D-1}$. Analyzing the gap equation we show the critical lines which divide the symmetric phase and the asymmetric phase. Section IV gives the concluding remarks.

II. SIMPLE MODEL OF DSB

We want to apply the mechanism of DSB to the symmetry breaking at the GUT era. One of the simplest models of DSB is the four-fermion interaction model. It may be interpreted as the low energy effective theory which is steaming from the more fundamental theory at the GUT era.

The four-fermion interaction model in curved spacetime is characterized by the action [10]

$$S=\int \sqrt{-g}d^Dx\left[\sum_{k=1}^{N}\bar{\psi}_k i\gamma^\mu\nabla_\mu\psi_k+\frac{\lambda_0}{2N}\left(\sum_{k=1}^{N}\bar{\psi}_k\psi_k\right)^2\right], \tag{2.1}$$

where the index k represents the flavors of the fermion field ψ, N is the number of fermion species, g the determinant of the metric tensor $g_{\mu\nu}$, γ^μ the Dirac matrix in curved spacetime, and $\nabla_\mu\psi$ the covariant derivative of the fermion field ψ. For simplicity, we neglect the flavor index below. The action (2.1) is invariant under the discrete transformation $\bar{\psi}\psi\rightarrow-\bar{\psi}\psi$. This discrete chiral symmetry prohibits the fermion mass term.

We introduce an auxiliary field σ for convenience [10] and rewrite the action (2.1) in the form

$$S_y=\int \sqrt{-g}d^Dx\left(\bar{\psi}i\gamma_\mu\nabla^\mu\psi-\frac{N}{2\lambda_0}\sigma^2-\bar{\psi}\sigma\psi\right). \tag{2.2}$$

Using the equation of motion for the field σ we easily know that the action (2.2) is equivalent to Eq. (2.1). If the auxiliary field σ develops the nonvanishing vacuum expectation value, $\langle\sigma\rangle=m\neq0$, there appears a mass term for the fermion field ψ and the discrete chiral symmetry is eventually broken.

To study the phase structure of the theory we want to find a ground state of the system. For this purpose we evaluate an effective potential for the field σ. As is known in the leading order of the $1/N$ expansion the effective potential of the model (2.2) is described by [5]

$$V(\sigma)=\frac{1}{2\lambda_0}\sigma^2-i\ \mathrm{tr}\int_0^\sigma ds S(x,x;s)+O\left(\frac{1}{N}\right), \tag{2.3}$$

where $S(x,x;s)$ is the spinor two-point function which satisfies the Dirac equation

$$(i\gamma^\mu\nabla_\mu-s)S(x,y;s)=\frac{1}{\sqrt{-g}}\delta^D(x,y), \tag{2.4}$$

where $\delta^D(x,y)$ is Dirac's delta function in curved spacetime. It should be noted that the effective potential is normalized so that $V(0)=0$. The ground state is determined by observing the minimum of the effective potential.

The two-point function $S(x,x;s)$ in Minkowski spacetime at $T=0$ is given by

$$S(x,x;s)=\int\frac{d^Dk}{(2\pi)^D}\frac{1}{\not{k}-s},\qquad \sqrt{-g}=1. \tag{2.5}$$

Inserting Eq. (2.5) into Eq. (2.3) and performing the integration by using the dimensional regularization method, we obtain the effective potential in Minkowski spacetime:

$$\begin{aligned}V(\sigma)&=\frac{1}{2\lambda_0}\sigma^2-i\ \mathrm{tr}\int_0^\sigma ds\int\frac{d^Dk}{(2\pi)^D}\frac{1}{\not{k}-s}\\&=\frac{1}{2\lambda_0}\sigma^2-\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{D/2}}\frac{1}{D}\Gamma\left(1-\frac{D}{2}\right)\sigma^D.\end{aligned} \tag{2.6}$$

It is divergent in two and four dimensions. Performing the renormalization procedure by imposing the renormalization condition

$$\left.\frac{d^2V}{d\sigma^2}\right|_{\sigma=\mu}=\frac{\mu^{D-2}}{\lambda_R}, \tag{2.7}$$

we obtain the renormalized coupling constant

$$\frac{1}{\lambda_0}=\left[\frac{1}{\lambda_R}+\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{D/2}}\Gamma\left(1-\frac{D}{2}\right)(D-1)\right]\mu^{D-2}. \tag{2.8}$$

Replacing the bare coupling constant λ_0 with the renormalized one λ_R, we obtain the renormalized effective potential which is no longer divergent in the spacetime dimensions, $2\leqslant D<4$. In four dimensions four-fermion theory is not renormalizable and the finite effective potential cannot be defined. Here we regard the effective potential for $D=4-\epsilon$ with ϵ sufficiently small positive as a regularization of the one in four dimensions.

Evaluating the renormalized effective potential $V(\sigma)$ we find the phase structure of the four-fermion model. In Minkowski space the shape of the effective potential is of a single and a double well for $\lambda\leqslant\lambda_{\mathrm{cr}}$ and $\lambda>\lambda_{\mathrm{cr}}$, respectively. The critical value of the coupling constant λ_{cr} is given by [10]

$$\lambda_{\mathrm{cr}}=\frac{(4\pi)^{D/2}}{\mathrm{tr}\mathbf{1}}\left[(1-D)\Gamma\left(1-\frac{D}{2}\right)\right]^{-1}. \tag{2.9}$$

Thus the ground state is invariant under the discrete chiral transformation for $\lambda\leqslant\lambda_{\mathrm{cr}}$. On the other hand the chiral symmetry is broken down for $\lambda>\lambda_{\mathrm{cr}}$ and the fermion acquires the dynamical mass m_0:

$$m_0=\mu\left[\frac{(4\pi)^{D/2}}{\mathrm{tr}\mathbf{1}\Gamma(1-D/2)}\frac{1}{\lambda_R}+D-1\right]^{1/(D-2)}. \tag{2.10}$$

In the following sections we will apply the similar analysis at finite temperature and curvature. We fix the coupling constant λ_R above the critical one and see whether the broken chiral symmetry is restored in an environment of the high temperature and/or large curvature.

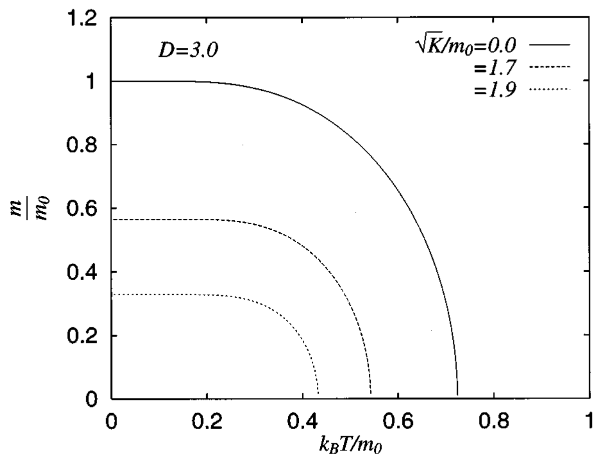

FIG. 1. Dynamical fermion mass m at $D=3.0$ in $R\otimes S^2$ as a function of the temperature T with the curvature K fixed.

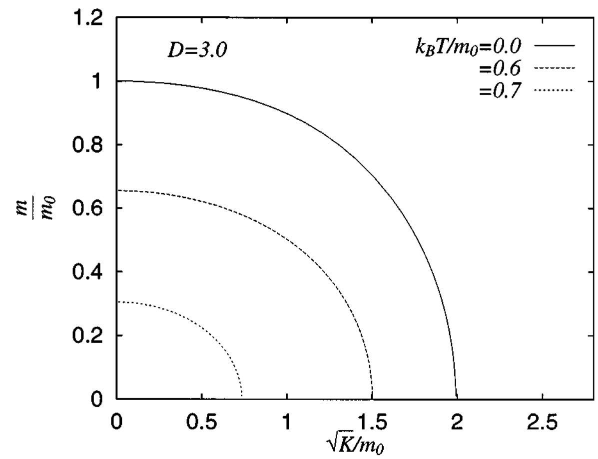

FIG. 2. Dynamical fermion mass m at $D=3.0$ in $R\otimes S^2$ as a function of the curvature K with the temperature T fixed.

III. THERMAL AND CURVATURE EFFECTS

Here we introduce the temperature and curvature in the theory and investigate the phase structure with varying the temperature and curvature.

First we introduce the effect of the finite temperature. As we have seen in the previous section, the effective potential is expressed by the two-point function $S(x,x;s)$ of a massive free fermion. The two-point function at finite temperature is defined by

$$S^T(x,x;s)=\frac{\Sigma_\alpha e^{-\beta E_\alpha}\langle\alpha|T(\psi(x)\overline{\psi(x)})|\alpha\rangle}{\Sigma_\alpha e^{-\beta E_\alpha}}, \qquad (3.1)$$

where E_α is the energy in the state specified by quantum number α, respectively, $\beta=1/k_BT$ with k_B the Boltzmann constant and T the temperature

Following the standard procedure of the Matsubara Green's function, the two-point function at finite temperature is obtained from the one at $T=0$ by the Wick rotation and the replacements [11]

$$\int_{-\infty}^{\infty}\frac{dk^0}{2\pi i}\to\frac{1}{\beta}\sum_{n=-\infty}^{\infty}, \qquad (3.2)$$

$$k^0\to i\omega_n\equiv i\frac{2n+1}{\beta}\pi,$$

$$\gamma^0\to i\gamma^0.$$

In Minkowski space the effective potential at finite temperature in the leading order of the $1/N$ expansion reads [3,4]

$$V^T(\sigma)=\frac{1}{2\lambda_0}\sigma^2+\mathrm{tr}\int_0^\sigma ds\frac{1}{\beta}\sum_n\int\frac{d^{D-1}k}{(2\pi)^{D-1}}\frac{1}{\not{k}+s}. \qquad (3.3)$$

If we perform the integration over k, we obtain

$$V^T(\sigma)=\frac{1}{2\lambda_0}\sigma^2-\int_0^\sigma ds\frac{\mathrm{tr}\mathbf{1}\Gamma[(3-D)/2]}{(4\pi)^{(D-1)/2}}\frac{1}{\beta}\times\sum_{n=-\infty}^{\infty}s(s^2+\omega_n^2)^{(D-3)/2}. \qquad (3.4)$$

Performing a summation and integrating over angle variables and s in Eq. (3.3), we get

$$V^T(\sigma)=\frac{1}{2\lambda_0}\sigma^2-\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{D/2}}\frac{1}{D}\Gamma\left(1-\frac{D}{2}\right)\sigma^D-\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{(D-1)/2}}\frac{1}{\Gamma[(D-1)/2]}\frac{1}{\beta}\int_0^\infty dt\ t^{(D-3)/2}\ln\frac{1+e^{-\beta\sqrt{t+\sigma^2}}}{1+e^{-\beta\sqrt{t}}}. \qquad (3.5)$$

Comparing Eq. (3.4) with Eq. (3.5) we find the following relation:

$$\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{D/2}}\frac{1}{D}\Gamma\left(1-\frac{D}{2}\right)\sigma^2+\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{(D-1)/2}}\frac{1}{\Gamma[(D-1)/2]}\frac{1}{\beta}\int_0^\infty dt\ t^{(D-3)/2}\ln\frac{1+e^{-\beta\sqrt{t+\sigma^2}}}{1+e^{-\beta\sqrt{t}}}$$
$$=\int_0^\sigma ds\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{(D-1)/2}}\Gamma\left(\frac{3-D}{2}\right)\frac{1}{\beta}\sum_{n=-\infty}^{\infty}s(s^2+\omega_n^2)^{(D-3)/2}. \qquad (3.6)$$

This relation will be used for numerical calculation of the effective potential at finite temperature in curved spacetime.

Starting from the theory with the broken chiral symmetry at vanishing T and evaluating the effective potential at finite temperature, it is known that the broken chiral symmetry is restored at a critical temperature $T_{\rm cr}$ through the second-order phase transition. The critical temperature is given by [3,4]

$$\frac{k_B T_{\rm cr}}{m_0}=\frac{1}{2\pi}\left[\frac{2\Gamma[(3-D)/2]}{\sqrt{\pi}\Gamma[(2-D)/2]}(2^{3-D}-1)\zeta(3-D)\right]^{1/(2-D)}. \tag{3.7}$$

Next we consider the constant curvature space $R\otimes S^{D-1}$ and $R\otimes H^{D-1}$ as Euclidean analogs of the Einstein universe and discuss the effect of the spacetime structure. The manifold $R\otimes S^{D-1}$ is defined by the metric

$$ds^2=dt^2+a^2(d\theta^2+\sin^2\theta d\Omega_{D-2}), \tag{3.8}$$

where $d\Omega_{D-2}$ is the metric on a unit sphere S^{D-2} while the manifold $R\otimes H^{D-1}$ is defined by

$$ds^2=dt^2+a^2(d\theta^2+\sinh^2\theta d\Omega_{D-2}). \tag{3.9}$$

The manifold $R\otimes S^{D-1}$ and $R\otimes H^{D-1}$ are constant curvature spacetimes with positive and negative curvature

$$R=\pm(D-1)(D-2)a^{-2}, \tag{3.10}$$

respectively $(2\leqslant D<4)$.

The effect of the spacetime structure is introduced to the effective potential through the two-point function appeared. As is shon in the Appendix the two-point functions are given by

$$\mathrm{tr}S(x,x:s)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\mathrm{tr}\mathbf{1}sK^{(D-3)/2}}{(4\pi)^{(D-1)/2}}\frac{\Gamma[(D-1)/2+i\alpha]\Gamma[(D-1)/2-i\alpha]}{\Gamma(1+i\alpha)\Gamma(1-i\alpha)}\Gamma\left(\frac{3-D}{2}\right)\quad \text{on}\ \ R\otimes S^{D-1}\ \ [12],$$

$$\mathrm{tr}S(x,x:s)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\mathrm{tr}\mathbf{1}sK^{(D-3)/2}}{(4\pi)^{(D-1)/2}}\frac{\Gamma[(D-1)/2+\alpha]}{\alpha\Gamma[(3-D)/2+\alpha]}\Gamma\left(\frac{3-D}{2}\right),\quad \text{on}\ \ R\otimes H^{D-1}, \tag{3.11}$$

where $K=1/a^2$ and α is defined in Eq. (A21). Substituting Eq. (3.11) to Eq. (2.3) the effective potential at $T=0$ reads

$$V^R(\sigma)=\frac{1}{2\lambda_0}\sigma^2-\int_0^{\sigma}ds\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\mathrm{tr}\mathbf{1}sK^{(D-3)/2}}{(4\pi)^{(D-1)/2}}\frac{\Gamma[(D-1)/2+i\alpha]\Gamma[(D-1)/2-i\alpha]}{\Gamma(1+i\alpha)\Gamma(1-i\alpha)}\Gamma\left(\frac{3-D}{2}\right)\quad \text{on}\ \ R\otimes S^{D-1}\ \ [7], \tag{3.12}$$

$$V^R(\sigma)=\frac{1}{2\lambda_0}\sigma^2-\int_0^{\sigma}ds\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\mathrm{tr}\mathbf{1}sK^{(D-3)/2}}{(4\pi)^{(D-1)/2}}\frac{\Gamma[(D-1)/2+\alpha]}{\alpha\Gamma[(3-D)/2+\alpha]}\Gamma\left(\frac{3-D}{2}\right)\quad \text{on}\ \ R\otimes H^{D-1}. \tag{3.13}$$

The weak curvature limit of Eq. (3.13) is consistent with the result in Ref. [6], but is different from the result in Ref. [9]. Evaluating the effective potential in curved spacetime it is known that the broken chiral symmetry is restored for a sufficiently large positive curvature. In the Einstein universe ($R\otimes S^{D-1}$) the critical curvature is given by [7]

$$R_{\rm cr}=(D-1)(D-2)m_0^2\left[\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{D-1}{2}\right)\Gamma\left(\frac{D}{2}\right)\right]^{2/(2-D)}. \tag{3.14}$$

On the other hand only a broken phase is realized for an arbitrary negative curvature irrespective of the coupling constant λ [6]. Thus there is no critical point where the chiral symmetry is restored for the case considered here, $\lambda>\lambda_{\rm cr}$. The chiral symmetry is broken down at $R<0$ even for $\lambda\leqslant\lambda_{\rm cr}$.

Below, we investigate the combined effects of the temperature and curvature on $R\otimes S^{D-1}$ and $R\otimes H^{D-1}$.

A. Positive curvature space ($R\otimes S^{D-1}$)

In the positive curvature space, $R\otimes S^{D-1}$, the effective potential is given by Eq. (3.12). According to the definition of the two-point function at finite temperature (3.1), we obtain the effective potential in the space at finite temperature by the replacements (3.2):

$$V^{TR}(\sigma)=\frac{1}{2\lambda_0}\sigma^2-\int_0^{\sigma}ds\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{(D-1)/2}}\frac{sK^{(D-3)/2}}{\beta}\sum_{n=-\infty}^{\infty}\frac{\Gamma[(D-1)/2+i\alpha_n]\Gamma[(D-1)/2-i\alpha_n]}{\Gamma(1+i\alpha_n)\Gamma(1-i\alpha_n)}\Gamma\left(\frac{3-D}{2}\right), \tag{3.15}$$

where α_n is defined by

$$\alpha_n \equiv \sqrt{\frac{s^2+\omega_n^2}{K}}. \tag{3.16}$$

Evaluating the effective potential (3.15) we will find the phase structure of the model at finite temperature in positive curvature space.

For numerical calculations we need the finite expression of the effective potential in summation and integration. Inserting Eq. (2.8) and Eq. (3.6) into Eq. (3.15) the renormalized effective potential reads

$$V_R^{TR}(\sigma)=\frac{1}{2}\left[\frac{1}{\lambda_R}+\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{D/2}}\Gamma\left(1-\frac{D}{2}\right)(D-1)\right]\mu^{D-2}\sigma^2-\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{D/2}}\frac{1}{D}\Gamma\left(1-\frac{D}{2}\right)\sigma^2-\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{(D-1)/2}}\frac{1}{\Gamma[(D-1)/2]}\frac{1}{\beta}$$
$$\times\int_0^\infty dt\; t^{(D-3)/2}\ln\frac{1+e^{-\beta\sqrt{t+\sigma^2}}}{1+e^{-\beta\sqrt{t}}}+\int_0^\sigma ds\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{(D-1)/2}}\frac{1}{\beta}\Gamma\left(\frac{3-D}{2}\right)\sum_{n=-\infty}^{\infty}s\left[(s^2+\omega_n^2)^{(D-3)/2}\right.$$
$$\left.-K^{(D-3)/2}\frac{\Gamma[(D-1)/2+i\alpha_n]\Gamma[(D-1)/2-i\alpha_n]}{\Gamma(1+i\alpha_n)\Gamma(1-i\alpha_n)}\right]. \tag{3.17}$$

In this representation of the effective potential the divergence is canceled out in the summation.

The phase structure of the theory is obtained by observing the minimum of the effective potential. The necessary condition for the minimum of the effective potential is given by the gap equation

$$\left.\frac{\partial V_R^{TR}(\sigma)}{\partial\sigma}\right|_{\sigma=m}=0. \tag{3.18}$$

If the gap equation has a nontrivial solution which corresponds to the minimum of the effective potential, the chiral symmetry is broken down and the dynamical fermion mass is generated. The nontrivial solution of the gap equation is given by

$$\left(\frac{1}{\lambda_R}-\frac{1}{\lambda_{\mathrm{cr}}}\right)\mu^{D-2}-\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{D/2}}\Gamma\left(1-\frac{D}{2}\right)m^{D-2}+\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{(D-1)/2}}\frac{1}{\Gamma[(D-1)/2]}\int_0^\infty dt\; t^{(D-3)/2}\frac{1}{\sqrt{t+m^2}}\frac{e^{-\beta\sqrt{t+m^2}}}{1+e^{-\beta\sqrt{t+m^2}}}$$
$$+\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{(D-1)/2}}\frac{1}{\beta}\Gamma\left(\frac{3-D}{2}\right)\sum_{n=-\infty}^{\infty}\left[(m^2+\omega_n^2)^{(D-3)/2}-K^{(D-3)/2}\frac{\Gamma[(D-1)/2+i\alpha_n]\Gamma[(D-1)/2-i\alpha_n]}{\Gamma(1+i\alpha_n)\Gamma(1-i\alpha_n)}\right]=0, \tag{3.19}$$

where $\alpha_n=\sqrt{(m^2+\omega_n^2)/K}$, λ_{cr} is defined in Eq. (2.9) and m corresponds to the dynamically generated fermion mass.

In Figs. 1 and 2 we plot the typical behaviors of the dynamical fermion mass m at $D=3.0$ as a function of temperature T or curvature K. Since no mass gap is observed at the critical point in Figs. 1 and 2, only the second-order phase transition occurs with varying temperature and/or curvature. By the same analysis we find that the broken chiral symmetry is restored for a sufficiently high temperature and large curvature. Only the second-order phase transition is realized for $2\leqslant D<4$.

Since the dynamical fermion mass smoothly disappears at the critical point for second-order phase transition, the critical line on the T-$\sqrt{K}$ plane is given by the massless limit of Eq. (3.19). To find the equation for the critical line in an analytic form we take the limit $m\to 0$ in Eq. (3.19) and find

$$\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{D/2}}\Gamma\left(1-\frac{D}{2}\right)m_0^{D-2}-\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{(D-1)/2}}\frac{2}{\beta_{\mathrm{cr}}}\Gamma\left(\frac{3-D}{2}\right)\left(\frac{2\pi}{\beta_{\mathrm{cr}}}\right)^{D-3}\zeta(3-D,1/2)+\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{(D-1)/2}}\frac{1}{\beta_{\mathrm{cr}}}\Gamma\left(\frac{3-D}{2}\right)$$
$$\times\sum_{n=-\infty}^{\infty}\left[|\omega_{\mathrm{cr}_n}|^{D-3}-K_{\mathrm{cr}}^{(D-3)/2}\frac{\Gamma[(D-1)/2+i\alpha_{\mathrm{cr}_n}]\Gamma[(D-1)/2-i\alpha_{\mathrm{cr}_n}]}{\Gamma(1+i\alpha_{\mathrm{cr}_n})\Gamma(1-i\alpha_{\mathrm{cr}_n})}\right]=0, \tag{3.20}$$

where $\zeta(z,a)$ is the generalized zeta function, $\omega_{\mathrm{cr}_n}=(2n+1)\pi/\beta_{\mathrm{cr}}$, $\alpha_{\mathrm{cr}_n}=|\omega_{\mathrm{cr}_n}|/\sqrt{K_{\mathrm{cr}}}$ and m_0 is the dynamical fermion mass in Minkowski spacetime at $T=0$. The critical lines are shown in Fig. 3.

The two-dimensional spacetime $R\otimes S^1$ is a flat compact spacetime, $R=0$. Thus the symmetry restoration which caused by increasing K is induced by the finite size effect of the compact space. In four dimensions the effective potential at $T=0$ is divergent. However the thermal effect gives only a finite correction to the effective potential. Thus the critical temperature T_{cr}

at $K=0$ is divergent at the four-dimensional limit. Near four dimensions the curvature effect seems to mainly contribute to the phase transition. But it comes from the nonrenormalizability of the four-fermion model. In a renormalizable theory the situation must be changed.

B. Negative curvature space ($R\otimes H^{D-1}$)

In a negative curvature spacetime the chiral symmetry is always broken down irrespective of λ at $T=0$. Can the thermal effect restore the symmetry in the negative curvature spacetime? Here we consider the model at finite temperature in the negative curvature spacetime, $R\otimes H^{D-1}$, for both $\lambda>\lambda_{\rm cr}$ and $\lambda\leqslant\lambda_{\rm cr}$.

According to the method used in the previous subsection the effective potential at finite temperature in $R\otimes H^{D-1}$ can be obtained from Eq. (3.13) by the replacements (3.2):

$$V^{TR}(\sigma)=\frac{1}{2\lambda_0}\sigma^2-\int_0^\sigma ds\,\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{(D-1)/2}}\frac{sK^{(D-3)/2}}{\beta}\sum_{n=-\infty}^{\infty}\frac{\Gamma[(D-1)/2+\alpha_n]}{\alpha_n\Gamma[(3-D)/2+\alpha_n]}\Gamma\left(\frac{3-D}{2}\right). \tag{3.21}$$

$R\otimes H^1$ is equivalent to the two-dimensional Minkowski space R^2. At the two-dimensional limit of Eq. (3.21) the effective potential in two-dimensional Minkowski space is reproduced. Because of the convenience for numerical calculations we rewrite the effective potential (3.21) in the same form described in the previous subsection. Inserting Eq. (2.8) and Eq. (3.6) into Eq. (3.21) we get

$$\begin{aligned}V_R^{TR}(\sigma)=&\frac{1}{2}\left[\frac{1}{\lambda_R}+\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{D/2}}\Gamma\left(1-\frac{D}{2}\right)(D-1)\right]\mu^{D-2}\sigma^2-\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{D/2}}\frac{1}{D}\Gamma\left(1-\frac{D}{2}\right)\sigma^2-\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{(D-1)/2}}\frac{1}{\Gamma[(D-1)/2]}\frac{1}{\beta}\\&\times\int_0^\infty dt\ t^{(D-3)/2}\ln\frac{1+e^{-\beta\sqrt{t+\sigma^2}}}{1+e^{-\beta\sqrt{t}}}+\int_0^\sigma ds\,\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{(D-1)/2}}\frac{1}{\beta}\Gamma\left(\frac{3-D}{2}\right)\\&\times\sum_{n=-\infty}^{\infty}s\left[(s^2+\omega_n^2)^{(D-3)/2}-K^{(D-3)/2}\frac{\Gamma[(D-1)/2+\alpha_n]}{\alpha_n\Gamma[(3-D)/2+\alpha_n]}\right].\end{aligned} \tag{3.22}$$

To find the minimum of the effective potential (3.22) we analyze the nontrivial solution of the gap equation. Substituting Eq. (3.22) to Eq. (3.18) the gap equation reads

$$\begin{aligned}&\left(\frac{1}{\lambda_R}-\frac{1}{\lambda_{\rm cr}}\right)\mu^{D-2}-\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{D/2}}\Gamma\left(1-\frac{D}{2}\right)m^{D-2}+\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{(D-1)/2}}\frac{1}{\Gamma[(D-1)/2]}\int_0^\infty dt\ t^{(D-3)/2}\frac{1}{\sqrt{t+m^2}}\frac{e^{-\beta\sqrt{t+m^2}}}{1+e^{-\beta\sqrt{t+m^2}}}\\&+\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{(D-1)/2}}\frac{1}{\beta}\Gamma\left(\frac{3-D}{2}\right)\sum_{n=-\infty}^{\infty}\left[(m^2+\omega_n^2)^{(D-3)/2}-K^{(D-3)/2}\frac{\Gamma[(D-1)/2+\alpha_n]}{\alpha_n\Gamma[(3-D)/2+\alpha_n]}\right]=0.\end{aligned} \tag{3.23}$$

Evaluating the gap equation (3.23) numerically we obtain the dynamical fermion mass m.

In Figs. 4–7 we draw the typical behaviors of the dynamical fermion mass m in $R\otimes H^2$ with varying the temperature or curvature. In drawing figures the normalization scale m_0 is taken to the value defined in Eq. (2.10) for $\lambda>\lambda_{\rm cr}$ and

$$m_0=\mu\left[-\frac{(4\pi)^{D/2}}{\mathrm{tr}\mathbf{1}\Gamma(1-D/2)}\frac{1}{\lambda_R}-D+1\right]^{1/(D-2)}, \tag{3.24}$$

for $\lambda\leqslant\lambda_{\rm cr}$. As is shown in Figs. 4 and 5 the dynamical fermion mass smoothly disappears as the temperature increases with the curvature fixed for both $\lambda>\lambda_{\rm cr}$ and $\lambda\leqslant\lambda_{\rm cr}$. Then the broken chiral symmetry is restored for a sufficiently high temperature. On the other hand the dynamical fermion mass becomes heavier as the curvature K increases with the temperature fixed as can be seen in Figs. 6 and 7. Calculating Eq. (3.7) in three dimensions the critical temperature for $K=0$ is given by

$$k_BT_{\rm cr}=\frac{1}{2\ln 2}. \tag{3.25}$$

The curvature effects enhance the symmetry breaking on $R\otimes H^2$. Hence there is only the broken phase for $k_BT<1/(2\ln 2)$ in the model $\lambda>\lambda_{\rm cr}$. For $k_BT\geqslant 1/(2\ln 2)$ or $\lambda\leqslant\lambda_{\rm cr}$ the dynamical fermion mass is smoothly generated as the curvature K increases and the chiral symmetry is broken down by the curvature effect. After the same analysis in arbitrary dimensions $2<D<4$ we find the same behavior for the dynamical fermion mass. The thermal effect restores the broken chiral symmetry while the negative curvature effect breaks the chiral symmetry. Only the second-order phase transition occurs with varying the temperature and the curvature in $2<D<4$. In two dimensions Eq. (3.23) has the same behavior in Minkowski space.

For the second-order phase transition the critical point is obtained by the massless limit of the gap equation. Taking the massless limit $m\rightarrow 0$ in Eq. (3.23) we find the equation that gives the relation between critical temperature $\beta_{\rm cr}=1/(k_B T_{\rm cr})$ and critical curvature $K_{\rm cr}$:

$$\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{D/2}}\Gamma\left(1-\frac{D}{2}\right)m_0^{D-2}-\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{(D-1)/2}}\frac{2}{\beta_{\rm cr}}\Gamma\left(\frac{3-D}{2}\right)\times\left(\frac{2\pi}{\beta_{\rm cr}}\right)^{D-3}\zeta(3-D,1/2)+\frac{\mathrm{tr}\mathbf{1}}{(4\pi)^{(D-1)/2}}\frac{1}{\beta_{\rm cr}}\Gamma\left(\frac{3-D}{2}\right)\times\sum_{n=-\infty}^{\infty}\left[|\omega_{{\rm cr}_n}|^{D-3}-K_{\rm cr}^{(D-3)/2}\times\frac{\Gamma[(D-1)/2+\alpha_{{\rm cr}_n}]}{\alpha_{{\rm cr}_n}\Gamma[(3-D)/2+\alpha_{{\rm cr}_n}]}\right]=0. \tag{3.26}$$

Evaluating Eq. (3.26) numerically we draw the phase diagram of the four-fermion model with varying the temperature and/or curvature on $R\otimes H^{D-1}$ at $D=2.5$, 3.0, 3.5 in Figs. 8 and 9. A ratio $T_{\rm cr}/\sqrt{K_{\rm cr}}$ takes larger value for higher dimension. At a scale $T\sim\sqrt{K}\sim m_0$ the thermal effect gives the main contribution to the phase structure for $D\leqslant 3$ and the symmetric phase is realized. At the four-dimensional limit $T_{\rm cr}$ is divergent for $\lambda>\lambda_{\rm cr}$. It comes from the divergence that appears at the four-dimensional limit. Thus it may be a result obtained especially from the nonrenormalizability of the theory.

IV. CONCLUSION AND DISCUSSIONS

We have investigated the phase structure of DSB in the four-fermion model at finite temperature and curvature in arbitrary dimensions $(2<D<4)$.

Evaluating the effective potential and the gap equation in the leading order of the $1/N$ expansion we found the thermal and curvature induced phase transition. The dynamically generated fermion mass is calculated numerically with varying the temperature and curvature. Only the second-order phase transition is observed with varying the temperature and curvature. We found the lines dividing the symmetric phase and asymmetric phase in the T-$\sqrt{K}$ plane. In positive curvature space the curvature effect restores the broken chiral symmetry. On the contrary the curvature effect enhances the chiral symmetry breaking in negative curvature space. At finite temperature the lower limit of the momentum for the fermion field $(k^0<\pi/\beta)$ appears from the antiperiodicity and then the long range effect is suppressed. Thus the thermal effect restores the broken symmetry even in a negative curvature spacetime.

In two spacetime dimensions it is not able to introduce the combined effects of the temperature and curvature within our method because of the assumption of the equilibrium. This assumption is not accepted in an inflationary expanding universe, but we may discuss the phase transition at an early universe within our results.

Under the discrete chiral symmetry Z_2 two kinds of states which are labeled by $\sigma=\pm\langle\sigma\rangle$ have the same potential energy. Thus the grand states of the present model are doubly degenerate. If the transition rate between these degenerate vacua is not negligibly small, a nonstatic field configuration is realized. We cannot deal with the nonstatic configurations in the effective potential approach.

We consider the static Einstein universe, $R\otimes S^{D-1}$, as an example of the positive curvature spacetime. The spatial volume for $R\otimes S^{D-1}$ is finite. For a small volume system it is expected that the transition rate between the degenerate vacua may be large enough to induce the nonstatic field configuration. As is known for $D=2$ at finite temperature the kink and antikink configurations which are one kind of the nonstatic configurations restore the broken chiral symmetry [4]. Thus our results are useful only in some finite region of the space for large K in $R\otimes S^{D-1}$. When summing contributions from different regions we would obtain the qualitatively different results. There is a possibility that the chiral symmetry is resorted for a smaller curvature than the critical curvature $K_{\rm cr}$ obtained in Eq. (3.20) and the phase diagram shown in Fig. 3 may be modified. We will need a new idea and further investigations to evaluate the nonstatic configurations in $R\otimes S^{D-1}$.

We are only looking the four-fermion model in the leading order of the $1/N$ expansion but there may be some fundamental properties of DSB. Decreasing the temperature, the phase transition may occur from the negative curvature

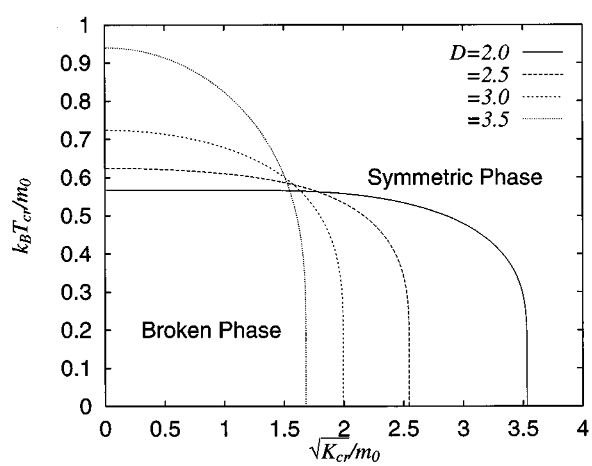

FIG. 3. The phase diagram at $D=2.0$, 2.5, 3.0, 3.5 in $R\otimes S^{D-1}$.

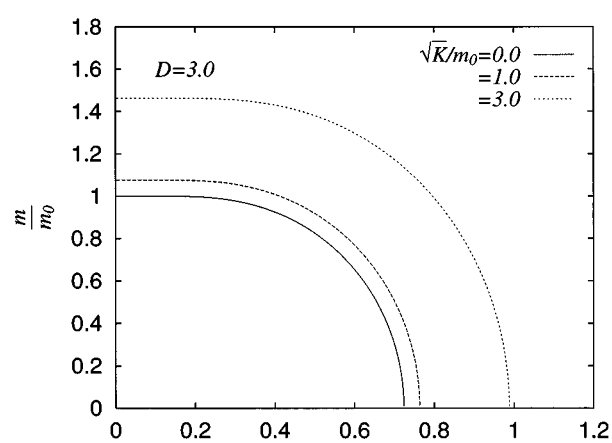

FIG. 4. Dynamical fermion mass m at $D=3.0$ for $\lambda>\lambda_{\rm cr}$ in $R\otimes H^2$ as a function of the temperature T with the curvature K fixed.

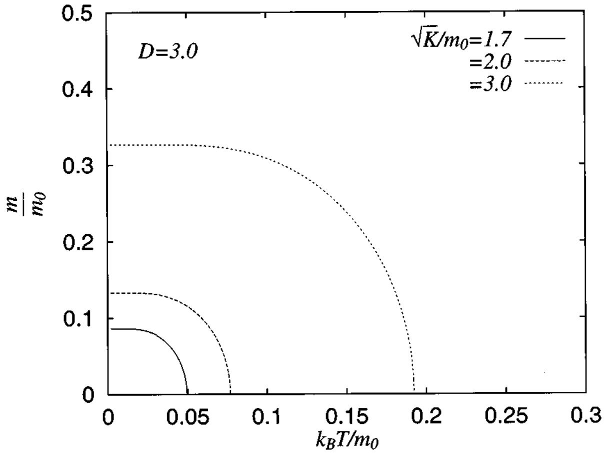

FIG. 5. Dynamical fermion mass m at $D=3.0$ for $\lambda \leqslant \lambda_{cr}$ in $R\otimes H^2$ as a function of the temperature T with the curvature K fixed.

places of the universe. After that the chiral symmetry is broken down at the flat places and then at the positive curvature places. It may have something for the evolution of the universe but it is difficult to cause the inflationary evolution of the universe in the four-fermion model [13] without a new idea.

To investigate the phenomena at the period of the phase transition we cannot avoid considering the nonequilibrium state. We will continue our work further and hope to extend our analysis to a nonequilibrium state.

ACKNOWLEDGMENTS

We would like to thank Kenji Fukazawa, Taizo Muta, and Kazuhiro Yamamoto for useful conversations.

APPENDIX: TWO-POINT FUNCTIONS IN $R\otimes S^{D-1}$ AND $R\otimes H^{D-1}$

In maximally symmetric spacetime S^D and H^D the exact expression of the two-point functions are known without making any approximation in dealing with the spacetime curvature [12,14–17]. On $R\otimes S^{D-1}$ and $R\otimes H^{D-1}$ the spinor two-point functions are obtained from those on S^D and H^D. Here we closely follow [12] and show the spinor two-point functions on $R\otimes S^{D-1}$ and $R\otimes H^{D-1}$.

The spinor two-point function $S(x,y;s)$ is defined by the Dirac equation[1]

$$(\gamma^\mu \nabla_\mu + s)S(x,y;s) = \frac{1}{\sqrt{-g}}\delta^D(x,y). \tag{A1}$$

We introduce the bispinor function G defined by

$$(\gamma^\mu \nabla_\mu - s)G(x,y;s) = S(x,y). \tag{A2}$$

According to Eq. (A1) in $R\otimes S^D$ and $R\otimes H^D$ $G(x,y;s)$ satisfies the following equation:

$$\left[(\partial_0)^2 + \Box_{D-1} - \frac{R}{4} - s^2\right] G(x,y;s) = -\frac{1}{\sqrt{g}}\delta^D(x,y), \tag{A3}$$

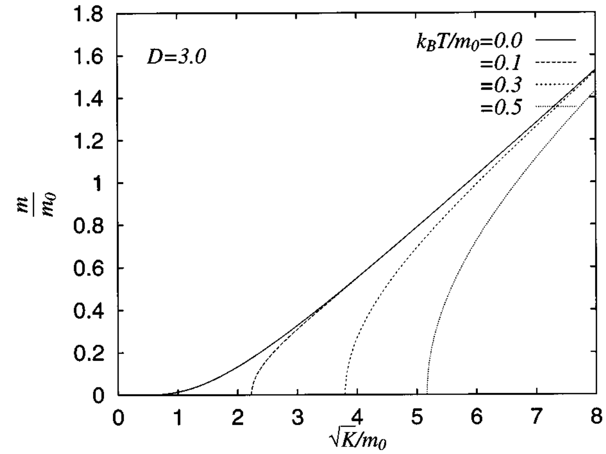

FIG. 7. Dynamical fermion mass m at $D=3.0$ for $\lambda \leqslant \lambda_{cr}$ in $R\otimes H^2$ as a function of the curvature K with the temperature T fixed.

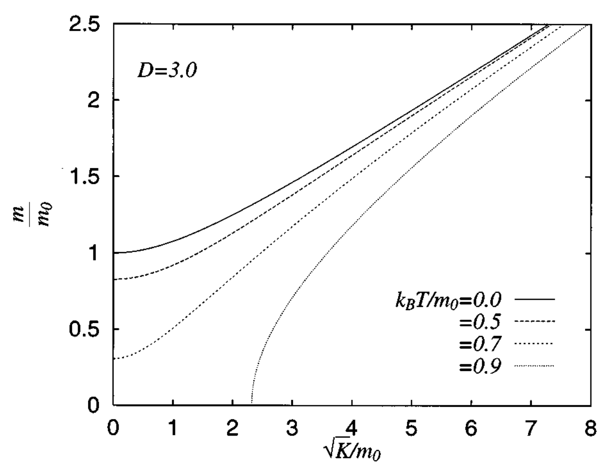

FIG. 6. Dynamical fermion mass m at $D=3.0$ for $\lambda > \lambda_{cr}$ in $R\otimes H^2$ as a function of the curvature K with the temperature T fixed.

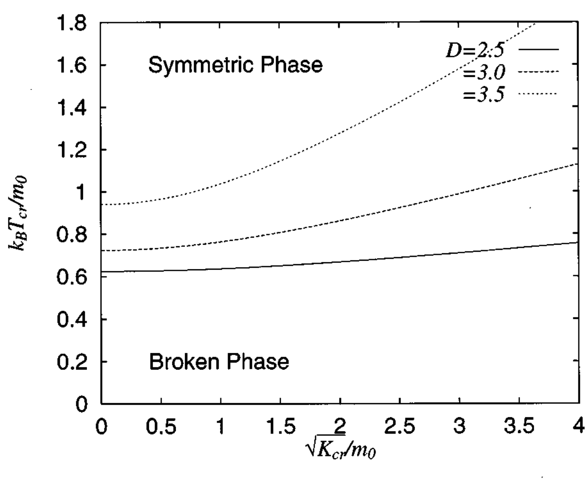

FIG. 8. The phase diagram at $D=2.5$, 3.0, 3.5 for $\lambda > \lambda_{cr}$ in $R\otimes H^{D-1}$.

[1]The Euclidean metric $(+,+,+,+)$ is used here.

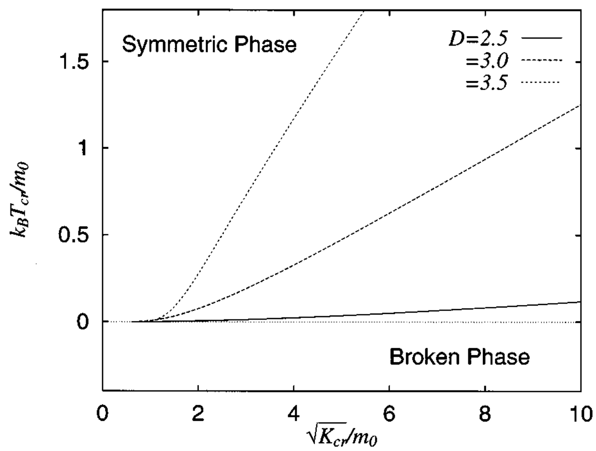

FIG. 9. The phase diagram at $D=2.5$, 3.0, 3.5 for $\lambda\leqslant\lambda_{\rm cr}$ in $R\otimes H^{D-1}$.

where $\Box_{D-1}$ is the Laplacian for a bispinor function on $R\otimes S^D$ and $R\otimes H^D$. Performing the Fourier transformation

$$G(x,y;s)=\int\frac{d\omega}{2\pi}e^{-i\omega(y-x)^0}\tilde{G}(\omega,\mathbf{x},\mathbf{y};s),\qquad\text{(A4)}$$

we rewrite Eq. (A3) in the form

$$\left[\Box_{D-1}-\frac{R}{4}-(s^2+\omega^2)\right]\tilde{G}(\omega,\mathbf{x},\mathbf{y};s)=-\frac{1}{\sqrt{g}}\delta^{D-1}(\mathbf{y}).\qquad\text{(A5)}$$

Equation (A5) is of the same form as the one for the spinor Green's function with mass $\sqrt{s^2+\omega^2}$ on the maximally symmetric space S^{D-1} or H^{D-1}.

The general form of the Green's function $\tilde{G}(\omega,\mathbf{x},\mathbf{y};s)$ is written as [17]

$$\tilde{G}(\omega,\mathbf{x},\mathbf{y};s)=U(\mathbf{x},\mathbf{y})g(l),\qquad\text{(A6)}$$

where U is a matrix in the spinor indices, g is a scalar function only of l, $l=a\theta$ which is the geodesic distance between $\mathbf{x}$ and $\mathbf{y}$ on S^{D-1} or H^{D-1}, n_i is a unit vector tangent to the geodesic $n_i=\nabla_i l$. Inserting Eq. (A6) into Eq. (A5) we get

$$\left[U\Box_{D-1}g+2(\nabla_jU)\nabla^jg+(\Box_{D-1}U)g-\left(\frac{R}{4}+s^2+\omega^2\right)Ug\right]=0,\qquad\text{(A7)}$$

where we restrict ourselves to the region $l\neq 0$. To evaluate Eq. (A7) we have to calculate the covariant derivative of U and n_i. U is the operator which makes parallel transport of the spinor at point $\mathbf{x}$ along the geodesic to point $\mathbf{y}$. Thus the operator U must satisfy the following parallel transport equations [15]:

$$n^i\nabla_iU=0,\qquad U(\mathbf{x},\mathbf{x})=\mathbf{1}.\qquad\text{(A8)}$$

To evaluate the second derivative of U we set

$$\nabla_iU\equiv V_iU.\qquad\text{(A9)}$$

From the integrability condition [14] on V_i,

$$\nabla_iV_j-\nabla_jV_i-[V_i,V_j]=\frac{R}{(D-1)(D-2)}\sigma_{ij},\qquad\text{(A10)}$$

and the parallel transport equation (A8) we easily find that

$$V_i=-\frac{1}{a}\tan\left(\frac{l}{2a}\right)\sigma_{ij}n^j\quad\text{on }S^{D-1},\qquad\text{(A11)}$$

$$V_i=\frac{1}{a}\tanh\left(\frac{l}{2a}\right)\sigma_{ij}n^j\quad\text{on }H^{D-1},\qquad\text{(A12)}$$

where σ_{ij} are the antisymmetric tensors constructed by the Dirac gamma matrices, $\sigma_{ij}=[\gamma_i,\gamma_j]/4$. To find V_i we have used the fact that the maximally symmetric bitensors are represented as a sum of products of n_i and g_{ij} with coefficients which are functions only of l [16]. After some calculations we get the Laplacian acting on U:

$$\Box_{D-1}U=-\frac{D-2}{4a^2}\tan^2\left(\frac{l}{2a}\right)U\quad\text{on }S^{D-1},$$
$$\Box_{D-1}U=-\frac{D-2}{4a^2}\tanh^2\left(\frac{l}{2a}\right)U\quad\text{on }H^{D-1}.\qquad\text{(A13)}$$

The derivative of n_i is also the maximally symmetric bitensor [16] and found to be

$$\nabla_in_j=\frac{1}{a}\cot\left(\frac{l}{a}\right)(g_{ij}-n_in_j)\quad\text{on }S^{D-1},$$
$$\nabla_in_j=\frac{1}{a}\coth\left(\frac{l}{a}\right)(g_{ij}-n_in_j)\quad\text{on }H^{D-1}.\qquad\text{(A14)}$$

Therefore Eq. (A7) reads

$$\left[\partial_l^2+\frac{D-2}{a}\cot\left(\frac{l}{a}\right)\partial_l-\frac{D-2}{4a^2}\tan^2\left(\frac{l}{2a}\right)-\frac{R}{4}-(s^2+\omega^2)\right]g=0\quad\text{on }S^{D-1},\qquad\text{(A15)}$$

$$\left[\partial_l^2+\frac{D-2}{a}\coth\left(\frac{l}{a}\right)\partial_l-\frac{D-2}{4a^2}\tanh^2\left(\frac{l}{2a}\right)-\frac{R}{4}-(s^2+\omega^2)\right]g=0\quad\text{on }H^{D-1}.\qquad\text{(A16)}$$

We define the functions $h_{SD}(l)$ and $h_{HD}(l)$ by $g(l)=\cos(l/2a)h_{SD}(l)$ and $g(l)=\cosh(l/2a)h_{HD}(l)$, respectively, and make a change of variable by $z=\cos^2(l/2a)$ in Eq. (A15) and $z'=\cosh^2(l/2a)$ in Eq. (A16). We then find that Eqs. (A15) and (A16) are rewritten in the forms of hypergeometric differential equations:

$$\left[z(1-z)\partial_z^2+\left(\frac{D+1}{2}-Dz\right)\partial_z-\frac{(D-1)^2}{4}-(s^2+\omega^2)a^2\right]h_{SD}(z)=0, \tag{A17}$$

$$\left[z'(1-z')\partial_{z'}^2+\left(\frac{D+1}{2}-Dz'\right)\partial_{z'}-\frac{(D-1)^2}{4}+(s^2+\omega^2)a^2\right]h_{HD}(z')=0. \tag{A18}$$

Noting that the Green's functions are regular at the point $l=a\pi$ and fall off for $l\to\infty$ we write the solutions of Eqs. (A17) and (A18) by the hypergeometric function

$$h_{SD}(z)=c_{SD}F\left(\frac{D-1}{2}+i\alpha,\ \frac{D-1}{2}-i\alpha,\ \frac{D+1}{2};z\right), \tag{A19}$$

$$h_{HD}(z')=c_{HD}(-z')^{(1-D)/2-\alpha}F\left(\frac{D-1}{2}+\alpha,\alpha,2\alpha+1;\frac{1}{z'}\right), \tag{A20}$$

where $\alpha=a\sqrt{s^2+\omega^2}$. As we remained in the region where $l\neq 0$ the normalization constants c_{SD} and c_{HD} are yet undetermined. To obtain c_{SD} and c_{HD} we consider the singularity of $\widetilde{G}(\omega,\mathbf{x},\mathbf{y};s)$ in the limit $l\to 0$,

$$\widetilde{G}\to c_{SD}\frac{\Gamma[(D+1)/2]\Gamma[(D-3)/2]}{\Gamma[(D-1)/2+i\alpha]\Gamma[(D-1)/2-i\alpha]}\left(\frac{l}{2a}\right)^{3-D},$$

$$\widetilde{G}\to c_{HD}(-1)^{(1-D)/2-\alpha}\frac{\Gamma(2\alpha+2)\Gamma[(D-3)2]}{\Gamma[(D-1)/2+\alpha]\Gamma(\alpha)}\left(\frac{l}{2a}\right)^{3-D}, \tag{A21}$$

and compare them with the singularity of the Green's function in flat spacetime. This procedure is justified because the singularity on a curved spacetime background has the same structure as that in the flat spacetime. For $l\sim 0$ the Green's function in the flat spacetime behaves as [12,17]

$$\widetilde{G}^{\mathrm{flat}}(l)\sim\frac{1}{4\pi^{(D-1)/2}}\Gamma\left(\frac{D-3}{2}\right)l^{3-D}. \tag{A22}$$

Comparing Eq. (A21) with Eq. (A22), the overall factors c_{SD} and c_{HD} are obtained:

$$c_{SD}=\frac{a^{3-D}}{(4\pi)^{(D-1)/2}}\frac{\Gamma[(D-1)/2+i\alpha]\Gamma[(D-1)/2-i\alpha]}{\Gamma[(D+1)/2]},$$

$$c_{HD}=\frac{(-1)^{(1-D)/2-\alpha}a^{3-D}}{(4\pi)^{(D-1)/2}}\frac{\Gamma[(D-1)/2+\alpha]\Gamma(\alpha)}{\Gamma(2\alpha+1)}. \tag{A23}$$

Inserting Eqs. (A19), (A20), and (A23) into Eq. (A6) we find on S^{D-1}

$$\widetilde{G}(\omega,\mathbf{x},\mathbf{y};s)=U(\mathbf{x},\mathbf{y})\frac{a^{3-D}}{(4\pi)^{(D-1)/2}}\frac{\Gamma[(D-1)/2+i\alpha]\Gamma[(D-1)/2-i\alpha]}{\Gamma[(D+1)/2]}$$
$$\times\cos\left(\frac{l}{2a}\right)F\left(\frac{D-1}{2}+i\alpha,\frac{D-1}{2}-i\alpha,\ \frac{D+1}{2};\cos^2\left(\frac{l}{2a}\right)\right), \tag{A24}$$

and on H^{D-1}

$$\widetilde{G}(\omega,\mathbf{x},\mathbf{y};s)=U(x,y)(-1)^{(1-D)/2-\alpha}\frac{a^{3-D}}{(4\pi)^{(D-1)/2}}\frac{\Gamma[(D-1)/2+\alpha]\Gamma(\alpha)}{\Gamma(2\alpha+1)}\left[\cosh\left(\frac{l}{2a}\right)\right]^{2-D-2\alpha}$$
$$\times F\left(\frac{D-1}{2}+\alpha,\alpha,2\alpha+1;\cosh^{-2}\left(\frac{l}{2a}\right)\right). \tag{A25}$$

Thus the Green's functions $\widetilde{G}(\omega,\mathbf{x},\mathbf{y};s)$ on the maximally symmetric spacetime are obtained.

The spinor two-point function $S(x,y;s)$ is derived from the Green's function $\widetilde{G}(\omega,\mathbf{x},\mathbf{y};s)$. From Eq. (A2) we get

$$S=(-i\omega\gamma^0+\gamma^i\nabla_i-s)\int\frac{d\omega}{2\pi}e^{-i\omega(y-x)^0}Ug$$
$$=\begin{cases}\displaystyle\int\frac{d\omega}{2\pi}e^{-i\omega(y-x)^0}\left[(s+i\omega\gamma^0)Ug-\gamma_in^iU\left(\partial_l-\frac{D-2}{2a}\tan\left(\frac{l}{2a}\right)\right)g\right] & \text{on } R\otimes S^{D-1},\\ \displaystyle\int\frac{d\omega}{2\pi}e^{-i\omega(y-x)^0}\left[(s+i\omega\gamma^0)Ug-\gamma_in^iU\left(\partial_l+\frac{D-2}{2a}\tanh\left(\frac{l}{2a}\right)\right)g\right] & \text{on } R\otimes H^{D-1}.\end{cases} \tag{A26}$$

Substituting Eqs. (A24) and (A25) in Eq. (A26) the spinor two-point function $S(x,y;s)$ is obtained:

$$S(x,y;s)=-\frac{a^{3-D}}{(4\pi)^{(D-1)/2}}\int\frac{d\omega}{2\pi}e^{-i\omega(y-x)^0}\frac{\Gamma[(D-1)/2+i\alpha]\Gamma[(D-1)/2-i\alpha]}{\Gamma[(D+1)/2]}$$
$$\times\left[(s+i\omega\gamma^0)U(\mathbf{x},\mathbf{y})\cos\left(\frac{l}{2a}\right)F\left(\frac{D-1}{2}+i\alpha,\frac{D-1}{2}-i\alpha,\frac{D+1}{2};\cos^2\left(\frac{l}{2a}\right)\right)+\gamma_in^iU(\mathbf{x},\mathbf{y})\frac{D-1}{2a}\right.$$
$$\left.\times\sin\left(\frac{l}{2a}\right)F\left(\frac{D-1}{2}+i\alpha,\frac{D-1}{2}-i\alpha,\frac{D-1}{2};\cos^2\left(\frac{l}{2a}\right)\right)\right], \tag{A27}$$

on $R\otimes S^{D-1}$ [12] and

$$S(x,y;s)=-\frac{a^{2-D}}{(4\pi)^{(D-1)/2}}\int\frac{d\omega}{2\pi}e^{-i\omega(y-x)^0}\frac{\Gamma[(D-1)/2+\alpha]\Gamma(\alpha)}{\Gamma(2\alpha+1)}$$
$$\times\left[\cosh\left(\frac{l}{2a}\right)\right]^{2-D-2\alpha}\left[a(s+i\omega\gamma^0)U(\mathbf{x},\mathbf{y})\cosh\left(\frac{l}{2a}\right)F\left(\frac{D-1}{2}+\alpha,\alpha,2\alpha+1;\cosh^{-2}\left(\frac{l}{2a}\right)\right)\right.$$
$$\left.+\alpha\gamma_in^iU(\mathbf{x},\mathbf{y})\sinh\left(\frac{l}{2a}\right)F\left(\frac{D-1}{2}+\alpha,\alpha+1,2\alpha+1;\cosh^{-2}\left(\frac{l}{2a}\right)\right)\right], \tag{A28}$$

on $R\otimes H^{D-1}$. According to the anticommutation relation of spinor fields the two-point function (A27) satisfies the antiperiodic boundary condition $S(l)=-S(l+2\pi na)$ where n is an arbitrary integer.

[1] I. L. Buchbinder, S. D. Odintsov, and I. L. Shapiro, *Effective Action in Quantum Gravity* (IOP Publishing, Bristol, 1992).
[2] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961).
[3] H. O. Wada, Lett. Nuovo Cimento **11**, 697 (1974); L. Jacobs, Phys. Rev. D **10**, 3956 (1974); D. J. Harrington and A. Yildiz, *ibid.* **11**, 779 (1975); W. Dittrich and B. G. Englert, Nucl. Phys. **B179**, 85 (1981); T. Inagaki, T. Kouno, and T. Muta, Int. J. Mod. Phys. A **10**, 2241 (1995).
[4] R. F. Dashen, S. K. Ma, and R. Rajaraman, Phys. Rev. D **11**, 1499 (1975); R. F. Dashen, B. Hasslacher, and A. Neveu, *ibid.* **12**, 2443 (1975).
[5] H. Itoyama, Prog. Theor. Phys. **64**, 1886 (1980); I. L. Buchbinder and E. N. Kirillova, Int. J. Mod. Phys. A **4**, 143 (1989); E. Elizalde, S. D. Odintsov, and Yu. I. Shil'nov, Mod. Phys. Lett. A **9**, 913 (1994); T. Inagaki, S. Mukaigawa, and T. Muta, Phys. Rev. D **52**, 4267 (1995); E. Elizalde, S. Leseduarte, S. D. Odintsov, and Yu. I. Shil'nov, *ibid.* **53**, 1917 (1996); T. Inagaki, T. Muta, and S. D. Odintsov, Prog. Theor. Phys. Suppl. **127**, 93 (1997).
[6] T. Inagaki, T. Muta, and S. D. Odintsov, Mod. Phys. Lett. A **8**, 2117 (1993); T. Inagaki, Int. J. Mod. Phys. A **11**, 4561 (1996).
[7] K. Ishikawa, T. Inagaki, and T. Muta, Mod. Phys. Lett. A **11**, 939 (1996).
[8] M. B. Altaie and J. S. Dowker, Phys. Rev. D **18**, 3557 (1978).
[9] S. Kanemura and H.-T. Sato, Mod. Phys. Lett. A **10**, 1777 (1995); **11**, 785 (1996).
[10] D. J. Gross and A. Neveu, Phys. Rev. D **10**, 3235 (1974).
[11] S. Weinberg, Phys. Rev. D **9**, 3357 (1974); L. Dolan and R. Jackiw, Phys. Rev. D **9**, 3320 (1974).
[12] T. Inagaki, K. Ishikawa, and T. Muta, Prog. Theor. Phys. **96**, 847 (1996).
[13] A. M. Matheson, R. D. Ball, A. C. Davis, and R. H. Brandenberger, Nucl. Phys. **B328**, 223 (1989).
[14] P. Candelas and D. J. Raine, Phys. Rev. D **12**, 965 (1975).
[15] B. Allen and C. A. Lütken, Commun. Math. Phys. **106**, 201 (1986).
[16] B. Allen and T. Jacobson, Commun. Math. Phys. **103**, 669 (1986).
[17] R. Camporesi, Commun. Math. Phys. **148**, 283 (1992).