Induced Chern-Simons-type terms in general metric nonlinear σ models

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We compute explicitly Chern-Simons-type terms induced by fermions coupled to external non-Abelian gauge fields in metric nonlinear σ models in three dimensions. We investigate the supersymmetric models defined on general Riemannian and Kähler manifolds. The diagrammatic calculation is performed by means of the interaction picture Dyson-Wick perturbation theory. [S0556-2821(97)01516-6]

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I. INTRODUCTION

Since the seminal work of Deser, Jackiw, and Templeton [1] the study of models in three-dimensional space-time has received considerable attention [2–10]. In that work the authors studied the gauge vector and gravity models and showed that a topological term, the Chern-Simons term, arises leading a large number of special and interesting features such as a massive gauge field and dynamical gravity. More recently the interest in studying models formulated in this space-time has been motivated by their promising connection to the fractional quantum Hall effect and to high-temperature superconductivity [6–8]. In fact, the Chern-Simons term can affect the spin and statistics of charged particles and a fermion-boson transmutation takes place [7]. In this context nonlinear σ models have played an important role. For instance, the $O(3)$ or its CP^1 variant without or with a Chern-Simons term [7] and, recently, a SO(5) quantum nonlinear σ sigma model [11] have been very useful in describing physical properties of superconducting materials. Also, when defined on a Riemannian manifold with a metric tensor g_{ij}. But now in another context, they have been proven to be very useful. Indeed, such models have been examined in two and three dimensions and it has been argued that they can describe the effective low energy of string and membrane theories [12–15]. However, the issue concerning generated Chern-Simons terms by fermions and their calculations, which has been investigated in other models by several methods [1,5], has not been considered yet in general metric nonlinear σ models.

As a matter of fact, such terms can be included in a particular model by hand with an adjustable coefficient, or by the existence of a fermion. Nevertheless, we consider the latter case much more natural and investigate here the situation in which we couple supersymmetric fermions to external gauge fields so that a so-called induced Chern-Simons-type term can arise. The main goal of this work, therefore, is to show how such terms can be generated in these models and how to compute them. We do that by using the covariant background field method combined with the Dyson-Wick algorithm [13] to generate the pertinent Feynman graphs to calculate them perturbatively.

The paper is organized as follows. In Sec. II we start with the three-dimensional $\mathcal{N}=1$ supersymmetric nonlinear σ model defined on a general Riemannian manifold and then we show how a Chern-Simons-type term induced by fermions coupled to external non-Abelian gauge fields can arise. We also compute it by means of the above mentioned technique. In Sec. III we consider the $\mathcal{N}=2$ supersymmetric extension of the previous model, that is, a model defined on a Kähler manifold and, by applying the same procedure as before, we compute a Chern-Simons-type term generated by the fermions. Section IV contains our conclusions.

II. CHERN-SIMONS-TYPE TERM IN THE $\mathcal{N}=1$ SUPERSYMMETRIC σ MODEL

The model we shall begin to consider here is the three-dimensional σ model defined on a general Riemannian manifold M with metric tensor $g_{ij}(\Phi^k)$, where Φ^k, for $k=1,\ldots,n$, are a set of coordinates on this manifold taken to be functions of x^μ, $\mu=0,1,2$, and θ_α, $\alpha=1,2$, where the latter form a two-component Majorana anticommuting spinor. This model is invariant under $\mathcal{N}=1$ supersymmetry and can be described by the action

$$S[\Phi]=\frac{1}{4i}\int d^3x d^2\theta g_{ij}(\Phi^k)\overline{D\Phi}^i D\Phi^j, \tag{1}$$

where D_α is the supercovariant derivative, $D_\alpha=\partial/\partial\overline{\theta}^\alpha - i(\partial\!\!\!/ \theta)_\alpha$, and Φ^i a scalar superfield whose expansion in terms of the component fields ϕ^i (the scalar fields), ψ^i (the Majorana spinor which are the fermionic partners), and F^i (the auxiliary fields), in the Majorana represention for the γ matrices, is given by

$$\Phi^i=\phi^i(x)+\overline{\theta}\psi^i(x)+\frac{1}{2}\overline{\theta}\theta F^i(x). \tag{2}$$

Therefore the action (1) can be alternatively written in terms of component fields by substituting the above relation back into it, integrating over the Grassmann variable θ by means

*Present address: Instituto de Ciências, Escola Federal de Engenharia de Itajubá, C.P. 50, Itajubá - MG 37500-000, Brazil. Electronic address: farnezio@cpd.efei.br of the standard rules of Berezin integration, and eliminating the auxiliary fields. Doing so, we get

$$S=\frac{1}{2}\int d^3x\bigg[g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j+ig_{ij}(\phi)\overline{\psi^i}\gamma^\mu D_\mu\psi^j+\frac{1}{6}R_{ijkl}(\overline{\psi^i}\psi^k)(\overline{\psi^j}\psi^l)\bigg], \tag{3}$$

where $(D_\mu\psi)^j=\partial_\mu\psi^j+\Gamma^j_{kl}\partial_\mu\phi^k\psi^l$, Γ^j_{kl} being the usual Christoffel symbol and R_{ijkl}, the curvature tensor of the manifold.

Several remarks on Eq. (3) are now in order. First, one can notice that it contains as its bosonic sector the usual purely bosonic metric nonlinear σ model. Second, even though it is renormalizable in three dimensions within the large N expansion approach, it is not in the Dyson power-counting sense. However, it has been recently examined in the covariant background field formalism and proved to be renormalizable at least in one-loop order in a more general sense in which the geometry of classical action changes due to quantum corrections [15]. In this paper we are going to deal with perturbative calculations only up to this level, where physical information can be extracted.

Let us now consider how a Chern-Simons-type term can be induced by the fermions of our model (1),(3). To extract and compute it in the effective action we must calculate the one-loop amplitudes. We shall begin this calculation by applying the covariant background field method [13]. This method is a powerful computational tool of the effective action in quantum field theory [16–18]. Indeed, it allows us to compute radiative corrections in a manifestly covariant way preserving the symmetries of the model under consideration [18]. For the present model it is already known [12,13,15,17] and consists in splitting the field ϕ^i into a classical (background) field φ^i and a quantum field π^i. A further step is then taken considering π^i as a function of a new covariant quantum field ξ^i in terms of which the expansion in the Riemann normal coordinate system [19] is defined. The fermion field ψ^i, also transformed to normal coordinates, may be considered as only a quantum field so that there is no background-quantum splitting for anticommuting variables. This can be justified since we are just interested in the quantal effect of the fermions introduced supersymmetrically in Eqs. (1),(3).

Therefore, by applying the above technique, we arrive at the expansion for the action (3), which, for a one-loop calculation, is all we need (we refer to [13] for details):

$$\begin{aligned}S(\varphi+\pi,\psi)=&\frac{1}{2}\int d^3x g_{ij}(\varphi)\partial^\mu\varphi^i\partial_\mu\varphi^j\\&+\frac{1}{2}\int d^3x g_{ij}(\varphi)\partial^\mu\varphi^i D_\mu\xi^i\\&+\frac{1}{2}\int d^3x[g_{ij}(\varphi)D^\mu\xi^i D_\mu\xi^j\\&+R_{iklj}\partial^\mu\varphi^i\partial_\mu\varphi^j\xi^k\xi^l]+\frac{i}{2}\int d^3x g_{ij}(\varphi)\\&\times[\overline{\psi^i}\gamma^\mu\partial_\mu\psi^j+\overline{\psi^i}\gamma^\mu\Gamma^j_{kl}\partial_\mu\varphi^k\psi^l].\end{aligned} \tag{4}$$

We notice that because of the $g_{ij}(\varphi)$ in the kinetic terms, the above expansion is not yet in a useable standard form to obtain Feynman rules for the quantum fields. However, we can deal with it moving to tangent spaces (labeled by the Latin indices $a,b,c,\ldots,h$) on the manifold by introducing a vielbein $e^a_i(\varphi)$ such that $e^a_i e_{ja}=g_{ij}$, and the spin connection satisfying the relation $D_i e^a_j\equiv e^a_{j;i}=\partial_i e^a_j+\omega^{ab}_i(e)e_{bj}-\Gamma^k_{ji}e^a_k=0$, that is, $\omega^{ab}_i=-e^{bj}\nabla_i e^a_j=-e^{bj}\partial_i e^a_j+e^{bj}\Gamma^k_{ij}e^a_k$. Therefore, we can define $\xi^i=e^i_a\xi^a$ and $\psi^a(x)=e^a_i\psi^i(x)$ so that, $D^\mu\xi^a=\partial^\mu\xi^a+\omega^{ab}_i\partial^\mu\varphi^i\xi^b$ and $D^\mu\psi^a=\partial^\mu\psi^a+\omega^{ab}_i\partial^\mu\varphi^i\psi^b$.

Now we can define the operator $A^{ab}_\mu\equiv\omega^{ab}_i\partial_\mu\varphi^i$ which transforms as an SO(n) Yang-Mills gauge potential under local tangent frame rotations in the manifold. This procedure then leads to the action

$$\begin{aligned}S(\varphi+\pi,\psi)=&\frac{1}{2}\int d^3x g_{ij}(\varphi)\partial^\mu\varphi^i\partial_\mu\varphi^j\\&+\frac{1}{2}\int d^3x[D^\mu\xi^a D_\mu\xi^b+R_{iabj}\partial^\mu\varphi^i\partial_\mu\varphi^j\xi^a\xi^b]\\&+\frac{i}{2}\int d^3x[\overline{\psi^a}\gamma^\mu\partial_\mu\psi^b+\overline{\psi^a}\gamma^\mu A^{ab}_\mu\psi^b].\end{aligned} \tag{5}$$

The last term in this expression will be denoted by S_F, the fermionic action, and the others by S_B, corresponding to the bosonic part. Hence, the one-loop fermion effects may be viewed as equivalent to those of the SO(n) gauge theory in which the fermions are coupled to external gauge fields. Therefore, we are going to consider the induced Chern-Simons term in this gauge theory. It should be noticed that the linear term has been discarded in this expansion. Indeed, it vanishes if we use the equation of motion of φ and, besides, it only contributes to a field redefinition.

We are now able to compute the Feynman propagators for the quantum fields and perform the calculation of the one-loop contribution of the effective action. The method which we shall employ to generate one-loop Feynman graphs will be the interaction picture Dyson-Wick perturbation theory in which the de Witt functional [16] is written in a convenient way to generate the diagrammatic algorithm. For example, in our case the one-loop graphs are generated by writing $\Omega[\varphi]=\langle 0|e^{iS_{\rm int}[\xi,\psi]}|0\rangle$, with $S_{\rm int}$ including all terms of Eq. (5) after extracting the propagators. In fact, in this approach all the vacuum diagrams are obtained simply computing all possible Wick contractions involving the quantum fields ξ^a and ψ^a. The propagators are extracted from the quadratic terms in these fields in Eq. (5). The bosonic and fermionic ones are, respectively,

$$\langle 0|T\xi^a(x)\xi^b(y)|0\rangle=\delta^{ab}\int\frac{d^3k}{(2\pi)^3}\frac{i}{(k^2-m^2)}e^{ik(x-y)}$$
$$=\delta^{ab}\Delta(x-y),$$
$$\langle 0|T\psi^a(x)\overline{\psi^b}(y)|0\rangle=\delta^{ab}\int\frac{d^3k}{(2\pi)^3}\frac{i(\gamma^\mu k_\mu+m)}{(k^2-m^2)}e^{ik(x-y)}$$
$$=\delta^{ab}S_F(x-y). \tag{6}$$

In the above expressions we have assumed that a term such as

$$S_m=\frac{m}{2i}\int d^3x d^2\theta g_{ij}(\Phi)\Phi^i\Phi^j \tag{7}$$

has been incorporated in Eq. (1) and also expanded in normal coordinates to combine with the kinetic terms leading to a massive theory.

Since we are interested only in graphs with A_μ^{ab} external legs and internal fermion lines there is no need to study the whole action (5). Actually, the bosonic interaction part does not obviously lead to any contribution to the Chern-Simons term. Moreover, as can be seen in Ref. [15] its linear divergence is canceled by adding a counterterm which, according to Friedan's interpretation [12], requires a change in the initial geometry due to renormalization. Also, it vanishes for Ricci-flat manifolds. So we have nothing to worry about in the bosonic sector of our model. The Chern-Simons term is really derived from the fermionic integral S_F in Eq. (5) which is given by

$$S_F=\frac{1}{2}\int d^3x[\overline{\psi^a}(i\gamma^\mu\partial_\mu-m)\psi^a+\overline{\psi^a}\gamma^\mu A_\mu^{ab}\psi^b]. \tag{8}$$

Gathering together all this information we have the following objects to consider:

$$\int d^3x\langle 0|T[(\overline{\psi^a}\gamma^\mu A_\mu^{ab}\psi^b)(x)|0\rangle, \tag{9}$$

$$\int d^3xd^3y\langle 0|T[(\overline{\psi^a}\gamma^\mu A_\mu^{ab}\psi^b)(x)(\overline{\psi^c}\gamma^\nu A_\nu^{cd}\psi^d)(y)]|0\rangle, \tag{10}$$

$$\int d^3xd^3yd^3z\langle 0|T[(\overline{\psi^a}\gamma^\mu A_\mu^{ab}\psi^b)(x)(\overline{\psi^c}\gamma^\nu A_\nu^{cd}\psi^d)(y)$$
$$\times(\overline{\psi^e}\gamma^\rho A_\rho^{ef}\psi^f)(z)]|0\rangle. \tag{11}$$

Upon contracting the ψ and moving to momentum space, we have to calculate several integrals. The first contribution, corresponding to the tadpole graph, as is easy to verify, is canceled. So we are going to concentrate on those ones that are really important: the vacuum polarization and triangle graphs.

Therefore, we have to calculate

$$\Pi^{\mu\nu}_{\rm pol}=\int\frac{d^3k}{(2\pi)^3}\mathrm{tr}[\gamma^\mu S_F(k)\gamma^\nu S_F(k+p)],$$

$$\Pi^{\mu\nu\rho}_{\rm tri}=\int\frac{d^3k}{(2\pi)^3}\mathrm{tr}[\gamma^\mu S_F(k)\gamma^\nu S_F(k+p)$$
$$\times\gamma^\rho S_F(k+p+q)]. \tag{12}$$

Explicitly,

$$\Pi^{\mu\nu}_{\rm pol}=\int\frac{d^3k}{(2\pi)^3}\frac{\mathrm{tr}[\gamma^\mu\gamma^\rho k_\rho+m)\gamma^\nu((k+p)^\sigma\gamma_\sigma+m)]}{(k^2-m^2)[(k+p)^2-m^2]}, \tag{13}$$

$$\Pi^{\mu\nu}_{\rm tri}=\int\frac{d^3k}{(2\pi)^3}\frac{\mathrm{tr}\{\gamma^\mu(\gamma^\alpha k_\alpha+m)\gamma^\nu[(k+p)^\sigma\gamma_\sigma+m]\gamma^\rho[(k+p+q)^\lambda\gamma_\lambda+m]\}}{(k^2-m^2)[(k+p)^2-m^2][(k+p+q)^2-m]}. \tag{14}$$

Before proceeding with our calculation we should first note that in three-dimensional space-time, the spinor structure is the same as that in the two-dimensional one and the Dirac matrices are 2×2 obeying the relations $\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}$, $C\gamma_\mu C^{-1}=-\gamma_\mu^T$, $C=-C^T$, $\gamma^\mu\gamma^\nu=g^{\mu\nu}-i\epsilon^{\mu\nu\rho}\gamma_\rho$, where $\eta^{\mu\nu}=(+--)$ is the metric tensor, C is the charge conjugation matrix, and $\epsilon^{\mu\nu\rho}$ is the Levi-Civita tensor which is totally antisymmetric. An explicit representation satisfying the above is $\gamma^0=\sigma^3$, $\gamma^1=i\sigma^1$, $\gamma^2=i\sigma^2$, where σ^i are the Pauli matrices. Notice that no γ_5 appears here.

The integrals (13) and (14) may be separated into two main contributions: one arising from a product of an even number of γ matrices and another one which can lead to a Chern-Simons-type term. We notice, however, that these integrals contain linear and logarithmic divergences and thus must be regularized. To do that, we introduce a heavy Pauli-Villars regulator field with mass M and use the same regularization prescription as in Refs. [4,10], that is, $S_{\rm eff}[A]\rightarrow S_{\rm eff}[A]-\lim_{M\rightarrow\infty}S_{\rm eff}[A,M]$. Following this procedure, the above mentioned divergences are eliminated in a gauge-invariant way and we get a finite result. As matter of fact, to extract the Chern-Simons-type term from Eqs. (13) and (14) we are going to consider only those terms which involve a product of an odd number of γ matrices and also have a proportionality factor of $m/|m|$. In other words, we shall investigate only terms which have the Levi-Civita tensor structure and are finite as $m\rightarrow\infty$.

Upon evaluating the trace over the Dirac matrices, using the Feynman parametrization, we can write down the Chern-

Simons part of the regulated vacuum polarization and the three-vertex tensors as

$$\Pi_{\rm CS}^{\mu\nu}=i\mu(p^2,m^2)\epsilon^{\mu\nu\rho}p_\rho, \tag{15}$$

$$\Pi_{\rm CS}^{\mu\nu\rho}=\mu(p^2,m^2)\epsilon^{\mu\nu\rho}, \tag{16}$$

where

$$\mu(p^2,m^2)=-\frac{1}{4\pi}m\int_0^1\frac{d\alpha}{\sqrt{-\alpha(1-\alpha)p^2+m^2}}+\frac{1}{4\pi}\frac{M}{|M|}, \tag{17}$$

that, by elementary integration, becomes

$$\mu(p^2,m^2)=\frac{1}{4\pi}\left[\frac{M}{|M|}-\frac{2m}{\sqrt{-p^2}}\arctan\sqrt{\frac{-p^2}{4m^2}}\right]. \tag{18}$$

Notice that this generated Chern-Simons-type term does not have a simple form. However, for massless fields, it is due to the regulator mass M only and its action may be written as

$$S_{\rm eff}^{\rm CS}[A]=\frac{1}{16\pi}\frac{M}{|M|}\int d^3x\,\epsilon^{\mu\nu\rho}\left[F_{\mu\nu}^{ab}A_{\rho ab}-\frac{2}{3}A_{\mu ab}A_\nu^{bc}A_{\rho c}^{a}\right], \tag{19}$$

where the gauge field strength $F_{\mu\nu}^{ab}$ is formed from the potential A_μ^{ab}.

For massive fermions, for a small momentum behavior, we can also have an ordinary Chern-Simons coefficient. In this case, the tensors are expanded around $p=0$ and we get the same expression as in Eq. (19) but now with a regularization-dependent constant given by $\varepsilon=M/|M|-m/|m|$ instead of $M/|M|$. For instance, if $\varepsilon=0$, the Chern-Simons-type term vanishes.

The action (19) closely resembles the Chern-Simons action for a Yang-Mills theory. Indeed, to be more precise, we have been treating from the very beginning, the field A_μ^{ab} in our model as a non-Abelian one.

III. THE CHERN-SIMONS TERM IN THE KÄHLERIAN MODEL

Now we are going to apply the previous technique to show how a Chern-Simons-type term can also be generated in the model (1) extended to accommodate $\mathcal{N}=2$ supersymmetry. As we have already seen, in three dimensions and $\mathcal{N}=1$, any Riemannian manifold can occur as target space. However, in this space-time, according to Zumino's theorem [20], $\mathcal{N}=2$ supersymmetry induces a complex structure and the manifold is Kählerian. In other words, the bosonic model given by the action

$$\int d^3x g_{i\bar{j}}(\phi,\bar{\phi})\partial_\mu\phi^i\partial^\mu\bar{\phi} \tag{20}$$

has a $\mathcal{N}=2$ supersymmetric extension, with n complex spinor fields, if and only if the mentric tensor $g_{i\bar{j}}(\phi,\bar{\phi})$ of its target space which is a complex manifold $\mathcal{M}$ with holomorphic and antiholomorhic coordinates ϕ and $\bar{\phi}^{\bar{i}}=\overline{\phi^{i}}$, $i,\bar{i}=1,\cdots,n$, respectively, restricted to be a Hermitian, satifies the Kähler condition

$$\partial_k g_{i\bar{j}}(\phi,\bar{\phi})=\partial_i g_{k\bar{j}}(\phi,\bar{\phi}),$$
$$\partial_{\bar{k}} g_{i\bar{j}}(\phi,\bar{\phi})=\partial_{\bar{j}} g_{i\bar{k}}(\phi,\bar{\phi}). \tag{21}$$

The action of this model in its component form can be written as

$$\mathcal{S}=\frac{1}{2}\int d^3x\left[g_{i\bar{j}}(\phi)\partial_\mu\phi^i\partial^\mu\bar{\phi}^{\bar{j}}+ig_{i\bar{j}}(\phi)\bar{\chi}^i\gamma^\mu D_\mu\chi^j+\frac{1}{6}R_{\bar{i}k\bar{j}l}(\bar{\chi}^i\chi^k)(\bar{\chi}^j\chi^l)\right]. \tag{22}$$

The covariant background method works well as in the previous case and for the one-loop calculation all we need is

$$\mathcal{S}=\int d^3x[R_{i\bar{a}b\bar{j}}(\varphi,\bar{\varphi})\partial_\mu\phi^i\partial^\mu\bar{\phi}^{\bar{j}}\bar{\xi}^{\bar{a}}\xi^b-A_{\mu\bar{a}b}\bar{\xi}^{\bar{a}}\overleftrightarrow{\partial^\mu}\xi^b+A_{\mu\bar{b}}^{\bar{a}}A_{\bar{a}c}^{\mu}\bar{\xi}^{\bar{b}}\xi^c+i\bar{\chi}^a\gamma^\mu\partial_\mu\chi_a-m\bar{\chi}^a\chi_a+i\bar{\chi}^a\gamma^\mu A_{\mu\bar{a}b}\chi^b]. \tag{23}$$

Notice that in the above action all the quantities have already moved to tangent space and $A_\mu^{\bar{a}b}=\omega_i^{\bar{a}b}\partial_\mu\varphi^i$ is a potential-like field which transforms as a gauge connection under the tangent frame.

This model is also one-loop renormalizable, at least in Friedan's sense [21], and for our purpose we are going to concentrate on the fermionic part of Eq. (23). From now on the calculation follows the same steps as in the previous section. So using the Dyson-Wick diagrammatic algorithm, in which we make all possible Wick contractions involving the quantum fields χ, we find the following objects to calculate:

$$\int d^3x\langle 0|T[(\bar{\chi}^a\gamma^\mu A_\mu^{\bar{a}b}\chi^b)(x)|0\rangle, \tag{24}$$

$$\int d^3xd^3y\langle 0|T[(\bar{\chi}^a\gamma^\mu A_\mu^{\bar{a}b}\chi^b)(x)(\bar{\chi}^c\gamma^\nu A_\nu^{\bar{c}d}\chi^d)(y)]|0\rangle, \tag{25}$$

$$\int d^3xd^3yd^3z\langle 0|T[(\bar{\chi}^a\gamma^\mu A_\mu^{\bar{a}b}\chi^b)(x)(\bar{\chi}^c\gamma^\nu A_\nu^{\bar{c}d}\chi^d)(y)\times(\bar{\chi}^e\gamma^\rho A_\rho^{\bar{e}f}\psi^f)(z)]|0\rangle, \tag{26}$$

with the propagators given by $\langle 0|\mathrm{T}\underline{\xi}^a(x)\bar{\xi}^{b}(y)|0\rangle=i\delta^{\bar{a}b}\Delta(x-y)$ and $\langle 0|\mathrm{T}\chi^a(x)\overline{\chi^b(y)}|0\rangle=i\delta^{ab}S_F(x-y)$.

Again, using the representation of the γ matrices given in Sec. II, we obtain the same integrals as in Eqs. (13) and (14). We have not regarded the "tadpole" contraction since it vanishes. As before, to derive a Chern-Simons-type term we select only those terms which can generate it and apply the Pauli-Villars prescription of regularization. In the present case we obtain

$$\mathcal{S}_{\rm eff}^{\rm CS}[A] = -\frac{1}{8}\int \frac{d^3k}{(2\pi)^3}\frac{d^3p}{(2\pi)^3} A_{\mu}^{a\bar{b}}(-k)A_{\nu a\bar{b}}(-p) \times \epsilon^{\mu\nu\rho}p_{\rho}\mu(p^2,m^2) - \frac{1}{24}\int \frac{d^3k}{(2\pi)^3}\frac{d^3p}{(2\pi)^3} \times A_{\mu a\bar{b}}(-k-p)A_{\nu}^{b\bar{c}}(k)A_{\rho\bar{c}}^{\bar{a}}(p)\epsilon^{\mu\nu\rho}\mu(p^2,m^2). \tag{27}$$

Naturally, if we were considering since the beginning a massless theory, then the mass m in the propagators would serve merely to separate the ultraviolet and infrared divergences, and thus should be put equal to zero at the end of the calculation. In this case the Chern-Simons-type action would be given by

$$\mathcal{S}_{\rm eff}^{\rm CS}[A] = \frac{1}{32\pi}\frac{M}{|M|}\int d^3x\,\epsilon^{\mu\nu\rho}\left[A_{\mu}^{a\bar{b}}\partial_{\nu}A_{\rho a\bar{b}} - \frac{1}{3}A_{\mu a\bar{b}}A_{\nu}^{b\bar{c}}A_{\rho\bar{c}}^{\bar{a}}\right]. \tag{28}$$

The same analysis for the infrared regime can also be applied to Eq. (27) and the interesting structure of the Chern-Simons term remains.

IV. CONCLUSION

In this paper we have shown how Chern-Simons-type terms can be generated by fermions in general metric nonlinear σ models in three space-time dimensions. We have also suceeded in computing them in the $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetric models which have, as target spaces, general Riemannian and Kähler manifolds, respectively. In both cases, we have considered the supersymmetric fermions coupled to external fields which transform as SO(n) Yang-Mills gauge potentials under local tangent frame rotations in the manifolds. We calculated them explicitly in the framework of the covariant background field method and by using the interaction picture Dyson-Wick perturbation theory. Indeed, this approach allows us to express, in a simple way, the de Witt functional from which we can obtain a diagrammatic algorithm and investigate the one-loop fermion effects. As one should expect, the induced Chern-Simons-type terms obtained in the models studied here have the same structure as those found in usual three-dimensional Yang-Mills theories. As a matter of fact, we have been treating from the very beginning the fields A_{μ}^{ab} and $A_{\mu}^{a\bar{b}}$ as non-Abelian ones. Moreover, if we express them in terms of the spin connection ω that, in turn, is a function (a curl) of the vielbein e_i^a our results, even though containing objects refered to a Riemannian target spaces, are closely analogous to those ocurring in gravity theories in three dimensions [1,22]. Finally, an interesting and attractive issue to be investigated is the one concerning the topological effects of the Chern-Simons-type terms found on the physics of the models studied here.

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