

Dynamical generation of extended objects in a (1 + 1)-dimensional chiral field theory: Nonperturbative Dirac operator resolvent analysis

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We analyze the (1 + 1)-dimensional Nambu–Jona-Lasinio (NJL) model nonperturbatively. In addition to its simple ground-state saddle points, the effective action of this model has a rich collection of nontrivial saddle points in which the composite fields $\sigma(x) = \langle \bar{\psi}\psi \rangle$ and $\pi(x) = \langle \bar{\psi}i\gamma_5\psi \rangle$ form static space-dependent configurations because of nontrivial dynamics. These configurations may be viewed as one-dimensional chiral “bags.” We start our analysis of such configurations by asking what kind of initially static $\{\sigma(x), \pi(x)\}$ background configurations will remain so under fermionic back reaction. By simply looking at the asymptotic spatial behavior of the expectation value of the fermion number current we show, independently of the large- N limit, that a necessary condition for this situation to occur is that $\{\sigma(x), \pi(x)\}$ give rise to a reflectionless Dirac operator. We provide an explicit formula for the diagonal resolvent of the Dirac operator in a reflectionless $\{\sigma(x), \pi(x)\}$ background which produces a prescribed number of bound states. We analyze in detail the cases of a single as well as two bound states. We explicitly check that these reflectionless backgrounds may be tuned such that the large- N saddle-point condition is satisfied. Thus, in the case of the NJL model, reflectionlessness is also sufficient to assure the time independence of the background. In our view, these facts make our work conceptually simpler than the previous work of Shei and of Dashen, Hasslacher, and Neveu which were based on the inverse scattering formalism. Our method of finding such nontrivial static configurations may be applied to other (1 + 1)-dimensional field theories. [S0556-2821(97)00620-6]

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I. INTRODUCTION

Over the last 30 years or so, physicists have gradually learned about the behavior of quantum field theory in the nonperturbative regime. In (1 + 1)-dimensional spacetime, some models are exactly soluble [1]. Another important approach involves the large- N expansion [2]. In particular, in the mid-1970s, Dashen, Hasslacher, and Neveu [3] used the inverse scattering method [4] to determine the spectrum of the Gross-Neveu model [5]. Recently, one of us developed an alternative method, based on the Gel’fand–Dikii equation [6], to study the same problem [7] as well as other problems [8]. As will be explained below, we feel that this method has certain advantages over the inverse scattering method.

In this paper we study the (1 + 1)-dimensional Nambu–Jona-Lasinio¹ (NJL) model [9] which is a renormalizable field theory defined by the action [5,10]

$$S = \int d^2x \left\{ \sum_{a=1}^N \bar{\psi}_a i \not{\partial} \psi_a + \frac{g^2}{2} \left[\left(\sum_{a=1}^N \bar{\psi}_a \psi_a \right)^2 - \left(\sum_{a=1}^N \bar{\psi}_a \gamma_5 \psi_a \right)^2 \right] \right\} \quad (1.1)$$

describing N self-interacting massless Dirac fermions ψ_a ($a = 1, \dots, N$). This action is invariant under $SU(N)_f \otimes U(1) \otimes U(1)_A$: namely, under

$$\psi_a \rightarrow U_{ab} \psi_b, \quad U \in SU(N)_f,$$

$$\psi_a \rightarrow e^{i\alpha} \psi_a,$$

and

$$\psi_a \rightarrow e^{i\gamma_5\beta} \psi_a. \quad (1.2)$$

We rewrite Eq. (1.1) as

$$S = \int d^2x \left\{ \bar{\psi} [i \not{\partial} - (\sigma + i\pi\gamma_5)] \psi - \frac{1}{2g^2} (\sigma^2 + \pi^2) \right\}, \quad (1.3)$$

where $\sigma(x)$ and $\pi(x)$ are the scalar and pseudoscalar auxiliary fields, respectively,² which are both of mass dimension one. These fields are singlets under $SU(N)_f \otimes U(1)$, but transform as a vector under the axial transformation in Eq. (1.2): namely,

$$\sigma + i\gamma_5\pi \rightarrow e^{-2i\gamma_5\beta} (\sigma + i\gamma_5\pi). \quad (1.4)$$

Thus, the partition function associated with Eq. (1.3) is

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¹This model is also dubbed in the literature as the “chiral Gross-Neveu model” as well as the “multiflavor Thirring model.”

²From this point to the end of this paper flavor indices are suppressed. Thus $i\bar{\psi}\not{\partial}\psi$ should be understood as $i\sum_{a=1}^N \bar{\psi}_a \not{\partial} \psi_a$. Similarly $\bar{\psi}\Gamma\psi$ stands for $\sum_{a=1}^N \bar{\psi}_a \Gamma \psi_a$, where $\Gamma = 1, \gamma_5$.

$$\mathcal{Z} = \int \mathcal{D}\sigma \mathcal{D}\pi \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left[i \int d^2x \left(\bar{\psi} [i\partial - (\sigma + i\pi\gamma_5)] \psi - \frac{1}{2g^2} (\sigma^2 + \pi^2) \right) \right]. \quad (1.5)$$

Integrating over the Grassmannian variables leads to $\mathcal{Z} = \int \mathcal{D}\sigma \mathcal{D}\pi \exp\{iS_{\text{eff}}[\sigma, \pi]\}$ where the bare effective action is

$$S_{\text{eff}}[\sigma, \pi] = -\frac{1}{2g^2} \int d^2x (\sigma^2 + \pi^2) - iN \text{Tr} \ln [i\partial - (\sigma + i\pi\gamma_5)]. \quad (1.6)$$

and the trace is taken over both functional and Dirac indices.

This theory has been studied in the limit $N \rightarrow \infty$ with Ng^2 held fixed [5]. In this limit Eq. (1.5) is governed by saddle points of Eq. (1.6) and the small fluctuations around them. The most general saddle-point condition reads

$$\begin{aligned} \frac{\delta S_{\text{eff}}}{\delta \sigma(x,t)} &= -\frac{\sigma(x,t)}{g^2} + iN \text{tr} \left[\left\langle x, t \left| \frac{1}{i\partial - (\sigma + i\pi\gamma_5)} \right| x, t \right\rangle \right] \\ &= 0, \\ \frac{\delta S_{\text{eff}}}{\delta \pi(x,t)} &= -\frac{\pi(x,t)}{g^2} - N \text{tr} \left[\gamma_5 \left\langle x, t \left| \frac{1}{i\partial - (\sigma + i\pi\gamma_5)} \right| x, t \right\rangle \right] \\ &= 0. \end{aligned} \quad (1.7)$$

In particular, the nonperturbative vacuum of Eq. (1.1) is governed by the simplest large- N saddle points of the path integral associated with it, where the composite scalar operator $\bar{\psi}\psi$ and the pseudoscalar operator $i\bar{\psi}\gamma_5\psi$ develop spacetime-independent expectation values.

These saddle points are extrema of the effective potential V_{eff} associated with Eq. (1.3), namely, the value of $-S_{\text{eff}}$ for spacetime-independent σ, π configurations per unit time per unit length. The effective potential V_{eff} depends only on the combination $\rho^2 = \sigma^2 + \pi^2$ as a result of chiral symmetry. V_{eff} has a minimum as a function of ρ at $\rho = m \neq 0$ that is fixed by the (bare) gap equation [5]

$$-m + iNg^2 \text{tr} \int \frac{d^2k}{(2\pi)^2} \frac{1}{\mathbf{k} - m} = 0 \quad (1.8)$$

which yields the dynamical mass

$$m = \Lambda e^{-\pi/Ng^2(\Lambda)}. \quad (1.9)$$

Here Λ is an ultraviolet cutoff. The mass m must be a renormalization-group invariant. Thus, the model is asymptotically free. We can get rid of the cutoff at the price of introducing an arbitrary renormalization scale μ . The renormalized coupling $g_R(\mu)$ and the cutoff-dependent bare coupling are then related through $\Lambda e^{-\pi/Ng^2(\Lambda)} = \mu e^{1-\pi/Ng_R^2(\mu)}$ in a convention where $Ng_R^2(m) = 1/\pi$. Trading the dimen-

sionless coupling g_R^2 for the dynamical mass scale m represents the well-known phenomenon of dimensional transmutation.

The vacuum manifold of Eq. (1.3) is therefore a circle $\rho = m$ in the σ, π plane, and the equivalent vacua are parametrized by the chiral angle $\theta = \arctan(\pi/\sigma)$. Therefore, small fluctuations of the Dirac fields around the vacuum manifold develop dynamical³ chiral mass $m \exp(i\theta\gamma_5)$.

Note in passing that the massless fluctuations of θ along the vacuum manifold decouple from the spectrum [11] so that the axial U(1) symmetry does not break dynamically in this two-dimensional model [12].

Nontrivial excitations of the vacuum, on the other hand, are described semiclassically by large- N saddle points of the path integral over Eq. (1.1) at which σ and π develop spacetime-dependent expectation values [13,14]. These expectation values are the spacetime-dependent solution of Eq. (1.7). Saddle points of this type are important also in discussing the large-order behavior [15,16] of the $1/N$ expansion of the path integral over Eq. (1.1).

Shei [17] has studied the saddle points of the NJL model by applying the inverse scattering method following Dashen *et al.* [3]. These saddle points describe sectors of Eq. (1.1) that include scattering states of the (dynamically massive) fermions in Eq. (1.1), as well as a rich collection of bound states thereof.

These bound states result from the strong infrared interactions, which polarize the vacuum inhomogeneously, causing the composite scalar $\bar{\psi}\psi$ and pseudoscalar $i\bar{\psi}\gamma_5\psi$ fields to form finite action spacetime-dependent condensates. These condensates are stable because of the binding energy released by the trapped fermions and therefore cannot form without such binding. This description agrees with the general physical picture drawn in [18]. We may regard these condensates as one dimensional chiral bags [19,20] that trap the original fermions (“quarks”) into stable finite action extended entities (“hadrons”).

In this paper we develop further the method of [7,8], applying it to the NJL model (1.1) as an alternative to the inverse scattering investigations in [17]. We focus on static extended configurations providing explicit expressions for the profiles of these objects and calculate their masses. Our analysis of these static saddle points is based on an explicit representation we find for the diagonal resolvent of the Dirac operator in a $\sigma(x), \pi(x)$ background which produces a prescribed number of bound states. This explicit construction of the diagonal resolvent can actually be carried out for finite N . It is based on elementary Sturm-Liouville theory as well as on simple dimensional analysis. All our manipulations involve the space-dependent scalar and pseudoscalar condensates directly. In our view, these facts make the method presented here simpler than inverse scattering calculations previously employed in this problem because we do not need to work with the scattering data and the so-called trace identities that relate them to the space-dependent condensates. Our

³Note that the axial U(1) symmetry in Eq. (1.2) protects the fermions from developing a mass term to any order in perturbation theory.

method of finding such nontrivial static configurations may be applied to other two-dimensional field theories.

It is worth mentioning at this point that the NJL model (1.1) is completely integrable for any number of flavors⁴ N . Its spectrum and completely factorized S matrix were determined in a series of papers [21] by a Bethe ansatz diagonalization of the Hamiltonian for any number N of flavors. The large- N spectrum obtained here as well as in [17] is consistent with the exact solution of [21]. Note, however, that the large- N analysis in this paper concerns only dynamics of the interactions between fermions and extended objects. We do not address issues like scattering of one extended object on another, which is discussed in the exact analysis of [21]. Consistency of our approximate large- N results and the exact results of [21] reassures us of the validity of our calculations.

Rather than treating nontrivial excitations as abstract vectors in Hilbert space, which is inevitable in [21], our analysis draws almost a “mechanical” picture of how “hadrons” arise in the NJL model. This description of “hadron” formation as a result of inhomogeneous polarizations of the vacuum due to strong infrared interactions may have some restricted similarity to dynamics of QCD in the real world. Furthermore, our resolvent method is potentially applicable for nonintegrable models in 1+1 dimensions. In contrast, Bethe ansatz and factorizable S -matrix techniques are limited in principle to 1+1 dimensions because of the Coleman-Mandula theorem [22], whereas large- N saddle-point techniques may provide powerful tools in analysis of more realistic higher dimensional field theories [23].

If we set $\pi(x)$ in Eq. (1.3) to be identically zero, we recover the Gross-Neveu model, defined by

$$S_{\text{GN}} = \int d^2x \left\{ \bar{\psi} [i \not{\partial} - \sigma] \psi - \frac{\sigma^2}{2g^2} \right\}. \quad (1.10)$$

In spite of their similarities, these two field theories are quite different, as is well known from the field-theoretic literature of the 1970s. The crucial difference is that the Gross-Neveu model possesses a discrete symmetry, $\sigma \rightarrow -\sigma$, rather than the continuous symmetry (1.2) in the NJL model studied here. This discrete symmetry is dynamically broken by the nonperturbative vacuum, and thus there is a kink solution [24,3,7], the so-called Callan-Coleman-Gross-Zee (CCGZ) kink $\sigma(x) = m \tanh(mx)$, interpolating between $\pm m$ at $x = \pm\infty$, respectively. Therefore, topology ensures the stability of these kinks. The kink may bind any number $n \leq N$ of fermions in its single zero energy bound state, without affecting its mass, because its stability is guaranteed by topology already.

In contrast, the NJL model, with its continuous symmetry, does not have a topologically stable soliton solution. The solitons arising in the NJL model and studied in this paper can only be stabilized by binding fermions. They are stable not due to topology, but to dynamics.

⁴For $N=1$ a simple Fierz transformation shows that Eq. (1.1) is simply the massless Thirring model, which is a conformal quantum field theory having no mass gap. A mass gap appears dynamically only for $N \geq 2$.

This paper is organized as follows: In Sec. II we prove that the static condensates $\sigma(x)$ and $\pi(x)$ in Eq. (1.3) must be such that the resulting Dirac operator is reflectionless. Our proof of this strong restriction on the Dirac operator involves basic field theoretic arguments and has nothing to do with the large- N approximation. We next show in Sec. III that if we fix in advance the number of bound states in the spectrum of the reflectionless Dirac operator, then simple dimensional analysis determines the diagonal resolvent of this operator explicitly in terms of the background fields and their derivatives. We then construct the resolvent assuming the background fields support a single bound state in Sec. IV A. We are able to determine the profile of the background fields up to a finite number of parameters: the relative chiral rotation of the two vacua at the two ends of the one-dimensional space and the bound state energies. In Sec. IV B we provide partial analysis of the case of two bound states. We stress again that our construction of these background fields has nothing to do with the large- N approximation.

We determine these parameters in Secs. V A and V B. From the topological considerations in [25] (and independently of the large- N approximation) we can determine the relative chiral rotation of the asymptotic vacua as being proportional to the number of fermions trapped in the bound states. In order to determine the bound-state energies we have to impose the large- N saddle-point conditions.

Some technical details are left to two appendices. In Appendix A we derive the spatial asymptotic behavior of the static Dirac operator Green’s function. In order to make our paper self-contained we derive the Gel’fand–Dikii equation in Appendix B.

II. ABSENCE OF REFLECTIONS IN THE DIRAC OPERATOR WITH STATIC BACKGROUND FIELDS

As we explained in the Introduction, we are interested in static space-dependent solutions of the extremum condition on S_{eff} . To this end we need to invert the Dirac operator

$$D = i \not{\partial} - [\sigma(x) + i\pi(x)\gamma_5] \quad (2.1)$$

in a given background of static field configurations $\sigma(x)$ and $\pi(x)$. In particular, we have to find the diagonal resolvent of Eq. (2.1) in that background. The extremum condition on S_{eff} relates this resolvent, which in principle is a complicated and generally unknown functional of $\sigma(x)$, $\pi(x)$ and of their derivatives, to $\sigma(x)$ and $\pi(x)$ themselves. This complicated relation is the source of all difficulties that arise in any attempt to solve the model under consideration. It turns out, however, that basic field theoretic considerations, that are unrelated to the extremum condition, imply that Eq. (2.1) must be reflectionless. This spectral property of Eq. (2.1) sets rather powerful restrictions on the static background fields $\sigma(x)$ and $\pi(x)$ which are allowed dynamically. In the next section we show how this special property of Eq. (2.1) allows us to write explicit expressions for the resolvent in some restrictive cases, that are interesting enough from a physical point of view.

Inverting Eq. (2.1) has nothing to do with the large- N approximation, and consequently our results in this section are valid for any value of N .

Here $\sigma(x)$ and $\pi(x)$ are our static background field configurations, for which we assume asymptotic behavior dictated by simple physical considerations. The overall energy deposited in any relevant static σ , π configuration must be finite. Therefore, these fields must approach constant vacuum asymptotic values, while their derivatives vanish asymptotically. Then the axial U(1) symmetry implies that

$$\sigma^2 + \pi^2 \xrightarrow{x \rightarrow \pm\infty} m^2,$$

where m is the dynamically generated mass, and therefore we arrive at the asymptotic boundary conditions for σ and π :

$$\begin{aligned} \sigma &\xrightarrow{x \rightarrow \pm\infty} m \cos\theta_{\pm}, & \sigma' &\xrightarrow{x \rightarrow \pm\infty} 0, \\ \pi &\xrightarrow{x \rightarrow \pm\infty} m \sin\theta_{\pm}, & \pi' &\xrightarrow{x \rightarrow \pm\infty} 0, \end{aligned} \quad (2.2)$$

where θ_{\pm} are the asymptotic chiral alignment angles. Only the difference $\theta_+ - \theta_-$ is meaningful, of course, and henceforth we use the axial symmetry to set $\theta_- = 0$, such that $\sigma(-\infty) = m$ and $\pi(-\infty) = 0$. We also omit the subscript from θ_+ and denote it simply by θ from now on. It is in the background of such fields that we wish to invert (2.1).

In this paper we use the Majorana representation $\gamma^0 = \sigma_2$, $\gamma^1 = i\sigma_3$, and $\gamma^5 = -\gamma^0\gamma^1 = \sigma_1$ for γ matrices. In this representation Eq. (2.1) becomes

$$D = \begin{pmatrix} -\partial_x - \sigma & -i\omega - i\pi \\ i\omega - i\pi & \partial_x - \sigma \end{pmatrix}. \quad (2.3)$$

Inverting Eq. (2.3) is achieved by solving

$$\begin{aligned} &\begin{pmatrix} -\partial_x - \sigma(x) & -i\omega - i\pi(x) \\ i\omega - i\pi(x) & \partial_x - \sigma(x) \end{pmatrix} \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & d(x,y) \end{pmatrix} \\ &= -i\mathbf{1}\delta(x-y) \end{aligned} \quad (2.4)$$

for the Green's function of Eq. (2.3) in a given background $\sigma(x)$, $\pi(x)$. By dimensional analysis, we see that the quantities a , b , c , and d are dimensionless.

The diagonal elements $a(x,y)$ and $d(x,y)$ in Eq. (2.4) may be expressed in term of the off-diagonal elements as

$$\begin{aligned} a(x,y) &= \frac{i[\partial_x - \sigma(x)]c(x,y)}{\omega - \pi(x)}, \\ d(x,y) &= \frac{i[\partial_x + \sigma(x)]b(x,y)}{\omega + \pi(x)} \end{aligned} \quad (2.5)$$

which in turn satisfy the second-order partial differential equations

$$\begin{aligned} &-\partial_x \left[\frac{\partial_x b(x,y)}{\omega + \pi(x)} \right] + \left[\sigma(x)^2 + \pi(x)^2 - \sigma'(x) - \omega^2 \right. \\ &\quad \left. + \frac{\sigma(x)\pi'(x)}{\omega + \pi(x)} \right] \frac{b(x,y)}{\omega + \pi(x)} = \delta(x-y), \\ &-\partial_x \left[\frac{\partial_x c(x,y)}{\omega - \pi(x)} \right] + \left[\sigma(x)^2 + \pi(x)^2 + \sigma'(x) - \omega^2 \right. \\ &\quad \left. + \frac{\sigma(x)\pi'(x)}{\omega - \pi(x)} \right] \frac{c(x,y)}{\omega - \pi(x)} = -\delta(x-y). \end{aligned} \quad (2.6)$$

Thus, $b(x,y)$ and $-c(x,y)$ are simply the Green's functions of the corresponding second order Sturm-Liouville operators in Eq. (2.6),

$$\begin{aligned} b(x,y) &= \frac{\theta(x-y)b_2(x)b_1(y) + \theta(y-x)b_2(y)b_1(x)}{W_b}, \\ c(x,y) &= -\frac{\theta(x-y)c_2(x)c_1(y) + \theta(y-x)c_2(y)c_1(x)}{W_c}. \end{aligned} \quad (2.7)$$

Here $b_1(x)$ and $b_2(x)$ are the Jost functions of the first equation in Eq. (2.6) and

$$W_b = \frac{b_2(x)b_1'(x) - b_1(x)b_2'(x)}{\omega + \pi(x)} \quad (2.8)$$

is their Wronskian. The latter is independent of x , since b_1 and b_2 share a common value of the spectral parameter ω^2 . Similarly, c_1 , c_2 are the Jost functions of the second equation in Eq. (2.6) and W_c is their Wronskian. We leave the precise definition of these Jost functions in terms of their spatial asymptotic behavior to Appendix A, where we also derive the spatial asymptotic behavior of the static Dirac operator Green's function. Substituting Eq. (2.7) into Eq. (2.5) we obtain the appropriate expressions for $a(x,y)$ and $d(x,y)$, which we do not write explicitly.⁵

We define the diagonal resolvent $\langle x | iD^{-1} | x \rangle$ symmetrically as

⁵It is useful however to note, that despite the ∂_x operation in Eq. (2.5), neither $a(x,y)$ nor $d(x,y)$ contain pieces proportional to $\delta(x-y)$. Such pieces cancel one another due to the symmetry of Eq. (2.7) under $x \leftrightarrow y$.

$$\begin{aligned} \langle x | -iD^{-1} | x \rangle &\equiv \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} \\ &= \frac{1}{2} \lim_{\epsilon \rightarrow 0^+} \begin{pmatrix} a(x,y) + a(y,x) & b(x,y) + b(y,x) \\ c(x,y) + c(y,x) & d(x,y) + d(y,x) \end{pmatrix}_{y=x+\epsilon}. \end{aligned} \tag{2.9}$$

Here $A(x)$ through $D(x)$ stand for the entries of the diagonal resolvent, which following Eqs. (2.5) and (2.7) have the compact representation⁶

$$\begin{aligned} B(x) &= \frac{b_1(x)b_2(x)}{W_b}, \quad D(x) = \frac{i}{2} \frac{[\partial_x + 2\sigma(x)]B(x)}{\omega + \pi(x)}, \\ C(x) &= -\frac{c_1(x)c_2(x)}{W_c}, \quad A(x) = \frac{i}{2} \frac{[\partial_x - 2\sigma(x)]C(x)}{\omega - \pi(x)}. \end{aligned} \tag{2.10}$$

A simplifying observation is that the two linear operators on the left-hand side of the equations (2.6) transform one into the other under a simultaneous sign flip⁷ of $\sigma(x)$ and $\pi(x)$. Therefore $c(\sigma, \pi) = -b(-\sigma, -\pi)$, and, in particular,

$$C(\sigma, \pi) = -B(-\sigma, -\pi), \tag{2.11}$$

and thus all four entries of the diagonal resolvent (2.9) may be expressed in terms of $B(x)$.

The spatial asymptotic behavior of Eq. (2.9) is derived in Appendix A and given by Eq. (A6). A more compact form of that result is

$$\begin{aligned} \langle x | -iD^{-1} | x \rangle &\sim_{x \rightarrow \pm\infty} \frac{1 + R(k)e^{2ik|x|}}{2k} [i\gamma_5 \pi(x) - \sigma(x) - \gamma^0 \omega] \\ &\quad + \frac{R(k)e^{2ik|x|}}{2} \gamma^1 \operatorname{sgn} x, \end{aligned} \tag{2.12}$$

where $k = \sqrt{\omega^2 - m^2}$ and $R(k)$ is the reflection coefficient of the first equation in Eq. (2.6).

Note that for $\omega^2 > m^2$, i.e., in the continuum part of the spectrum of Eq. (2.3), the piece of the resolvent (2.12) that is proportional to $R(k)$ oscillates persistently as a function of x . This observation has a far reaching result that we now derive. Consider the expectation value of fermionic vector current operator j^μ in the static $\sigma(x)$, $\pi(x)$ background⁸

$$\langle \sigma(x), \pi(x) | j^\mu | \sigma(x), \pi(x) \rangle$$

⁶ $A, B, C,$ and D are obviously functions of ω as well. For notational simplicity we suppress their explicit ω dependence.

⁷This is merely a reflection of the fact that coupling the fermions to $\pi\gamma_5$ does not respect charge conjugation invariance.

⁸In the following it is enough to discuss only the vector current, because the axial current $j_5^\mu = \epsilon^{\mu\nu} j_\nu$.

$$= N \int \frac{d\omega}{2\pi} \operatorname{tr} \left[\gamma^\mu \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} \right]. \tag{2.13}$$

Therefore, we find from Eq. (2.12) that the asymptotic behavior of the current matrix elements is

$$\langle \sigma(x), \pi(x) | j^0 | \sigma(x), \pi(x) \rangle \xrightarrow{x \rightarrow \pm\infty} 0$$

and

$$\langle \sigma(x), \pi(x) | j^1 | \sigma(x), \pi(x) \rangle \sim_{x \rightarrow \pm\infty} N \int \frac{d\omega}{2\pi} R(k) e^{2ik|x|} \operatorname{sgn} x, \tag{2.14}$$

where we used the fact that $\int (d\omega/2\pi)(\omega/k)f(k) = 0$ because $k(\omega)$ is an even function of ω .

Thus, an arbitrary static background $\sigma(x)$, $\pi(x)$ induces space-dependent fermion currents that do not decay fast enough as $x \rightarrow \pm\infty$, unless $R(k) \equiv 0$. Clearly, we cannot have such currents in our static problem and we conclude that as far as the field theory (1.3) is concerned, the fields $\sigma(x)$, $\pi(x)$ must be such that the Sturm-Liouville operators in Eq. (2.6) and therefore the Dirac operator (2.3) are reflectionless.

The absence of reflections emerges here from basic principles of field theory, and not merely as a large- N saddle-point condition, as in [3,17]. Indeed, reflectionlessness of Eq. (2.3) must hold whatever the value of N is. Therefore, the fact that reflectionlessness of Eq. (2.3) appeared in [3,17] as a saddle-point condition in the inverse scattering formalism simply indicates consistency of the large- N approximation in analyzing space-dependent condensations $\sigma(x)$, $\pi(x)$. The absence of reflections also restores asymptotic translational invariance. What we mean by this statement is that if $R(k) \equiv 0$ then Eq. (2.12) is simply the result of inverting Eq. (2.3) in Fourier space with constant asymptotic background (2.2), namely,

$$\langle x | -iD^{-1} | x \rangle = \frac{1}{2\sqrt{m^2 - \omega^2}} \begin{pmatrix} im \cos \theta & \omega + m \sin \theta \\ -\omega + m \sin \theta & im \cos \theta \end{pmatrix}, \tag{2.15}$$

which therefore yields the asymptotic behavior of Eq. (2.9) for properly chosen chiral alignment angles. Note that in the absence of reflections, Eq. (2.12) attains its asymptotic value (2.15) by simply following the asymptotic behavior of $\sigma(x)$ and of $\pi(x)$, which are the exclusive sources of any asymptotic x dependence of the resolvent. This expression (2.15) has cuts in the complex ω plane stemming from scat-

tering states of fermions of mass m . These cuts must obviously persist in A , B , C , and D away from the asymptotic region, and we make use of this fact in the next section.

We used the asymptotic matrix elements (2.13) of the vector current operator in the background of static $\sigma(x)$, $\pi(x)$ to establish the absence of reflections in the static Dirac operator. We can now make use of this result to examine its general dynamical implications on matrix elements of other interesting operators, namely, the scalar $\bar{\psi}\psi$ and pseudoscalar $\bar{\psi}i\gamma_5\psi$ density operators. Their matrix elements in the background of $\sigma(x)$, $\pi(x)$ are

$$\langle \sigma(x), \pi(x) | \bar{\psi}\psi | \sigma(x), \pi(x) \rangle = -N \int \frac{d\omega}{2\pi} \text{tr} \langle x | iD^{-1} | x \rangle$$

and

$$\begin{aligned} & \langle \sigma(x), \pi(x) | \bar{\psi}i\gamma_5\psi | \sigma(x), \pi(x) \rangle \\ &= -N \int \frac{d\omega}{2\pi} \text{tr} [i\gamma_5 \langle x | iD^{-1} | x \rangle]. \end{aligned} \quad (2.16)$$

Therefore, from Eq. (2.12) their asymptotic behavior is simply

$$\begin{aligned} & \langle \sigma(x), \pi(x) | \bar{\psi}\psi | \sigma(x), \pi(x) \rangle \\ & \sim_{x \rightarrow \pm\infty} -N\sigma(x) \int \frac{d\omega}{2\pi} \frac{1+R(k)e^{2ik|x|}}{k} \end{aligned}$$

and

$$\begin{aligned} & \langle \sigma(x), \pi(x) | \bar{\psi}i\gamma_5\psi | \sigma(x), \pi(x) \rangle \\ & \sim_{x \rightarrow \pm\infty} -N\pi(x) \int \frac{d\omega}{2\pi} \frac{1+R(k)e^{2ik|x|}}{k}. \end{aligned} \quad (2.17)$$

Clearly, in the absence of reflections, the asymptotic x dependence of these matrix elements follows the profiles of $\sigma(x)$ and $\pi(x)$, respectively. Otherwise, if $R(k) \neq 0$, these matrix elements will have further powerlike decay in x superimposed on these profiles, which is not related directly to the typical length scales appearing in $\sigma(x)$ and in $\pi(x)$. We close this section by investigating implications of Eq. (2.17) for extremal background configurations. For such configurations the matrix element of the scalar density is equal to $-\sigma(x)/g^2$ and that of the pseudoscalar density is equal to $-\pi(x)/g^2$. Such background fields must obviously correspond to a reflectionless Dirac operator, but let us for the

moment entertain ourselves with the assumption that $R(k)$ in Eq. (2.17) is arbitrary and see how the absence of reflections appears as a saddle-point condition. Thus, for extremal configurations, as $x \rightarrow \pm\infty$, $\sigma(x)$ cancels off both sides of the first equation in Eq. (2.17) and $\pi(x)$ cancels off both sides of the other equation. This leaves us with a common dispersion integral:

$$\frac{1}{Ng^2} = \int \frac{d\omega}{2\pi} \frac{1+R(k)e^{2ik|x|}}{k}.$$

It turns out [see Eq. (5.3) below] that the integral over the first x -independent term on the right-hand side cancels precisely the constant term on the left-hand side. This is simply a reformulation of Eq. (1.8) in Minkowski space. Therefore, the remaining x -dependent integral must vanish for any (large) $|x|$. It follows then, that $R(k)$ must vanish. Thus absence of reflections appears here as a saddle-point requirement, in a rather simple elegant manner, without ever invoking the inverse scattering transform. The whole purpose of this section is to prove that one cannot consider static reflectionful backgrounds to begin with, and thus the emergence of absence of reflections as a saddle-point condition is simply a successful consistency check for the validity of the large- N approximation applied to space-dependent condensates.

III. THE DIAGONAL RESOLVENT FOR A FIXED NUMBER OF BOUND STATES

The requirement that the static Dirac operator (2.3) be reflectionless is by itself quite restrictive, since most $\sigma(x)$, $\pi(x)$ configurations will not lead to a reflectionless static Dirac operator. Construction of explicit expressions for the resolvent in terms of $\sigma(x)$, $\pi(x)$ and their derivatives is a formidable task even under such severe restrictions on these fields. We now show how to accomplish such a construction at the price of posing further restrictions on $\sigma(x)$ and $\pi(x)$ in function space. However, even under these further restrictions the results we obtain are still quite interesting from a physical point of view.

In the following we concentrate on the $B(x)$ component of Eq. (2.9). The other entries in Eq. (2.9) may be deduced from $B(x)$ through Eqs. (2.10) and (2.11).

Our starting point here is the observation that one can derive from the representation of $B(x)$ in Eq. (2.10) a functional identity in the form of a differential equation relating $B(x)$ to $\sigma(x)$ and $\pi(x)$ without ever knowing the explicit form of the Jost functions $b_1(x)$ and $b_2(x)$. We leave the details of derivation to Appendix B, where we show that the identity mentioned above is

$$\begin{aligned} & \partial_x \left\{ \frac{1}{\omega + \pi(x)} \partial_x \left[\frac{\partial_x B(x)}{\partial + \pi(x)} \right] \right\} - \frac{4}{\omega + \pi(x)} \left\{ \partial_x \left[\frac{B(x)}{\omega + \pi(x)} \right] \right\} \left[\sigma(x)^2 + \pi(x)^2 - \sigma'(x) - \omega^2 + \frac{\sigma(x)\pi'(x)}{\omega + \pi(x)} \right] \\ & - \frac{2B(x)}{[\omega + \pi(x)]^2} \partial_x \left[\sigma(x)^2 + \pi(x)^2 - \sigma'(x) + \frac{\sigma(x)\pi'(x)}{\omega + \pi(x)} \right] \equiv 0 \end{aligned} \quad (3.1)$$

with a similar expression for $C(x)$ in which $\sigma \rightarrow -\sigma$, $\pi \rightarrow -\pi$ that we do not write down explicitly.

Here we denote derivatives with respect to x either by primes or by partial derivatives. This equation is a linear form of what is referred to in the mathematical literature as the ‘‘Gel’fand–Dikii’’ identity [6]. This identity merely reflects the fact that $B(x)$ is the diagonal resolvent of the Sturm-Liouville operator discussed above and sets no restrictions on $\sigma(x)$ and $\pi(x)$.

If we were able to solve Eq. (3.1) for $B(x)$ in a closed form for any static configuration of $\sigma(x), \pi(x)$, we would then be able to express $\langle x | iD^{-1} | x \rangle$ in terms of the latter fields and their derivatives, and therefore to integrate Eq. (1.7) back to find an expression for the effective action (1.6) explicitly in terms of $\sigma(x)$ and $\pi(x)$. Invoking at that point Lorentz invariance of Eq. (1.6) we would then actually be able to write down the full effective action for spacetime-dependent σ and π . Note, moreover, that in principle such a procedure would yield an exact expression for the effective action, regardless of what N is.

Unfortunately, deriving such an expression for $B(x)$ is a difficult task, and thus we set ourselves a simpler goal in this paper, by determining the desired expression for $B(x)$ with $\sigma(x), \pi(x)$ restricted to a specific sectors in the space of all possible static configurations. To specify these sectors consider the Dirac equation associated with Eq. (2.1), $D\psi=0$. For a given configuration of $\sigma(x), \pi(x)$ (such that D is reflectionless), this equation has n bound states at energies $\omega_1, \dots, \omega_n$ as well as scattering states. A given sector is then defined by specifying the number of bound states the Dirac equation has.⁹

As we saw above, $B(x)$ must have a cut in the ω plane with branch points at $\omega = \pm m$. If in addition to scattering states $\sigma(x), \pi(x)$ support n bound states at energies $\omega_1, \dots, \omega_n$ (which must all lie in the real interval $-m < \omega < m$) then¹⁰ the corresponding B must contain a simple pole for each of these bound states. Therefore, $B(x)$ must contain the purely ω -dependent factor

$$\frac{1}{\sqrt{m^2 - \omega^2} \prod_{k=1}^n (\omega - \omega_k)} \tag{3.2}$$

of mass dimension $-n-1$. Any other singularity $B(x)$ may have in the complex ω plane cannot be directly related to the spectrum of the Dirac operator, and therefore must involve x dependence as well. Based on our discussion in Appendix A, the only possible combination that mixes these variables is

⁹The effect of scattering states on $B(x)$ is rigidly fixed by spatial asymptotics, as Eq. (2.15) indicates, so only bound states are used to specify such a sector.

¹⁰The Gross-Neveu model [5,3,7] is a theory of Majorana (real) fermions. Therefore, its spectrum is invariant under charge conjugation, i.e., it is symmetric under $\omega \rightarrow -\omega$. Thus, in that case the bound states are paired symmetrically around $\omega=0$ and $B(x)$ is really a function of ω^2 . The chiral NJL model on the other hand is a theory of Dirac (complex) fermions, charge conjugation symmetry of the spectrum is broken by the π field and bound states are not paired.

$\exp(i\sqrt{\omega^2 - m^2}x)$. But such a combination is ruled out as we elaborated in the previous section, by the requirement that the Dirac operator be reflectionless. The factor (3.2) then exhausts all allowed singularities of $B(x)$ in the complex ω plane. Recall further that $B(x)$ is a dimensionless quantity, and thus the negative dimension of the ω -dependent factor (3.2) must be balanced by a polynomial of degree $n+1$ in ω (with x -dependent coefficients) of mass dimension $n+1$: namely,¹¹

$$B(x, \omega) = \frac{B_{n+1}(x)\omega^{n+1} + \dots + B_1(x)\omega + B_0(x)}{\sqrt{m^2 - \omega^2} \prod_{k=1}^n (\omega - \omega_k)}. \tag{3.3}$$

The mass dimension of $B_k(x) (k=0, \dots, n+1)$ is $n+1-k$.

The main point here is that simple dimensional analysis in conjunction with the prescribed analytic properties of $B(x)$ fix its ω dependence completely, up to $n+1$ unknown bound-state energies, and $n+2$ unknown functions of $\sigma(x), \pi(x)$ and their derivatives. These functions are by no means arbitrary. They have to be such that Eq. (3.3) and the resulting expression for $C(x)$ are indeed the resolvents of the appropriate Sturm-Liouville operators. These expressions for $B(x)$ and $C(x)$ must be therefore subjected to the Gel’fand–Dikii identity (3.1) and the corresponding identity for $C(x)$.

Substituting $B(x)$ into Eq. (3.1) we obtain an equation of the form

$$Q_{n+5}^{(B)}(\omega, x) / [\omega + \pi(x)]^4 \equiv 0, \tag{3.4}$$

where $Q_{n+5}^{(B)}(\omega, x)$ is a polynomial of degree $n+5$ in ω with x -dependent coefficients that are linear combinations of the functions $B_k(x)$ and their first three derivatives. Note that because of the linearity and homogeneity of Eq. (3.1), the purely ω -dependent denominator of Eq. (3.3) with its explicit dependence on the bound-state energies drops out from Eq. (3.4). This is actually the main advantage¹² of working with the linear form of the Gel’fand–Dikii identity rather than with its nonlinear form (B10) [7].

Setting to zero each of the x -dependent coefficients in $Q_{n+5}^{(B)}$ we obtain an overdetermined system of $n+6$ linear differential equations in the $n+2$ functions $B_k(x)$. Using $n+2$ of the equations we fix all the functions $B_k(x)$ in terms of $\sigma(x), \pi(x)$ and their derivatives, up to $n+2$ integration constants b_k . These integration constants are completely determined once we enforce on the resulting expression for

¹¹One may argue that Eq. (3.3) should be further multiplied by a dimensionless bounded function $f(\omega/m)$. However, such a function must be entire, otherwise it will change the prescribed singularity properties of $B(x)$, but the only bounded entire functions are constant.

¹²The nonlinear version of the Gel’fand–Dikii identity (B8) [or Eq. (B10)] contains further information about the normalization of $B(x)$, but the latter may be readily determined from the asymptotic behavior (2.15) of $B(x)$.

$B(x)$ the asymptotic behavior (2.2) and (2.15). The integration constants b_k turn out to be polynomials in m^2 and the bound-state energies ω_k .

At this stage we are left, independently of n , with four nonlinear differential equations in $\sigma(x)$ and $\pi(x)$.¹³

A similar analysis applies for $C(x)$, leading to an equation of the form $Q_{n+5}^{(C)}(\omega, x) \equiv 0$, where following Eq. (2.11) $Q_{n+5}^{(C)}(\omega, \sigma(x), \pi(x)) = -Q_{n+5}^{(B)}(\omega, -\sigma(x), -\pi(x))$. Setting the first $n+2$ coefficients in $Q_{n+5}^{(C)}(\omega, x)$ to zero we verify that $C(x)$ is related to $B(x)$ as in Eq. (2.11), but that the remaining four equations for $\sigma(x)$ and $\pi(x)$ are different from their counterparts associated with $B(x)$ as the explicit relation between $Q_{n+5}^{(C)}$ and $Q_{n+5}^{(B)}$ suggests. We are thus left with an overdetermined set of eight nonlinear differential equations for the two functions $\sigma(x)$ and $\pi(x)$. Observing that $Q_{n+5}^{(C)} \pm Q_{n+5}^{(B)}$ is odd (even) in σ and π , we note that these eight equations are equivalent to breaking each of the four remaining equations associated with $B(x)$ into a part even in σ and π and a part odd in σ and π and setting each of these parts to zero separately.

Mathematical consistency of our analysis requires that the six most complicated equations of the total eight be redundant relative to the remaining two equations, because we have only two unknown functions, $\sigma(x)$ and $\pi(x)$. This requirement must be fulfilled, because otherwise we are compelled to deduce that there can be no $\sigma(x)$ and $\pi(x)$ configurations for which the Dirac equation $D\psi=0$ has precisely n bound states, with $n=0,1,2, \dots$, which is presumably an erroneous conclusion.

Therefore $\sigma(x)$ and $\pi(x)$ are uniquely determined from the two independent equations given the asymptotic boundary conditions (2.2) they satisfy. This leaves only the bound-state energies undetermined, but the latter cannot be determined by the resolvent identity, which does not really care what their values are. These energies are determined by imposing the saddle-point conditions (1.7), i.e., by dynamical aspects of the model under investigation.

In the preceding paragraphs we laid down the mathematical aspects of our analysis. We now add to these a symmetry argument which will simplify our solution of the differential equations for $\sigma(x)$ and $\pi(x)$ a great deal. The two non-redundant coupled differential equations for $\sigma(x)$ and $\pi(x)$ allow us to eliminate one of these functions in terms of the other. We choose to eliminate¹⁴ $\pi(x)$ in terms of $\sigma(x)$:

$$\pi_\alpha(x) = G_\alpha[\sigma_\alpha(x)], \quad (3.5)$$

where α is a global chiral alignment angle. This relation is clearly covariant under axial rotations $\alpha \rightarrow \alpha + \Delta\alpha$, because $\sigma(x)$ and $\pi(x)$ transform as the two components of a vector under $U(1)_A$ as Eq. (1.4) shows. We expect Eq. (3.5) to be a linear relation. Imposing the boundary conditions (2.2) we have

¹³Coefficients of the various terms in these equations are also polynomials in m^2 and the bound-state energies ω_k .

¹⁴We prefer to eliminate $\pi(x)$ in terms of $\sigma(x)$ because the latter never vanishes identically.

$$\pi(x) = -[\sigma(x) - m] \cot \frac{\theta}{2}. \quad (3.6)$$

In this way we reduce the problem into finding the single function $\sigma(x)$. Condition (3.6) is an external supplement to the coupled differential equations for $\sigma(x)$ and $\pi(x)$ stemming from the Gel'fand–Dikii equation. We thus have to make sure that the resulting solution for $\sigma(x)$ and Eq. (3.6) are indeed solutions of these coupled differential equations.

We now provide the details of such calculations in the case of a single bound state, as well as partial results concerning two bound states.

IV. EXTENDED OBJECT PROFILES

A. A single bound state

In this case Eq. (3.3) becomes

$$B(x) = \frac{B_2(x)\omega^2 + B_1(x)\omega + B_0(x)}{(\omega - \omega_1)\sqrt{m^2 - \omega^2}}, \quad (4.1)$$

where the single bound-state energy is ω_1 . Then setting to zero the coefficients of ω^6 through ω^4 in the degree six polynomial (3.4), we find

$$B_2(x) = b_2, \quad B_1(x) = b_2\pi(x) + b_1$$

and

$$B_0(x) = b_1\pi(x) + \frac{b_2}{2} [\sigma^2(x) + \pi^2(x) - \sigma'(x)] + b_0, \quad (4.2)$$

where b_2 , b_1 , and b_0 are integration constants. We then impose the asymptotic boundary conditions

$$B(x) \xrightarrow{x \rightarrow \pm\infty} \frac{1}{2\sqrt{m^2 - \omega^2}} (\omega + m \sin\theta_\pm)$$

to fix the latter,

$$b_2 = \frac{1}{2}, \quad b_1 = -\frac{\omega_1}{2}, \quad b_0 = -\frac{m^2}{4} \quad (4.3)$$

and therefore

$$B(x) = \frac{\omega + \pi(x)}{2\sqrt{m^2 - \omega^2}} + \frac{\sigma^2(x) + \pi^2(x) - \sigma'(x) - m^2}{4(\omega - \omega_1)\sqrt{m^2 - \omega^2}}. \quad (4.4)$$

Relation (2.11) then immediately leads to

$$C(x) = -\frac{\omega - \pi(x)}{2\sqrt{m^2 - \omega^2}} - \frac{\sigma^2(x) + \pi^2(x) + \sigma'(x) - m^2}{4(\omega - \omega_1)\sqrt{m^2 - \omega^2}}. \quad (4.5)$$

Having the coefficients of ω^6 through ω^4 in the degree six polynomial (3.4) set to zero, we are left with a cubic polynomial

$$4\partial_x \left[\left([m^2 - \pi^2(x) - \sigma^2(x)]\pi(x) - \omega_1 \sigma'(x) + \frac{1}{2} \pi''(x) \right) \right. \\ \left. + \{ \omega_1 [\pi^2(x) + \sigma^2(x)] - \sigma(x)\pi'(x) + \sigma'(x)\pi(x) \} \right] \\ \times \omega^3 + \dots \\ \equiv 0, \quad (4.6)$$

where the ellipsis stands for lower powers of ω . The cubic (4.6) has to be set to zero identically, producing eight coupled differential equations in $\sigma(x)$ and $\pi(x)$ as we discussed above. The simplest equation of these is obtained by setting to zero the part of the ω^3 coefficient in Eq. (4.6) that is even in σ and π , namely,

$$\partial_x \{ \omega_1 [\pi^2(x) + \sigma^2(x)] - \sigma(x)\pi'(x) + \sigma'(x)\pi(x) \} = 0$$

which we immediately integrate into

$$\omega_1 [\pi^2(x) + \sigma^2(x) - m^2] = \sigma(x)\pi'(x) - \sigma'(x)\pi(x). \quad (4.7)$$

Here we have used the boundary conditions (2.2) to determine the integration constant. The next simplest equation is obtained by setting to zero the part of the ω^3 coefficient in Eq. (4.6) that is odd in σ and π , and so on.

Following our general discussion we solve the system of coupled equations (3.6) and (4.7), which leads to

$$\frac{d}{dx} \left[\frac{1}{\sigma(x) - m} \right] - \frac{2\omega_1 \tan(\theta/2)}{\sigma(x) - m} = \frac{2\omega_1}{m \sin\theta}. \quad (4.8)$$

Solving Eq. (4.8) we find

$$\sigma(x) = m - \frac{m \sin\theta \tan(\theta/2)}{1 + \exp[2\omega_1 \tan(\theta/2)(x - x_0)]}, \\ \pi(x) = \frac{m \sin\theta}{1 + \exp[2\omega_1 \tan(\theta/2)(x - x_0)]}, \quad (4.9)$$

where we have chosen the integration constant (parametrized by x_0) such that $\sigma(x)$ and $\pi(x)$ would be free of poles. Note that the boundary conditions at $x \rightarrow +\infty$ require

$$\omega_1 \tan \frac{\theta}{2} < 0. \quad (4.10)$$

Substituting the expressions (4.9) into Eq. (4.4) one finds that the resulting $B(x)$ is indeed a solution of the corresponding Gel'fand–Dikii equation (3.1), verifying the consistency of our solution. Our results (4.9) for $\sigma(x)$ and $\pi(x)$ agree with those of [17]. They have the profile of an extended object, a lump or a chiral “bag,” of size of the order $\cot(\theta/2)/\omega_1$ centered around an arbitrary point x_0 . Note that the profiles in Eq. (4.9) satisfy

$$\rho^2(x) = \sigma^2(x) + \pi^2(x) \\ = m^2 - m^2 \sin^2(\theta/2) \operatorname{sech}^2 \left[\omega_1 \tan \frac{\theta}{2} (x - x_0) \right]. \quad (4.11)$$

Thus, as expected by construction, this configuration interpolates between two different vacua at $x = \pm\infty$. As x increases from $-\infty$, the vacuum configuration becomes distorted. The distortion reaches its maximum at the location of the “bag,” where $m^2 - \rho^2(x_0) = m^2 \sin^2(\theta/2)$ and then relaxes back into the other vacuum state at $x = \infty$. The arbitrariness of x_0 is, of course, a manifestation of translational invariance.

B. Two bound states

In this case Eq. (3.3) becomes

$$B(x) = \frac{B_3(x)\omega^3 + B_2(x)\omega^2 + B_1(x)\omega + B_0(x)}{(\omega - \omega_1)(\omega - \omega_2)\sqrt{m^2 - \omega^2}} \quad (4.12)$$

where the bound-state energies are ω_1 and ω_2 . [Obviously, $B(x)$ in this subsection should not be confused with its counterpart in the previous subsection.] In this case the polynomial (3.4) is of degree seven in ω . Following the procedure outlined in Sec. III we find after imposing the boundary conditions (2.2) that

$$B(x) = \frac{\omega + \pi(x)}{2\sqrt{m^2 - \omega^2}} + \frac{[\sigma^2(x) + \pi^2(x) - \sigma'(x) - m^2][\pi(x) + \omega - \omega_1 - \omega_2]}{4(\omega - \omega_1)(\omega - \omega_2)\sqrt{m^2 - \omega^2}} - \frac{\pi''(x) - 2\sigma(x)\pi'(x)}{8(\omega - \omega_1)(\omega - \omega_2)\sqrt{m^2 - \omega^2}}. \quad (4.13)$$

Then, from Eq. (2.11) we find that

$$C(x) = -\frac{\omega - \pi(x)}{2\sqrt{m^2 - \omega^2}} - \frac{[\sigma^2(x) + \pi^2(x) + \sigma'(x) - m^2][-\pi(x) + \omega - \omega_1 - \omega_2]}{4(\omega - \omega_1)(\omega - \omega_2)\sqrt{m^2 - \omega^2}} - \frac{\pi''(x) + 2\sigma(x)\pi'(x)}{8(\omega - \omega_1)(\omega - \omega_2)\sqrt{m^2 - \omega^2}}. \quad (4.14)$$

Note that if we set $\omega_1 + \omega_2 = 0$ and $\pi(x) = 0$ the resolvents (4.13) and (4.14), and therefore the whole spectrum, become invariant under $\omega \rightarrow -\omega$, and we obtain the equation appropriate to the Gross-Neveu model.

Setting to zero the coefficients of ω^7 through ω^4 in the degree seven polynomial (3.4), we are left with a cubic poly-

nomial in ω which we do not write down explicitly. This polynomial must vanish identically, producing eight coupled differential equations in $\sigma(x)$ and $\pi(x)$ as we discussed above. The simplest equation of these is obtained by setting to zero the part of the ω^3 coefficient in that polynomial which unlike the previous case, is now odd in σ and π ,

$$2\partial_x\{4(\omega_1 + \omega_2)[\pi^2(x) + \sigma^2(x) - m^2]\pi(x) + 6[\pi^2(x) + \sigma^2(x)]\sigma'(x) + 2(2\omega_1\omega_2 - m^2)\sigma'(x) - 2(\omega_1 + \omega_2)\pi''(x) - \sigma'''(x)\} = 0. \quad (4.15)$$

As in the previous case, this is a complete derivative which we readily integrate into

$$4(\omega_1 + \omega_2)[\pi^2(x) + \sigma^2(x) - m^2]\pi(x) + 6[\pi^2(x) + \sigma^2(x)]\sigma'(x) + 2(2\omega_1\omega_2 - m^2)\sigma'(x) - 2(\omega_1 + \omega_2)\pi''(x) - \sigma'''(x) = 0. \quad (4.16)$$

Here we have used the boundary conditions (2.2) to determine the integration constant. Note that Eq. (4.16) is of third order in derivatives and cubic, whereas its single bound-state counterpart (4.7) is only first order in derivatives and quadratic.

The next simplest equation is obtained by setting to zero the part of the ω^3 coefficient in the cubic polynomial that is even in σ and π :

$$2\partial_x\{2(m^2 - 2\omega_1\omega_2)[\pi^2(x) + \sigma^2(x)] - 3[\pi^2(x) + \sigma^2(x)]^2 + 4(\omega_1 + \omega_2)[\sigma(x)\pi'(x) - \sigma'(x)\pi(x)]\} + 4[\pi(x)\pi'''(x) + \sigma(x)\sigma'''(x)] = 0, \quad (4.17)$$

and so on.

Following our general discussion we have to solve the system of coupled equations (3.6) and (4.16) which is equivalent to

$$2\lambda(\omega_1 + \omega_2)[4my^2 + 2(1 + \lambda^2)y^3 - y''] + \partial_x\{4(m^2 + \omega_1\omega_2)y + 6my^2 + 2(\lambda^2 + 1)y^3 - y''\} = 0, \quad (4.18)$$

where $\lambda = -\cot(\theta/2)$ and $y(x) = \sigma(x) - m$. We have not succeeded in solving this nonlinear ordinary differential equation in closed form.

V. THE SADDLE-POINT CONDITIONS

Derivation of the explicit expressions of $\sigma(x)$ and $\pi(x)$ does not involve the saddle-point equations (1.7). Rather, it tells us independently of the large- N approximation that $\sigma(x)$ and $\pi(x)$ must have the form given in Eq. (4.9) in order for the associated Dirac operator to be reflectionless and to have a single bound state at a prescribed energy ω_1 in addition to scattering states. Thus, for the solution (4.9) we have yet to determine the values of ω_1 and θ allowed by the saddle-point condition. More generally, our discussion in Sec. III will lead us to the $\sigma(x)$ and $\pi(x)$ configurations which correspond to reflectionless Dirac operators with a prescribed number of bound states at some prescribed energies $\omega_1, \omega_2, \dots$ in addition to scattering states. As emphasized earlier, this result is independent of the large- N limit.

The allowed values of $\theta, \omega_1, \omega_2, \dots$ must then be determined by the saddle-point condition (1.7). It is this dynamical feature that we can analyze only in the large- N limit.

For static background fields the general saddle-point condition (1.7) assumes the simpler form

$$\sigma(x) + Ng^2 \int \frac{d\omega}{2\pi} [A(x) + D(x)] = 0,$$

$$\pi(x) + iNg^2 \int \frac{d\omega}{2\pi} [B(x) + C(x)] = 0. \quad (5.1)$$

In Sec. V A we impose this condition on the explicit single bound-state background we found in the previous section and calculate the mass of such ‘‘bags.’’ The two-bound-state case is discussed in Sec. V B.

A. A single bound state

Substituting Eqs. (4.4) and (4.5) into the saddle-point equations (5.1) we obtain¹⁵

¹⁵In the following formula we omit explicit x dependence of the fields. The number π also appears in the formula, but only in the combination $d\omega/2\pi$. Therefore, there is no longer danger of confusing the field $\pi(x)$ and the number π .

$$\frac{\sigma}{Ng^2 + i} \int_{-\Lambda}^{\Lambda} \frac{d\omega}{2\pi} \frac{1}{4\sqrt{m^2 - \omega^2}(\omega - \omega_1)(\omega^2 - \pi^2)} \left\{ 4\sigma\omega^3 + 2(\pi' - 2\omega_1\sigma)\omega^2 + 2 \left[\sigma(\sigma^2 - \pi^2 - m^2) - \omega_1\pi' - \frac{1}{2}\sigma'' \right] \omega - 2\pi^2(\pi' - 2\omega_1\sigma) \right\} = 0,$$

$$\frac{\pi}{Ng^2 + i} \int_{-\Lambda}^{\Lambda} \frac{d\omega}{2\pi} \frac{2\pi\omega - 2\omega_1\pi - \sigma'}{2\sqrt{m^2 - \omega^2}(\omega - \omega_1)} = 0. \quad (5.2)$$

These equations are dispersion relations among the various x -dependent parts and ω_1 . Clearly, both dispersion integrals in Eq. (5.2) are logarithmically divergent in Λ , but subtracting each of the integrals once we can get rid of these divergences. The required subtractions are actually already built in Eq. (5.2). In order to see this consider the (bare) gap equation (1.8) in Minkowski space. This equation is equivalent to the (logarithmically divergent) dispersion relation¹⁶

$$\frac{i}{Ng^2} = \int_{-\Lambda}^{\Lambda} \frac{d\omega}{2\pi} \frac{1}{\sqrt{m^2 - \omega^2} + \epsilon}. \quad (5.3)$$

If we now replace each of the $1/Ng^2$ coefficients in Eq. (5.2) by the integral on the right-hand side of Eq. (5.3) [7], the divergent parts of each pair of integrals cancel and Eq. (5.2) become

$$\int_{\mathcal{C}} \frac{d\omega}{2\pi} \frac{\pi' + [\omega/(\omega^2 - \pi^2)]F(\sigma, \pi)}{\sqrt{m^2 - \omega^2}(\omega - \omega_1)} = 0,$$

$$\int_{\mathcal{C}} \frac{d\omega}{2\pi} \frac{\sigma'}{\sqrt{m^2 - \omega^2}(\omega - \omega_1)} = 0, \quad (5.4)$$

where

$$F(\sigma, \pi) = \sigma(\sigma^2 + \pi^2 - m^2) - \omega_1\pi' - \frac{1}{2}\sigma'' \quad (5.5)$$

and \mathcal{C} is the contour in the complex ω plane depicted in Fig. 1.

Expression (5.5) is the residue of the x -dependent poles at $\omega = \pm \pi(x)$ in the first equation in Eq. (5.4). The quantization condition (5.4) on ω_1 cannot be x dependent. Therefore Eq. (5.5) must vanish as a consistency requirement. Substituting¹⁷ Eq. (4.9) in Eq. (5.5) we find that

$$F(x) = \frac{m \tan^2(\theta/2)}{2} \operatorname{sech}^3 \left[\omega_1 x \tan \frac{\theta}{2} \right] \left[\cos \theta e^{-\omega_1 x \tan(\theta/2)} + e^{\omega_1 x \tan(\theta/2)} \right] \left(\omega_1^2 - m^2 \cos^2 \frac{\theta}{2} \right).$$

Thus, $F(\sigma, \pi)$ vanishes for the configurations (4.9) provided

$$\omega_1^2 = m^2 \cos^2 \left(\frac{\theta}{2} \right). \quad (5.6)$$

which sets an interesting relation between the bound-state energy and the chiral alignment angle of the vacuum at $x \rightarrow +\infty$. This relation actually leaves θ the only free parameter in the problem with respect to which we have to extremize the action. Condition (4.10) then picks out one branch of Eq. (5.6).

We still have to determine ω_1 and θ separately. Now the saddle-point condition simply boils down to the single equation

$$I(\omega_1) = \int_{\mathcal{C}} \frac{d\omega}{2\pi} \frac{1}{\sqrt{m^2 - \omega^2}(\omega - \omega_1)} = 0. \quad (5.7)$$

The contour integral in Eq. (5.7) is most conveniently calculated by deforming the contour \mathcal{C} into the contour \mathcal{C}' shown in Fig. 2. The ‘‘hairpin’’ wing of \mathcal{C}' picks up the contribution of the filled Fermi sea, and the little circle around the simple pole at $\omega = \omega_1$ is the contribution of fermions populating the bound state of the ‘‘bag.’’

Assuming that the ‘‘bag’’ traps n fermions in that state, and recalling that each state in the Fermi sea continuum accommodates N fermions, we see that Eq. (5.7) becomes

$$I(\omega_1) = \frac{iN}{\sqrt{m^2 - \omega_1^2}} \left[\frac{2}{\pi} \arctan \sqrt{\frac{m + \omega_1}{m - \omega_1}} + \frac{n}{N} - 1 \right] = 0. \quad (5.8)$$

The solution of this equation yields the quantization condition

$$\omega_1 = m \cos \left(\frac{n\pi}{N} \right) \quad (5.9)$$

in agreement with [17].

It follows from Eqs. (4.9), (4.10), and (5.6) that

$$\theta = -\frac{2\pi n}{N}, \quad (5.10)$$

that is, the relative chiral rotation of the vacua at $\pm\infty$ is proportional to the number of fermions trapped in the ‘‘bag,’’ as one should expect due to the fact that the fermion number current in a soliton background can be determined in

¹⁶To see this equivalence simply perform (contour) integration over spatial momentum first.

¹⁷Here we have set $x_0 = 0$ for simplicity.

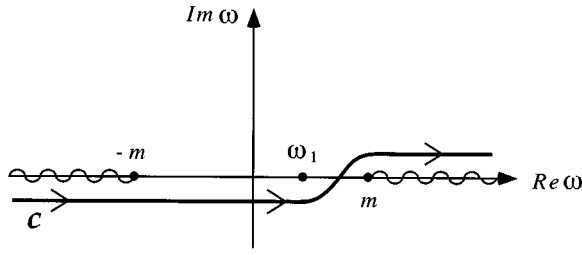


FIG. 1. The contour \mathcal{C} in the complex ω plane in Eq. (5.4). The continuum states appear as the two cuts along the real axis with branch points at $\pm m$ (wiggly lines), and the bound state is the pole at ω_1 .

some cases by topological considerations [25]. Note from Eq. (5.10), that in the large- N limit, θ (and therefore ω_1) take on nontrivial values only when the number of the trapped fermions scales as a finite fraction of N .

As we already mentioned in the Introduction, “bags” formed in the NJL model are not stable because of topology. They are stabilized by releasing binding energy of the fermions trapped in them. To see this more explicitly, we calculate now the mass of the “bag” corresponding to Eqs. (4.9), (5.9), and (5.10). The effective action (1.6) for background fields (4.9) is an ordinary function of the chiral angle θ .¹⁸ Let us denote this action per unit time by $S(\theta)/T$. Then, from Eqs. (2.9) and (5.3) we find

$$\begin{aligned} \frac{1}{T} \frac{\partial S}{\partial \theta} = & \int \frac{dx d\omega}{2\pi} \left\{ \left[\frac{i\sigma(x)}{\sqrt{m^2 - \omega^2}} - (A+D) \right] \frac{\partial \sigma(x)}{\partial \theta} \right. \\ & \left. + i \left[\frac{\pi(x)}{\sqrt{m^2 - \omega^2}} - (B+C) \right] \frac{\partial \pi(x)}{\partial \theta} \right\}. \end{aligned} \quad (5.11)$$

Then as in our analysis of the saddle-point condition (which is simply the condition for $\partial S/\partial \theta = 0$) we use Eqs. (4.4), (4.5), (2.10), and the fact that Eq. (5.5) vanishes to find

$$\frac{1}{T} \frac{\partial S}{\partial \theta} = i \int \frac{dx d\omega}{2\pi} \frac{\partial_x \sigma(x) \partial_\theta \pi(x) - \partial_x \pi(x) \partial_\theta \sigma(x)}{2(\omega - \omega_1) \sqrt{m^2 - \omega^2}}.$$

Then, Eq. (3.6) leads to the factorized expression

$$\begin{aligned} \frac{1}{T} \frac{\partial S}{\partial \theta} = & \frac{i}{4 \sin^2(\theta/2)} \int_{-\infty}^{\infty} dx (\sigma - m) \sigma' \\ & \times \int_{\mathcal{C}'} \frac{d\omega}{2\pi} \frac{1}{(\omega - \omega_1) \sqrt{m^2 - \omega^2}}, \end{aligned} \quad (5.12)$$

where \mathcal{C}' is the contour in Fig. 2. The space integral is immediate and is essentially fixed by the boundary conditions (2.2). The spectral integral is given by the left-hand side of Eq. (5.8), but with a generic ω_1 given by $\omega_1 = m \cos(\theta/2)$. Here we have chosen the particular branch of Eq. (5.6) that contains all the extremal values of ω_1 . Putting everything together we finally arrive at

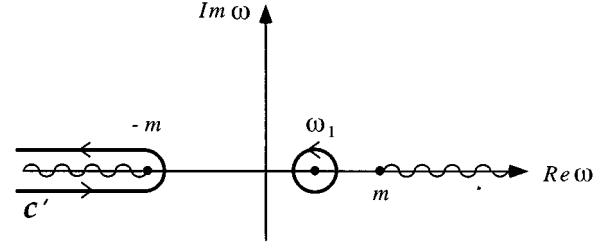


FIG. 2. The deformed integration contour \mathcal{C}' leading to Eq. (5.7).

$$\frac{1}{NT} \frac{\partial S}{\partial \theta} = \frac{m}{2} \left(\frac{n}{N} + \frac{\theta}{2\pi} \right) \sin \frac{\theta}{2}. \quad (5.13)$$

The zeros of Eq. (5.13) are extremal points of $S(\theta)$. The zero of the factor linear in θ in Eq. (5.13) coincides with Eq. (5.10) we deduced from the saddle-point condition (5.8). The other zeros of Eq. (5.13) are $\theta_k = 2\pi k$, $k \in \mathbf{Z}$, and it is clear from Eq. (4.9) that they do not correspond to extended objects at all.¹⁹ We therefore discard them, and concentrate on the extremum given by Eq. (5.10).

Integrating Eq. (5.13) with respect to θ we finally find

$$\frac{-1}{NTm} S(\theta) = \left(\frac{n}{N} + \frac{\theta}{2\pi} \right) \cos \frac{\theta}{2} - \frac{1}{\pi} \sin \frac{\theta}{2}. \quad (5.14)$$

Note that Eq. (5.14) is not manifestly periodic in θ because the Pauli exclusion principle limits θ to be between 0 and 2π .

The mass of a “bag” containing n fermions in a single bound state is given by the energy $E(\theta) = -S(\theta)/T$ evaluated at the appropriate chiral angle (5.10). We thus find that this mass is simply

$$M_n = \frac{Nm}{\pi} \sin \frac{\pi n}{N} \quad (5.15)$$

in accordance with [17,21]. It is easy to check that Eq. (5.15) is a minimum of $E(\theta)$ for $0 < n/N < 1$. These “bags” are stable because

$$\sin \frac{\pi(n_1 + n_2)}{N} < \sin \frac{\pi n_1}{N} + \sin \frac{\pi n_2}{N} \quad (5.16)$$

for $n_1 + n_2$ less than N , such that a “bag” with $n_1 + n_2$ fermions cannot decay into two “bags” each containing a lower number of fermions.

Entrapment of a small number of fermions cannot distort the homogeneous vacuum considerably, so we expect that M_n will be roughly the mass of n free massive fermions for $n \ll N$. As a matter of fact we used this expectation to fix the integration constant in Eq. (5.14). For $n \ll N$ we have $M_n \sim nm [1 - \frac{1}{6}(\pi n/N)^2 + \dots]$, so the binding energy released,

¹⁹Note from Eq. (5.11) that every solution of the saddle-point equations (5.1) yields an extremal $S(\theta)$. The converse, however, is not true. Clearly, only extrema of $S(\theta)$ associated with the saddle-point equations (5.1) are related to masses of stable extended objects.

¹⁸Recall that ω_1 is a function of θ and not a free parameter.

$$B_n \sim \frac{nm}{6} \left(\frac{\pi n}{N} \right)^2 + \dots, \quad (5.17)$$

is indeed very small. However, as the number of fermions trapped in the ‘‘bag’’ approaches N , M_n vanishes and the fermions release practically all their rest mass Nm as binding energy, to achieve maximum stability [18]. In a weakly coupled field theory containing solitons, the mass of these extended objects is a measure of $1/g^2$, the inverse square of the coupling constant. Here we have $1/g^2 = N$. It is amusing to speculate that these maximally stable massless solitons may teach us something about the strong-coupling regime of the NJL model.

Note from Eq. (5.10) that the soliton twists all the way around as the number of fermions approaches N . In this case $\omega_1 \rightarrow -m$, and the pole the resolvent has at $\omega = \omega_1$ pinches the branch point $\omega = -m$ at the edge of the filled Dirac sea. One may wonder whether this enhanced singularity is a mathematical artifact, as the bound state simply tries to plunge into the filled Dirac sea. But this is clearly not the case. Indeed, ω_1 is occupied by N fermions (in a flavor singlet). Their common spinor wave function must still be part of the discrete spectrum of the Dirac operator, because the highest lying state of the sea at $\omega = -m$ is already occupied by a flavor singlet made of N fermions, sharing a continuum spinor wave function, and therefore Pauli’s exclusion principle protects the bound state from ‘‘dissolving’’ into the sea.

B. Two bound states

We concluded Sec. IV B short of an explicit solution of Eq. (4.18), namely, short of an explicit expression for the two bound-state background fields $\sigma(x)$ and $\pi(x)$. In the following we make the eminently reasonable assumption that such a background exists, and pursue our analysis of its saddle-point condition as far as we can without having its explicit form in hand.

As in the previous subsection, we substitute Eqs. (4.13) and (4.14) into the saddle-point equations (5.1). We then make use of Eq. (5.3) to eliminate the ultraviolet logarithmic divergences and to write the saddle-point conditions as

$$\begin{aligned} & \int_{C''} \frac{d\omega}{2\pi} \left[\frac{i\sigma(x)}{\sqrt{m^2 - \omega^2}} - (A + D) \right] \\ &= i \int_{C''} \frac{d\omega}{2\pi} \frac{K + \omega \{L/[\omega^2 - \pi^2(x)]\}}{2\sqrt{m^2 - m^2}(\omega - \omega_1)(\omega - \omega_2)} \end{aligned}$$

and

$$\begin{aligned} & \int_{C''} \frac{d\omega}{2\pi} \left[\frac{\pi(x)}{\sqrt{m^2 - \omega^2}} - (B + C) \right] \\ &= \int_{C''} \frac{d\omega}{2\pi} \frac{M + \omega\sigma'(x)}{2\sqrt{m^2 - m^2}(\omega - \omega_1)(\omega - \omega_2)}, \quad (5.18) \end{aligned}$$

where C'' is a contour similar to the contour C' in Fig. 2 that encircles the additional pole at ω_2 as well, and $K(x)$, $L(x)$, and $M(x)$ are given by

$$K(\sigma, \pi) = -\sigma(\sigma^2 + \pi^2 - m^2) + (\omega_1 + \omega_2)\pi' + \frac{1}{2}\sigma''$$

$$\begin{aligned} L(\sigma, \pi) &= (\omega_1 + \omega_2) \left[\sigma(\sigma^2 + \pi^2 - m^2) - \frac{1}{2}\sigma'' \right] \\ &\quad - \left(\frac{\sigma^2 + \pi^2 - m^2}{2} + \omega_1\omega_2 + \sigma^2 \right) \pi' + \frac{\pi'''}{4} \end{aligned}$$

and

$$M(\sigma, \pi) = -\pi(\sigma^2 + \pi^2 - m^2) - (\omega_1 + \omega_2)\pi' + \frac{1}{2}\pi'' \quad (5.19)$$

Note that $K(\sigma, \pi)$ differs from $-F(\sigma, \pi)$ in Eq. (5.5) only by the additional term $\omega_2\pi'$. The expression $L(\sigma, \pi)$ is the residue of the x -dependent poles at $\omega = \pm\pi(x)$ in the first equation in Eq. (5.18). The quantization conditions (5.18) on ω_1 and ω_2 cannot be x dependent. Therefore $L(\sigma, \pi)$ must vanish as a consistency requirement. As we do not have the explicit expressions of $\sigma(x)$ and $\pi(x)$, we assume from now on that L indeed vanishes. This is the only extra assumption we make. Then, for the case in which the ‘‘bag’’ traps n_1 fermions in ω_1 and n_2 fermions in ω_2 , Eq. (5.18) boils down to the simple conditions

$$I(\omega_1) = I(\omega_2) = 0, \quad (5.20)$$

where $I(\omega)$ is given in Eq. (5.8). Therefore,

$$\omega_1 = m \cos\left(\frac{n_1\pi}{N}\right), \quad \omega_2 = m \cos\left(\frac{n_2\pi}{N}\right) \quad (5.21)$$

which are identical in form to single bound-state energy levels. From the general considerations of [25] we expect that the chiral angle θ will be proportional to the total number of fermions trapped by the ‘‘bag,’’ so Eq. (5.10) must now read

$$\theta = -\frac{2\pi(n_1 + n_2)}{N}. \quad (5.22)$$

The soliton mass is clearly a function of m and of the chiral angle θ . However, if a ‘‘bag’’ with two bound states is distinguishable from a ‘‘bag’’ with a single bound state (which contains the same number of fermions), then this mass must depend on n_1 and n_2 separately, and not only on their sum through θ .

VI. CONCLUSION

We have studied nonperturbatively the emergence of fermion bags in the (1+1)-dimensional Nambu–Jona-Lasinio model (focusing on static bags.) These bags are stable due to dynamics and not because of topological reasons: as they bind fermions they release binding energy. The more fermions the bag bind, the more stable it becomes, which is a salient feature of bag models in any number of dimensions.

In this paper we developed further the method of [7,8], applying it to Eq. (1.1) as an alternative to the inverse scattering investigations in [17]. Our method is based on elementary Sturm-Liouville theory as well as on simple dimensional

analysis. Moreover, at all stages of our analysis we deal with the space-dependent condensates $\sigma(x)$ and $\pi(x)$ directly. In our view, these facts make the method presented here simpler than inverse scattering calculations previously employed in this problem because we do not need to work with the scattering data and the so-called trace identities that relate them to the space-dependent condensates.

It is important to stress that the analysis in Secs. II–IV has nothing to do with the large- N limit and is also independent of dynamics. It is a purely mathematical construction of the diagonal resolvent of the Dirac operator in a given reflectionless static background $\sigma(x)$ and $\pi(x)$ with a given number of bound states.

Also, given the number of fermions trapped in the bag, the total chiral angle $\theta = [\arctan(\pi(x)/\sigma(x))]_{-\infty}^{\infty}$ [see Eq. (3.5)] is fixed by topological considerations [the Goldstone-Wilczek relation, Eq. (5.10)], which holds for any value of N .

Dynamical considerations enter only in Sec. V, where we determine the dependence of the bound-state energies $\omega_1, \omega_2, \dots$ and the mass of the bag on the dynamical fermion mass m and the chiral angle θ by imposing the large- N saddle-point conditions.

It would be interesting to extend the resolvent method presented here to analyze effectively one-dimensional static condensates in quantum field theories in higher spatial dimensions.

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APPENDIX A

In this appendix we provide precise definitions of the Jost functions b_1 through c_2 in terms of their spatial asymptotic behavior and derive the spatial asymptotic behavior of the static Dirac operator Green's function.

We concentrate for the moment on the first equation in Eq. (2.6). The boundary conditions (2.2) lead to the following simple spatial asymptotic behavior:

$$[-\partial_x^2 + m^2 - \omega^2]b(x) = 0$$

of the homogeneous part of that equation. Thus, solutions of that homogeneous equation assume the generic asymptotic form

$$b(x, \omega) \sim \begin{cases} M_+ e^{ikx} + N_+ e^{-ikx}, & x \rightarrow +\infty, \\ M_- e^{ikx} + N_- e^{-ikx}, & x \rightarrow -\infty \end{cases} \quad (\text{A1})$$

where

$$k(\omega) = +\sqrt{\omega^2 - m^2}. \quad (\text{A2})$$

On the real ω axis $k(\omega)$ is real for $|\omega| > m$, which corresponds to scattering states of Eq. (2.3). Bound states of Eq.

(2.3) reside in the domain $|\omega| < m$, where $k(\omega) = +i\sqrt{m^2 - \omega^2} = +i\kappa(\omega)$ is purely imaginary and lies in the upper half plane.

The Jost functions b_1 and b_2 alluded to in Sec. II form a particular pair of linearly independent solution of the homogeneous equation mentioned above, specified by their asymptotic behavior. Let the asymptotic amplitudes of $b_r(x)$ ($r=1,2$) in Eq. (A1) be $M_{r\pm}, N_{r\pm}$. The asymptotic form (A1) of $b_1(x)$ has by definition $M_{1-} = 0$, and that of $b_2(x)$ has $N_{2+} = 0$. One may summarize our definitions of b_1 and b_2 , by saying that b_1 corresponds to a one-dimensional scattering situation where the source is at $+\infty$ emitting waves to the left (the term $N_{1+}e^{-ikx}$) and that b_2 corresponds to a one-dimensional scattering situation where the source is at $-\infty$ emitting waves to the right (the term $M_{2-}e^{ikx}$). Note also that outside the continuum, $b_1(x)$ decays to the left while $b_2(x)$ decays to the right. With these definitions the Wronskian (2.8) becomes

$$\begin{aligned} W_b(+\infty) &= -2ik \frac{M_{2+}N_{1+}}{\omega + \pi(+\infty)} \\ &= -2ik \frac{M_{2-}N_{1-}}{\omega + \pi(-\infty)} \\ &= W_b(-\infty). \end{aligned} \quad (\text{A3})$$

Therefore, it follows from Eq. (2.10) and from Eq. (2.11) that the entries A , B , C , and D in Eq. (2.9) have the asymptotic form

$$A(x) = -\frac{1}{2k} \{[\sigma(x) - ik \operatorname{sgn}x]R(k)e^{2ik|x|} + \sigma(x)\},$$

$$B(x) = \frac{\omega + \pi(x)}{-2ik} [1 + R(k)e^{2ik|x|}],$$

$$C(x) = \frac{\omega - \pi(x)}{2ik} [1 + R(k)e^{2ik|x|}],$$

$$D(x) = -\frac{1}{2k} \{[\sigma(x) + ik \operatorname{sgn}x]R(k)e^{2ik|x|} + \sigma(x)\} \quad (\text{A4})$$

as $x \rightarrow \pm\infty$, where

$$R(k) = \frac{M_{1+}}{N_{1+}} = \frac{N_{2-}}{M_{2-}} \quad (\text{A5})$$

is the reflection coefficient of the Sturm-Liouville operator in the first equation in Eq. (2.6). The diagonal resolvent of the Dirac operator is therefore

$$\langle x | -iD^{-1} | x \rangle \xrightarrow{x \rightarrow \pm\infty} \frac{i}{2k} \begin{pmatrix} i\sigma(x) & \omega + \pi(x) \\ -\omega + \pi(x) & i\sigma(x) \end{pmatrix} + \frac{iR(k)e^{2ik|x|}}{2k} \begin{pmatrix} i\sigma(x) + k \operatorname{sgn} x & \omega + \pi(x) \\ -\omega + \pi(x) & i\sigma(x) - k \operatorname{sgn} x \end{pmatrix}. \quad (\text{A6})$$

APPENDIX B

Consider the Sturm-Liouville problem

$$-[p(x)\psi'(x)]' + [V(x) - E\rho(x)]\psi(x) = 0, \quad -\infty < x < \infty. \quad (\text{B1})$$

We assume that the ‘‘metric’’ $p(x)$ does not vanish anywhere and that the weight function $\rho(x)$ is positive everywhere. E is a complex number, called the spectral parameter.

As in our discussion in the main text and in the previous appendix, let $\psi_1(x)$ be the Jost function which decays as $x \rightarrow -\infty$ for values of E below the continuum threshold. Similarly, let $\psi_2(x)$ be the Jost function which decays as $x \rightarrow +\infty$. Then, the Green’s function of the operator in Eq. (B1) is

$$G(x, y) = \frac{\theta(x-y)\psi_2(x)\psi_1(y) + \theta(y-x)\psi_2(y)\psi_1(x)}{W}, \quad (\text{B2})$$

where

$$W = p(x)[\psi_2(x)\psi_1'(x) - \psi_1(x)\psi_2'(x)] \quad (\text{B3})$$

is the (x -independent) Wronskian of these two functions. Note that Eq. (B2) decays (at a rate dictated by the Jost functions) as either one of its argument diverges in absolute value, holding the other one finite, as long as E does not hit one of the eigenvalues of the Sturm-Liouville operator.

As in the main text the diagonal resolvent $R(x) = G(x, x)$ is defined as

$$R(x) = \frac{1}{2} \lim_{\epsilon \rightarrow 0} [G(x, x + \epsilon) + G(x + \epsilon, x)] = \frac{\psi_1(x)\psi_2(x)}{W}. \quad (\text{B4})$$

We then use Eqs. (B3) and (B4) to show that

$$\frac{\psi_1'}{\psi_1} = \frac{pR' + 1}{2pR}, \quad \frac{\psi_2'}{\psi_2} = \frac{pR' - 1}{2pR}. \quad (\text{B5})$$

Finally, using Eqs. (B1) and (B5) we find

$$(pR')' = 2(V - E\rho)R + \frac{(pR')^2 - 1}{2pR}. \quad (\text{B6})$$

and

$$[p(pR')']' = [2p(V - E\rho)R]' + 2p(V - E\rho)R'. \quad (\text{B7})$$

Note that the nonlinearity of Eq. (B6) in R has disappeared after one more differentiation with respect to x .

Multiplying Eq. (B6) through by $2pR$ we find

$$-2pR(pR')' + (pR')^2 + 4pR^2(V - \rho E) = 1 \quad (\text{B8})$$

which is the Gel’fand–Dikii equation [6]. Equation (B7) is the linear form of the Gel’fand–Dikii equation we use in the text [Eq. (3.1)].

The quantities corresponding to the discussion of the Dirac operator in the text are

$$p(x) = \rho(x) = \frac{1}{\omega + \pi(x)}, \quad E = \omega^2$$

and

$$V(x) = \frac{1}{\omega + \pi(x)} \left[\sigma(x)^2 + \pi(x)^2 - \sigma'(x) + \frac{\sigma(x)\pi'(x)}{\omega + \pi(x)} \right]. \quad (\text{B9})$$

The Gel’fand–Dikii equation (B8) then reads

$$-2 \frac{B(x)}{\omega + \pi(x)} \partial_x \left[\frac{\partial_x B(x)}{\omega + \pi(x)} \right] + \left[\frac{\partial_x B(x)}{\omega + \pi(x)} \right]^2 + \left[\frac{2B(x)}{\omega + \pi(x)} \right]^2 \left[\sigma(x)^2 + \pi(x)^2 - \sigma'(x) - \omega^2 + \frac{\sigma(x)\pi'(x)}{\omega + \pi(x)} \right] = 1. \quad (\text{B10})$$

Strictly speaking, Sturm-Liouville theory requires that $p(x) = \rho(x) = 1/[\omega + \pi(x)] > 0$. Our solution for $\pi(x)$ turns out to be bounded, so all formulas are valid *a posteriori* for large positive ω . Such a restriction on ω , though mathematically required, is unphysical.

Note, however, that because of the relation (2.11), we may view $C(x)$ as a continuation of $B(x)$ to large negative ω .

An important application of the Gel’fand–Dikii identities (B7) and (B8) is that they generate an asymptotic expansion [6] of R in negative odd powers of \sqrt{E} . The explicit ω (and therefore E) dependence of our specific $p(x)$, $\rho(x)$, and $V(x)$ complicates this expansion.

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