Operator cutoff regularization and renormalization group in Yang-Mills theory

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The symmetry-preserving nature of the operator cutoff regularization and its analogy with the invariant Slavnov regularization are demonstrated at one loop order for pure Yang-Mills theory. The presence of momentum cutoff scales in our regularization offers a direct application of the Wilson-Kadanoff renormalization group to the theory. In particular, via the Schwinger-Dyson self-consistency argument, the one-loop perturbative equation is dressed into a nonlinear renormalization group evolution equation which takes into consideration the contributions of higher dimensional operators and provides a systematic way of exploring the influence of these operators as the strong coupling, infrared limit is approached. [S0556-2821(97)04920-5]

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I. INTRODUCTION

An important technical issue in quantum field theory is regularization, the removal of divergences which arise from incorporating the effects of quantum fluctuations. Examples of the most frequently encountered prescriptions in perturbative calculations include the sharp momentum cutoff, dimensional regularization [1], ζ function regularization [2], Pauli-Villars procedure [3], and the proper-time regularization [4]. Although numerous regularization schemes are available for making the theory finite and well defined, it is crucial that the chosen procedure preserves all the symmetries of the original theory. For example, when considering gauge theories, not all of the above-mentioned prescriptions can be utilized due to the symmetry constraint.

The construction of a regularization scheme that respects gauge symmetry has proven to be a difficult task. By far, the most popular invariant method is dimensional regularization which is based on the analytic continuation in the number of space-time dimension. The concept of analytic continuation, however, was first conceived by Speer *et al.* in their pioneering work of analytic regularization [5], which unfortunately, does not preserve gauge symmetry. Even though dimensional regularization is applicable to gauge theories, it remains meaningful only within the context of perturbative framework and is known to become problematic when the system under investigation has additional symmetry properties that are dimensionality dependent (e.g., chiral symmetry or supersymmetry). In the alternative Slavnov regularization [6], both the method of higher covariant derivatives and an additional auxiliary regulator, usually of Pauli-Villars type, must be employed to completely regulate all the divergences. In addition, the prescription must be exercised with caution in certain gauges in order to reproduce the standard renormalization group (RG) coefficient functions for the Yang-Mills theory [7]. Needless to say, a sharp momentum cutoff is clearly unsuitable for gauge theories owing to its noninvariant nature.

When treating nonperturbative phenomena such as confinement in the strong-coupling regime or the high-temperature electroweak phase transition, the constraint from the gauge symmetry becomes even more stringent. A nonperturbative continuum regularization program based on stochastic quantization was developed by Bern *et al.* [8] a decade ago. Recently, a nonlocal nonperturbative regularization has also been employed to study QED [9]. However, the most promising nonperturbative method has been the lattice regularization in which the space-time is discretized. Unfortunately, the regulator also has its drawback in causing the doubling of the fermionic degrees of freedom. Thus, it would be desirable to formulate a new invariant scheme which complements the lattice approach.

The enormous success of the Wilson-Kadanoff renormalization group [10] in analyzing the behavior of the scalar field theory in the nonperturbative regime has recently prompted much activity in understanding how the same technique can be implemented in gauge theories [11,12]. The prominent feature of the Wilson-Kadanoff RG is the use of an effective infrared (IR) cutoff which provides a systematic separation of the large and small momentum modes; an effective low-energy theory is obtained upon integrating out the large momentum modes that correspond to the irrelevant microscopic details [13]. Although no divergence is encountered when imposing these cutoff scales, their presence, as mentioned before, is in conflict with gauge or Becchi-Rouet-Stora (BRS) symmetry, and one can only hope to restore the symmetry in the limit when the cutoff is removed. Thus, a gauge invariant formulation of the nonperturbative Wilson-Kadanoff RG program would entail two essential steps: (i) the introduction of some momentum cutoff scales without spoiling gauge symmetry and (ii) the derivation of the corresponding invariant differential RG flow equations, usually nonlinear in nature, based on the infinitesimal variation of the IR cutoff scale. In the present work, we demonstrate how step (i) can be achieved at the one loop level. Since it is rather difficult to carry out step (ii) rigorously, we address how the one-loop result can help generate nonperturbative RG evolution equation.

To achieve the first step, we apply the operator cutoff regularization [14,15]. The novel feature of this prescription

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is that it resembles the conventional momentum cutoff regulator, yet preserves gauge symmetry despite the presence of an IR scale. In the operator cutoff formalism, the one-loop contribution to the effective action is written as

$$\mathrm{Tr}_{\mathrm{oc}}(\ln\mathcal{H}-\ln\mathcal{H}_0)=-\int_0^\infty \frac{ds}{s}\,\rho_k^{(d)}(\Lambda,s)\mathrm{Tr}(e^{-\mathcal{H}s}-e^{-\mathcal{H}_0 s}),\tag{1.1}$$

where $\mathcal{H}$ is an arbitrary fluctuation operator governing the quadratic fluctuations of the fields and $\mathcal{H}_0$ its corresponding limit of zero background field. The subscript oc implies that the trace sum is to be operator cutoff regularized using the d-dimensional smearing function $\rho_k^{(d)}(s,\Lambda)$ which contains both the ultraviolet (UV) cutoff Λ and the IR cutoff k. Since the usual one-loop quantum correction is obtained by evaluating a Gaussian integral which corresponds to solving for the eigenvalues of the fluctuation operator, the role of $\rho_k^{(d)}(s,\Lambda)$ can also been seen as to provide an upper and lower cutoff on the eigenvalues. We require $\rho_k^{(d)}(s,\Lambda)$ to satisfy the following conditions: (1) $\rho_k^{(d)}(s=0,\Lambda)=0$, i.e., it must vanish sufficiently fast near $s=0$ to eliminate the unwanted UV divergence; (2) $\rho_{k=0}^{(d)}(s\to\infty,\Lambda)=1$ since the physics in the IR ($s\sim\infty$) regime is to remain unmodified; and (3) $\rho_{k=\Lambda}^{(d)}(s,\Lambda)=0$ so that the one-loop correction to the effective action vanishes at the UV cutoff. In addition, we have

$$\rho_{k=0}^{(d)}(s,\Lambda\to\infty)=1,\tag{1.2}$$

which reduces the operator cutoff regularization to the original Schwinger's proper-time formalism [4]

$$\mathrm{Tr}(\ln\mathcal{H}-\ln\mathcal{H}_0)=-\int_0^\infty \frac{ds}{s}\,\mathrm{Tr}(e^{-\mathcal{H}s}-e^{-\mathcal{H}_0 s}).\tag{1.3}$$

Thus, the operator cutoff may be regarded as a special case of proper-time regularization whose invariant nature is based on the transfer of the space-time singularity into a singularity in s which is independent of gauge transformation. In fact, various other prescriptions such as sharp proper-time cutoff, point-splitting method, Pauli-Villars regulator, operator cutoff, dimensional regularization, and ζ function regularization all belong to the generalized class of proper time and can be represented by a suitably chosen smearing function [16].

The presence of an arbitrary IR scale k scale makes the operator cutoff regularization an ideal candidate for examining the RG flow of the theory using the Wilson-Kadanoff approach, with the procedure of blocking transformation [17] being taken over by the smearing function $\rho_k^{(d)}(s,\Lambda)$. Once an IR cutoff is introduced, the corresponding RG equation can be obtained by varying the cutoff infinitesimally. For scalar theory, it suffices to employ a sharp momentum cutoff and the RG improved equation for the blocked potential $U_k(\Phi)$ takes on the following form (see the form in Appendix A):

$$k\partial_k U_k(\Phi)=-S_d k^d \ln\left(\frac{k^2+U_k''(\Phi)}{k^2+U_k''(0)}\right),$$

$$S_d=\frac{2}{(4\pi)^{d/2}\Gamma(d/2)}.\tag{1.4}$$

One can readily verify [15] that when choosing the smearing function to be of the form

$$\rho_k^{(d)}(s,\Lambda)=\rho^{(d)}(\Lambda^2 s)-\rho^{(d)}(k^2 s)=\frac{2s^{d/2}}{\Gamma(d/2)}\int_k^\Lambda dz z^{d-1}e^{-z^2 s}$$
$$=\frac{2s^{d/2}}{S_d\Gamma(d/2)}\int_z' e^{-z^2 s}=\frac{1}{\Gamma(d/2)}\,\Gamma\left[\frac{d}{2};k^2 s,\Lambda^2 s\right],\tag{1.5}$$

with

$$\int_z=\int\frac{d^d z}{(2\pi)^d},\quad \int_z'=S_d\int_k^\Lambda dz\,z^{d-1},\tag{1.6}$$

the same sharp momentum cutoff RG equation is obtained. As for coefficients such as the wave function renormalization constant $\mathcal{Z}_k$ that are associated with the higher order (covariant) derivative terms, $\rho_k^{(d)}(s,\Lambda)$ corresponds to a smooth regulator.

While the Wilson-Kadanoff RG equation can be easily obtained in the operator cutoff formalism for the scalar theory, it turns out to be a notorious difficult task for gauge theories due to the symmetry constraint. In fact, all the attempts made so far have either broken the symmetry or have lead to difficulties such as the dependence of the blocked action on two gauge fields: a classical average and a background [12]. Because of the difficulty in providing a rigorous derivation of an evolution equation which manifestly preserves gauge symmetry, we propose in this paper a more modest alternative. The methodology we adopt here is to first apply the operator cutoff prescription to regularize the one-loop contribution of the non-Abelian Yang-Mills theory [18]. A linear differential flow equation is then obtained by an infinitesimal variation of the IR cutoff from k to $k-\Delta k$. Albeit gauge symmetry is completely preserved, the equation only incorporates the perturbative one-loop contribution. As an improvement, we use the Schwinger-Dyson self-consistency argument and turn the equation into a nonlinear RG evolution equation which can provide a resummation over the higher order nonoverlapping loop diagrams. As demonstrated by Wegner and Houghton [19], since the higher loop contributions are suppressed by additional powers of Δk, the equation which is based on the one-loop functional form may be regarded as being ''exact.'' Indeed, RG equations obtained in this manner have been used for exploring the critical behavior of scalar field theories and other nonperturbative phenomena with enormous success. In particular, the three-dimensional exponents extracted from this approach agree remarkably well with that of ϵ expansion [20]. Thus, despite the fact that our nonlinear RG equation for gauge theories cannot be derived from the first principle, we believe that its solution can offer important insights into the nonperturbative nature of gauge theories in the IR regime.

The organization of the paper is as follows. In Sec. II, we review the essential features of the operator cutoff regularization and illustrate how it can be used in conjunction with covariant derivative expansion. An analogy between our formalism and the Slavnov regularization is drawn. We also show how the smearing function can be modified to provide a faster convergence. In Sec. III we construct the operator cutoff effective Lagrangian which reproduces the effective propagators. Details of the perturbative computation of the non-Abelian Yang-Mills blocked action in the covariant background method are given in Sec. IV. We illustrate how the symmetry is spoiled when the theory is expanded in power series of the background field $\overline{A}_\mu^a$ which is a noninvariant quantity. The RG pattern of the blocked action is investigated in Sec. V. Working in Feynman gauge, $\alpha=1$, we demonstrate how operator cutoff regularization completely regularizes the one-loop divergences and leads to the correct β function which governs the evolution of the gauge coupling constant. In Sec. VI we apply the operator cutoff prescription to examine the SU(2) Yang-Mills theory in a constant chromomagnetic field configuration. Since the theory develops an imaginary part which signals instability of the vacuum as the momentum scale falls below $\sqrt{gB}$, we choose the IR cutoff to be such that $k^2>gB$, thereby eliminating the difficulties associated with an unstable vacuum. Our analysis shows that if only the real part of the potential is considered, the existence of the ''Savvidy vacuum'' is intimately linked to the existence of a nontrivial fixed point in the chromomagnetic field background. Our search for such a fixed point gives a negative result. Section VII is reserved for summary and discussions. In Appendix A we provide the details of calculating the blocked potentials for $\lambda\phi^4$ theory and scalar electrodynamics in $d=4$ using operator cutoff regularization. In Appendix B, we compare and contrast various prescriptions that belong to the generalized class of proper-time regularization. In particular, we show how dimensional regularization can be modified to incorporate cutoff scales. The connection between momentum regulator and dimensional regularization is readily established in our ''dimensional cutoff'' scheme. Momentum cutoff scales can also be brought into the ζ function regularization in a similar manner.

II. OPERATOR CUTOFF REGULARIZATION

As mentioned in the Introduction, operator cutoff regularization not only allows us to bypass the complications of dealing with divergences, it also encompasses the features of momentum space blocking transformation in a symmetry-preserving manner. With the smearing function $\rho_k^{(d)}(s,\Lambda)$ given in Eq. (1.5), the operator cutoff regularized propagator and the one-loop contribution of the blocked action become, respectively,

$$\begin{aligned}\frac{1}{\mathcal{H}^n}\bigg|_{\rm oc}&=\frac{1}{\Gamma(n)}\int_0^\infty ds\, s^{n-1}\rho_k^{(d)}(s,\Lambda)e^{-\mathcal{H}s}\\&=\frac{1}{\mathcal{H}^n}\,\frac{2\Gamma(n+d/2)}{d\Gamma(n)\Gamma(d/2)}\left\{\left(\frac{\Lambda^2}{\mathcal{H}}\right)^{d/2}\right.\\&\quad\times F\left(\frac{d}{2},\frac{d}{2}+n,1+\frac{d}{2};-\frac{\Lambda^2}{\mathcal{H}}\right)-\left(\frac{k^2}{\mathcal{H}}\right)^{d/2}\\&\quad\left.\times F\left(\frac{d}{2},\frac{d}{2}+n,1+\frac{d}{2};-\frac{k^2}{\mathcal{H}}\right)\right\},\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}{\rm Tr}_{\rm oc}(\ln\mathcal{H}-\ln\mathcal{H}_0)&=-\int_0^\infty\frac{ds}{s}\,\rho_k^{(d)}(s,\Lambda){\rm Tr}(e^{-\mathcal{H}s}-e^{-\mathcal{H}_0s})\\&=-\frac{2}{d}\,{\rm Tr}\left\{\left(\frac{\Lambda^2}{\mathcal{H}}\right)^{d/2}F\left(\frac{d}{2},\frac{d}{2},1+\frac{d}{2};-\frac{\Lambda^2}{\mathcal{H}}\right)\right.\\&\quad-\left(\frac{\Lambda^2}{\mathcal{H}_0}\right)^{d/2}F\left(\frac{d}{2},\frac{d}{2},1+\frac{d}{2};-\frac{\Lambda^2}{\mathcal{H}_0}\right)\\&\quad-\left(\frac{k^2}{\mathcal{H}}\right)^{d/2}F\left(\frac{d}{2},\frac{d}{2},1+\frac{d}{2};-\frac{k^2}{\mathcal{H}}\right)\\&\quad\left.+\left(\frac{k^2}{\mathcal{H}_0}\right)^{d/2}F\left(\frac{d}{2},\frac{d}{2},1+\frac{d}{2};-\frac{k^2}{\mathcal{H}_0}\right)\right\},\end{aligned}\tag{2.2}$$

where

$$\begin{aligned}F(a,b,c;\beta)&=B^{-1}(b,c-b)\int_0^1dx\,x^{b-1}(1-x)^{c-b-1}\\&\quad\times(1-\beta x)^{-a}\quad({\rm Re}c>{\rm Re}b>0)\end{aligned}\tag{2.3}$$

is the hypergeometric function symmetric under the exchange between a and b, and

$$\begin{aligned}B(x,y)&=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\\&=\int_0^1dt\,t^{x-1}(1-t)^{y-1}\quad({\rm Re}x,{\rm Re}y>0)\end{aligned}\tag{2.4}$$

is the Euler β function. For $n=1$ and 2, Eq. (2.1) gives, respectively,

$$\begin{aligned}\frac{1}{\mathcal{H}}\bigg|_{\rm oc}&=\int_0^\infty ds\,\rho_k^{(d)}(s,\Lambda)e^{-\mathcal{H}s}\\&=\frac{1}{\mathcal{H}}\left\{\left(\frac{\Lambda^2}{\mathcal{H}+\Lambda^2}\right)^{d/2}-\left(\frac{k^2}{\mathcal{H}+k^2}\right)^{d/2}\right\},\end{aligned}\tag{2.5}$$

and

$$\left.\frac{1}{\mathcal{H}^2}\right|_{\rm oc}=\int_0^\infty ds\, s\rho_k^{(d)}(s,\Lambda)e^{-\mathcal{H}s}$$
$$=\frac{1}{\mathcal{H}^2}\left\{\left(\frac{\Lambda^2}{\mathcal{H}+\Lambda^2}\right)^{d/2}\left[1+\frac{d}{2}\frac{\mathcal{H}}{\mathcal{H}+\Lambda^2}\right]-\left(\frac{k^2}{\mathcal{H}+k^2}\right)^{d/2}\left[1+\frac{d}{2}\frac{\mathcal{H}}{\mathcal{H}+k^2}\right]\right\}. \tag{2.6}$$

Consider $d=4$, where

$$\rho_k^{(4)}(s,\Lambda)=(1+k^2s)e^{-k^2s}-(1+\Lambda^2s)e^{-\Lambda^2s}. \tag{2.7}$$

We have

$$\left.\frac{1}{\mathcal{H}}\right|_{\rm oc}=\frac{1}{\mathcal{H}+k^2}-\frac{1}{\mathcal{H}+\Lambda^2}-\frac{\Lambda^2}{(\mathcal{H}+\Lambda^2)^2}+\frac{k^2}{(\mathcal{H}+k^2)^2}, \tag{2.8}$$

and

$$\mathrm{Tr}_{\rm oc}(\ln\mathcal{H}-\ln\mathcal{H}_0)=\mathrm{Tr}\left\{\ln\left[\frac{\mathcal{H}+k^2}{\mathcal{H}_0+k^2}\frac{\mathcal{H}_0+\Lambda^2}{\mathcal{H}+\Lambda^2}\right]-\frac{\Lambda^2(\mathcal{H}-\mathcal{H}_0)}{(\mathcal{H}+\Lambda^2)(\mathcal{H}_0+\Lambda^2)}+\frac{k^2(\mathcal{H}-\mathcal{H}_0)}{(\mathcal{H}+k^2)(\mathcal{H}_0+k^2)}\right\}, \tag{2.9}$$

which shows that Λ may be interpreted as the mass of some ghost particles. This interpretation follows from the relative negative sign in the modified propagator. Equivalently, one may also say that the effect of the extra Λ-dependent terms is to make the theory superrenormalizable by incorporating higher order derivative terms. For example, Eq. (2.8) implies that the kinetic term in the scalar theory is to be modified as

$$-\frac{1}{2}\phi\partial^2\phi\to\frac{1}{2}\phi\left[-\partial^2+\frac{2}{\Lambda^2}(-\partial^2)^2+\frac{1}{\Lambda^4}(-\partial^2)^3\right]\phi. \tag{2.10}$$

On the other hand, the IR scale k may be thought of as an additional mass which makes an overall shift on the mass parameter $\mu^2\to\mu^2_{\rm eff}=\mu^2+k^2$. Thus, the scale k is useful not only for the purpose of studying RG, but can also be employed as an IR regulator for the theory containing massless modes.

Our operator cutoff scheme actually resembles the invariant Slavnov regularization which involves both the higher covariant derivative method and a secondary Pauli-Villars regulator. The analogy can be readily seen from Eqs. (2.8) and (2.9), where we see how it changes the propagator and the fluctuation operators; namely, the modification on the former is similar to that of the higher covariant derivative method, and the latter to the Pauli-Villars regularization. In other words, both ingredients of the Slavnov regularization are encapsulated by a single smearing function $\rho_k^{(d)}(s,\Lambda)$ in the operator cutoff prescription. Moreover, as we shall demonstrate later, the presence of cutoff scales in the proper-time integration is not only guaranteed to give finite results, but also lead to the correct RG coefficient functions. In the case when stronger divergences are encountered, one may choose the smearing function to be of the form

$$\rho_k^{(d,m)}(s,\Lambda)=\frac{2s^{m+d/2}}{S_d\Gamma(m+d/2)}\int_z'(z^2)^me^{-z^2s}=\frac{1}{\Gamma(m+d/2)}\Gamma\left[m+\frac{d}{2};k^2s,\Lambda^2s\right], \tag{2.11}$$

where $\rho_k^{(d,0)}(s,\Lambda)=\rho_k^{(d)}(s,\Lambda)$, and obtain a faster convergence [15]. The connection between $\rho_k^{(d,m)}(s,\Lambda)$ and the momentum regularization can be seen from

$$\int_p'\frac{(p^2)^m}{(p^2+a)^n}=\frac{1}{\Gamma(n)}\int_0^\infty ds\, s^{n-1}e^{-as}\int_p'(p^2)^me^{-p^2s}=\frac{\Gamma(m+d/2)}{(4\pi)^{d/2}\Gamma(d/2)\Gamma(n)}\times\int_0^\infty ds\, s^{n-1-m-d/2}e^{-as}\rho_k^{(d,m)}(s,\Lambda). \tag{2.12}$$

Before examining Yang-Mills theory, we first consider the following covariant fluctuation kernel:

$$\mathcal{H}=-D^2+\mu^2+Y(x), \tag{2.13}$$

where D_μ is the covariant derivative for the gauge group, μ^2 the mass for the scalar field interacting with the gauge field $A_\mu^a(x)$, and $Y(x)$ a matrix-valued function of x describing the interaction between the scalar particles. The index a runs over the dimension of the gauge (color) group. One may also write $Y=Y^aT^a$, where the T^a's are the anti-Hermitian generators of the gauge group satisfying

$$[T^a,T^b]=f^{abc}T^c,\quad \mathrm{tr}(T^aT^b)=-\frac{1}{2}\delta^{ab}, \tag{2.14}$$

with f^{abc} being the structure constants and tr the summation over only the internal indices. In the fundamental SU(N) representation, we have

$$T^a=\begin{cases}\sigma^a/2i, & a=1,\dots,3,\ n=2,\\ \lambda^a/2i, & A=1,\dots,8,\ N=3,\end{cases} \tag{2.15}$$

where σ^a and λ^a are, respectively, the Pauli and the Gell-Mann matrices. When operating on Y with the covariant derivative, we have

$$D_\mu^{ab}Y=(g^{ab}\partial_\mu-gf^{abc}A_\mu^c)Y, \tag{2.16}$$

or $D_\mu Y=\partial_\mu Y+[A_\mu,Y]$, where $A_\mu=gA_\mu^aT^a$ and g is the renormalized coupling constant.

The unregularized one-loop contribution to the effective action is

$$\widetilde{S}^{(1)}=\frac{1}{2}\,\mathrm{Tr}(\ln\mathcal{H}-\ln\mathcal{H}_0)$$

$$=-\frac{1}{2}\int_x\int_0^\infty\frac{ds}{s}\,\mathrm{tr}\langle x|(e^{-\mathcal{H}s}-e^{-\mathcal{H}_0 s})|x\rangle,$$

$$\int_x=\int d^dx, \tag{2.17}$$

where the diagonal part of the ''heat kernel'' is written as

$$h(s;x,x)=\langle x|e^{-\mathcal{H}s}|x\rangle=\int_p\langle x|p\rangle e^{-\mathcal{H}_x s}\langle p|x\rangle$$

$$=\int_p e^{-ipx}e^{-\mathcal{H}_x s}e^{ipx}=\int_p e^{-(p^2-2ip\cdot D+\mathcal{H}_x)s}\mathbf{1}$$

$$=\int_p e^{-(p^2+\mu^2)s}e^{(2ip\cdot D+D^2-Y)s}\mathbf{1},\quad \int_p=\int\frac{d^dp}{(2\pi)^d}. \tag{2.18}$$

The above expression is derived by employing the plane wave basis $|p\rangle$ with $\langle x|p\rangle=e^{-ipx}$ and the commutation relations [16,21]

$$[D_\mu,e^{ipx}]=e^{ipx}ip_\mu,\quad [\mathcal{H}_x,e^{ipx}]=e^{ipx}(p^2-2ip\cdot D). \tag{2.19}$$

The factor $\mathbf{1}$ indicates that the operator D_μ acts on the identity. Splitting the s integral into two parts $s>s_0$ and $s<s_0$, for the latter region one can use the Baker-Campbell-Hausdorf formulas to expand the operators in the exponential in a power series of s:

$$h(s;x,x)=e^{-(\mu^2+Y)s}\int_p e^{-p^2s}\Bigg\{1+D^2s+\frac{D^4}{2}s^2-\frac{[D^2,Y]}{2}s^2$$
$$-\frac{2p^2}{d}\Bigg[D^2s^2+\frac{1}{3}\,([[D^2,D_\mu],D_\mu]$$
$$+3D_\mu[D^2,D_\mu]+3D^4-[D^2,Y]$$
$$-[D_\mu,Y]D_\mu)s^3\Bigg]+\frac{2(p^2)^2}{3d(d+2)}[D^4+(D_\mu D_\nu)^2$$
$$+D_\mu D^2D_\mu]s^4+\cdots\Bigg\}\mathbf{1}, \tag{2.20}$$

where we have used the O(d)-invariant property of the momentum integrals

$$\int_p p_{\mu_1}p_{\mu_2}\cdots p_{\mu_{2m}}e^{-p^2s}=\frac{T^m_{\mu_1\mu_2\cdots\mu_{2m}}\Gamma(d/2)}{2^m\Gamma(m+d/2)}\int_p(p^2)^me^{-p^2s}$$
$$=\frac{T^m_{\mu_1\mu_2\cdots\mu_{2m}}}{(4\pi s)^{d/2}(2s)^m}, \tag{2.21}$$

$$T^m_{\mu_1\mu_2\cdots\mu_{2m}}=\delta_{\mu_1,\mu_2}\cdots\delta_{\mu_{2m-1},\mu_{2m}}+\text{permutations}. \tag{2.22}$$

The approximation, often referred to as the Seeley-DeWitt expansion [22], allows us to parametrize the theory with symmetry-preserving local operators. Since the singularity arising from taking the space-time trace is transferred to the proper-time integration, we insert the regulating smearing function $\rho_k^{(d)}(s,\Lambda)$ into Eq. (2.20) and obtain the corresponding ''blocked'' heat kernel

$$h_k(s;x,x)=\frac{e^{-(\mu^2+Y)s}}{(4\pi s)^{d/2}}\,\rho_k^{(d)}(s,\Lambda)\Bigg\{1+\frac{1}{12}[F_{\mu\nu}F_{\mu\nu}$$
$$-2(D^2Y)]s^2+O(s^3)\Bigg\}, \tag{2.23}$$

which is in agreement with the result found in [21] for $\rho_k^{(d)}(s,\Lambda)=1$. Gauge symmetry is easily seen to be preserved by noting that $h_k(s;x,x)$ consists of gauge invariant quantities only. Had we used momentum cutoff regularization instead, there would be contribution from noninvariant operators D^2, $D_\mu YD_\mu$, YD^2, D^4, and $D_\mu D^2D_\mu$ [15]. Higher order invariant contributions to Eq. (2.23) can also be included, and the details can be found in [16].

III. OPERATOR CUTOFF YANG-MILLS LAGRANGIAN

In this section, we apply the formalism discussed earlier to the Yang-Mill theory. For simplicity, we neglect the matter fields and consider the pure Yang-Mills Lagrangian

$$\mathcal{L}_\Lambda=\frac{Z_\Lambda^{-1}}{4}\,G^a_{\mu\nu}G^a_{\mu\nu}, \tag{3.1}$$

where Z_Λ is the bare Λ-dependent wave function renormalization constant. The field strength $G^a_{\mu\nu}$ is given by

$$G^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu \tag{3.2}$$

or

$$G_{\mu\nu}=gG^a_{\mu\nu}T^a=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu] \tag{3.3}$$

in the matrix-valued representation. To set up the Wilson-Kadanoff RG formalism, the conventional approach is to first go to the momentum space and divide the modes into

$$A^a_\mu(p)=\begin{cases}\overline{A^a_\mu}(p), & 0\le p\le k,\\ \xi^a_\mu(p), & k<p<\Lambda,\end{cases} \tag{3.4}$$

where $\overline{A^a_\mu}$ and ξ^a_μ are, respectively, the slowly varying background and the fast-fluctuating fields. The difficulty associated with such a sharp separation is that it inherently breaks gauge symmetry. On the other hand, in the symmetry-preserving operator cutoff prescription, $\rho_k^{(d)}(s,\Lambda)$ generally corresponds to a smooth regulator with no sharp boundary between the modes. Thus, our interpretation of k as the IR scale is only approximate.

In the next step of RG, we integrate out the irrelevant short-distance (fast-fluctuating) modes ξ having momenta between k and Λ and obtain a low-energy effective blocked action which depends only on the slowly varying background fields $\overline{A}$ with momenta below k. In the background field formalism [23], one introduces

$$\mathcal{L}_{\rm GF}=\frac{1}{2\alpha}\,[D_\mu(\overline{A})A_\mu^a]^2 \tag{3.5}$$

and

$$\mathcal{L}_{\rm FPG}=\chi^\dagger D^2(\overline{A})\chi, \tag{3.6}$$

as the desired gauge-fixing condition and the Faddeev-Popov ghost term, respectively. Adding up the above, the Lagrangian takes on the form

$$\begin{aligned}\mathcal{L}(\overline{A_\mu}+\xi_\mu,\widetilde{\chi}^\dagger+\eta^\dagger,\widetilde{\chi}+\eta)&\\ =\mathcal{L}_\Lambda+\mathcal{L}_{\rm GF}+\mathcal{L}_{\rm FPG}&=\frac{Z_\Lambda^{-1}}{4}\,F_{\mu\nu}^aF_{\mu\nu}^a+\frac{1}{2}\,\xi_\mu^a\\ &\times\left[-Z_\Lambda^{-1}D^2(\overline{A})g_{\mu\nu}+\left(Z_\Lambda^{-1}-\frac{1}{\alpha}\right)D_\mu(\overline{A})D_\nu(\overline{A})\right]^{ab}\xi_\nu^b\\ &-Z_\Lambda^{-1}gf^{acb}\xi_\mu^aF_{\mu\nu}^c\xi_\nu^b+\widetilde{\chi}^{\dagger a}D^2(\overline{A})^{ab}\widetilde{\chi}^b\\ &+\eta^{\dagger a}D^2(\overline{A})^{ab}\eta^b+\delta\mathcal{L}(\overline{A_\mu},\xi_\mu),\end{aligned} \tag{3.7}$$

where $F_{\mu\nu}^a=G_{\mu\nu}^a(\overline{A})$ denotes the background field strength, and $\eta^\dagger$ and η are the fast-fluctuating modes for the ghost fields having momenta between k and Λ. The higher order self-interaction is represented by

$$\delta\mathcal{L}(\overline{A_\mu},\xi_\mu)=Z_\Lambda^{-1}\left[gf^{abc}(D_\mu(\overline{A})\xi_\nu)^a\xi_\mu^b\xi_\nu^c+\frac{1}{4}\,g^2f^{abc}f^{ade}\xi_\mu^b\xi_\nu^c\xi_\mu^d\xi_\nu^e\right]+\cdots\,. \tag{3.8}$$

Notice that $\mathcal{L}$ is now invariant under the simultaneous BRS transformations of A_μ^a, χ, and $\chi^\dagger$. The partition function can be written as

$$\begin{aligned}Z&=\int\mathcal{D}[A_\mu]\mathcal{D}[\chi]\mathcal{D}[\chi^\dagger]e^{-S[\overline{A_\mu}+\xi_\mu,\widetilde{\chi}^\dagger+\eta^\dagger,\widetilde{\chi}+\eta]}\\ &=\int\mathcal{D}[\overline{A}]\mathcal{D}[\widetilde{\chi}]\mathcal{D}[\widetilde{\chi}^\dagger]e^{-\widetilde{S}_k[\overline{A_\mu},\widetilde{\chi}^\dagger,\widetilde{\chi}]},\end{aligned} \tag{3.9}$$

where

$$e^{-\widetilde{S}_k[\overline{A_\mu},\widetilde{\chi}^\dagger,\widetilde{\chi}]}=\int\mathcal{D}[\xi_\mu]\mathcal{D}[\eta]\mathcal{D}[\eta^\dagger]e^{-S[\overline{A_\mu}+\xi_\mu,\widetilde{\chi}^\dagger+\eta^\dagger,\widetilde{\chi}+\eta]}. \tag{3.10}$$

By substituting Eq. (3.7) into Eq. (3.10) and dropping higher order fluctuating terms, the operator cutoff regularized blocked action up to the one-loop order reads

$$\begin{aligned}\widetilde{S}_k[\overline{A},\widetilde{\chi}^\dagger,\widetilde{\chi}]=\frac{Z_\Lambda^{-1}}{4}\int_xF_{\mu\nu}^aF_{\mu\nu}^a+\frac{1}{2}\,{\rm Tr_{oc}}[\ln\mathcal{K}_{\overline{A}}-\ln\mathcal{K}_0]\\ -{\rm Tr_{oc}}[\ln\mathcal{O}_{\overline{A}}-\ln\mathcal{O}_0]+\widetilde{\chi}^{\dagger a}D^2(\overline{A})^{ab}\widetilde{\chi}^b,\end{aligned} \tag{3.11}$$

where, by the help of

$$D_\mu^{ab}(\overline{A})D_\nu^{bc}(\overline{A})-D_\nu^{ab}(\overline{A})D_\mu^{bc}(\overline{A})=gf^{abc}F_{\mu\nu}^b, \tag{3.12}$$

the gauge and the ghost kernels become

$$\begin{aligned}\mathcal{K}_{\mu\nu,\overline{A}}^{ab}&=\frac{\partial^2S}{\partial A_\mu^a(x)\partial A_\nu^b(y)}\bigg|_{\overline{A},\widetilde{\chi}^\dagger=\widetilde{\chi}=0}\\ &=\left\{-\left[D^2g_{\mu\nu}-\left(1-\frac{1}{\alpha}\right)D_\mu D_\nu\right]^{ab}+2gf^{abc}F_{\mu\nu}^c\right\}\delta^4(x-y)\end{aligned} \tag{3.13}$$

and

$$\mathcal{O}_{\overline{A}}^{ab}=-\frac{\partial^2S}{\partial\chi^{\dagger a}\partial\chi^b}\bigg|_{\overline{A},\widetilde{\chi}^\dagger=\widetilde{\chi}=0}=-D^2(\overline{A})^{ab}\delta^4(x-y), \tag{3.14}$$

respectively. In the above, the covariant derivatives are understood to be defined at $\overline{A}$. Within the accuracy of the one-loop approximation, we also set Z_Λ^{-1} in $\mathcal{K}_{\mu\nu}^{ab}$ to unity. Here ${\rm Tr_{oc}}$ denotes the trace sum over the (operator cutoff regularized) space-time, Lorentz indices, as well as the color indices and tr is for the latter two only. When no confusion arises, internal indices shall be suppressed for brevity.

The fluctuation operator (3.13) in momentum space can be written as

$$\mathcal{K}_{\mu\nu,0}^{ab}=p^2\delta^{ab}\left(T_{\mu\nu}+\frac{1}{\alpha}\,L_{\mu\nu}\right), \tag{3.15}$$

where

$$T_{\mu\nu}=g_{\mu\nu}-\frac{p_\mu p_\nu}{p^2},\quad L_{\mu\nu}=\frac{p_\mu p_\nu}{p^2}, \tag{3.16}$$

with $(T_{\mu\nu})^2=T_{\mu\nu}$ and $(L_{\mu\nu})^2=L_{\mu\nu}$. The propagator, defined as the inverse of $\mathcal{K}_{\mu\nu,0}^{ab}$, is

$$d_{\mu\nu}^{ab}=\frac{\delta^{ab}}{p^2}\,(T_{\mu\nu}+\alpha L_{\mu\nu}). \tag{3.17}$$

Similarly, for the ghost propagator, one has

$$\Delta^{ab}=\frac{\delta^{ab}}{p^2}. \tag{3.18}$$

Using Eqs. (2.8) and (2.9) and setting $k=0$, we now have the operator cutoff regularized propagators (denoted with a tilde)

$$d_{\mu\nu}^{ab}\to\widetilde{d}_{\mu\nu}^{ab}=\frac{\delta^{ab}}{p^2}\,(a_1T_{\mu\nu}+a_2L_{\mu\nu}) \tag{3.19}$$

and

$$\Delta^{ab}\to\widetilde{\Delta}^{ab}=a_1\,\frac{\delta^{ab}}{p^2}, \tag{3.20}$$

where

$$a_1=\left(1+\frac{p^2}{\Lambda^2}\right)^{-2},\quad a_2=\left(1+\frac{1}{\alpha}\frac{p^2}{\Lambda^2}\right)^{-2}. \tag{3.21}$$

This choice also leads to a ''factorization'' of the propagators. We also see that in the Feynman gauge where $\alpha=1$, the gauge field propagator simplifies to $d_{\mu\nu}^{ab}=a_1\delta^{ab}g_{\mu\nu}/p^2$. In addition, the original propagators are recovered in the limit $\Lambda\to\infty$.

What is the form of the effective Lagrangian which reproduces these effective propagators? Certainly the original gauge sector must be changed to

$$\mathcal{L}_\Lambda\to\widetilde{\mathcal{L}}_\Lambda=\frac{Z_\Lambda^{-1}}{4}G_{\mu\nu}^a\left(1-\frac{D^2(\overline{A})}{\Lambda^2}\right)^2G_{\mu\nu}^a. \tag{3.22}$$

Similarly, one may show that

$$\mathcal{L}_{\rm GF}\to\widetilde{\mathcal{L}}_{\rm GF}=\frac{1}{2\alpha}[D_\mu(\overline{A})A_\mu^a]\left(1-\frac{1}{\alpha}\frac{D^2(\overline{A})}{\Lambda^2}\right)^2[D_\mu(\overline{A})A_\mu^a], \tag{3.23}$$

and

$$\mathcal{L}_{\rm FPG}\to\widetilde{\mathcal{L}}_{\rm FPG}=\chi^\dagger\left(1-\frac{D^2(\overline{A})}{\Lambda^2}\right)^2D^2(\overline{A})\chi. \tag{3.24}$$

Adding these all up, the operator cutoff regularized Lagrangian becomes

$$\mathcal{L}_\Lambda^{\rm oc}=\frac{Z_\Lambda^{-1}}{4}G_{\mu\nu}^a\left(1-\frac{D^2(\overline{A})}{\Lambda^2}\right)^2G_{\mu\nu}^a+\frac{1}{2\alpha}[D_\mu(\overline{A})A_\mu^a]\times\left(1-\frac{1}{\alpha}\frac{D^2(\overline{A})}{\Lambda^2}\right)^2[D_\mu(\overline{A})A_\mu^a]+\chi^\dagger\left(1-\frac{D^2(\overline{A})}{\Lambda^2}\right)^2\times D^2(\overline{A})\chi. \tag{3.25}$$

One may verify that $\mathcal{L}^{\rm eff}$ indeed yields the desired propagators $\widetilde{d}_{\mu\nu}^{ab}$ and $\widetilde{\Delta}^{ab}$ given in Eqs. (3.19) and (3.20), and the original symmetry is fully preserved.

In the large p limits, both $\widetilde{d}_{\mu\nu}^{ab}$ and $\widetilde{\Delta}^{ab}$ behave as $1/p^6$, which makes the theory superrenormalizable. However, the regularization is only partial and the one-loop divergences remain unregularized, as can be seen from the power counting of the superficial degree of divergence ω_Λ [7]:

$$\omega_\Lambda=4-4(L-1)-E_A-\frac{7}{2}E_g, \tag{3.26}$$

where L, E_A, and E_g are the numbers of loops, external gauge field lines, and external ghost field lines, respectively. A complete removal of the one-loop divergences would require an auxiliary regulator which was taken to be Pauli-Villars by Slavnov [6]. The infinities in the Pauli-Villars procedure are controlled by modifying the one-loop contribution as (see Appendix B):

$$\mathrm{Tr}\ln\left(\frac{\mathcal{H}}{\mathcal{H}_0}\right)\to\mathrm{Tr}_{\rm pv}\ln\left(\frac{\mathcal{H}}{\mathcal{H}_0}\right)=\mathrm{Tr}\ln\left[\left(\frac{\mathcal{H}+k^2}{\mathcal{H}_0+k^2}\right)\left(\frac{\mathcal{H}_0+\Lambda^2}{\mathcal{H}+\Lambda^2}\right)\right], \tag{3.27}$$

which, apart from some higher-order contributions, is precisely what one obtains in Eq. (2.9) using the operator cutoff regularization. In fact, it is possible to choose $\rho_k^{(d)}(s,\Lambda)$ which completely simulates the Pauli-Villars. Thus, operator cutoff regularization encompasses both features of higher covariant derivative method and the Pauli-Villars. The freedom to choose $\rho_k^{(d)}(s,\Lambda)$ makes it a more general invariant prescription as that by Slavnov.

IV. PERTURBATIVE EXPANSIONS

Since the Yang-Mills blocked action is generally a complicated object even at the one-loop level, an approximate solution exists only in certain energy regime. The renown property of asymptotic freedom in the large momentum limit allows for the small s expansion of the heat kernel. However, to preserve gauge symmetry, the expansion should be carried out in a manner such that all the coefficients consist of gauge-invariant quantities only. This is achieved by employing the Schwinger-DeWitt covariant derivative expansion method [24]. On the other hand, a noncovariant prescription such as expanding the theory in power series of the noninvariant background field A is incompatible with gauge symmetry. Below we describe both methods and illustrate how the symmetry is violated by the latter.

A. Covariant derivative expansion

To approximate the one-loop blocked action in Eq. (3.11) using covariant derivative expansion, we apply the procedures outlined in Sec. II. The details have been worked out by D'yakonov *et al.* in [25], and we recapitulate here to elucidate its connection with the operator cutoff regularization.

For the ghost kernel, one has

$$\begin{aligned}\mathrm{Tr}_{\rm oc}\ln\left(\frac{\mathcal{O}_{\overline{A}}}{\mathcal{O}_0}\right)&=-\int_x\int_0^\infty\frac{ds}{s}\rho_k^{(d)}(s,\Lambda)\mathrm{tr}\langle x|(e^{-\mathcal{O}_{\overline{A}}s}-e^{-\mathcal{O}_0s})|x\rangle\\&=-\int_x\int_0^\infty\frac{ds}{s}\rho_k^{(d)}(s,\Lambda)\mathrm{tr}\\&\quad\times\int_p e^{-p^2s}\sum_{n=1}^\infty\frac{s^n}{n!}(2ip\cdot D+D^2)^n\mathbf{1}\\&=-\frac{1}{12(4\pi)^{d/2}}\int_x\int_0^\infty ds\,s^{1-d/2}\rho_k^{(d)}\\&\quad\times(s,\Lambda)\mathrm{tr}\left\{[D_\mu,D_\nu]^2+\frac{s}{15}[-6O_1+O_2+4O_3+3O_4\right.\\&\quad\left.+3O_5]+O(s^2)\right\},\end{aligned} \tag{4.1}$$

where

$$O_1=[D_\mu,D_\nu][D_\nu,D_\rho][D_\rho,D_\mu],$$
$$O_2=[D_\mu,[D_\mu,D_\nu]][D_\rho,[D_\rho,D_\nu]],$$
$$O_3=[D_\mu,[D_\nu,D_\rho]][D_\mu,[D_\nu,D_\rho]],$$
$$O_4=[D_\mu,[D_\mu,[D_\nu,D_\rho]]][D_\nu,D_\rho],$$
$$O_5=[D_\nu,D_\rho][D_\mu,[D_\mu,[D_\nu,D_\rho]]]. \tag{4.2}$$

Noting that $[D_\mu,D_\nu]^{ab}=gf^{acb}F^c_{\mu\nu}$ and $f^{abc}f^{abd}=\delta^{cd}C_2(G)$ where $C_2(G)$ is a Casimir operator with $C_2(G)=N$ for $G=\mathrm{SU}(N)$, the trace over the internal indices can now be summed and one obtains

$$\mathrm{tr}[D_\mu,D_\nu][D_\mu,D_\nu]=-C_2(G)\mathcal{F}_2,$$
$$\mathrm{tr}O_1=-\frac{C_2(G)}{2}\mathcal{F}_3,\quad \mathrm{tr}O_2=-C_2(G)\mathcal{I}_3,$$
$$\mathrm{tr}O_3=-\mathrm{tr}O_4=-\mathrm{tr}O_5=2C_2(G)(g\mathcal{F}_3-\mathcal{I}_3), \tag{4.3}$$

where

$$\mathcal{F}_2=g^2F^a_{\mu\nu}F^a_{\mu\nu},$$
$$\mathcal{F}_3=g^3f^{abc}F^a_{\mu\nu}F^b_{\nu\rho}F^c_{\rho\mu},$$
$$\mathcal{I}_3=g^2(D^{ab}_\mu F^b_{\mu\nu})(D^{ac}_\rho F^c_{\rho\nu}), \tag{4.4}$$

are the leading-order local gauge-invariant operators. Notice that $\mathcal{I}_3$ is identically zero if $F^b_{\mu\nu}$ satisfies the classical equation of motion $D^{ab}_\mu F^b_{\mu\nu}=0$. Substituting the above expressions into Eq. (4.1) gives

$$\mathrm{Tr}_{\mathrm{oc}}\ln\left(\frac{\mathcal{O}_{\overline{A}}}{\mathcal{O}_0}\right)=\frac{C_2(G)}{12(4\pi)^{d/2}}\int_x\int_0^\infty ds\, s^{1-d/2}\rho_k^{(d)}(s,\Lambda)$$
$$\times\left\{\mathcal{F}_2+\frac{1}{15}(\mathcal{F}_3-3\mathcal{I}_3)s+O(s^2)\right\}. \tag{4.5}$$

Repeating the same procedure for the gauge kernel, we have

$$\mathrm{Tr}_{\mathrm{oc}}\ln\left(\frac{\mathcal{K}_{\overline{A}}}{\mathcal{K}_0}\right)=-\frac{C_2(G)}{3(4\pi)^{d/2}}\int_x\int_0^\infty ds\, s^{1-d/2}\rho_k^{(d)}(s,\Lambda)$$
$$\times\left\{5\mathcal{F}_2-\frac{1}{15}(\mathcal{F}_3+27\mathcal{I}_3)s+O(s^2)\right\}. \tag{4.6}$$

Thus, the one-loop contribution to $\widetilde{S}_k$ can be written as

$$\widetilde{S}_k^{(1)}=-\frac{C_2(G)}{12(4\pi)^{d/2}}\int_x\int_0^\infty ds\, s^{1-d/2}\rho_k^{(d)}(s,\Lambda)$$
$$\times\left\{11\mathcal{F}_2-\frac{1}{15}(\mathcal{F}_3+57\mathcal{I}_3)s+O(s^2)\right\}. \tag{4.7}$$

For $d=4$, the s integration can be carried out using

$$\int_0^\infty ds\, s^n\rho_k^{(4)}(s,\Lambda)=\begin{cases}(n+2)\Gamma(n+1)(k^{-2(n+1)}-\Lambda^{-2(n+1)}), & n\geqslant 0,\\ \ln\left(\frac{\Lambda^2}{k^2}\right), & n=-1,\end{cases} \tag{4.8}$$

and the invariant blocked action up to the one-loop order can be written as $\widetilde{S}_k=\int_x\widetilde{\mathcal{L}}_k$, where the effective blocked Lagrangian reads

$$\widetilde{\mathcal{L}}_k=\frac{1}{4g^2}\left[Z_\Lambda^{-1}-\frac{11g^2C_2(G)}{48\pi^2}\ln\left(\frac{\Lambda^2}{k^2}\right)\right]\mathcal{F}_2+\frac{C_2(G)}{1440\pi^2}\frac{1}{k^2}$$
$$\times(\mathcal{F}_3+57\mathcal{I}_3)+\widetilde{\chi}^{\dagger a}D^2(\overline{A})^{ab}\widetilde{\chi}^b+\cdots$$
$$=\frac{\mathcal{Z}_k^{-1}}{4g^2}\mathcal{F}_2+\frac{C_2(G)}{1440\pi^2}\frac{1}{k^2}(\mathcal{F}_3+57\mathcal{I}_3)+\widetilde{\chi}^{\dagger a}D^2(\overline{A})^{ab}\widetilde{\chi}^b$$
$$+\cdots, \tag{4.9}$$

with

$$\mathcal{Z}_k^{-1}=Z_\Lambda^{-1}-\frac{11g^2C_2(G)}{48\pi^2}\ln\left(\frac{\Lambda^2}{k^2}\right)$$
$$=1+\frac{11g^2C_2(G)}{48\pi^2}\ln\left(\frac{k^2}{\mu^2}\right). \tag{4.10}$$

Notice that the logarithmic divergence associated with $\mathcal{F}_2$ is now cancelled by setting

$$Z_\Lambda^{-1}=1+\frac{11g^2C_2(G)}{48\pi^2}\ln\left(\frac{\Lambda^2}{\mu^2}\right). \tag{4.11}$$

Several comments are in order. First of all, since the gauge fields are massless, the renormalization condition is defined at an off-shell subtraction point $\widetilde{\mu}=\mu/p$, i.e., $g_{k=\mu}=g$. The effective blocked Lagrangian obtained in Eq. (4.9) is completely local, with the higher dimensional invariant operators being suppressed by at least k^2 in the large k limit. In fact, our local approximation is analogous to the inverse mass expansion considered in [16]. Equation (4.9) can also be compared with the low-energy blocked Lagrangian of the scalar theory in powers of derivatives

$$\widetilde{\mathcal{L}}_k=\sum_{n=0}^\infty Z_k^{(2n)}(\Phi)(\partial_\mu\Phi)^{(2n)}=U_k(\Phi)+\frac{\mathcal{Z}_k(\Phi)}{2}(\partial_\mu\Phi)^2$$
$$+Y_k^1(\Phi)(\partial_\mu\Phi)^4+Y_k^2(\Phi)(\Phi\partial^2\Phi)^2$$
$$+Y_k^3(\Phi)\Phi^2(\partial_\mu\partial_\nu\Phi)(\partial_\mu\partial_\nu\Phi)+O(\partial^6), \tag{4.12}$$

with an important distinction that the latter is applicable only in the small k limit.

B. Noncovariant expansion

Instead of using the covariant derivative expansion which allows us to approximate Eqs. (3.13) and (3.14) with gauge-invariant operators, let us inquire what happens when the expansion parameter is the background field $\overline{A^a_\mu}$ which is a noninvariant quantity. Following the perturbative formalism developed in [4] and working in momentum space where $\partial_\mu \to ip_\mu$, Eqs. (3.13) and (3.14) can be rewritten as [26]

$$\mathcal{K}^{ab}_{\mu\nu} = \left\{ -\left[D^2 g_{\mu\nu} - \left(1-\frac{1}{\alpha}\right) D_\mu D_\nu \right]^{ab} + 2gf^{abc}F^c_{\mu\nu} \right\} \delta^4(x-y)$$
$$\to g_{\mu\nu}[p^2\delta^{ab} - igf^{acb}(p\cdot\overline{A^c} + \overline{A^c}\cdot p) - g^2 f^{amc}f^{clb}\overline{A^m_\lambda}\,\overline{A^l_\lambda}] + 2gf^{abc}F^c_{\mu\nu} - \left(1-\frac{1}{\alpha}\right)$$
$$\times [p_\mu p_\nu \delta^{ab} - igf^{acb}(p_\mu \overline{A^c_\nu} + \overline{A^c_\mu} p_\nu) - g^2 f^{amc}f^{clb}\overline{A^m_\mu}\,\overline{A^l_\nu}], \tag{4.13}$$

and

$$\mathcal{O}^{ab} = -(D^2)^{ab}\delta^4(x-y) \to p^2\delta^{ab} - igf^{acb}(p\cdot\overline{A^c} + \overline{A^c}\cdot p) - g^2 f^{amc}f^{clb}\overline{A^m_\lambda}\,\overline{A^l_\lambda}. \tag{4.14}$$

By splitting the kernel as $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$ where $\mathcal{H}_I$ accounts for the interactions, an approximation in powers of $\mathcal{H}_I$ can be obtained as

$$\mathrm{Tr}(e^{-\mathcal{H}s}) = \mathrm{Tr}(e^{-\mathcal{H}_0 s}) + \int_0^\infty d\lambda\ \mathrm{Tr}(\mathcal{H}_I e^{-(\mathcal{H}_0 + \lambda\mathcal{H}_I)s})$$
$$= \mathrm{Tr}\left\{ e^{-\mathcal{H}_0 s} + (-s)e^{-\mathcal{H}_0 s}\mathcal{H}_I + \frac{(-s)^2}{2}\int_0^1 du_1 e^{-(1-u_1)\mathcal{H}_0 s}\mathcal{H}_I e^{-u_1\mathcal{H}_0 s}\mathcal{H}_I + \frac{(-s)^3}{3}\int_0^1 du_1 u_1 \int_0^1 du_2 \right.$$
$$\left. \times e^{-(1-u_1)\mathcal{H}_0 s}\mathcal{H}_I e^{-u_1(1-u_2)\mathcal{H}_0 s}\mathcal{H}_I e^{-u_1 u_2 \mathcal{H}_0 s}\mathcal{H}_I + \cdots \right\}. \tag{4.15}$$

Applying the above expansion formula to Eqs. (4.13) and (4.14) and keeping only the terms which are quadratic in $\overline{A}$ gives

$$\mathrm{Tr}(e^{-\mathcal{O}^{ab}s}) = \mathrm{Tr}\,\delta^{ab}\left\{ e^{-p^2 s} - s e^{-p^2 s}(-g^2 f^{amc}f^{clb}\overline{A^m_\lambda}\,\overline{A^l_\lambda}) + \frac{s^2}{2}\int_0^1 du e^{(1-u)p^2 s}[-igf^{acl}(p\cdot\overline{A^c} + \overline{A^c}\cdot p)] \right.$$
$$\left. \times e^{-up^2 s}[-igf^{lmb}(p\cdot\overline{A^m} + \overline{A^m}\cdot p)] + \cdots \right\}, \tag{4.16}$$

and, in Feynman gauge where $\alpha = 1$,

$$\mathrm{Tr}(e^{-\mathcal{K}^{ab}_{\mu\nu}s}) = \mathrm{Tr}\,\delta^{ab}\left\{ g_{\mu\nu}e^{-p^2 s} - s e^{-p^2 s} \times (-g^2 f^{amc}f^{clb}g_{\mu\nu}\overline{A^m_\lambda}\,\overline{A^l_\lambda}) + \frac{s^2}{2}\int_0^1 du\, e^{(1-u)p^2 s} \right.$$
$$\times [-igf^{acl}g_{\mu\rho}(p\cdot\overline{A^c} + \overline{A^c}\cdot p) - 2gf^{acl}F^c_{\mu\rho}(\overline{A})]e^{-up^2 s}[-igf^{lmb}g_{\rho\nu}(p\cdot\overline{A^m}$$
$$\left. + \overline{A^m}\cdot p) - 2gf^{lmb}F^m_{\rho\nu}(\overline{A})] + \cdots \right\}. \tag{4.17}$$

Although the above expressions are no longer gauge invariant, let us proceed and see how the symmetry can be recovered.

Concentrating on the $d=4$ case and inserting a complete orthonormal set of momentum states $|p\rangle$ satisfying

$$\int_p |p\rangle\langle p| = 1, \quad \langle p|p'\rangle = (2\pi)^4\delta^4(p-p'),$$
$$\langle x|p\rangle = e^{ip\cdot x}, \quad \langle p|A_\mu|p'\rangle = A_\mu(p-p'), \tag{4.18}$$

we obtain

$$\mathrm{Tr}\int_0^1 du\ e^{-(1-u)p^2 s}(p\cdot\overline{A^c} + \overline{A^c}\cdot p)e^{-up^2 s}(p\cdot\overline{A^c} + \overline{A^c}\cdot p)$$
$$= \mathrm{tr}\int_0^1 du \int_{p,q} e^{-[(1-u)p^2 + uq^2]s}(p+q)_\mu (p+q)_\nu \times \overline{A^c_\mu}(p-q)\overline{A^c_\nu}(q-p)$$
$$= \frac{1}{16\pi^2 s^2}\,\mathrm{tr}\int_0^1 du \int_p \overline{A^c_\mu}(p)\overline{A^c_\nu}(-p)e^{-u(1-u)p^2 s} \times \left[\frac{2g_{\mu\nu}}{s} + (2u-1)^2 p_\mu p_\nu\right]. \tag{4.19}$$

The above expression is arrived at by first shifting the variable $p \to p+q$ followed by $q \to q-(1-u)p$, and the q integration using O(4) invariance. Regularizing the integral with $\rho_k^{(4)}(s,\Lambda)$ in the operator cutoff formalism then gives

$$\mathrm{Tr}_{\mathrm{oc}}\ln\mathcal{O}=\frac{g^2C_2(G)}{16\pi^2}\int_p\overline{A^c_\mu}(p)\overline{A^c_\nu}(-p)\bigg\{g_{\mu\nu}\int_0^\infty\frac{ds}{s^2}\,\rho_k^{(4)}(s,\Lambda)-\frac{1}{2}\int_0^\infty\frac{ds}{s}\,\rho_k^{(4)}(s,\Lambda)\times\int_0^1 du\,e^{-u(1-u)p^2s}\bigg[\frac{2g_{\mu\nu}}{s}+(2u-1)^2p_\mu p_\nu\bigg]\bigg\}$$

$$=\frac{g^2C_2(G)}{32\pi^2}\int_p F^a_{\mu\nu}(p)F^a_{\mu\nu}(-p)\int_0^\infty\frac{ds}{s^2}\,\rho_k^{(4)}(s,\Lambda)\,\frac{1}{p^2}\left\{1-\frac{\sqrt{\pi}}{ps^{1/2}}\,e^{-p^2s/4}\mathrm{Erfi}\left(\frac{ps^{1/2}}{2}\right)\right\},\tag{4.20}$$

where $\mathrm{Erfi}(x)=(2/\sqrt{\pi})\int_0^x dt\ e^{t^2}$, and we have used the approximation

$$\frac{1}{2}\,F^c_{\mu\nu}(p)F^c_{\mu\nu}(-p)=(p^2g_{\mu\nu}-p_\mu p_\nu)\overline{A^c_\mu}(p)\overline{A^c_\nu}(-p)+\cdots\ .\tag{4.21}$$

The substitution can be justified by noting that the differences are of higher order in $\overline{A}$. Similarly, the gauge field contribution reads

$$\mathrm{Tr}_{\mathrm{oc}}\ln\mathcal{K}=\frac{g^2C_2(G)}{4\pi^2}\int_p\overline{A^c_\mu}(p)\overline{A^c_\nu}(-p)g_{\mu\nu}\int_0^\infty\frac{ds}{s^2}\,\rho_k^{(4)}(s,\Lambda)-\frac{1}{2}\,\mathrm{Tr}\int_0^\infty ds\,s\rho_k^{(4)}(s,\Lambda)\times\bigg(\int_0^1 du\,e^{-(1-u)p^2s}\delta^{ab}\{-igf^{acl}[(p\cdot\overline{A^c}+\overline{A^c}\cdot p)g_{\mu\rho}-2iF^c_{\mu\rho}]\}\times e^{-up^2s}\{-igf^{lmb}[(p\cdot\overline{A^m}+\overline{A^m}\cdot p)g_{\rho\nu}-2iF^m_{\rho\nu}]\}\bigg)$$

$$=4\,\mathrm{Tr}_{\mathrm{oc}}\ \ln\,\mathcal{O}(\overline{A})-\frac{g^2C_2(G)}{8\pi^2}\int_p F^c_{\mu\rho}(p)F^c_{\mu\rho}(-p)\int_0^1 du\int_0^\infty\frac{ds}{s}\,\rho_k^{(4)}(s,\Lambda)e^{-u(1-u)p^2s}$$

$$=\frac{g^2C_2(G)}{8\pi^2}\int_p F^a_{\mu\nu}(p)F^a_{\mu\nu}(-p)\int_0^\infty\frac{ds}{s}\,\rho_k^{(4)}(s,\Lambda)\left\{1-\left(1+\frac{1}{p^2s}\right)\frac{\sqrt{\pi}}{ps^{1/2}}\,e^{-p^2s/4}\,\mathrm{Erfi}\left(\frac{ps^{1/2}}{2}\right)\right\}.\tag{4.22}$$

Adding up these terms, the perturbative Yang-Mills blocked action becomes

$$\widetilde{S}_k=\frac{Z_\Lambda^{-1}}{4}\int_p F^a_{\mu\nu}F^a_{\mu\nu}+\frac{1}{2}\,\mathrm{Tr}_{\mathrm{oc}}\ln\mathcal{K}(\overline{A})-\mathrm{Tr}_{\mathrm{oc}}\ln\mathcal{O}(\overline{A})$$

$$=\frac{1}{4}\int_p\widetilde{\mathcal{Z}}_k^{-1}(p)F^a_{\mu\nu}(p)F^a_{\mu\nu}(-p),\tag{4.23}$$

where

$$\widetilde{\mathcal{Z}}_k^{-1}(p)=Z_\Lambda^{-1}+\frac{g^2C_2(G)}{8\pi^2}\int_0^\infty\frac{ds}{s}\,\rho_k^{(4)}(s,\Lambda)\times\left\{\frac{1}{p^2s}-\left(2+\frac{1}{p^2s}\right)\frac{\sqrt{\pi}}{ps^{1/2}}\,e^{-p^2s/4}\,\mathrm{Erfi}\left(\frac{ps^{1/2}}{2}\right)\right\}.\tag{4.24}$$

Here it becomes apparent that the singularities arising from evaluating the space-time trace is being transformed into divergences in the s integral and can be readily regulated by $\rho_k^{(4)}(s,\Lambda)$. Once more, this illustrates the importance of retaining the s integration till the end in order not to break gauge symmetry during the course of regularization [15]. However, the appearance of the integral function $\mathrm{Erfi}(ps^{1/2}/2)$ not only forbids the s integral to be completed, but is also indicative of nonlocality, as can be seen from the p dependence in the argument.

Since the theory is dominated by local operators in the high-energy regime, one must recover the standard perturbative RG coefficient functions from Eq. (4.24) in that limit. By making use of the relation

$$\int_0^1 du\ e^{-u(1-u)p^2s}=\frac{\sqrt{\pi}}{ps^{1/2}}\,e^{-p^2s/4}\,\mathrm{Erfi}\left(\frac{ps^{1/2}}{2}\right),\tag{4.25}$$

and exchanging the order of integrations between u and s, Eq. (4.24) can be rewritten as

$$\widetilde{\mathcal{Z}}_k^{-1}(p)=Z_\Lambda^{-1}-\frac{g^2C_2(G)}{48\pi^2}\bigg\{11\,\ln\left(\frac{\widetilde{\Lambda}^2}{\widetilde{k}^2}\right)-4(\widetilde{\Lambda}^2-\widetilde{k}^2)-(8\widetilde{\Lambda}^4+22\widetilde{\Lambda}^2+11)f(\widetilde{\Lambda})+(8\widetilde{k}^4+22\widetilde{k}^2+11)f(\widetilde{k})\bigg\},\tag{4.26}$$

where Z_Λ^{-1} is given by Eq. (4.11), $\tilde{\Lambda}=\Lambda/p$, $\tilde{k}=k/p$ and

$$f(x)=\frac{1}{\sqrt{4x^2+1}}\ln\left(\frac{\sqrt{4x^2+1}-1}{\sqrt{4x^2+1}+1}\right)\rightarrow\begin{cases}-\dfrac{1}{2x^2}, & x\rightarrow\infty,\\ \ln(x^2), & x\rightarrow 0.\end{cases}\tag{4.27}$$

In the limit $\tilde{\Lambda}\rightarrow\infty$, the expression simplifies to

$$\tilde{\mathcal{Z}}_k^{-1}=1-\frac{g^2C_2(G)}{48\pi^2}\left\{11\ln\left(\frac{\tilde{\mu}^2}{\tilde{k}^2}\right)+\frac{31}{3}+4\tilde{k}^2+(8\tilde{k}^4+22\tilde{k}^2+11)f(\tilde{k})\right\}=\mathcal{Z}_k^{-1}+\delta\tilde{\mathcal{Z}}_k^{1},\tag{4.28}$$

where $\mathcal{Z}_k^{-1}$ is the same as that derived in Eq. (4.10) using covariant expansion, and

$$\delta\tilde{\mathcal{Z}}_k^{-1}=-\frac{g^2C_2(G)}{48\pi^2}\left[\frac{31}{3}+4\tilde{k}^2+(8\tilde{k}^4+22\tilde{k}^2+11)f(\tilde{k})\right],\tag{4.29}$$

represents the additional nonlocal contribution. The appearance of the Λ^2-dependent contributions in Eq. (4.26) is a consequence of using a noninvariant regularization. However, in the limit $\Lambda\rightarrow\infty$, the quadratic divergences cancel each other. From considering $\mathcal{Z}_k^{-1}$ alone, one also recognizes the familiar factor of $-11g^2C_2(G)/48\pi^2$ associated with the $\ln k^2$ term, with $-10g^2C_2(G)/48\pi^2$ coming from the gauge kernel $\mathrm{Tr}_{\mathrm{oc}}\ln\mathcal{K}/2$ and $-g^2C_2(G)/48\pi^2$ from the ghost sector $\mathrm{Tr}_{\mathrm{oc}}\ln\mathcal{O}$.

In passing, we remark that the presence of nonlocality is generally characteristic of low-energy effective theories irrespective of how it is regularized. Equation (4.29) contains a nonlocal sector $\delta\tilde{\mathcal{Z}}_k^{-1}$ due to the use of the noninvariant expansion parameter A. This can be contrasted with the method of covariant derivative expansion in which the blocked action is parametrized by local gauge-invariant operators. However, in the large $\tilde{k}$ limit, $\delta\tilde{\mathcal{Z}}_k^{-1}$ becomes identically zero and the nonlocal effect completely disappears. This is not to say that nonlocal operators are incompatible with gauge invariance. In fact, the Wilson loops commonly encountered in lattice gauge theories are nonlocal invariant operators.

V. RENORMALIZATION GROUP EQUATIONS

A. Coupling constant and the wave function renormalization

We now examine the RG flow pattern of the Yang-Mills blocked action derived in the last section. Let us first focus on the result obtained from the noncovariant expansion. In the large k limit where $\delta\tilde{\mathcal{Z}}_k^{-1}$ vanishes, by using the Slavnov-Taylor identities, $\mathcal{Z}_k^{-1}$ can be related to the coupling constant renormalization by

$$Z_\Lambda^{-1}g_\Lambda^2=\mathcal{Z}_k^{-1}g_k^2=g^2,\tag{5.1}$$

with g_Λ being the cutoff-dependent bare coupling constant. This implies the following running behavior for g_k:

$$\frac{1}{g_k^2}=\frac{\mathcal{Z}_k^{-1}}{g^2}=\frac{1}{g^2}+\frac{11C_2(G)}{48\pi^2}\ln\left(\frac{k^2}{\mu^2}\right).\tag{5.2}$$

At $k=\Lambda$ and $k=\mu$, g_k readily reduces to g_Λ and the off-shell renormalized g, respectively. The interpolation is expected from perturbation theory. Similarly, the one-loop β function reads

$$\beta(g_k)=k\,\frac{\partial g_k}{\partial k}=-\frac{11C_2(G)}{48\pi^2}\,g_k^3,\tag{5.3}$$

exhibiting the well-known property of asymptotic freedom. From Eq. (5.1), one also obtains

$$\gamma(g_k)=k\,\frac{\partial\ln\mathcal{Z}_k}{\partial k}=\frac{2\beta(g_k)}{g_k}=-\frac{11C_2(G)}{24\pi^2}\,g_k^2.\tag{5.4}$$

While the contribution from $\delta\tilde{Z}_k^{-1}$ is vanishingly small in the large k limit, the contribution it generates in general cannot be neglected in the small k regime where g_k becomes large. Thus, if we consider not just Z_k^{-1} but $\tilde{Z}_k^{-1}$, we then have the following nonlocal running coupling constant (denoted with a tilde):

$$\frac{1}{\tilde{g}_{\tilde{k}}^2}=\frac{\tilde{\mathcal{Z}}_k^{-1}}{g^2}=\frac{1}{g^2}-\frac{C_2(G)}{48\pi^2}\left[11\ln\left(\frac{\mu^2}{k^2}\right)+\frac{31}{3}+4\tilde{k}^2+(8\tilde{k}^4+22\tilde{k}^2+11)f(\tilde{k})\right],\tag{5.5}$$

which implies

$$\tilde{\beta}(\tilde{g}_{\tilde{k}},\tilde{k})=\tilde{k}\,\frac{\partial\tilde{g}_{\tilde{k}}}{\partial\tilde{k}}=\frac{C_2(G)}{8\pi^2}\,\tilde{g}_{\tilde{k}}^3\,\frac{\tilde{k}^2}{4\tilde{k}^2+1}\times\{-3+4\tilde{k}^2+2\tilde{k}^2(5+4\tilde{k}^2)f(\tilde{k})\}.\tag{5.6}$$

The above flow equation illustrates how nonlocality in $\tilde{g}_{\tilde{k}}$ is developed as k decreases. Notice that the expression inside the braces is always negative, as required by asymptotic freedom. In the large $\tilde{k}$ regime, however, Eqs. (5.5) and (5.6) reduce to the usual perturbative results (5.2) and (5.3), respectively. On the other hand, in the covariant derivative expansion approach, nonlocal effect is completely absent and only Eq. (5.3) is obtained. It remains an interesting issue to explore how the nonperturbative, strong-coupling physics in the low-energy regime is influenced by nonlocal operators.

We comment that in calculating the RG coefficient functions using either covariant or noncovariant expansion in the operator cutoff formalism, no unphysical contributions arise to modify the expected results. This is in accord with the observation that the Slavnov regularization which combines the method of higher covariant derivatives and Pauli-Villars regularization is a consistent prescription when working in the covariant background gauge with $\alpha\neq 0$ [27].

B. Blocked action—noninvariant approach

In probing the RG evolution of the blocked action, a naive differentiation of Eq. (4.23) yields

$$k\partial_k\tilde{S}_k=\frac{1}{4}\int_p\left(k\,\frac{\partial\tilde{\mathcal{Z}}_k^{-1}}{\partial k}\right)F^a_{\mu\nu}(p)F^a_{\mu\nu}(-p). \tag{5.7}$$

From Eqs. (4.28) and (5.5), one immediately sees that Eq. (5.7) is equivalent to the $\tilde{\beta}$ function (5.6) which governs the flow of $\tilde{g}_{\tilde{k}}$. The result is to be expected on the ground that $\tilde{g}_{\tilde{k}}$ is the only free relevant parameter we have in the theory, apart from $\tilde{Z}_k$ which can be related to $\tilde{g}_{\tilde{k}}$ by the Slavnov-Taylor identity. This also justifies the truncation of the background field $\bar{A}$ beyond quadratic order in our noninvariant consideration of the evaluation of $\tilde{S}_k$.

How can we improve Eq. (5.7) so that it takes into account higher dimensional operators as well? We first turn to the simplest momentum cutoff regularization which can provide valuable insights into the structure of the Wilson-Kadanoff RG flow equation despite its gauge-noninvariant nature. A straightforward differentiation of Eq. (3.11) with respect to the IR cutoff k gives

$$\begin{aligned}k\partial\tilde{S}_k&=\frac{1}{2}\,k\partial_k\{\mathrm{Tr}'[\ln\mathcal{K}^{ab}_{\mu\nu,\bar{A}}-\ln\mathcal{K}^{ab}_{\mu\nu,0}]\\&\quad-2\,\mathrm{Tr}'[\ln\mathcal{O}^{ab}_{\bar{A}}-\ln\,\mathcal{O}^{ab}_0]\}\\&=\frac{1}{2}\,\mathrm{tr}\,k\partial_k\int_x\int_p'\left\{\ln\left(\frac{\mathcal{K}_{\bar{A}}}{\mathcal{K}_0}\right)^{ab}_{\mu\nu}(\partial_\alpha\to\partial_\alpha+ip_\alpha)\right.\\&\quad\left.-2\,\ln\left(\frac{\mathcal{O}_{\bar{A}}}{\mathcal{O}_0}\right)^{ab}(\partial_\alpha\to\partial_\alpha+ip_\alpha)\right\}\mathbf{1}\\&=-\frac{S_dk^d}{2}\,\mathrm{tr}\,\int_x\left\{\ln\left(\frac{k^2-2ik\cdot D-D^2+2gF}{k^2-2ik\cdot\partial-\partial^2}\right)^{ab}_{\mu\nu}\right.\\&\quad\left.-2\,\ln\left(\frac{k^2-2ik\cdot D-D^2}{k^2-2ik\cdot\partial-\partial^2}\right)^{ab}\right\}\mathbf{1},\end{aligned}\tag{5.8}$$

where Tr' implies a summation over the restricted space-time, i.e., cutoff scales are present in the momentum integration. In going beyond the simple one-loop approximation to probe the physics near the energy scale $\sim k$ in the manner of Wilson-Kadanoff, one first divides the momentum integration volume defined between k and the UV cutoff Λ into a large number of thin shells each having a width Δk. Lowering the cutoff infinitesimally from $\Lambda\to\Lambda-\Delta k\to\Lambda-2\Delta k$ until reaching the desired scale k allows us to incorporate the continuous feedbacks from the higher modes to the lower ones as they get integrated over [13]. Following this prescription, we obtain the following equation for the RG improved action S_k^{eff}:

$$\begin{aligned}k\partial S_k^{\mathrm{eff}}&=\frac{1}{2}\,k\partial_k\left\{\mathrm{Tr}'\left[\ln\left(\frac{\partial^2S_k^{\mathrm{eff}}}{\partial\tilde{A}^a_\mu(x)\partial\bar{A}^b_\nu(y)}\right)_{\bar{A}}\right.\right.\\&\quad\left.-\ln\left(\frac{\partial^2S_k^{\mathrm{eff}}}{\partial\bar{A}^a_\mu(x)\partial\bar{A}^b_\nu(y)}\right)_0\right]\\&\quad-2\,\mathrm{Tr}'\left[\ln\left(-\frac{\partial^2S_k^{\mathrm{eff}}}{\partial\tilde{\chi}^{\dagger a}(x)\partial\tilde{\chi}^b(y)}\right)_{\bar{A}}\right.\\&\quad\left.\left.-\ln\left(-\frac{\partial^2S_k^{\mathrm{eff}}}{\partial\tilde{\chi}^{\dagger a}(x)\partial\tilde{\chi}^b(y)}\right)_0\right]\right\}\mathbf{1}.\end{aligned}\tag{5.9}$$

Equation (5.9) is analogous to that derived by Reuter *et al.* in [12], and can be contrasted with the linear differential equation (5.8) which accounts only for the one-loop contributions. It may also be regarded as a self-consistent type of Schwinger-Dyson equation. The dressing provided by the nonlinear differential RG equation is equivalent to summing over all possible higher order nonoverlapping graphs such as the daisies and the superdaisies that are frequently encountered in finite temperature theory [13]. For the effective scalar theory written in Eq. (4.12), the corresponding improved coupled differential RG equations read

$$\begin{aligned}k\partial_kU_k&=f_1(U_k,Z_k,Y_k,\ldots),\\k\partial_kZ_k&=f_2(U_k,Z_k,Y_k,\ldots),\\k\partial_kY_k&=f_3(U_k,Z_k,Y_k,\ldots),\end{aligned}\tag{5.10}$$

where f_i are functions of the coefficients in the derivative expansion.

C. Blocked action—invariant operator cutoff approach

Although we now have an RG improved equation (5.9), it is obtained by imposing a momentum cutoff that manifestly breaks gauge symmetry. Let us see how a similar equation can be obtained in the operator cutoff prescription.

In the operator cutoff approach, one observes that the k dependence is contained entirely in the regulating smearing function $\rho_k^{(d)}(s,\Lambda)$. Therefore, after differentiating the blocked action with respect to k, and replacing $\tilde{S}_k$ and S_k^{eff} in the spirit of Schwinger-Dyson, one arrives at

$$\begin{aligned}k\partial_kS_k^{\mathrm{eff}}&=-\frac{1}{2}\,\mathrm{Tr}\int_0^\infty\frac{ds}{s}\,k\,\frac{\partial\rho_k^{(d)}(s,\Lambda)}{\partial k}\\&\quad\times\left\{\exp\left(-\frac{\partial^2S_k^{\mathrm{eff}}}{\partial\bar{A}^a_\mu\partial\bar{A}^b_\nu}\cdot s\right)_{\bar{A}}-\exp\left(-\frac{\partial^2S_k^{\mathrm{eff}}}{\partial\bar{A}^a_\mu\partial\bar{A}^b_\nu}\cdot s\right)_0\right.\\&\quad\left.-2\left[\exp\left(\frac{\partial^2S_k^{\mathrm{eff}}}{\partial\tilde{\chi}^{\dagger a}\partial\tilde{\chi}^b}\cdot s\right)_{\bar{A}}-\exp\left(\frac{\partial^2S_k^{\mathrm{eff}}}{\partial\tilde{\chi}^{\dagger a}\partial\tilde{\chi}^b}\cdot s\right)_0\right]\right\}\mathbf{1},\end{aligned}\tag{5.11}$$

which again takes on the form of a nonlinear partial differential equation. This is the RG improved equation which governs the flow of the theory. As we mentioned in the Introduction, Eq. (5.11) cannot be rigorously justified from first principles; instead, it is based on a prescription (of the Schwinger-Dyson type) which has been successful in describing many nonperturbative field theories. In spite of this, the remarkable feature of our RG equation is that it incorporates contributions from higher order operators and allows for numerical or analytical approximation in the IR limit.

If one approximates Eq. (5.11) by expanding the integrand in power of s and keeping only contributions up to $O(\bar{A}^2)$, the flow would then reduce to the usual β function shown in Eq. (5.3). However, treating s as a small expansion parameter corresponds to exploring the high-energy regime of the theory and it is not surprising after all that the short-distance property of asymptotic freedom is easily recovered from perturbative approximations. However, if one is interested in the IR behavior of the theory, it would be desirable to incorporate as much higher order effect as possible. Actually a complete s integration of Eq. (5.11) can be done and one obtains

$$k\partial_k S_k^{\rm eff}=\mathrm{Tr}\Bigg\{\Bigg[\hat{1}^{ab}_{\mu\nu}+\frac{1}{k^2}\left(\frac{\partial^2 S_k^{\rm eff}}{\partial\bar{A}^a_\mu\partial\bar{A}^b_\nu}\right)_{\bar{A}}\Bigg]^{-d/2} -\Bigg[\hat{1}^{ab}_{\mu\nu}+\frac{1}{k^2}\left(\frac{\partial^2 S_k^{\rm eff}}{\partial\bar{A}^a_\mu\partial\bar{A}^b_\nu}\right)_0\Bigg]^{-d/2} -2\Bigg[\delta^{ab}+\frac{1}{k^2}\left(\frac{\partial^2 S_k^{\rm eff}}{\partial\tilde{\chi}^{\dagger a}\partial\tilde{\chi}^b}\right)_{\bar{A}}\Bigg]^{-d/2} +2\Bigg[\delta^{ab}+\frac{1}{k^2}\left(\frac{\partial^2 S_k^{\rm eff}}{\partial\tilde{\chi}^{\dagger a}\partial\tilde{\chi}^b}\right)_0\Bigg]^{-d/2}\Bigg\}\mathbf{1}, \quad (5.12)$$

where $\hat{1}^{ab}_{\mu\nu}$ is the unit matrix in the Lorentz and color space. The role played by higher dimensional operators at the energy scale k can now be elucidated by solving Eq. (5.12) explicitly; nonlocal effects too are taken into consideration by this nonlinear partial differential equation. In fact, the two equations (5.12) and (5.9) are structurally quite similar as can be demonstrated using the simple scalar theory in $d=4$:

$$k^4\ \mathrm{Tr}\left[\left(\frac{k^2+p^2+V''(\Phi)}{k^2}\right)^{-2}-\left(\frac{k^2+p^2}{k^2}\right)^{-2}\right]\mathbf{1}$$
$$=k^4\int_x\int_p\{[k^2+p^2+V''(\Phi)]^{-2}-(k^2+p^2)^{-2}\}$$
$$=-\frac{k^4}{16\pi^2}\ln\left(\frac{k^2+V''(\Phi)}{k^2}\right)$$
$$=k\partial_k\int_p'\ln\left(\frac{p^2+V''(\Phi)}{p^2}\right)$$
$$=k\partial_k\ \mathrm{Tr}'\{\ln[-\partial^2+V''(\Phi)]-\ln(-\partial^2)\}\mathbf{1}. \quad (5.13)$$

In the large k regime, Eq. (5.12) can be expanded in Taylor series as

$$k\partial_k S_k^{\rm eff}=\mathrm{Tr}\sum_{n=0}^{\infty}(-1)^n(n+1)\frac{1}{k^{2n}}\Bigg[\left(\frac{\partial^2 S_k^{\rm eff}}{\partial\bar{A}^a_\mu\partial\bar{A}^b_\nu}\right)^n_{\bar{A}} -2\left(\frac{\partial^2 S_k^{\rm eff}}{\partial\tilde{\chi}^{\dagger a}\partial\tilde{\chi}^b}\right)^n_{\bar{A}}\Bigg]+\cdots, \quad (5.14)$$

for $d=4$, and the theory is characterized by the leading order local gauge-invariant operators. On the other hand, as $k\to 0$, we have

$$k\partial_k S_k^{\rm eff}=k^d\mathrm{Tr}\Bigg[\left(\frac{\partial^2 S_k^{\rm eff}}{\partial\bar{A}^a_\mu\partial\bar{A}^b_\nu}\right)^{-d/2}_{\bar{A}}-2\left(\frac{\partial^2 S_k^{\rm eff}}{\partial\tilde{\chi}^{\dagger a}\partial\tilde{\chi}^b}\right)^{-d/2}_{\bar{A}}\Bigg]+\cdots, \quad (5.15)$$

which is manifestly nonlocal. The above expression may be compared with

$$k\partial_k U_k(\Phi)=-S_d k^d\left[\ln U_k''+\frac{k^2}{U_k''}-\frac{1}{2}\left(\frac{k^2}{U_k''}\right)^2+\cdots\right], \quad (5.16)$$

which is the expansion of Eq. (1.4) for scalar theory in the small k limit.

VI. CONSTANT CHROMOMAGNETIC FIELD

Enormous efforts have been devoted to the study of the vacuum structure of SU(2) gauge theory in a constant chromomagnetic background since the pioneering work of Matinyan and Savvidy [28]. With the help of Eq. (5.12), we now explore the RG evolution associated with this configuration.

For simplicity, we choose the background to be a constant chromomagnetic field B in the $\hat{z}$ direction produced by

$$\bar{A}^a_\mu=\delta^{a3}\delta_{\mu 2}Bx, \quad (6.1)$$

with

$$F^a_{\mu\nu}F^a_{\mu\nu}=2B^2. \quad (6.2)$$

An alternative choice $\bar{A}^a_\mu=\frac{1}{2}B\delta^{a3}(x\delta_{\mu 2}-y\delta_{\mu 1})$ has also been used in [29]. Working in the background gauge, the eigenvalues for the kernels $\mathcal{K}$ and $\mathcal{O}$ can be obtained by a diagonalization in the color space, which then reduces the equation of motion into a harmonic oscillator equation and yields the Landau energy levels labeled by n, where $n=0,1,2,\ldots$ [30,31]. Thus, the Yang-Mills blocked potential can be written as

$$U_k(B)\sim\int\frac{dp}{2\pi}\sum_{n=0}^{\infty}\sum_{S_z=\pm 1}\times\sqrt{p^2+k^2+2gB\left(n+\frac{1}{2}\right)-2gBS_z}, \quad (6.3)$$

where S_z is the $\hat{z}$ component of the gluon spin along the direction of the chromomagnetic field, and the factor 2 in

$\vec{B}\cdot\vec{S}$ comes from the gyromagnetic ratio $g_L=2$ for the gluon fields. Since gluons are massless vector particles, $S_z=0$ is an unphysical degree of freedom and the associated contribution will be cancelled by the Faddeev-Popov ghost [31]. For $n=0$ and $S_z=1$, we notice that U_k becomes complex below certain momentum scale. The unstable mode gives an imaginary contribution to the blocked potential and signals an instability for the vacuum. To stabilize the theory, we therefore choose the IR scale k to be such that $k^2>gB$.

Using

$$\int \frac{dp}{2\pi}\sqrt{p^2+E^2}=\int \frac{d^2p}{(2\pi)^2}\ln(p^2+E^2), \tag{6.4}$$

which holds up to an E-independent constant, the trace sum in the operator cutoff formalism may be represented as

$$\mathrm{Tr}_{\mathrm{oc}}=\Omega\,\frac{gB}{2\pi}\,\rho_k^{(2)}(s,\Lambda)\int \frac{d^2p}{(2\pi)^2}\sum_n , \tag{6.5}$$

where Ω is the space-time volume. Taking into account the multiplicity factors for the eigenvalues, it follows from Eq. (5.12) that the RG equation for the theory reads

$$\begin{aligned}
k\partial_k U_k=&\frac{k^2gB}{2\pi}-2\int \frac{d^2p}{(2\pi)^2}\Bigg\{\left[\frac{1}{p^2+k^2-gB}-\frac{1}{p^2+k^2}\right]\\
&+3\left[\frac{1}{p^2+k^2+gB}-\frac{1}{p^2+k^2}\right]\\
&+4\sum_{n=1}^{\infty}\left[\frac{1}{p^2+k^2+(2n+1)gB}-\frac{1}{p^2+k^2}\right]\\
&-2\sum_{n=0}^{\infty}\left[\frac{1}{p^2+k^2+(2n+1)gB}-\frac{1}{p^2+k^2}\right]\Bigg\}\\
=&-\frac{k^2gB}{4\pi^2}\Bigg\{\ln\left(\frac{k^2-gB}{k^2}\right)+3\ln\left(\frac{k^2+gB}{k^2}\right)\\
&+4\sum_{n=1}^{\infty}\ln\left(\frac{k^2+(2n+1)gB}{k^2}\right)\\
&-2\sum_{n=0}^{\infty}\ln\left(\frac{k^2+(2n+1)gB}{k^2}\right)\Bigg\},
\end{aligned} \tag{6.6}$$

where the overall factor of 2 accounts for the color charge degeneracy in the SU(2) gauge group. While the last term in the first braces represents the contribution from the Faddeev-Popov ghost kernel, the first term is due to the mode which becomes unstable for $k^2<gB$. Notice that the multiplicity factors for the eigenvalues gB and $(2n+1)gB$ for $n>1$ were incorrect in [30]; the correct factors should be 3 and 4, respectively. The reason is because of the negligence of the unphysical $S_z=0$ sector which yields eigenvalues $(2n+1)gB$ for $n=0,1,\dots$. As explained before, this mode must be considered fully in the presence of Faddeev-Popov ghosts.

Using Eq. (5.1) and

$$\mathcal{Z}_k^{-1}=\frac{\partial U_k}{\partial\mathcal{F}}=\frac{1}{B}\frac{\partial U_k}{\partial B}, \tag{6.7}$$

the β function can be rewritten as

$$\begin{aligned}
\beta(g_k,\tau)=&k\partial_k g_k=-\frac{g}{2}\mathcal{Z}_k^{3/2}k\partial_k\mathcal{Z}_k^{-1}=-g\mathcal{Z}_k^{3/2}k\partial_k\left(\frac{\partial U_k}{\partial B^2}\right)\\
=&-\frac{g\mathcal{Z}_k^{3/2}}{2B}\frac{\partial}{\partial B}(k\partial_k U_k)\\
=&\frac{g_k^3}{8\pi^2\tau}\Bigg\{\ln(1-\tau)+3\ln(1+\tau)\\
&+4\sum_{n=1}^{\infty}\ln[1+(2n+1)\tau]\\
&-2\sum_{n=0}^{\infty}\ln[1+(2n+1)\tau]-\tau\Bigg[\frac{1}{1-\tau}-\frac{3}{1+\tau}\\
&-4\sum_{n=1}^{\infty}\frac{(2n+1)}{1+(2n+1)\tau}+2\sum_{n=0}^{\infty}\frac{(2n+1)}{1+(2n+1)\tau}\Bigg]\Bigg\},
\end{aligned} \tag{6.8}$$

where $\tau=gB/k^2$. In the presence of B field which defines another characteristic length scale, one naturally would expect the β function to depend not only on g_k, but also on the dimensionless parameter τ. With the help of the Euler formula [31]

$$\sum_{n=0}^{\infty}h\left(n+\frac{1}{2}\right)=\int_0^{\infty}dx\,h(x)-\frac{1}{24}h'(x)\Big|_0^{\infty}+\cdots, \quad h(\infty)=0, \tag{6.9}$$

the β function reads

$$\begin{aligned}
\beta(g_k,\tau)=&\frac{g_k^3}{8\pi^2\tau}\Bigg\{\ln(1-\tau)-\ln(1+\tau)+\frac{\tau}{3}-\frac{\tau}{6}+\cdots\\
&-\tau\Bigg[\frac{1}{1-\tau}+\frac{1}{1+\tau}-\frac{1}{3}+\frac{1}{6}+\cdots\Bigg]\Bigg\},
\end{aligned} \tag{6.10}$$

which in the limit of large k or vanishing τ, gives

$$\beta(g_k)=-\frac{g_k^3}{4\pi^2}\left\{1+\frac{2}{3}+\frac{1}{6}\right\}=-\frac{11g_k^3}{24\pi^2}, \tag{6.11}$$

in complete agreement with that obtained from Eq. (5.3) for SU(2). Notice that the contributions to the β function from the ''unstable mode'' (the first term) and the ghost kernel (the third term) are, respectively, $-g_k^3/4\pi^2$ and $-g_k^3/24\pi^2$, in accord with the analyses of Nielsen and Olsen [31].

We mentioned before that two multiplicity factors used in [30] were incorrect due to the negligence of the unphysical $S_z=0$ sector albeit the correct β function was given. The way it was obtained is as follows. The original expression which makes no reference of the unphysical sector $S_z=0$ actually gives

$$\beta_M(g_k) = -\frac{g_k^3}{4\pi^2}\left\{1+\frac{3}{4}+\frac{1}{6}\right\} = -\frac{23g_k^3}{48\pi^2}, \quad (6.12)$$

instead of Eq. (6.11). Removing the contribution from the would-be unstable mode entirely by the subtraction

$$\int \frac{d^2p}{(2\pi)^2}\ln(p^2+gB) - \int_{p^2>gB}\frac{d^2p}{(2\pi)^2}\ln(p^2-gB), \quad (6.13)$$

followed by the substitution

$$\int \frac{ds}{s} = \ln s \to 2\ln\left(\frac{m^2}{gB}\right) \quad (6.14)$$

in the proper-time representation with m being some appropriate renormalization point then leads to the correct result

$$\beta_M(g_k) \to -\frac{g_k^3}{4\pi^2}\left\{0+\frac{3}{4}+\frac{1}{6}\right\}2 = -\frac{11g_k^3}{24\pi^2}. \quad (6.15)$$

In other words, eliminating the contribution from the would-be unstable mode completely followed by multiplying the extra factor of 2 as appeared on the right-hand side of Eq. (6.14) is how the expected β function was arrived at in [30].

Taking into account the imaginary contribution in the $p^2 < gB$ region, the complete complex effective potential reads

$$U(B) = \frac{B^2}{2} + \frac{11g^2}{48\pi^2}B^2\left[\ln\left(\frac{gB}{m^2}\right) - \frac{1}{2}\right] - i\,\frac{g^2B^2}{8\pi^2}, \quad (6.16)$$

upon imposing the renormalization condition [28]

$$\left.\frac{\partial(\mathrm{Re}U)}{\partial\mathcal{F}}\right|_{\tilde{m}^2} = 1, \quad (6.17)$$

with $\mathcal{F}=B^2/2$ being the gauge-invariant quantity of the theory. The condition is readily fulfilled by choosing $\tilde{m}^4 = 2g^2\mathcal{F}$.

The arbitrary scale m and the IR cutoff k arising from the operator cutoff regularization can be related to each other by noting that while

$$m\partial_m(\mathrm{Re}U) = -\frac{11g^2B^2}{24\pi^2}, \quad (6.18)$$

one has

$$k\partial_k U_k = -\frac{k^2 gB}{4\pi^2}\left\{\ln\left(\frac{k^2-gB}{k^2+gB}\right) + 2\sum_{n=0}^{\infty}\ln\left(\frac{k^2+(2n+1)gB}{k^2}\right)\right\} \to \frac{11g^2B^2}{24\pi^2} + \frac{1}{6\pi^2}\frac{g^4B^4}{k^4} + \cdots \quad (6.19)$$

in the large k limit. This implies that we must have $m\partial_m \equiv -k\partial_k$, i.e., the two scales run in the opposite manner. This connection can also be seen from

$$\frac{1}{g_k^2} = \frac{1}{g^2} + \frac{11}{24\pi^2}\ln\left(\frac{k^2}{\mu^2}\right) \quad (6.20)$$

given in Eq. (5.2) and

$$\frac{1}{g_m^2} = \frac{1}{g^2} + \frac{11}{24\pi^2}\ln\left(\frac{gB}{m^2}\right). \quad (6.21)$$

We readily see that while k^2 represents the general IR cutoff (squared) for the theory in the operator cutoff formalism, in the discussion of a constant chromomagnetic field, the role of IR scale is taken over by gB. Thus, the perturbative large momentum region $k^2/\mu^2 \gg 1$ corresponds to the intense field limit $gB/m^2 \gg 1$. By replacing then right-hand side of Eq. (6.19) with the corresponding k-dependent running parameters, the RG evolution equation for U_k becomes

$$k\partial_k U_k = -\frac{2\beta(g_k)}{g_k}U_k\left(1+\frac{8g_k^2}{11}\frac{U_k}{k^4}\right) + \cdots . \quad (6.22)$$

Solving this differential equation by retaining only the leading order contribution, we have the following RG improved blocked potential:

$$\ln U_k = -2\int \frac{dk}{k}\frac{\beta(g_k)}{g_k}, \quad (6.23)$$

which is similar to that obtained in [28]. However, it only takes into consideration the effect the $\mathcal{F}=B^2/2$ term. In order to explore the influence of the higher order operators, one must solve Eq. (6.22) completely without truncation.

We emphasize that the above perturbative treatments are limited to the regime where k is large and the theory is asymptotically free. Continuing to evolve the system to a lower k will result in a complicated blocked action which invalidates perturbation theory. Furthermore, in the IR region where τ is large, Eq. (6.22) is no longer a good approximation. Serious difficulties are encountered for $\tau > 1$ where β function becomes complex. The source of the singularity is undoubtedly due to the unstable mode which becomes unsuppressed for $k \leq \sqrt{gB}$ $(\tau \geq 1)$. In [28], Savvidy considered only the real part of the one-loop potential given in Eq. (6.18) and obtained a nontrivial minimum B_m

$$\frac{gB_m}{m^2} = e^{-24\pi^2/11g^2}, \quad (6.24)$$

which has a maximum value of 0.6053 at $g^2 = 48\pi^2/11$. Since the condition $gB_m/m^2 < 1$ is just like $k^2 < \mu^2$ in the operator (momentum) cutoff case, one naturally would argue that such a configuration is unreliable since it lies in the deep IR regime where perturbation is known to break down. Moreover, the running of the momentum in the RG trajectory is generally restricted to be between the UV cutoff and μ, where one defines the physical renormalized coupling constant. Further complication arising from the persistence of the unstable mode in the IR region then lead Maiani *et al.* [30] to argue that the problem associated with unstable configurations can only be treated nonperturbatively. On the other hand, lattice calculations seem to support the formation of such chromomagnetic condensate [29,32].

Let us examine the physical consequence if a nontrivial vacuum state should exist. From Eq. (6.8), we find that a minimum of the potential B_0 must satisfy

$$0 = gk\partial_k\left(\frac{\partial U_k}{\partial B}\right)_{B_0} = -2\mathcal{Z}_k^{-3/2}B_0 \cdot k\partial_k g_k = -2\mathcal{Z}_k^{-3/2}B_0\beta(g_k,\tau_0), \tag{6.25}$$

where $\tau_0 = gB_0/k^2$. While $B_0=0$ trivially satisfies the requirement, a nontrivial configuration $B_0 \neq 0$ necessarily implies $\beta(g_k,\tau_0)=0$, i.e., $g_k(\tau_0)$ is a fixed point of the theory. Thus, the existence of a nontrivial vacuum state in the presence of a background chromomagnetic field B is intimately related to the existence of a nontrivial fixed point, and to locate such a fixed point generally would require a nonperturbative prescription to track the evolution of the coupling constant. In analogy to the Schwinger-Dyson self-consistent procedure, we replace g on the right-hand side of Eq. (6.8) by g_k and obtain

$$\beta(g_k,\tau_k) = k\partial_k g_k = \frac{g_k^3}{8\pi^2\tau_k}\left\{\ln\left(\frac{1-\tau_k}{1+\tau_k}\right) + 2\sum_{n=0}^{\infty}\ln[1+(2n+1)\tau_k] - \tau_k\left[\frac{1}{1-\tau_k}+\frac{1}{1+\tau_k} - 2\sum_{n=0}^{\infty}\frac{(2n+1)}{1+(2n+1)\tau_k}\right]\right\}, \tag{6.26}$$

where $\tau_k = g_kB/k^2$. In the above, since the expression inside the braces has the form $F(\tau_k) = f(\tau_k) + \tau_k f'(\tau_k)$, with

$$f(\tau_k) = \ln\left(\frac{1-\tau_k}{1+\tau_k}\right) + 2\sum_{n=0}^{\infty}\ln[1+(2n+1)\tau_k], \tag{6.27}$$

one may readily identify the solution to $F(\tau_k)=0$, or equivalently, $f(\tau_k)=0$, as a fixed point of the theory. For $f(\tau_k)$ to vanish, one must have

$$0 = 1-(1-\tau_k^2)(1+3\tau_k)^2(1+5\tau_k)^2\cdots = 1-(1-\tau_k^2)\prod_{n=1}^{\infty}[1+(2n+1)\tau_k]^2. \tag{6.28}$$

Besides $\tau_k=0$, one may readily verify that there is a second solution located near $\tau_k=1$. However, since $\tau_k=1$ is precisely the scale at which the imaginary contribution begins to set in, it is not legitimate to associate this with a fixed point. The absence of any other nontrivial fixed point when considering only the real sector of U_k seems to cast doubts on the existence of the Savvidy state. To investigate the full vacuum structure, the dynamics of the unstable mode must be taken into account.

VII. SUMMARY AND DISCUSSIONS

In this paper we demonstrate the invariant nature of the operator cutoff regularization whose regulating smearing function $\rho_k^{(d)}(s,\Lambda)$ simulates a momentum cutoff and is reminiscent to the invariant Slavnov regularization which involves both the method higher covariant derivatives and the Pauli-Villars. We also construct an RG equation based on the Schwinger-Dyson self-consistency argument which, albeit not completely rigorous, has been applied to scalar theory with remarkable success. In the covariant background formalism with $\alpha=1$, the resulting RG coefficient functions completely agree with the expected values, and no inconsistency is found. It would be interesting to see how the regularization can be carried out for an arbitrary α.

From the RG equation (5.12), we see that the blocked action S_k^{eff} provides a smooth interpolation between the bare action defined at $k=\Lambda$ and the effective action at an arbitrary scale k. In particular, in the large k limit where only the leading order gauge-invariant operator is kept, Eq. (5.12) reproduces the standard β function. On the other hand, as k is lowered, contributions from the higher dimensional operators that are generated in the course of blocking continue to pile up and are also accounted for by our RG prescription. Therefore, Eq. (5.12) allows us to probe the theory down to a smaller k regime compared with perturbation. Even though the complicated nonlinear partial differential RG equation seems to make the analytical form for the low-energy blocked action rather hopeless, its numerical solution may nevertheless provide a consistent check for the nonperturbative lattice method. For the simplest SU(2) theory in the presence of a static chromomagnetic field considered in Sec. V, a complete solution to the RG flow equation may yield additional insights on the role of the unstable mode. It may even help resolve the longstanding issue of the reliability of the energetically more favored ground state found in [28]. For realistic theories, the effects of matter fields too must be considered. Operator cutoff regularization is an ideal regulator for chiral theories since it is performed directly in d space-time dimensions, and no ambiguity in the definition of γ_5 arises. The generalization of this regularizing scheme to higher loops would also be helpful for computing Feynman graphs. Work along these directions is now in progress.

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APPENDIX A: SCALAR FIELD THEORY

In this appendix, we give the details of how blocked potentials for scalar field theory are computed using operator cutoff formalism. To be definite, the calculations will be carried out in $d=4$ dimensions. Consider for simplicity the following bare Lagrangian:

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\phi)^2+V(\phi). \tag{A1}$$

In the presence of a slowly varying background field $\Phi(x)$ whose Fourier modes are constrained by an upper cutoff scale k, by integrating out the fast-fluctuating modes, the one-loop contribution to the low-energy blocked potential is given by

$$U_k^{(1)}(\Phi)=-\frac{1}{2}\int_0^\infty\frac{ds}{s}\,\rho_k^{(4)}(s,\Lambda)\times\int_p e^{-p^2s}(e^{-V''(\Phi)s}-e^{-V''(0)s})$$
$$=-\frac{1}{32\pi^2}\int_0^\infty\frac{ds}{s^2}\,\rho_k^{(4)}(s,\Lambda)(e^{-V''(\Phi)s}-e^{-V''(0)s}). \tag{A2}$$

Notice that the scales set by momentum regularization are now taken over by $\rho_k^{(4)}(s,\Lambda)$. As shown in [15], smearing function of the form

$$\rho_k^{(4)}(s,\Lambda)=[1-(1+\Lambda^2s)e^{-\Lambda^2s}]-[1-(1+k^2s)e^{-k^2s}]$$
$$=\rho(\Lambda^2s)-\rho(k^2s) \tag{A3}$$

is equivalent to imposing sharp momentum cutoffs. That is, inserting Eq. (A3) into (A2) leads to the cutoff expression

$$U_k^{(1)}(\Phi)=\frac{1}{2}\int_p'\ln\left(\frac{p^2+V''(\Phi)}{p^2+V''(0)}\right)$$
$$=\frac{1}{64\pi^2}\Bigg\{(\Lambda^2-k^2)[V''(\Phi)-V''(0)]+\Lambda^4\ln\left(\frac{\Lambda^2+V''(\Phi)}{\Lambda^2+V''(0)}\right)-k^4\ln\left(\frac{k^2+V''(\Phi)}{k^2+V''(0)}\right)$$
$$-V''(\Phi)^2\ln\left(\frac{\Lambda^2+V''(\Phi)}{k^2+V''(\Phi)}\right)+\cdots\Bigg\}, \tag{A4}$$

up to some Φ-independent constant. Taking the $\lambda\phi^4$ theory as an example, the blocked potential up to the one-loop order becomes

$$U_k(\Phi)=V(\Phi)-\frac{1}{2}\int_p\int_0^\infty\frac{ds}{s}\,\rho_k^{(4)}(s,\Lambda)e^{-(p^2+\mu^2)s}\times(e^{-\lambda\Phi^2s/2}-1)$$
$$=\frac{1}{2}\left[\mu^2+\frac{\lambda}{32\pi^2}\left(\Lambda^2+\mu^2\ln\frac{\mu^2}{\Lambda^2}\right)-\frac{\lambda}{64\pi^2}k^2\right]\Phi^2+\frac{1}{4!}\left[\lambda+\frac{3\lambda^2}{32\pi^2}\left(1+\ln\frac{\mu^2}{\Lambda^2}\right)\right]\Phi^4$$
$$+\frac{1}{64\pi^2}\left[\left(\mu^2+\frac{\lambda}{2}\Phi^2\right)^2-k^4\right]\ln\left(\frac{k^2+\mu^2+\lambda\Phi^2/2}{\mu^2}\right)$$
$$=\frac{\mu^2}{2}\Phi^2\left[1-\frac{\lambda}{64\pi^2}\left(1+\frac{k^2}{\mu^2}\right)\right]+\frac{\lambda}{4!}\Phi^4\left(1-\frac{9\lambda}{64\pi^2}\right)$$
$$+\frac{1}{64\pi^2}\left[\left(\mu^2+\frac{\lambda}{2}\Phi^2\right)^2-k^4\right]\times\ln\left(1+\frac{k^2+\mu^2+\lambda\Phi^2/2}{\mu^2}\right), \tag{A5}$$

where the renormalized parameters can be related to the cutoff-dependent bare quantities by

$$\mu^2=\mu_\Lambda^2+\frac{\lambda}{32\pi^2}\left(\Lambda^2+\mu^2\ln\frac{\mu^2}{\Lambda^2}\right),$$
$$\lambda=\lambda_\Lambda+\frac{3\lambda^2}{32\pi^2}\left(1+\ln\frac{\mu^2}{\Lambda^2}\right). \tag{A6}$$

It is easily seen that in the limit $k=0$, Eq. (A5) reduces to the usual effective potential obtained in [33]. For this theory, the improved RG equation reads

$$k\partial_k U_k(\Phi)=-\frac{k^4}{16\pi^2}\ln\left(\frac{k^2+U_k''(\Phi)}{k^2+U_k''(0)}\right), \tag{A7}$$

which is a nonlinear differential equation that takes into account the coupling between the high and the low momentum modes.

The results obtained above can be readily extended to scalar electrodynamics. The Lagrangian is given by

$$\mathcal{L}_{\rm SQED}=-\frac{1}{4}F_{\mu\nu}F_{\mu\nu}-\frac{1}{2\alpha}(\partial_\mu A_\mu)^2+|(\partial_\mu+ie_0A_\mu)\phi(x)|^2+\frac{\mu_\Lambda^2}{2}\phi(x)^\dagger\phi(x)+\frac{\lambda_\Lambda}{6}[\phi(x)^\dagger\phi(x)]^2, \tag{A8}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ and α is the gauge-fixing parameter. The complex field $\phi(x)$ may be rewritten in terms of real fields ϕ_1 and ϕ_2 as $[\phi_1(x)+i\phi_2(x)]/\sqrt{2}$. Considering the special case where $A_c=0$ and $\Phi^a=\Phi\delta^{a,1}$ with Φ being the constant background configuration, the blocked potential in the Landau gauge with $\alpha=0$ becomes

$$U_k^{(1)}(\Phi)=-\frac{1}{32\pi^2}\int_0^\infty\frac{ds}{s^3}\{e^{-\mu^2s}[(1+k^2s)e^{-k^2s}-(1+\Lambda^2s)e^{-\Lambda^2s}][(e^{-\lambda\Phi^2s/2}-1)+(e^{-\lambda\Phi^2s/2}-1)]$$
$$+3[1-(1+\Lambda^2s)e^{-\Lambda^2s}](e^{-e_0^2\Phi^2s}-1)\}. \tag{A9}$$

Even though blocking is performed only for the scalar fields, it can be implemented in a similar fashion for gauge fields as well. Notice that the extra factor of three in the photon loop contribution arises from the trace of the propagator in the Landau gauge. We also comment that the form of $U_k(\Phi)$ is generally gauge dependent although physical quantities must be gauge independent [34].

The theory, however, is plagued by IR singularity due to the presence of massless photons. The problem could be avoided if blocking is also done for the gauge fields, i.e., using $\rho_k^{(4)}(s,\Lambda)$ instead of $\rho_{k=0}^{(4)}(s,\Lambda)$. The conventional regularization scheme is an off-shell subtraction condition for the coupling constant [33]

$$\lambda=\frac{\partial^4 U_k(\Phi)}{\partial\Phi^4}\bigg|_{\Phi=M,k=0}, \tag{A10}$$

which, in the language of operator cutoff, is equivalent to using the following coupling constant counterterm [35]:

$$\begin{aligned}\delta\lambda=&\frac{1}{32\pi^2}\int_0^\infty\frac{ds}{s}\,[1-(1+\Lambda^2 s)e^{-\Lambda^2 s}]\\&\times\bigg\{\lambda^2 e^{-(\mu^2+\lambda M^2/2)s}(3-6\lambda M^2 s+\lambda^2 M^4 s^2)\\&+\frac{\lambda^2}{81}\,e^{-(\mu^2+\lambda M^2/6)s}(27-18\lambda M^2 s+\lambda^2 M^4 s^2)\\&+12e_0^4 e^{-e_0^2 M^2 s}(3-12e_0^2 M^2 s+4e_0^4 M^4 s^2)\bigg\}\\=&-\frac{1}{64\pi^2}\bigg\{\frac{20}{3}\,\lambda^2+\frac{4\lambda^3 M^2(\lambda M^2+9\mu^2)}{81(\mu^2+\lambda M^2/6)^2}\\&+\frac{4\lambda^3 M^2(\lambda M^2+3\mu^2)}{(\mu^2+\lambda M^2/2)^2}+24e_0^4\bigg[11+3\ln\bigg(\frac{M^2}{\Lambda^2}\bigg)\bigg]\\&+\frac{2\lambda^2}{3}\ln\bigg(\frac{\mu^2+\lambda M^2/6}{\Lambda^2}\bigg)+6\lambda^2\ln\bigg(\frac{\mu^2+\lambda M^2/2}{\Lambda^2}\bigg)\bigg\}.\end{aligned} \tag{A11}$$

After removing the Λ dependence, the blocked potential becomes

$$\begin{aligned}U_k(\Phi)=&\frac{\mu^2}{2}\,\Phi^2+\frac{\lambda}{4!}\,\Phi^4+\frac{1}{64\pi^2}\bigg\{-\frac{2\lambda}{3}\,(k^2+\mu^2)\Phi^2\\&-\frac{5\lambda^2}{12}\,\Phi^4+\bigg[\bigg(\mu^2+\frac{\lambda}{2}\,\Phi^2\bigg)^2-k^4\bigg]\\&\times\ln\bigg(\frac{k^2+\mu^2+\lambda\Phi^2/2}{\mu^2}\bigg)+\bigg[\bigg(\mu^2+\frac{\lambda}{6}\,\Phi^2\bigg)^2-k^4\bigg]\\&\times\ln\bigg(\frac{k^2+\mu^2+\lambda\Phi^2/6}{\mu^2}\bigg)+\frac{\lambda^2}{4}\,\Phi^4\ln\bigg(\frac{\mu^2}{\mu^2+\lambda M^2/2}\bigg)\\&+\frac{\lambda^2}{36}\,\Phi^4\ln\bigg(\frac{\mu^2}{\mu^2+\lambda M^2/6}\bigg)\\&-\frac{\lambda^3 M^2}{6}\,\Phi^4\bigg[\frac{\lambda M^2+9\mu^2}{81(\mu^2+\lambda M^2/6)^2}+\frac{\lambda M^2+3\mu^2}{(\mu^2+\lambda M^2/2)^2}\bigg]\\&+3e_0^4\Phi^4\bigg[\ln\bigg(\frac{\Phi^2}{M^2}\bigg)-\frac{25}{6}\bigg]\bigg\},\end{aligned} \tag{A12}$$

which for $\mu^2=k^2=0$ reduces to

$$U_{k=0}(\Phi)=\frac{\lambda}{4!}\,\Phi^4+\frac{\Phi^4}{64\pi^2}\,\bigg(\frac{5}{18}\,\lambda^2+3e_0^4\bigg)\bigg[\ln\bigg(\frac{\Phi^2}{M^2}\bigg)-\frac{25}{6}\bigg]. \tag{A13}$$

The theory in this limit shows spontaneous symmetry breaking driven by radiative corrections [33]. Once more, the symmetry-preserving nature of the operator cutoff formalism is seen from the absence of cutoff scales in the p integration.

APPENDIX B: GENERALIZED PROPER-TIME CLASS

Numerous regularization schemes can all be shown to belong to the generalized class of proper time since they can be represented by a suitable definition of smearing function. A detailed discussion can be found in [16]. However, for completeness and comparative purpose, we recapitulate here various examples and examine how they modify the propagator and the corresponding one-loop kernel. We also show how cutoff scales can be implemented in dimensional regularization as well as ζ function regularization. The ‘‘hybrid’’ prescriptions of dimensional cutoff and ζ function cutoff allows us to establish a direct connection with the momentum regularization. To be specific, we apply each of these techniques to regularize the divergences encountered in the computation of the two- and the four-point vertex functions for $\mathcal{H}=p^2+\mu^2$ in $d=4$.

(1) Operator cutoff. For $d=4$, the smearing function becomes

$$\begin{aligned}\rho_k^{(4)}(s,\Lambda)&=(1+k^2 s)e^{-k^2 s}-(1+\Lambda^2 s)e^{-\Lambda^2 s}\\&=(e^{-k^2 s}-e^{-\Lambda^2 s})+s(k^2 e^{-k^2 s}-\Lambda^2 e^{-\Lambda^2 s}),\end{aligned} \tag{B1}$$

which leads to the following regularized propagator and one-loop kernel:

$$\begin{aligned}\frac{1}{\mathcal{H}^n}\bigg|_{\rm oc}=&\frac{1}{\Gamma(n)}\int_0^\infty ds\,s^{n-1}[(1+k^2 s)e^{-k^2 s}\\&-(1+\Lambda^2 s)e^{-\Lambda^2 s}]e^{-\mathcal{H}s}\\=&\frac{1}{(\mathcal{H}+k^2)^n}-\frac{1}{(\mathcal{H}+\Lambda^2)^n}+\frac{nk^2}{(\mathcal{H}+k^2)^{n+1}}\\&-\frac{n\Lambda^2}{(\mathcal{H}+\Lambda^2)^{n+1}},\end{aligned} \tag{B2}$$

and

$$\begin{aligned}{\rm Tr}_{\rm oc}\ln\bigg(\frac{\mathcal{H}}{\mathcal{H}_0}\bigg)=&{\rm Tr}\bigg\{\ln\bigg[\frac{\mathcal{H}+k^2}{\mathcal{H}_0+k^2}\,\frac{\mathcal{H}_0+\Lambda^2}{\mathcal{H}+\Lambda^2}\bigg]\\&-\frac{\Lambda^2(\mathcal{H}-\mathcal{H}_0)}{(\mathcal{H}+\Lambda^2)(\mathcal{H}_0+\Lambda^2)}+\frac{k^2(\mathcal{H}-\mathcal{H}_0)}{(\mathcal{H}+k^2)(\mathcal{H}_0+k^2)}\bigg\}.\end{aligned} \tag{B3}$$

As demonstrated in Sec. II, Eq. (B1) simulates a sharp cutoff at the level of blocked potential. Using Eq. (A2), the one-loop correction to the two- and four-point vertex functions for scalar theory can be written as

$$\delta\Gamma_{\rm oc}^{(2)}=\frac{\partial^2 U_k^{(1)}}{\partial\Phi^2}\bigg|_{\Phi=0}=\frac{\lambda}{32\pi^2}\int_0^\infty\frac{ds}{s^2}\,\rho_k^{(4)}(s,\Lambda)e^{-\mu^2 s}, \tag{B4}$$

and

$$\delta\Gamma^{(4)}_{\rm oc}=\frac{\partial^4 U^{(1)}_k}{\partial\Phi^4}\bigg|_{\Phi=0}=-\frac{3\lambda^2}{32\pi^2}\int_0^\infty\frac{ds}{s}\,\rho^{(4)}_k(s,\Lambda)e^{-\mu^2 s},\tag{B5}$$

which for $k=0$ become

$$\begin{aligned}\delta\Gamma^{(2)}_{\rm oc}&=\frac{\lambda}{2}\int_p\frac{1}{p^2+\mu^2}\left(\frac{\Lambda^2}{p^2+\mu^2+\Lambda^2}\right)^2\\&=\frac{\lambda}{32\pi^2}\int_0^\infty\frac{ds}{s^2}\,\rho^{(4)}_{k=0}(s,\Lambda)e^{-\mu^2 s}\\&=\frac{\lambda}{32\pi^2}\left[\Lambda^2-\mu^2\ln\left(\frac{\Lambda^2+\mu^2}{\mu^2}\right)\right],\end{aligned}\tag{B6}$$

and

$$\begin{aligned}\delta\Gamma^{(4)}_{\rm oc}&=-\frac{3\lambda^2}{2}\int_p\frac{1}{(p^2+\mu^2)^2}\left(\frac{\Lambda^2}{p^2+\mu^2+\Lambda^2}\right)^2\\&\quad\times\left[1+\frac{2(p^2+\mu^2)}{p^2+\mu^2+\Lambda^2}\right]\\&=-\frac{3\lambda^2}{32\pi^2}\int_0^\infty\frac{ds}{s}\,\rho^{(4)}_{k=0}(s,\Lambda)e^{-\mu^2 s}\\&=-\frac{3\lambda^2}{32\pi^2}\left[\ln\left(\frac{\Lambda^2+\mu^2}{\mu^2}\right)-\frac{\Lambda^2}{\mu^2+\Lambda^2}\right].\end{aligned}\tag{B7}$$

On the other hand, using the momentum cutoff procedure, one also has

$$\delta\Gamma^{(2)}_{\Lambda}=\frac{\lambda}{2}\int_p^\Lambda\frac{1}{p^2+\mu^2}=\frac{\lambda}{32\pi^2}\left[\Lambda^2-\mu^2\ln\left(\frac{\Lambda^2+\mu^2}{\mu^2}\right)\right],\tag{B8}$$

and

$$\begin{aligned}\delta\Gamma^{(4)}_{\Lambda}&=-\frac{3\lambda^2}{2}\int_p^\Lambda\frac{1}{(p^2+\mu^2)^2}\\&=-\frac{3\lambda^2}{32\pi^2}\left[\ln\left(\frac{\Lambda^2+\mu^2}{\mu^2}\right)-\frac{\Lambda^2}{\mu^2+\Lambda^2}\right].\end{aligned}\tag{B9}$$

(2) Pauli-Villars. The conventional Pauli-Villars scheme can be parametrized in the proper-time representation by taking the smearing function to be

$$\rho^{\rm pv}_k(s,\Lambda)=\sum_i\,(a_i e^{-k_i^2 s}-b_i e^{-\Lambda_i^2 s}),\tag{B10}$$

where Λ_i are the masses of some ghost states, and k_i the extra masses added to the spectra. To render the theory finite, the coefficients a_i and b_i as well as i, the number of ghost terms, are appropriately chosen. The physical limit, however, corresponds to taking $\Lambda_i\to\infty$ and $k_i\to 0$ since Λ_i and k_i control, respectively, the divergent behaviors of the theory in the UV and the IR regimes. Equation (B10) implies

$$\begin{aligned}\frac{1}{\mathcal{H}^n}\bigg|_{\rm pv}&=\sum_i\frac{1}{\Gamma(n)}\int_0^\infty ds\,s^{n-1}(a_i e^{-k_i^2 s}-b_i e^{-\Lambda_i^2 s})e^{-\mathcal{H}s}\\&=\sum_i\left[\frac{a_i}{(\mathcal{H}+k_i^2)^n}-\frac{b_i}{(\mathcal{H}+\Lambda_i^2)^n}\right],\end{aligned}\tag{B11}$$

and

$$\begin{aligned}{\rm Tr}_{\rm pv}\ln\left(\frac{\mathcal{H}}{\mathcal{H}_0}\right)&=-\sum_i\int_0^\infty\frac{ds}{s}\,(a_i e^{-k_i^2 s}-b_i e^{-\Lambda_i^2 s})\\&\quad\times{\rm Tr}(e^{-\mathcal{H}s}-e^{-\mathcal{H}_0 s})\\&={\rm Tr}\sum_i\ln\left[\left(\frac{\mathcal{H}+k_i^2}{\mathcal{H}_0+k_i^2}\right)^{a_i}\left(\frac{\mathcal{H}_0+\Lambda_i^2}{\mathcal{H}+\Lambda_i^2}\right)^{b_i}\right].\end{aligned}\tag{B12}$$

The similarity between the operator cutoff and the Pauli-Villars is now apparent. By choosing $a_i=b_i=i=1$, we notice that the two smearing functions differ from one another only by a higher order correction.

In computing $\delta\Gamma^{(2)}_{\rm pv}$ using the Pauli-Villars regulator, it is necessary to introduce two ghost terms since the integral in Eq. (B8) is quadratically divergent. Thus, we write [36]

$$\begin{aligned}\frac{1}{p^2+\mu^2}\bigg|_{\rm pv}&=\frac{1}{p^2+\mu^2}-\frac{b_1}{p^2+\mu^2+\Lambda_1^2}-\frac{b_2}{p^2+\mu^2+\Lambda_2^2}\\&=\frac{f(p^2,\mu^2,\Lambda_1^2,\Lambda_2^2)}{(p^2+\mu^2)(p^2+\mu^2+\Lambda_1^2)(p^2+\mu^2+\Lambda_2^2)},\end{aligned}\tag{B13}$$

where

$$\begin{aligned}&f(p^2,\mu^2,\Lambda_1^2,\Lambda_2^2)\\&\quad=(1-b_1-b_2)p^4+[2(1-b_1-b_2)\mu^2\\&\qquad+(1-b_1)\Lambda_2^2+(1-b_2)\Lambda_1^2]p^2+\mu^2[(1-b_1)\Lambda_2^2\\&\qquad+(1-b_2)\Lambda_1^2]+\Lambda_1^2\Lambda_2^2,\end{aligned}\tag{B14}$$

and demand that

$$\frac{1}{p^2+\mu^2}\bigg|_{\rm pv}\to\frac{1}{p^6},\quad\text{as }p^2\to\infty.\tag{B15}$$

The condition is satisfied if

$$b_1+b_2-1=0,\quad(1-b_2)\Lambda_1^2+(1-b_1)\Lambda_2^2=0,\tag{B16}$$

which implies

$$b_1=\frac{\Lambda_2^2}{\Lambda_2^2-\Lambda_1^2},\quad b_2=-\frac{\Lambda_1^2}{\Lambda_2^2-\Lambda_1^2}.\tag{B17}$$

The correction to the two-point function can now be obtained as

$$\delta\Gamma^{(2)}_{\rm pv}=\frac{\lambda}{2}\int_p \frac{1}{p^2+\mu^2}\bigg|_{\rm pv}$$
$$=\frac{\lambda}{2}\int_p \frac{\Lambda_1^2\Lambda_2^2}{(p^2+\mu^2)(p^2+\mu^2+\Lambda_1^2)(p^2+\mu^2+\Lambda_2^2)}$$
$$\to\frac{\lambda}{2}\int_p \frac{1}{p^2+\mu^2}\left(\frac{\Lambda^2}{p^2+\mu^2+\Lambda^2}\right)^2$$
$$=\frac{\lambda}{32\pi^2}\left[\Lambda^2-\mu^2\ln\left(\frac{\Lambda^2+\mu^2}{\mu^2}\right)\right] \quad (\Lambda_1,\Lambda_2\to\Lambda), \tag{B18}$$

which is in agreement with Eq. (B8). As for $\delta\Gamma^{(4)}_{\rm pv}$, since it is logarithmically divergent, only one ghost term is sufficient and we obtain

$$\delta\Gamma^{(4)}_{\rm pv}=-\frac{3\lambda^2}{2}\int_p \frac{1}{(p^2+\mu^2)^2}\bigg|_{\rm pv}=-\frac{3\lambda^2}{2}\int_p\bigg[\frac{1}{(p^2+\mu^2)^2}$$
$$-\frac{1}{(p^2+\mu^2+\Lambda^2)^2}\bigg]=-\frac{3\lambda^2}{32\pi^2}\ln\left(\frac{\Lambda^2+\mu^2}{\mu^2}\right). \tag{B19}$$

(3) Proper-time cutoff. Since divergences generated from taking the trace in space-time are transferred into singularities in the proper-time integration, one may regularize the theory by a direct truncation of the integration regime(s) to avoid singularity. For example, we may simply take the smearing function to be a sharp proper-time cutoff:

$$\rho_k^{\rm pc}(s,\Lambda)=\Theta\left(s-\frac{1}{\Lambda^2}\right)\Theta\left(\frac{1}{k^2}-s\right). \tag{B20}$$

In this manner, we have

$$\frac{1}{\mathcal{H}^n}\bigg|_{\rm pc}=\frac{1}{\Gamma(n)}\int_0^\infty ds\, s^{n-1}\Theta\left(s-\frac{1}{\Lambda^2}\right)\Theta\left(\frac{1}{k^2}-s\right)e^{-\mathcal{H}s}$$
$$=\frac{1}{\Gamma(n)}\int_{1/\Lambda^2}^{1/k^2} ds\, s^{n-1}e^{-\mathcal{H}s}$$
$$=\frac{1}{\mathcal{H}^n}\frac{1}{\Gamma(n)}\left(\Gamma\left[n,0,\frac{\mathcal{H}}{k^2}\right]-\Gamma\left[n,0,\frac{\mathcal{H}}{\Lambda^2}\right]\right) \tag{B21}$$

and

$$\mathrm{Tr}_{\rm pc}\ln\left(\frac{\mathcal{H}}{\mathcal{H}_0}\right)=-\int_{1/\Lambda^2}^{1/k^2}\frac{ds}{s}\,\mathrm{Tr}(e^{-\mathcal{H}s}-e^{-\mathcal{H}_0 s})$$
$$=\mathrm{Tr}\{-\mathrm{Ei}(-\mathcal{H}/k^2)+\mathrm{Ei}(-\mathcal{H}_0/k^2)$$
$$+\mathrm{Ei}(-\mathcal{H}/\Lambda^2)-\mathrm{Ei}(-\mathcal{H}_0/\Lambda^2)\}$$
$$=\mathrm{Tr}\ \ln\left(\frac{\mathcal{H}}{\mathcal{H}_0}\right)+\cdots, \tag{B22}$$

where we have employed the asymptotic forms of the exponential-integral function

$$\mathrm{Ei}(-s_0)=-\int_{s_0}^\infty \frac{ds}{s}e^{-s}$$
$$=\begin{cases}\ln s_0+\gamma+\sum_{n=1}^\infty \frac{(-s_0)^n}{n!n} & (s_0\to 0^+),\\ -\frac{e^{-s_0}}{s_0} & (s_0\to\infty).\end{cases} \tag{B23}$$

Correspondingly, we have

$$\delta\Gamma^{(2)}_{\rm pc}=\frac{\lambda}{2}\int_p \frac{e^{-(p^2+\mu^2)/\Lambda^2}}{p^2+\mu^2}$$
$$=\frac{\lambda}{32\pi^2}\left[\Lambda^2 e^{-\mu^2/\Lambda^2}+\mu^2\mathrm{Ei}\left(-\frac{\mu^2}{\Lambda^2}\right)\right]$$
$$=\frac{\lambda}{32\pi^2}\left[\Lambda^2-\mu^2\ \ln\left(\frac{\Lambda^2}{\mu^2}\right)\right]+\cdots, \tag{B24}$$

and

$$\delta\Gamma^{(4)}_{\rm pc}=-\frac{3\lambda^2}{2}\int_p \frac{e^{-(p^2+\mu^2)/\Lambda^2}}{(p^2+\mu^2)^2}\left[1+\frac{p^2+\mu^2}{\Lambda^2}\right]$$
$$=-\frac{3\lambda^2}{32\pi^2}\bigg[-\left(1-\frac{2\mu^2}{\Lambda^2}-\frac{2\mu^4}{\Lambda^4}\right)\mathrm{Ei}\left(-\frac{\mu^2}{\Lambda^2}\right)$$
$$+2\left(1+\frac{\mu^2}{\Lambda^2}\right)e^{-\mu^2/\Lambda^2}\bigg]$$
$$=-\frac{3\lambda^2}{32\pi^2}\ln\left(\frac{\Lambda^2}{\mu^2}\right)+\cdots. \tag{B25}$$

(4) Point-splitting. One may also choose the smearing function to be of the form

$$\rho_k^{\rm ps}(s,\Lambda)=e^{-1/\Lambda^2 s}-e^{-1/k^2 s}, \tag{B26}$$

which corresponds to the so-called point-splitting regularization scheme. This smearing function yields

$$\frac{1}{\mathcal{H}^n}\bigg|_{\rm ps}=\frac{1}{\Gamma(n)}\int_0^\infty ds\, s^{n-1}(e^{-1/\Lambda^2 s}-e^{-1/k^2 s})e^{-\mathcal{H}s}$$
$$=\frac{1}{\mathcal{H}^n}\frac{2}{\Gamma(n)}\left[\left(\frac{\mathcal{H}}{\Lambda^2}\right)^{n/2}K_n\left(\frac{2\mathcal{H}^{1/2}}{\Lambda}\right)-\left(\frac{\mathcal{H}}{k^2}\right)^{n/2}K_n\left(\frac{2\mathcal{H}^{1/2}}{k}\right)\right]$$
$$=\frac{1}{\mathcal{H}^n}+\cdots, \tag{B27}$$

and

$$\mathrm{Tr}_{\rm ps}\ \ln\left(\frac{\mathcal{H}}{\mathcal{H}_0}\right)=-\int_0^\infty\frac{ds}{s}\ (e^{-1/\Lambda^2 s}-e^{-1/k^2 s})\times\mathrm{Tr}(e^{-\mathcal{H}s}-e^{-\mathcal{H}_0 s})$$
$$=2\ \mathrm{Tr}\left[K_0\left(\frac{2\mathcal{H}_0^{1/2}}{\Lambda}\right)-K_0\left(\frac{2\mathcal{H}^{1/2}}{\Lambda}\right)-K_0\left(\frac{2\mathcal{H}_0^{1/2}}{k}\right)+K_0\left(\frac{2\mathcal{H}^{1/2}}{k}\right)\right]$$
$$=\mathrm{Tr}\ \ln\left(\frac{\mathcal{H}}{\mathcal{H}_0}\right)+\cdots\ , \tag{B28}$$

where we have expanded the modified Bessel function asymptotically as [37]

$$K_n(x)\sim\sqrt{\frac{\pi}{2x}}\ e^{-x}\left[1+\frac{(4n^2-1^2)}{1!8x}+\frac{(4n^2-1^2)(4n^2-3^2)}{2!(8x)^2}+\cdots\right]\quad(x\to\infty), \tag{B29}$$

and

$$K_n(x)\sim\begin{cases}2^{n-1}(n-1)!x^{-n}+\cdots & n\geqslant 1,\\ -\ln\dfrac{x}{2}-\gamma & n=0.\end{cases}\qquad (x\to 0^+). \tag{B30}$$

The two- and four-point functions in this scheme are

$$\delta\Gamma_{\rm ps}^{(2)}=\frac{\lambda}{2}\int_p\frac{1}{p^2+\mu^2}\bigg|_{\rm ps}$$
$$=\frac{\lambda}{2}\int_0^\infty dy\ e^{-y-\mu^2/\Lambda^2 y}\int_p\frac{e^{-p^2/\Lambda^2 y}}{p^2+\mu^2}$$
$$=\frac{\lambda}{32\pi^2}\int_0^\infty dy\ e^{-y-\mu^2/\Lambda^2 y}\left[\Lambda^2 y+\mu^2 e^{\mu^2/\Lambda^2 y}\times\mathrm{Ei}\left(-\frac{\mu^2}{\Lambda^2 y}\right)\right]$$
$$\approx\frac{\lambda\mu^2}{32\pi^2}\left\{2K_2\left(\frac{2\mu}{\Lambda}\right)+\int_0^\infty dy\ e^{-y}\ \ln\left(\frac{\mu^2}{\Lambda^2 y}\right)\right\}$$
$$=\frac{\lambda}{32\pi^2}\left[\Lambda^2-\mu^2\ln\left(\frac{\Lambda^2+\mu^2}{\mu^2}\right)\right]+\cdots\ , \tag{B31}$$

and

$$\delta\Gamma_{\rm ps}^{(4)}=-\frac{3\lambda^2}{2}\int_p\frac{1}{(p^2+\mu^2)^2}\bigg|_{\rm ps}$$
$$=-\frac{3\lambda^2}{2}\int_0^\infty dy\ y e^{-y-\mu^2/\Lambda^2 y}\int_p\frac{e^{-p^2/\Lambda^2 y}}{(p^2+\mu^2)^2}$$
$$=\frac{3\lambda^2}{32\pi^2}\int_0^\infty dy\ y e^{-y-\mu^2/\Lambda^2 y}\left\{1+\left(1+\frac{\mu^2}{\Lambda^2 y}\right)e^{\mu^2/\Lambda^2 y}\times\mathrm{Ei}\left(-\frac{\mu^2}{\Lambda^2 y}\right)\right\}$$
$$=-\frac{3\lambda^2}{32\pi^2}\ln\left(\frac{\Lambda^2+\mu^2}{\mu^2}\right)+\cdots\ . \tag{B32}$$

We remark that the four smearing functions presented so far in a certain sense can all be viewed as a special case of the generalized momentum regularization in which the regularized integral for an arbitrary momentum-dependent function $f(p)$ is written as

$$\int_{b(k)}^{a(\Lambda)}dp\ f(p), \tag{B33}$$

where $a(\Lambda)$ and $b(k)$ are arbitrary functions of the cutoffs Λ and k, respectively. This is readily seen by noting that the prescriptions presented previously can be related to the generalized momentum regularization via

$$\rho_k^{\rm reg}(s,\Lambda)\int_p e^{-p^2 s}=\frac{1}{(4\pi s)^{d/2}}\ \rho_k^{\rm reg}(s,\Lambda)=\int_p{}'\ e^{-p^2 s}=S_d\int_{b(k)}^{a(\Lambda)}dp\ p^{d-1}e^{-p^2 s}, \tag{B34}$$

or

$$\rho_k^{\rm reg}(s,\Lambda)=\frac{2s^{d/2}}{\Gamma(d/2)}\int_{b(k)}^{a(\Lambda)}dp\ p^{d-1}e^{-p^2 s}. \tag{B35}$$

For example, in the $d=4$ Pauli-Villars case, we have

$$\rho_k^{\rm pv}(s,\Lambda)=e^{-k^2 s}-e^{-\Lambda^2 s}=2s^2\int_{b(k)}^{a(\Lambda)}dp\ p^3 e^{-p^2 s}$$
$$=[1+b^2(k)s]e^{-b^2(k)s}-[1+a^2(\Lambda)s]e^{-a^2(\Lambda)s}, \tag{B36}$$

where $a(\Lambda)$ obeys the transcendental equation

$$e^{-\Lambda^2 s}=[1+a^2(\Lambda)s]e^{-a^2(\Lambda)s}. \tag{B37}$$

The IR cutoff function $b(k)$ can be obtained in a similar manner.

(5a) Dimensional regularization. One can also show that dimensional regularization falls into the generalized class of proper-time by taking the smearing function to be

$$\rho_\epsilon(s)=(4\pi s)^{\epsilon/2}, \tag{B38}$$

which follows from integrating the z variable in $d-\epsilon$ dimension without imposing the cutoffs, i.e.,

$$\rho_k^{(d)}(s,\Lambda)=\frac{2s^{d/2}}{S_d\Gamma(d/2)}\int_z' e^{-z^2s}\rightarrow\frac{2s^{d/2}}{S_d\Gamma(d/2)}\int\frac{d^{d-\epsilon}z}{(2\pi)^{d-\epsilon}}e^{-z^2s}=(4\pi s)^{\epsilon/2}. \tag{B39}$$

Since z can be regarded as the momentum variable p, the feature of dimensional regularization is completely encapsulated.

We comment that an alternative approach akin to dimensional regularization is the method of analytic regularization. The manner in which the theory is regulated is to increase the power of the propagator by $\tilde{\epsilon}/2$, e.g., $(p^2+\mu^2)^{-1}\rightarrow(p^2+\mu^2)^{-(1+\tilde{\epsilon}/2)}$, thereby decreasing the power of divergence. In the proper-time formulation, it is equivalent to choosing

$$\tilde{\rho}_{\tilde{\epsilon}}(s)=\frac{2s^{(d-\tilde{\epsilon})/2}}{S_{d-\tilde{\epsilon}}\Gamma[(d-\tilde{\epsilon})/2]}\int\frac{d^dz}{(2\pi)^d}e^{-z^2s}=(4\pi s)^{-\tilde{\epsilon}/2}, \tag{B40}$$

where the original space dimension of z is kept. It is interesting to notice the relative negative sign between ϵ in dimensional regularization and $\tilde{\epsilon}$ in the analytic regularization. While both dimensional and analytic regularizations give the same results at the one loop level, the latter is known to violate BRS symmetry at higher loop order. In other words, upon inverting the propagator given by the analytic regularization to obtain the kernel, one finds that the regularized theory is no longer BRS invariant. The details can be found in [5].

Equation (B38) suggests

$$\left.\frac{1}{\mathcal{H}^n}\right|_\epsilon=\frac{(4\pi)^{\epsilon/2}}{\Gamma(n)}\int_0^\infty ds\,s^{\epsilon/2+n-1}e^{-\mathcal{H}s}=\frac{\Gamma(n+\epsilon/2)}{\Gamma(n)}(4\pi)^{\epsilon/2}\mathcal{H}^{-(n+\epsilon/2)}, \tag{B41}$$

and

$$\mathrm{Tr}_\epsilon\ln\left(\frac{\mathcal{H}}{\mathcal{H}_0}\right)=-(4\pi)^{\epsilon/2}\int_0^\infty ds\,s^{-1+\epsilon/2}\mathrm{Tr}(e^{-\mathcal{H}s}-e^{-\mathcal{H}_0s})=-(4\pi)^{\epsilon/2}\Gamma(\epsilon/2)\mathrm{Tr}(\mathcal{H}^{-\epsilon/2}-\mathcal{H}_0^{-\epsilon/2}). \tag{B42}$$

The corrections to the two- and four-point functions are, respectively,

$$\delta\Gamma_\epsilon^{(2)}=\frac{\lambda}{2}\int_p\left.\frac{1}{p^2+\mu^2}\right|_\epsilon=\frac{\lambda}{2}(4\pi)^{\epsilon/2}\int_0^\infty ds\,s^{\epsilon/2}e^{-\mu^2s}\int_p e^{-p^2s}=\frac{\lambda}{2(4\pi)^{2-\epsilon/2}}(\mu^2)^{1-\epsilon/2}\Gamma(-1+\epsilon/2)=\frac{\lambda}{32\pi^2}\left[-\frac{2\mu^2}{\epsilon}-\mu^2\ln\left(\frac{4\pi}{\mu^2}\right)\right]+\cdots, \tag{B43}$$

and

$$\delta\Gamma_\epsilon^{(4)}=-\frac{3\lambda^2}{2}\int_p\left.\frac{1}{(p^2+\mu^2)^2}\right|_\epsilon=-\frac{3\lambda^2}{2(4\pi)^{2-\epsilon/2}}(\mu^2)^{-\epsilon/2}\Gamma(\epsilon/2)=-\frac{3\lambda^2}{32\pi^2}\left[\frac{2}{\epsilon}+\ln\left(\frac{4\pi}{\mu^2}\right)\right]+\cdots, \tag{B44}$$

where we have used [38]

$$\Gamma(-n+\epsilon/2)=\frac{(-1)^n}{n!}\left[\frac{2}{\epsilon}+\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\gamma\right)+O(\epsilon)\right]. \tag{B45}$$

The divergences now appear as poles for $\epsilon=0$ and 2 since

$$\Gamma(-1+\epsilon/2)=\frac{1}{(-1+\epsilon/2)(\epsilon/2)}\Gamma(1+\epsilon/2). \tag{B46}$$

These poles can be mapped onto the divergent expressions obtained by using the momentum cutoffs.

(5b) Dimensional cutoff regularization. The most direct way to establish the connection between dimensional regularization and the momentum cutoff regulator is by means of the ‘‘dimensional cutoff regularization’’ defined by

$$\rho_{\epsilon'}(s,\Lambda)=\rho_\epsilon(s)\rho_k^{(d)}(s,\Lambda)=\frac{(4\pi)^{\epsilon/2}}{S_d\Gamma(d/s)}s^{(d+\epsilon)/2}\int_z' e^{-z^2s}, \tag{B47}$$

which is simply the product of the two smearing functions taken from each scheme in d dimension. The modified propagator and kernel in this ϵ' scheme read

$$\left.\frac{1}{\mathcal{H}^n}\right|_{\epsilon'}=\frac{1}{\Gamma(n)}\int_0^\infty ds\,s^{n-1}\rho_k^{(d)}(s,\Lambda)e^{-\mathcal{H}s}=\frac{1}{\mathcal{H}^n}\frac{2(4\pi)^{\epsilon/2}\Gamma(n+d/2)}{d\Gamma(n)\Gamma(d/2)}\times\left\{\left(\frac{\Lambda^2}{\mathcal{H}}\right)^{d/2}F\left(\frac{d}{2},\frac{d+\epsilon}{2}+n,1+\frac{d}{2};-\frac{\Lambda^2}{\mathcal{H}}\right)-\left(\frac{k^2}{\mathcal{H}}\right)^{d/2}F\left(\frac{d}{2},\frac{d+\epsilon}{2}+n,1+\frac{d}{2};-\frac{k^2}{\mathcal{H}}\right)\right\}, \tag{B48}$$

and

$$\mathrm{Tr}_{\epsilon'}\ \ln\left(\frac{\mathcal{H}}{\mathcal{H}_0}\right)=-\int_0^\infty \frac{ds}{s}\ \rho_k^{(d)}(s,\Lambda)\mathrm{Tr}(e^{-\mathcal{H}s}-e^{-\mathcal{H}_0 s})$$
$$=-\frac{2(4\pi)^{\epsilon/2}}{d}\ \mathrm{Tr}\left\{\left(\frac{\Lambda^2}{\mathcal{H}}\right)^{d/2}\times F\left(\frac{d}{2},\frac{d+\epsilon}{2},1+\frac{d}{2};-\frac{\Lambda^2}{\mathcal{H}}\right)-\left(\frac{\Lambda^2}{\mathcal{H}_0}\right)^{d/2}\times F\left(\frac{d}{2},\frac{d+\epsilon}{2},1+\frac{d}{2};-\frac{\Lambda^2}{\mathcal{H}_0}\right)-\left(\frac{k^2}{\mathcal{H}}\right)^{d/2}F\left(\frac{d}{2},\frac{d+\epsilon}{2},1+\frac{d}{2};-\frac{k^2}{\mathcal{H}}\right)+\left(\frac{k^2}{\mathcal{H}_0}\right)^{d/2}F\left(\frac{d}{2},\frac{d+\epsilon}{2},1+\frac{d}{2};-\frac{k^2}{\mathcal{H}_0}\right)\right\}. \tag{B49}$$

Equations (B43) and (B44) are now modified as

$$\delta\Gamma^{(2)}_{\epsilon'}=\frac{\lambda}{2}\int_p \frac{1}{p^2+\mu^2}\bigg|_{\epsilon'}=\frac{\lambda}{2}\ (4\pi)^{\epsilon/2}\int_0^\infty ds\ s^{\epsilon/2}[1-(1+\Lambda^2 s)e^{-\Lambda^2 s}]e^{-\mu^2 s}\int_p e^{-p^2 s}$$
$$=\frac{\lambda}{2}\ \frac{\Gamma(-1+\epsilon/2)}{(4\pi)^{2-\epsilon/2}}\left\{(\mu^2)^{1-\epsilon/2}-\frac{\epsilon\Lambda^2/2+\mu^2}{(\Lambda^2+\mu^2)^{\epsilon/2}}\right\}, \tag{B50}$$

and

$$\delta\Gamma^{(4)}_{\epsilon'}=-\frac{3\lambda^2}{2}\int_p \frac{1}{(p^2+\mu^2)^2}\bigg|_{\epsilon'}=-\frac{3\lambda^2}{2}\ \frac{\Gamma(\epsilon/2)}{(4\pi)^{2-\epsilon/2}}\times\left\{(\mu^2)^{-\epsilon/2}-\frac{(1+\epsilon/2)\Lambda^2+\mu^2}{(\Lambda^2+\mu^2)^{1+\epsilon/2}}\right\}. \tag{B51}$$

To recover the cutoff results, we take the limit $\epsilon\to 0$ first and obtain

$$\delta\Gamma^{(2)}_{\epsilon'}=-\frac{\lambda\mu^2}{16\pi^2}\left\{\left[\frac{1}{\epsilon}-\frac{1}{2}\ \ln 4\pi\mu^2\right]-\left[\frac{1}{\epsilon}+\frac{\Lambda^2}{2\mu^2}-\frac{1}{2}\ \ln 4\pi(\Lambda^2+\mu^2)\right]\right\}+O(\epsilon)$$
$$=\frac{\lambda}{32\pi^2}\left[\Lambda^2-\mu^2\ln\left(\frac{\Lambda^2+\mu^2}{\mu^2}\right)\right]+O(\epsilon), \tag{B52}$$

and

$$\delta\Gamma^{(4)}_{\epsilon'}=-\frac{3\lambda^2}{32\pi^2}\left(\frac{2}{\epsilon}-1+\ln 4\pi\right)\left\{\left[1-\frac{\epsilon}{2}\ \ln\mu^2\right]-\left[1-\frac{\epsilon}{2}\ \ln(\Lambda^2+\mu^2)+\frac{\epsilon\Lambda^2/2}{\Lambda^2+\mu^2}\right]\right\}+\cdots$$
$$=-\frac{3\lambda^2}{32\pi^2}\left[\ln\left(\frac{\Lambda^2+\mu^2}{\mu^2}\right)-\frac{\Lambda^2}{\mu^2+\Lambda^2}\right]+O(\epsilon). \tag{B53}$$

The above equations explicitly demonstrate how with this hybrid dimensional cutoff regulator, the $1/\epsilon$ singular term coming from dimensional regularization and cutoff regularization cancels each other and gives back the Λ dependence of the cutoff theory shown in Eqs. (B8) and (B9). On the other hand, taking the limit $\Lambda\to\infty$ before $\epsilon\to 0$ allows us to recover the usual dimensional regularization scheme. In other words, depending on the order in which the limits $\Lambda\to\infty$ and $\epsilon\to 0$ are taken, different regularization schemes are actually achieved.

(6a) ζ-function regularization. ζ-function regularization has been discussed extensively by Elizalde *et al.* [39] and in the context of operator regularization by McKeon *et al.* [26].

In the ζ-function regularization, the logarithm of an operator is represented by

$$\ln\mathcal{H}=-\lim_{t\to 0}\frac{d}{dt}\ \mathcal{H}^{-t}. \tag{B54}$$

Noting that

$$\frac{1}{\mathcal{H}^t}=\frac{1}{\Gamma(t)}\int_0^\infty ds\ s^{t-1}e^{-\mathcal{H}s}, \tag{B55}$$

one may define the ζ function as

$$\zeta(t)=\frac{1}{\Gamma(t)}\int_0^\infty ds\ s^{t-1}\mathrm{Tr}\ e^{-\mathcal{H}s}, \tag{B56}$$

which implies

$$\det\ \mathcal{H}=\exp[\mathrm{Tr}\ \ln\mathcal{H}]=\exp\left\{\mathrm{Tr}\ \lim_{t\to 0}\left[-\frac{d}{dt}\ \mathcal{H}^{-t}\right]\right\}=\exp\left[-\lim_{t\to 0}\frac{d}{dt}\ \zeta(t)\right]. \tag{B57}$$

The equivalent of ζ-function regularization in the proper-time formulation can be obtained by choosing the smearing function

$$\rho_t^{\zeta}(s)=\lim_{t\to 0}\frac{d}{dt}\ \frac{1}{\Gamma(t)}\ s^t, \tag{B58}$$

which gives

$$\left.\frac{1}{\mathcal{H}^n}\right|_\zeta=\lim_{t\to0}\frac{d}{dt}\left\{\frac{1}{\Gamma(t)\Gamma(n)}\int_0^\infty ds\, s^{n+t-1}e^{-\mathcal{H}s}\right\}$$
$$=\lim_{t\to0}\frac{d}{dt}\left\{\frac{\Gamma(n+t)}{\Gamma(n)\Gamma(t)}\mathcal{H}^{-(n+t)}\right\}$$
$$=\lim_{t\to0}\left\{\frac{\Gamma(n+t)}{\Gamma(n)\Gamma(t)}\mathcal{H}^{-(n+t)}[\psi(n+t)-\psi(t)-\ln\mathcal{H}]\right\}$$
$$=\frac{1}{\mathcal{H}^n}\lim_{t\to0}\left\{\frac{\Gamma(n+t)}{\Gamma(n)\Gamma(t+1)}\mathcal{H}^{-t}\times\left[1+t\left(\sum_{l=1}^{n-1}\frac{1}{t+l}-\ln\mathcal{H}\right)\right]\right\}\to\frac{1}{\mathcal{H}^n},\tag{B59}$$

and

$$\mathrm{Tr}_\zeta\ln\left(\frac{\mathcal{H}}{\mathcal{H}_0}\right)=-\lim_{t\to0}\frac{d}{dt}\left\{\frac{1}{\Gamma(t)}\int_0^\infty ds\, s^{t-1}\mathrm{Tr}(e^{-\mathcal{H}s}-e^{-\mathcal{H}_0 s})\right\}$$
$$=\mathrm{Tr}\left[-\lim_{t\to0}\frac{d}{dt}(\mathcal{H}^{-t}-\mathcal{H}_0^{-t})\right]=\mathrm{Tr}\,\ln\left(\frac{\mathcal{H}}{\mathcal{H}_0}\right),\tag{B60}$$

where we have used [40]

$$\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}\tag{B61}$$

and

$$\psi(n+t)=\psi(t)=\sum_{l=0}^{n-1}\frac{1}{t+l}.\tag{B62}$$

(6b) ζ-function cutoff regularization. In an analogous manner to the dimensional cutoff regularization scheme, one can introduce cutoff scales to the ζ-function cutoff regularization as well. This again can be done with the product of two smearing functions:

$$\rho_{t,k}^{\zeta,(d)}(s,\Lambda)=\rho_t^\zeta(s)\rho_k^{(d)}(s,\Lambda)=\lim_{t\to0}\frac{d}{dt}\frac{1}{\Gamma(t)}s^t\rho_k^{(d)}(s,\Lambda).\tag{B63}$$

To show that the same $\rho_k^{(d)}(s,\Lambda)$ obtained in Eq. (1.5) can be used to reproduce the momentum cutoff structure, we consider again the simple scalar theory. In this ζ-function cutoff formalism, the one-loop correction to U_k is written as remove the Λ dependence. On the other hand, if Λ is first sent to infinity before the s integration, one arrives at

$$U_k^{(1)}(\Phi)=\frac{1}{2}\int_p'\ln\left(\frac{p^2+V''(\Phi)}{p^2+V''(0)}\right)$$
$$\to-\frac{1}{2(4\pi)^{d/2}}\lim_{t\to0}\frac{d}{dt}\left\{\frac{1}{\Gamma(t)}\int_0^\infty ds\, s^{t-1-d/2}\times\rho_k^{(d)}(s,\Lambda)(e^{-V''(\Phi)s}-e^{-V''(0)s})\right\}.\tag{B64}$$

By demanding that Eq. (B64) yields the same differential flow equation for U_k as that obtained from momentum cutoff regularization, we are led to

$$\lim_{t\to0}\frac{d}{dt}\left\{\frac{1}{\Gamma(t)}\int_0^\infty ds\, s^{t-1-d/2}\left(k\frac{\partial\rho_k^{(d)}(s,\Lambda)}{\partial k}\right)(e^{-V''(\Phi)s}-e^{-V''(0)s})\right\}$$
$$=-\frac{2k^d}{\Gamma(d/2)}\int_0^\infty\frac{ds}{s}e^{-k^2s}(e^{-V''(\Phi)s}-e^{-V''(0)s}).\tag{B65}$$

One can then verify by direct substitution that the above expression is indeed satisfied by the smearing function given in Eq. (1.5). Thus the same $\rho_k^{(d)}(s,\Lambda)$ can be used to bring the momentum cutoffs into ζ-function regularization although this may seem redundant since no divergence is encountered in this prescription. Nevertheless, by retaining the cutoff scales, the flow pattern of the theory may be explored in a lucid manner.

As an explicit demonstration of ζ function cutoff regularization, we compute the one-loop contribution of the blocked potential in $d=4$ and obtain

$$U_k^{(1)}(\Phi)=-\frac{1}{32\pi^2}\lim_{t\to0}\frac{d}{dt}\left\{\frac{1}{\Gamma(t)}\int_0^\infty ds\, s^{t-3}[(1+k^2s)e^{-k^2s}-(1+\Lambda^2s)e^{-\Lambda^2s}](e^{-V''(\Phi)s}-e^{-V''(0)s})\right\}$$
$$=-\frac{1}{32\pi^2}\lim_{t\to0}\frac{d}{dt}\left\{\frac{1}{(t-1)(t-2)}\{[k^2+V''(\Phi)]^{2-t}-[\Lambda^2+V''(\Phi)]^{2-t}\}+\frac{1}{t-1}\{k^2[k^2+V''(\Phi)]^{1-t}-\Lambda^2[\Lambda^2+V''(\Phi)]^{1-t}\}+\cdots\right\}$$
$$=\frac{1}{64\pi^2}\left\{V''(\Phi)(\Lambda^2-k^2)+\Lambda^4\ln\left(1+\frac{V''(\Phi)}{\Lambda^2}\right)-k^4\ln\left(1+\frac{V''(\Phi)}{k^2}\right)+V''(\Phi)^2\ln\left(\frac{k^2+V''(\Phi)}{\Lambda^2+V''(\Phi)}\right)\right\}+\cdots.\tag{B66}$$

Here we see that by keeping the UV cutoff Λ finite when integrating over s, counterterms have to be introduced to remove the Λ dependence. On the other hand, if Λ is first sent to infinity before the s integration, one arrives at

$$
\begin{aligned}
U_k^{(1)}(\Phi) &= -\frac{1}{32\pi^2}\lim_{t\to 0}\frac{d}{dt}\left\{\frac{1}{\Gamma(t)}\int_0^\infty ds\, s^{t-3}(1+k^2 s)\right.\\
&\quad\left.\times e^{-k^2 s}(e^{-V''(\Phi)s}-e^{-V''(0)s})\right\}\\
&= -\frac{1}{32\pi^2}\lim_{t\to 0}\frac{d}{dt}\left\{\frac{[k^2+V''(\Phi)]^{2-t}}{(t-1)(t-2)}\right.\\
&\quad\left.+\frac{k^2[k^2+V''(\Phi)]^{1-t}}{t-1}+\cdots\right\}\\
&= -\frac{1}{64\pi^2}\left\{k^2V''(\Phi)+\frac{3}{2}V''(\Phi)^2\right.\\
&\quad\left.+[k^4-V''(\Phi)^2]\ln\left(\frac{k^2+V''(\Phi)}{k^2}\right)\right\}+\cdots,
\end{aligned}
\tag{B67}
$$

which is precisely the finite one-loop contribution of the blocked potential. It is interesting to note that taking the limit $\Lambda\to\infty$ before and after the s integration actually yields different results. In fact, the two limits correspond to two different regularization procedures. The connection between the ζ-function cutoff formalism and the momentum cutoff regularization can actually be established by the following integral transformation:

$$
\begin{aligned}
U_k^{(1)}(\Phi) &= -\frac{1}{32\pi^2}\lim_{t\to 0}\frac{d}{dt}\left\{\frac{1}{\Gamma(t)}\int_0^\infty ds\, s^{t-3}\rho_k^{(d)}(s,\Lambda)(e^{-V''(\Phi)s}-e^{-V''(0)s})\right\}\\
&\to -\frac{1}{2}\int_z' \lim_{t\to 0}\frac{d}{dt}\left\{\frac{1}{\Gamma(t)}\int_0^\infty ds\, s^{t-1}e^{-z^2 s}(e^{-V''(\Phi)s}-e^{-V''(0)s})\right\}\\
&= -\frac{1}{2}\int_z' \lim_{t\to 0}\frac{d}{dt}\left\{\frac{1}{[z^2+V''(\Phi)]^t}-\frac{1}{[z^2+V''(0)]^t}\right\}\\
&= \frac{1}{2}\int_z' \ln\left(\frac{z^2+V''(\Phi)}{z^2+V''(0)}\right).
\end{aligned}
\tag{B68}
$$

In other words, equality with cutoff regularization can be obtained by keeping the z integration till the end and interpreting z as the momentum scale p.

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