PHYSICAL REVIEW D VOLUME 56, NUMBER 8 15 OCTOBER 1997

General rotating black holes in string theory: Greybody factors and event horizons

Mirjam Cvetič* and Finn Larsen†

Department of Physics and Astronomy, University of Pennsylvania, Philadelphia, Pennsylvania 19104

(Received 4 June 1997)

We derive the wave equation for a minimally coupled scalar field in the background of a general rotating five-dimensional black hole. It is written in a form that involves two types of thermodynamic variables, defined at the inner and outer event horizon, respectively. We model the microscopic structure as an effective string theory, with the thermodynamic properties of the left- and right-moving excitations related to those of the horizons. Previously known solutions to the wave equation are generalized to the rotating case, and their regime of validity is sharpened. We calculate the greybody factors and interpret the resulting Hawking emission spectrum microscopically in several limits. We find a *U*-duality-invariant expression for the effective string length that does not assume a hierarchy between the charges. It accounts for the universal low-energy absorption cross section in the general nonextremal case. [S0556-2821(97)08020-X]

PACS number(s): 04.70.Dy, 11.27.+d

I. INTRODUCTION

Hawking's seminal calculation of the black hole temperature allows for a surprising window to quantum gravity: It immediately yields the size of the underlying space of quantum states in quantitative detail [1]. The result relies only on a particular detail of the black hole geometry, namely, its limiting form close to the *outer* event horizon. We will argue that other geometric properties give similarly direct evidence on the microscopic structure of black holes. Specifically, we find an important role for the geometry in the vicinity of the *inner* event horizon, as well.

The discussion and the examples aim at the description of black holes as quantum states in string theory (for a review see [2,3]). It is a characteristic property of string models that the entropy is the sum of contributions from left- and right-moving excitations of the string, and the thermodynamic variables accordingly appear in duplicate versions. The black hole geometry exhibits an analogous structure: Standard thermodynamic variables, defined at the outer event horizon, are mirrored by an independent set of thermodynamic variables, defined at the inner event horizon. We find that the left- and right-moving thermodynamics of the string theory correspond to the sum and the difference of the outer and the inner horizon thermodynamics. This relation can be established by direct inspection for large classes of extremal and near-extremal black holes. Indeed, it is valid in all the cases where the correspondence between black holes and string theory, has been demonstrated. Ultimately we would like to find a microscopic description of *all* black holes within string theory, and our geometrical observations may be sufficiently robust to serve as guidance towards this goal (other attempts include [4–6]).

In the following we give an outline the paper and summarize the results in more detail.

We begin with an important motivating fact that concerns the entropy of general rotating black holes in five dimensions [7]:

$$S=2\pi\left[\sqrt{\frac{1}{4}\mu^3\left(\prod_{i=1}^{3}\cosh\delta_i+\prod_i\sinh\delta_i\right)^2-J_L^2}+\sqrt{\frac{1}{4}\mu^3\left(\prod_{i=1}^{3}\cosh\delta_i-\prod_i\sinh\delta_i\right)^2-J_R^2}\right]. \quad (1)$$

(As we explain in Sec. II the nonextremality parameter μ and the boosts δ_i parametrize the mass and the charges, and $J_{L,R}$ are angular momenta.) The form of the entropy may be interpreted as an indication that it derives from two independent microscopic contributions, and each of these may be attributed to a gas of strings [7–9]. We will consider the general case of rotating black holes because the crucial division into two terms becomes ambiguous in the limit of vanishing angular momenta. We develop the thermodynamics of this interpretation in detail, in Sec. II. An important feature is that we find two independent temperatures T_R and T_L, one for each gas. These two temperatures play central roles in subsequent sections.

In Sec. III we present our main technical result: We write the exact wave equation for a minimally coupled scalar in the most general black hole background in five dimensions [Eq. (36)]. The wave equation has a surprisingly symmetric structure, given the generality of the setting. A characteristic feature is that the outer and inner event horizons appear in a symmetric fashion. The modes in the vicinity of the outer horizon give rise to the Hawking radiation, with characteristic temperature $T_H^{-1}=\frac{1}{2}(T_R^{-1}+T_L^{-1})$. Analogously, from the modes in the vicinity of the inner horizon we infer a ''temperature'' given by $T_-^{-1}=\frac{1}{2}(T_R^{-1}-T_L^{-1})$. The temperatures T_R and T_L that appear in these formulas agree precisely with those that follow from thermodynamics. Similar results are derived for the other thermodynamic variables, i.e., rotational velocities and U(1) potentials.

The wave equation has an exact symmetry that interchanges the inner and outer event horizons. In Sec. IV we identify this discrete symmetry with the T duality of an un-

*Electronic address: cvetic@cvetic.hep.upenn.edu

†Electronic address: larsen@cvetic.hep.upenn.edu

derlying string theory. Moreover, we exhibit an approximate $SL(2,R)_R \times SL(2,R)_L$ symmetry group that is realized directly on the macroscopic fields. From the quantum numbers of the symmetry group we recover the temperatures T_R and T_L. Although the precise interpretation of these facts remains unclear, it is interesting that they point rather specifically towards a string theory description.

In Sec. V we find solutions to the radial wave equation in two regions, solving first in the asymptotic region and then in the near-horizon region. We also discuss the angular equation. These results generalize previously known results to the case of rotating black holes. We discuss the ranges of charge, angular momenta, and mass for which these solutions can be combined to approximate wave functions covering the entire spacetime, and so the black hole absorption cross sections can be calculated explicitly. The results presented in Sec. VI include the following.

The low-energy S-wave absorption cross section is

$$\sigma_{\rm abs}(\omega \to 0) = A, \tag{2}$$

where A is the area of the black hole. Our result shows that this holds for *all* five-dimensional black holes in toroidally compactified string theory.

For a range of parameters (that we specify) black holes exhibit the S-wave absorption spectrum:

$$\sigma^{(0)}_{\rm abs}(\omega) = A\ \frac{(\omega/2T_L)(\omega/2T_R)}{\omega/T_H}\ \frac{(e^{\omega/T_H}-1)}{(e^{\omega/2T_L}-1)(e^{\omega/2T_R}-1)}. \tag{3}$$

This spectrum is a precise indication that the Hawking emission process of the black hole can be described in an effective string theory as a simple two-body process [10–12]. In this dynamical model the distribution functions of the colliding quanta are thermal with the temperatures T_R and T_L. The freedom afforded by the angular momenta allows a demonstration of this characteristic behavior in several regions of parameter space that were previously out of reach. For example there is a parameter range with *no* hierarchy in the relative magnitudes of the charges.

For a larger range of black hole parameters, and for higher partial waves, an explicit solution can still be found [13,14]. In this case the absorption cross section has a more complicated form and the Hawking radiation cannot be interpreted as a two-body process. However, it is suggestive that the emission spectrum still takes a factorized form where each factor depends on T_R and T_L, respectively.

We complete the paper, in Sec. VII, with a discussion of the microscopic description of the dynamics. It is shown that, for the most general black holes, the two-body emission processes can be modeled by a simple value of the effective string length. However, we also stress that, for generic nonextremal black holes, the typical Hawking process cannot be described in this simple fashion.

II. THERMODYNAMICS OF ROTATING BLACK HOLES

We are interested in a class of black holes in five dimensions that are parametrized by their mass M, two angular momenta $J_{R,L}$, and three independent U(1) charges Q_i [15]. These are the most general solutions to the low-energy effective action of the heterotic and type-II string theories, toroidally compactified to five dimensions[1] [17]. The explicit expressions for these black holes are involved and given in detail in [15]. For the sake of completeness we present their spacetime metric in the Appendix. In this section we discuss their thermodynamical properties.

The mass and the charges of the black holes are conveniently given in the parametric form[2]

$$M = \frac{1}{2}\mu \sum_{i=1}^{3} \cosh 2\delta_i\,, \tag{4}$$

$$Q_i = \frac{1}{2}\mu\ \sinh 2\delta_i\,, \quad i = 1,2,3. \tag{5}$$

The Bogomol'nyi-Prasad-Sommerfield- (BPS-)saturated limit corresponds to $\mu \to 0$ and $\delta_i \to \infty$ with Q_i kept fixed, and so μ is a measure of the deviation from the BPS case. The parameters δ_i are referred to as boosts because of their role in the solution generating technique employed to find the charged black holes.

In five dimensions the rotation group is $SO(4) \simeq SU(2)_R \times SU(2)_L$. Therefore black holes are characterized by two independent projections of the angular momentum vector. These parameters are the two angular momenta that will be denoted J_R and J_L. Normalizations have been chosen such that $J_{R,L}$ are pure numbers (in units where $\hbar = 1$) that are quantized in the microscopic theory.[3] It is sometimes convenient to parametrize the angular momenta of the general black hole in terms of the $l_{1,2}$ defined through

$$J_{R,L} = \frac{1}{2}\mu (l_1 \pm l_2)\left(\prod_i \cosh\delta_i \mp \prod_i \sinh\delta_i\right). \tag{6}$$

The $l_{1,2}$ are the angular momenta of the Kerr black hole used as a starting point of the generating technique. We will give the formulas in terms of $l_{1,2}$ along with those using $J_{R,L}$, because both forms will be needed.

A. Entropy

The black hole entropy [Eq. (1)] was derived in [7]. As noted already in the Introduction the entropy clearly divides into two terms. We make this manifest by writing $S = S_L + S_R$ where

[1] We write formulas in their generating form, and so they are only the most general up to duality. However, they can be written in a manifestly duality invariant way [16].

[2] The notation here is $\mu = 2m$ where m is the notation in [15], or $\mu = r_0^2$ where r_0 is the notation of [4]. We choose duality invariant units where the five-dimensional gravitational coupling constant is $G_5 = \pi/4$. In string conventions this amounts to $(\alpha')^4 g^2 / (R_1 R_2 R_3 R_4 R_5) = 1$.

[3] The quantization condition is that $J_{R,L} = \frac{1}{2}(J_\phi \pm J_\psi)$ where J_ϕ and J_ψ are quantized as integers.

$$S_L = 2\pi\sqrt{\frac{1}{4}\mu^3\left(\prod_i \cosh\delta_i + \prod_i \sinh\delta_i\right)^2 - J_L^2}$$
$$= \pi\mu\left(\prod_i \cosh\delta_i + \prod_i \sinh\delta_i\right)\sqrt{\mu-(l_1-l_2)^2}, \quad (7)$$

$$S_R = 2\pi\sqrt{\frac{1}{4}\mu^3\left(\prod_i \cosh\delta_i - \prod_i \sinh\delta_i\right)^2 - J_R^2}$$
$$= \pi\mu\left(\prod_i \cosh\delta_i - \prod_i \sinh\delta_i\right)\sqrt{\mu-(l_1+l_2)^2}. \quad (8)$$

By now there are many hints from string theory that collective excitations of solitonic objects can be described by effective low-energy theories that are themselves string theories. The structure of the entropy as a sum of two terms may be an indication that *all* black holes can be described in this way and that the two terms in the entropy are the contributions from left- (L-) and right- (R-) moving modes, respectively. If true, it must be that the interactions between the two kinds of modes can be treated as weak. Motivated by the BPS-saturated case we assume that the relevant effective theory is a noncritical string theory with $c=6$ [18–20] and identify the levels of the effective string as

$$N_L = \frac{1}{4}\mu^3\left(\prod_i \cosh\delta_i + \prod_i \sinh\delta_i\right)^2 - J_L^2, \quad (9)$$

$$N_R = \frac{1}{4}\mu^3\left(\prod_i \cosh\delta_i - \prod_i \sinh\delta_i\right)^2 - J_R^2, \quad (10)$$

so that, for large levels,

$$S = S_L + S_R = 2\pi(\sqrt{N_L} + \sqrt{N_R}). \quad (11)$$

If these relations could be derived from first principles, we would have a microscopic interpretation of the entropy in the general nonextremal case. Some evidence in this direction was presented in [9].

Black holes in four dimensions have entropies of a very similar form [7]: the index $i=1,2,3 \rightarrow i=1,2,3,4$, the parameter $\mu^3 \rightarrow \mu^4$, and the angular momentum $J_L=0$. Therefore the thermodynamics and indeed most results presented in this paper immediately carry over to four dimensions. Note, however, that there is only one angular momentum in four dimensions, and so the symmetry between the two entropies $S_{R,L}$ is a special property of the five-dimensional case that hints at a particularly symmetric underlying structure. We will discuss rotating black holes in four dimensions in a separate paper [21].

B. Thermodynamics

Our assumption that the entropy is a sum of two independent contributions has consequences that can be derived from general principles. Consider the first law of thermodynamics:

$$dM = T_H dS + \Omega^R dJ_R + \Omega^L dJ_L + \sum_i \Phi^i dQ_i. \quad (12)$$

We write the inverse Hawking temperature as

$$\beta_H \equiv \tfrac{1}{2}(\beta_L + \beta_R), \quad (13)$$

and use $S = S_L + S_R$. Then we find

$$\left[-\frac{1}{2}\beta_R dM + dS_R + \beta_H\Omega^R dJ_R + \beta_H\sum_i \Phi_R^i dQ_i\right] + \left[-\frac{1}{2}\beta_L dM + dS_L + \beta_H\Omega^L dJ_L + \beta_H\sum_i \Phi_L^i dQ_i\right] = 0. \quad (14)$$

The two independent inverse temperatures follow directly from this relation:

$$\beta_L = \frac{\pi\mu^2(\Pi_i\cosh^2\delta_i - \Pi_i\sinh^2\delta_i)}{\sqrt{\frac{1}{4}\mu^3(\Pi_i\cosh\delta_i + \Pi_i\sinh\delta_i)^2 - J_L^2}} = \frac{2\pi\mu(\Pi_i\cosh\delta_i - \Pi_i\sinh\delta_i)}{\sqrt{\mu-(l_1-l_2)^2}}, \quad (15)$$

$$\beta_R = \frac{\pi\mu^2(\Pi_i\cosh^2\delta_i - \Pi_i\sinh^2\delta_i)}{\sqrt{\frac{1}{4}\mu^3(\Pi_i\cosh\delta_i - \Pi_i\sinh\delta_i)^2 - J_R^2}} = \frac{2\pi\mu(\Pi_i\cosh\delta_i + \Pi_i\sinh\delta_i)}{\sqrt{\mu-(l_1+l_2)^2}}. \quad (16)$$

In the string theory interpretation these are the physical temperatures of the left- and right-moving modes. For this to make sense we must assume that the modes are interacting in such a way that the thermal equilibrium is maintained in each of the two gasses independently, and so that the couplings between the two sectors are much weaker than the ones that act within each sector. Although this is perhaps surprising from the string theory point of view, it may be reasonable when considering the nature of black holes: Colliding left and right modes give rise to Hawking radiation, and we know that large black holes are exceedingly stable objects.

The angular velocities also follow from the first law of thermodynamics:

$$\beta_H\Omega^L = \frac{2\pi J_L}{\sqrt{\frac{1}{4}\mu^3(\Pi_i\cosh\delta_i + \Pi_i\sinh\delta_i)^2 - J_L^2}} = \frac{2\pi(l_1-l_2)}{\sqrt{\mu-(l_1-l_2)^2}}, \quad (17)$$

$$\beta_H \Omega^R = \frac{2\pi J_R}{\sqrt{\frac{1}{4}\mu^3(\Pi_i \cosh\delta_i - \Pi_i \sinh\delta_i)^2 - J_R^2}} = \frac{2\pi(l_1+l_2)}{\sqrt{\mu-(l_1+l_2)^2}}. \tag{18}$$

As before these potentials can be attributed to their respective independent sets of modes. Note, however, that the inverse temperature β_H is the sum of left and right contributions, and so the rotational velocities $\Omega^{L,R}$ cannot be unambiguously associated with a specific sector. It is only the combinations $\beta_H\Omega^{L,R}$ that can be interpreted in this way.

The U(1) potentials for general rotating black holes are

$$\beta_H \Phi_L^j = \frac{\pi\mu(\tanh\delta_j \Pi_i \cosh\delta_i - \coth\delta_j \Pi_i \sinh\delta_i)}{\sqrt{\mu-(l_1-l_2)^2}}, \tag{19}$$

$$\beta_H \Phi_R^j = \frac{\pi\mu(\tanh\delta_j \Pi_i \cosh\delta_i + \coth\delta_j \Pi_i \sinh\delta_i)}{\sqrt{\mu-(l_1+l_2)^2}}. \tag{20}$$

The potentials are important for the description of emission processes involving charged particles [9,22–24]. As in the case of rotational velocities we note that it is the combinations $\beta_H\Phi^j_{R,L}$ that can be attributed a given sector, rather than β_H and $\Phi^j_{R,L}$ individually.

Finally, from independent scaling symmetries in the two sectors we have the sum rules

$$\frac{1}{2}\beta_R M - \sum_j \beta_H \Phi_R^j Q_j - \frac{3}{2}\beta_H \Omega^R J_R = \frac{3}{2} S_R, \tag{21}$$

$$\frac{1}{2}\beta_L M - \sum_j \beta_H \Phi_L^j Q_j - \frac{3}{2}\beta_H \Omega^L J_L = \frac{3}{2} S_L, \tag{22}$$

which serve as useful checks on the algebra.

C. Spacetime geometry

In the preceding subsections the thermodynamic variables were derived from the entropy, but the standard thermodynamic quantities also have direct spacetime interpretations. The black hole entropy is given in terms of the area of the outer event horizon by the Bekenstein-Hawking formula

$$S = \frac{A}{4G_N}, \tag{23}$$

the physical inverse temperature is defined from the surface acceleration κ_+ at the outer event horizon as

$$\beta_H = \frac{2\pi}{\kappa_+}, \tag{24}$$

and the physical angular velocities are

$$\Omega^R = \frac{1}{2}\left(\frac{d(\phi+\psi)}{dt}\right)_{\text{outer horizon}}, \tag{25}$$

$$\Omega^L = \frac{1}{2}\left(\frac{d(\phi-\psi)}{dt}\right)_{\text{outer horizon}}. \tag{26}$$

Direct calculations from the metric indeed verify that these geometric definitions agree with thermodynamics. This will be shown in the subsequent section, as a by-product of a more detailed exploration.

It is remarkable that the natural division of thermodynamic potentials into independent L and R contributions also allows an interpretation in terms of spacetime geometry: This follows from the presence of both outer and inner event horizons. Indeed, from the area A_- of the inner horizon we can define an ''entropy''[4]

$$S_- \equiv \frac{A_-}{4G_N} = 2\pi\left[\sqrt{\frac{1}{4}\mu^3\left(\prod_i \cosh\delta_i + \prod_i \sinh\delta_i\right)^2 - J_L^2} - \sqrt{\frac{1}{4}\mu^3\left(\prod_i \cosh\delta_i - \prod_i \sinh\delta_i\right)^2 - J_R^2}\right]. \tag{27}$$

It follows that [9]

$$S_{R,L} = \frac{1}{2}\left(\frac{A_+}{4G_N} \mp \frac{A_-}{4G_N}\right). \tag{28}$$

Similarly,

$$\beta_{R,L} = \frac{2\pi}{\kappa_+} \pm \frac{2\pi}{\kappa_-}, \tag{29}$$

where $\kappa_\pm$ are the surface accelerations at the inner and outer event horizons, respectively:

$$\frac{1}{\kappa_\pm} = \frac{\frac{1}{4}\mu^2(\Pi_i \cosh^2\delta_i - \Pi_i \sinh^2\delta_i)}{\sqrt{\frac{1}{4}\mu^3(\Pi_i \cosh\delta_i - \Pi_i \sinh\delta_i)^2 - J_R^2}} \pm \frac{\frac{1}{4}\mu^2(\Pi_i \cosh^2\delta_i - \Pi_i \sinh^2\delta_i)}{\sqrt{\frac{1}{4}\mu^3(\Pi_i \cosh\delta_i + \Pi_i \sinh\delta_i)^2 - J_L^2}}. \tag{30}$$

It is suggestive that the spacetime geometry divides the entropy and the temperature in the *very* same way that the microscopic interpretation does.

Next we consider the angular velocities. They are usually defined from the geometry in the vicinity of the outer event horizon. Complementary rotational velocities can be introduced at the inner horizon through

$$\Omega_-^R = \frac{1}{2}\left(\frac{d(\phi+\psi)}{dt}\right)_{\text{inner horizon}}, \tag{31}$$

[4]Variables with index ''$-$'' always denote quantities measured at the inner horizon. The corresponding quantities at the outer horizon will sometimes be denoted with an index ''$+$'' and sometimes without an index.

$$\Omega_-^L = \frac{1}{2}\left(\frac{d(\phi-\psi)}{dt}\right)_{\text{inner horizon}}. \tag{32}$$

However, we have already defined angular momenta $J_{R,L}$ that couple only to their designated sectors, and so in this case it should not be expected that the rotational velocities would be further divided into two contributions. Indeed, in the next section we show that

$$\frac{1}{\kappa_-}\Omega_-^R = \frac{1}{\kappa_+}\Omega^R$$

and

$$\frac{1}{\kappa_-}\Omega_-^L = -\frac{1}{\kappa_+}\Omega^L,$$

and so the rotational velocities at the inner horizon are not independent thermodynamic parameters. [Similar comments apply to the U(1) potentials.]

In sum, we find that each thermodynamic variable is split into two parts. This is in accordance with the microscopic interpretation because the string supports both left- and right-moving excitations, and macroscopically it follows as a consequence of the two horizons. Note that some special cases have only one event horizon.[5] However, we can interpret these cases as limits that appear when the inner horizon coalesces with the curvature singularity, and hence continue referring to an inner horizon.

III. GENERAL WAVE EQUATION

A good way to explore the geometry of a black hole is to consider small perturbations of the background. The simplest possibility is a minimally coupled scalar, i.e., a scalar field that satisfies the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi)=0. \tag{33}$$

From the black hole background given in the Appendix it is straightforward to write out the equation explicitly. To present the result in a satisfying symmetric form we use the Killing symmetries deriving from stationarity, and the two axial symmetries of the rotation group in four spatial dimensions. Then the wave function can be written

$$\begin{aligned}\Phi &\equiv \Phi_0(r)\chi(\theta)\exp(-i\omega t + im_\phi\phi + im_\psi\psi)\\ &= \Phi_0(r)\chi(\theta)\exp[-i\omega t + im_R(\phi+\psi) + im_L(\phi-\psi)].\end{aligned} \tag{34}$$

The angular variables ϕ and ψ have period 2π, and so $m_{\phi,\psi}=m_R\pm m_L$ are integer valued. We also introduce a dimensionless radial coordinate x that is related to the standard radial coordinate r through[6]

$$x \equiv \frac{r^2 - \frac{1}{2}(r_+^2 + r_-^2)}{(r_+^2 - r_-^2)}. \tag{35}$$

In this coordinate system the outer and inner horizons at $r_\pm$ are at $x=\frac{1}{2}$ and $x=-\frac{1}{2}$, respectively, and the asymptotically flat region is at $x=\infty$. With this notation the wave equation can be written as

$$\begin{aligned}&\frac{\partial}{\partial x}\left(x^2-\frac{1}{4}\right)\frac{\partial}{\partial x}\Phi_0 + \frac{1}{4}\Bigg[x\Delta\omega^2 - \Lambda + M\omega^2\\ &\quad + \frac{1}{x-\frac{1}{2}}\left(\frac{\omega}{\kappa_+} - m_R\frac{\Omega^R}{\kappa_+} - m_L\frac{\Omega^L}{\kappa_+}\right)^2\\ &\quad - \frac{1}{x+\frac{1}{2}}\left(\frac{\omega}{\kappa_-} - m_R\frac{\Omega^R}{\kappa_+} + m_L\frac{\Omega^L}{\kappa_+}\right)^2\Bigg]\Phi_0 = 0.\end{aligned} \tag{36}$$

Here $\kappa_\pm$ is the surface acceleration at the inner and outer event horizons, $\Omega^{R,L}$ are the angular velocities conjugate to the two angular momenta, M is the mass, Λ is the eigenvalue of the angular Laplacian, and Δ can be expressed in terms of the entropy and the temperature as $\Delta=\beta_H^{-1}S$. The expressions for $\kappa_\pm$ and $\Omega^{R,L}$ are precisely those given in the preceding section [Eqs. (30) and (17), (18)]. We emphasize that this expression is the exact Klein-Gordon equation in the most general black hole background in five dimensions. Interestingly it is in fact no more complicated than special cases that have been considered previously [13,14].

The wave equation is much simpler than the metric it derives from, but it nevertheless remains rather involved. Fortunately each term has a simple interpretation, as follows.

Energy at infinity. The symbol Δ can be defined in the equivalent forms

$$\Delta \equiv \beta_H^{-1}S = r_+^2 - r_-^2. \tag{37}$$

When we use the latter form for Δ and the definition of x in terms of the radial variable r [Eq. (35)], the term $\frac{1}{4}x\Delta\omega^2$ and the derivative term in Eq. (36) (without the $\frac{1}{4}$) can be written as

$$\left(\frac{1}{r^3}\frac{\partial}{\partial r}r^3\frac{\partial}{\partial r} + \omega^2\right)\Phi_0 = 0. \tag{38}$$

This is simply the radial part of the Klein-Gordon equation in five *flat* spacetime dimensions. Evidently the term $\frac{1}{4}x\Delta\omega^2$ encodes properties of the perturbation that persist even in the absence of a black hole. It can be interpreted physically as the energy of the perturbation at infinity.

---

[5]These include the neutral black holes where one or more of the boost parameters vanish. An important case is the Schwarzschild black hole.

[6]More precisely the coordinate r is the five-dimensional analogue of the Boyer-Lindquist coordinate. It reduces to the Schwarzschild coordinate when charges and angular momenta vanish.

We can use the angular momentum parameters $l_{1,2}$ [defined in Eq. (6)] to write Δ as

$$\Delta=\sqrt{[\mu-(l_1-l_2)^2][\mu-(l_1+l_2)^2]}. \qquad (39)$$

It is curious that, in terms of $l_{1,2}$, Δ does not depend on the boost parameters δ_i. Note also that this relation shows that, in the absence of angular momentum, we have simply $\Delta=\mu$.

The screening terms. The term Λ reflects the angular momentum barrier. At large distances it is suppressed relative to the energy at infinity by one power of $x\propto r^2$ as expected. The mass term M is the long-range gravitational interaction. Coulomb-type potentials are of the $r^{-2}\propto x^{-1}$ form in five dimensions, and so it is reasonable that the gravitational screening and the angular momentum barrier are of the same order.

The precise form of the angular Laplacian is

$$\hat{\Lambda}=4\vec{K}^2+(l_1^2+l_2^2)\omega^2+(l_2^2-l_1^2)\omega^2\cos 2\theta, \qquad (40)$$

where

$$\vec{K}^2=-\frac{1}{4\sin 2\theta}\frac{\partial}{\partial\theta}\sin 2\theta\frac{\partial}{\partial\theta}-\frac{1}{4\sin^2\theta}\frac{\partial^2}{\partial\phi^2}-\frac{1}{4\cos^2\theta}\frac{\partial^2}{\partial\psi^2}. \qquad (41)$$

is the angular Laplacian in five *flat* spacetime dimensions. The rotation of the background modifies the angular momentum barrier experienced by a small perturbation, but the change is a very mild one. Specifically it is r independent so that separation of θ and r variables is still possible. Moreover, it is charge independent when the angular momenta are expressed in terms of $l_{1,2}$.

The outer event horizon. Consider the vicinity of the outer event horizon $x\sim\frac{1}{2}$, ignoring temporarily the angular velocities. On general grounds the geometry of the black hole must reduce to Rindler space:

$$ds^2=-\kappa_+^2\rho^2dt^2+d\rho^2. \qquad (42)$$

Here κ_+ is clearly identified as the surface acceleration. The proper radial coordinate ρ is related to the variable x as $\rho\sim\sqrt{x-\frac{1}{2}}$ for $x\sim\frac{1}{2}$ (with $x>\frac{1}{2}$). The solution to the radial wave equation in this regime is of the form

$$\begin{aligned}\Phi_0&\sim\exp[-i\omega(t\pm\kappa_+^{-1}\log\rho)]\\&\sim\exp\{-i\omega[t\pm\tfrac{1}{2}\kappa_+^{-1}\log(x-\tfrac{1}{2})]\}.\end{aligned} \qquad (43)$$

The full wave equation, Eq. (36), indeed supports solutions of this limiting form close to the outer horizon. In this way the Rindler space approximation explains the form of the singularity at $x=\frac{1}{2}$ in Eq. (36). Specifically it verifies that the κ_+ of Eq. (36) is indeed precisely the surface acceleration. Angular parameters can be restored by transforming to the comoving frame, using the definitions of rotational velocities [Eqs. (31) and (32)]. Then the full wave function in this regime becomes

$$\begin{aligned}\Phi&\sim\exp[-i\omega t+im_R(\phi+\psi)+im_L(\phi-\psi)]\\&\quad\times\exp\left[\mp\frac{i}{2}\left(\frac{\omega}{\kappa_+}-m_R\frac{\Omega^R}{\kappa_+}-m_L\frac{\Omega^L}{\kappa_+}\right)\log\left(x-\frac{1}{2}\right)\right]\chi(\theta).\end{aligned} \qquad (44)$$

Comparison with Eq. (36) shows that the rotational parameters $\Omega^{R,L}$ have been identified correctly. This constitutes the promised verification that the geometrical definition of the physical parameters agrees with the thermodynamical one.

For later reference we note that modes of the form

$$\Phi_0^{\rm in}\sim\left(x-\frac{1}{2}\right)^{-i(\omega/\kappa_+-m_R\Omega^R/\kappa_+-m_L\Omega^L/\kappa_+)/2}, \qquad (45)$$

are the infalling modes and those of the form

$$\Phi_0^{\rm out}\sim\left(x-\frac{1}{2}\right)^{i(\omega/\kappa_+-m_R\Omega^R/\kappa_+-m_L\Omega^L/\kappa_+)/2}, \qquad (46)$$

are outgoing. In general relativity these modes are sometimes referred to as left- and right-moving modes, respectively, as this is their direction in the Rindler diagram. We do not use this terminology here in order to avoid confusion with excitations of the effective string.

The inner event horizon. Similarly, in the vicinity of the inner event horizon the metric can be written

$$ds^2=\kappa_-^2\rho^2dt^2-d\rho^2. \qquad (47)$$

Here ρ and x are related as $\rho\sim\sqrt{x+\frac{1}{2}}$ for $x\sim-\frac{1}{2}$ (with $x>-\frac{1}{2}$). Note that the overall signature is opposite of the one close to the outer horizon [Eq. (42)]. However, the wave equation is of second order, and so it is unaffected by this change. The modes are

$$\begin{aligned}\Phi_0&\sim\exp[-i\omega(t\pm\kappa_-^{-1}\log\rho)]\\&\sim\exp\{-i\omega[t\pm\tfrac{1}{2}\kappa_-^{-1}\log(x+\tfrac{1}{2})]\}.\end{aligned} \qquad (48)$$

As before the full wave equation indeed supports modes with this limiting form close to the inner horizon. Hence, from the approximate metric close to the inner horizon we understand the form of the pole term in Eq. (36) at $x=-\frac{1}{2}$, and verify the physical meaning of the various symbols. This calculation therefore substantiates the advertised relations between thermodynamics and the geometry in the vicinity of the inner horizon. In particular the relations

$$\frac{1}{\kappa_-}\Omega_-^R=\frac{1}{\kappa_+}\Omega^R$$

and

$$\frac{1}{\kappa_-}\Omega_-^L=-\frac{1}{\kappa_+}\Omega^L$$

can be read off directly from the inner horizon term. This explains why the parameter κ_+, associated with the outer horizon, appears in the pole of the inner horizon: It is a consequence of the fact that the $\Omega^{R,L}$ refer to quantities at the outer horizon.

IV. SPACETIME SYMMETRIES AND STRING THEORY

As we have seen the black hole thermodynamics can be naturally organized into an R and an L sector that is related to the black hole event horizons, but it is not obvious why they are, roughly, the sum and the difference of inner and outer horizon contributions. In this section we indicate how this comes about, by exhibiting a symmetry of the spacetime geometry that singles out precisely these combinations.

The thermal behavior at the outer horizon can be thought of as a complex periodicity of the (real) Rindler time τ. In analogy, we introduce a new Rindler-type variable σ that encodes the complex periodicity close to the inner horizon. Just as the ''temperature'' $\kappa_-/2\pi$ of the inner horizon is not quite a temperature, because the signature is flipped and the variable σ is not quite a Rindler ''time,'' but rather an analogous spatial variable. Introducing these auxiliary variables τ and σ directly in the wave equation, and ignoring for the time being the energy at infinity, the radial part becomes the eigenvalue problem

$$\mathcal{H}_r\Phi_0=\tfrac{1}{4}(M\omega^2-\Lambda)\Phi_0, \tag{49}$$

where

$$\mathcal{H}_r=-\frac{1}{4\sinh 2\rho}\frac{\partial}{\partial\rho}\sinh 2\rho\frac{\partial}{\partial\rho}-\frac{1}{4\sinh^2\rho}\frac{\partial^2}{\partial\tau^2}+\frac{1}{4\cosh^2\rho}\frac{\partial^2}{\partial\sigma^2} \tag{50}$$

is written in terms of the radial variable ρ defined by $x=\frac{1}{2}\cosh 2\rho$ (ρ reduces to the proper radial coordinate close to the horizons).

This radial equation is closely related to an underlying $\mathrm{SL}(2,R)_R\times\mathrm{SL}(2,R)_L$ symmetry group. The generators $\vec{R}$ of the $\mathrm{SL}(2,R)_R$ group are

$$R_1=\frac{1}{2}\sin(\tau+\sigma)\frac{\partial}{\partial\rho}+\frac{1}{2}\cos(\tau+\sigma)\times\left(\coth\rho\frac{\partial}{\partial\tau}+\tanh\rho\frac{\partial}{\partial\sigma}\right), \tag{51}$$

$$R_2=-\frac{1}{2}\cos(\tau+\sigma)\frac{\partial}{\partial\rho}+\frac{1}{2}\sin(\tau+\sigma)\times\left(\coth\rho\frac{\partial}{\partial\tau}+\tanh\rho\frac{\partial}{\partial\sigma}\right), \tag{52}$$

$$R_3=\frac{1}{2}\left(\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\sigma}\right), \tag{53}$$

and the generators $\vec{L}$ of the $\mathrm{SL}(2,R)_L$ group are found by taking $\sigma\to-\sigma$. The $\vec{R}$ satisfy the algebra

$$[R_i,R_j]=i\epsilon_{ijk}(-)^{\delta_{k3}}R_k, \tag{54}$$

and similarly for $\vec{L}$. These are the appropriate commutation relations for $\mathrm{SL}(2,R)\simeq\mathrm{SO}(2,1,R)$. The two sets of generators commute $[R_i,L_j]=0$, as they should. It is an important fact that the quadratic Casimirs of the groups are identical $\vec{R}^2=\vec{L}^2$ and equal to

$$\vec{R}^2=-R_1^2-R_2^2+R_3^2=\mathcal{H}_r. \tag{55}$$

A maximal set of commuting operators for the $\mathrm{SL}(2,R)_R\times\mathrm{SL}(2,R)_L$ symmetry can be chosen as the two compact generators R_3 and L_3, and the quadratic Casimir. The wave function is an eigenfunction of all these operators. By abuse of notation we equate the operators and their eigenvalues:

$$2\pi R_3=\beta_R\frac{\omega}{2}-\beta_H m_R\Omega^R, \tag{56}$$

$$2\pi L_3=\beta_L\frac{\omega}{2}-\beta_H m_L\Omega^L, \tag{57}$$

$$\vec{R}^2=\vec{L}^2=\frac{1}{4}(M\omega^2-\Lambda). \tag{58}$$

Then the wave function is

$$\Phi\sim\Phi_0 e^{R_3(\tau+\sigma)+L_3(\tau-\sigma)}, \tag{59}$$

where, as before, Φ_0 denotes the radial wave function that depends only on ρ. The R_3 and L_3 eigenvalues are the complex periodicities of the variables $\tau+\sigma$ and $\tau-\sigma$. They can therefore be thought of as the world sheet temperatures, if we reinterpret τ and σ as the world sheet variables of an effective string theory.

In the calculation just presented we have ignored the term $\frac{1}{4}x\Delta\omega^2$ of the original wave equation [Eq. (36)]. This term is a property of the perturbing field, namely, its energy at infinity, and so it is possible that the description nevertheless indicates the internal structure of the black hole accurately. The role of the energy at infinity is to ensure that the geometry far from the black hole is indeed flat Minkowski space. In this sense the troublesome term encodes boundary conditions, and so indicates that the internal symmetry $\mathrm{SL}(2,R)_R\times\mathrm{SL}(2,R)_L$ is spontaneously broken. The precise role of the energy at infinity is a major concern that must eventually be elucidated.

We conclude this section by exhibiting another symmetry. The exact equation [Eq. (36)] is invariant under

$$x\to-x, \tag{60}$$

$$r_+^2\leftrightarrow r_-^2\quad(\Delta\to-\Delta), \tag{61}$$

$$2\pi R_3\to 2\pi R_3, \tag{62}$$

$$2\pi L_3\to-2\pi L_3. \tag{63}$$

Macroscopically this interchanges the role of the two horizons. In the microscopic interpretation the symmetry leaves R_3 invariant and acts as a parity transformation on the L_3. This is precisely the way T duality acts on conventional conformal field theories, and so the interchange of horizons can be identified with T duality. From this point of view the transformations in spacetime geometry generalize the usual $R \rightarrow \alpha'/R$ that accompanies T duality in the simplest case.

To avoid misunderstanding we emphasize that the arguments presented in this section are entirely in the context of the classical geometry. We interpret them as an indication of a strategy towards a comprehensive effective string model of black holes, but we do not yet have such a model.

V. SOLUTIONS OF THE WAVE EQUATION

In general Eq. (36) is a rather complicated differential equation. It has regular singularities at the horizons $x = \pm \frac{1}{2}$ and an irregular singularity at infinity. The singularity at infinity is not of the so-called normal kind, and so it can not be cured by absorption in a determining factor (see, e.g., [25]). The solutions to this kind of ordinary differential equation has an essential singularity and it is not known how to find them explicitly. However, the equation simplifies in various regions of the radial variable x. In the following we consider these cases, postponing the discussion of their combination into solutions covering all of space to Sec. VI.

We will omit the rotational parameters for simplicity in notation but this involves no loss of generality as they can be restored by the substitutions

$$\beta_R \frac{\omega}{2} \rightarrow \beta_R \frac{\omega}{2} - \beta_H m_R \Omega^R, \tag{64}$$

$$\beta_L \frac{\omega}{2} \rightarrow \beta_L \frac{\omega}{2} - \beta_H m_L \Omega^L, \tag{65}$$

$$\beta_H \omega \rightarrow \beta_H \omega - \beta_H m_R \Omega^R - \beta_H m_L \Omega_L . \tag{66}$$

The asymptotic region. At large $|x| \gg 1$ we approximate Eq. (36) by

$$\frac{\partial}{\partial x} x^2 \frac{\partial}{\partial x} \Phi_0 + \frac{1}{4}(x \Delta \omega^2 - \Lambda + M\omega^2)\Phi_0 = 0. \tag{67}$$

The horizon terms were omitted and we took $x^2 - \frac{1}{4} \simeq x^2$ in the kinetic energy. This equation can be solved exactly in terms of Bessel functions. The linearly independent solutions are[7]

$$\Phi_\infty^\pm = \frac{1}{x^{1/2}} J_{\pm(2\xi-1)}(\omega \sqrt{x\Delta}), \tag{68}$$

where ξ is

$$\xi = \tfrac{1}{2}(1 + \sqrt{1 + \Lambda - M\omega^2}). \tag{69}$$

The horizon region. In the horizon region the wave equation can be approximated by

$$\left[\frac{\partial}{\partial x}\left(x^2 - \frac{1}{4}\right)\frac{\partial}{\partial x} + \frac{1}{4}\left(-\Lambda + M\omega^2 + \frac{1}{x - \frac{1}{2}\kappa_+^2}\,\omega^2 - \frac{1}{x + \frac{1}{2}\kappa_-^2}\,\omega^2 \right) \right] \Phi_0 = 0. \tag{70}$$

The *only* approximation is the omission of the term $\frac{1}{4} x \Delta \omega^2 = \frac{1}{4} x (r_+^2 - r_-^2)\omega^2$. It is the divergence of this term for large x that is responsible for the irregular singularity at infinity in the general case, and so the approximate equation has three singularities that are all regular. This is a standard problem that is solved by the hypergeometric function [13,14]. One solution is

$$\Phi_0^{\rm in} = \left(\frac{x - \frac{1}{2}}{x + \frac{1}{2}}\right)^{-i\beta_H \omega/4\pi} \left(x + \frac{1}{2}\right)^{-\xi} \times F\left(\xi - i\frac{\beta_R \omega}{4\pi}, \xi - i\frac{\beta_L \omega}{4\pi}, 1 - i\frac{\beta_H \omega}{2\pi}, \frac{x - \frac{1}{2}}{x + \frac{1}{2}}\right), \tag{71}$$

where ξ is given in Eq. (69). The surface accelerations $\kappa_\pm$ were eliminated in terms of the temperatures $\beta_{R,L}$ and $\beta_H = \frac{1}{2}(\beta_R + \beta_L)$ [using Eq. (29)]. A linearly independent solution can be chosen as

$$\Phi_0^{\rm out} = \left(\frac{x - \frac{1}{2}}{x + \frac{1}{2}}\right)^{i\beta_H \omega/4\pi} \left(x + \frac{1}{2}\right)^{-\xi} \times F\left(\xi + i\frac{\beta_R \omega}{4\pi}, \xi + i\frac{\beta_L \omega}{4\pi}, 1 + i\frac{\beta_H \omega}{2\pi}, \frac{x - \frac{1}{2}}{x + \frac{1}{2}}\right). \tag{72}$$

The two solutions are related by time reversal. This can be seen directly by the substitution $\omega \rightarrow -\omega$.

The two independent solutions have been chosen in a form that reflects the physics in the vicinity of the outer horizon: They reduce to plane waves $(x - \frac{1}{2})^{\pm i\beta_H \omega/4\pi}$ for $x \sim \frac{1}{2}$. An alternative basis that is adapted to the behavior at infinity follows by the modular properties of the hypergeometric functions. For example $\Phi_0^{\rm in}$ of Eq. (71) can be written

$$\begin{aligned}\Phi_0^{\rm in} = \left(\frac{x - \frac{1}{2}}{x + \frac{1}{2}}\right)^{-i\beta_H \omega/4\pi} \Bigg[&\left(x + \frac{1}{2}\right)^{-\xi} \times \frac{\Gamma(1 - i\,\beta_H\omega/2\pi)\Gamma(1 - 2\xi)}{\Gamma(1 - \xi - i\,\beta_L\omega/4\pi)\Gamma(1 - \xi - i\,\beta_R\omega/4\pi)} \times F\left(\xi - i\frac{\beta_R\omega}{4\pi}, \xi - i\frac{\beta_L\omega}{4\pi}, 2\xi, \frac{1}{x + \frac{1}{2}}\right) \\ &+ \left(x + \frac{1}{2}\right)^{\xi - 1} \frac{\Gamma(1 - i\,\beta_H\omega/2\pi)\Gamma(2\xi - 1)}{\Gamma(\xi - i\,\beta_L\omega/4\pi)\Gamma(\xi - i\,\beta_R\omega/4\pi)} \times F\left(1 - \xi - i\frac{\beta_L\omega}{4\pi}, 1 - \xi - i\frac{\beta_R\omega}{4\pi}, 2 - 2\xi, \frac{1}{x + \frac{1}{2}}\right) \Bigg].\end{aligned} \tag{73}$$

[7]For approximate solutions at large distances we replace the index 0 of the radial wave functions with ∞.

In this form the asymptotic behavior for large x is manifest:

$$\Phi_0^{\rm in}\sim x^{-\xi}\frac{\Gamma(1-i\,\beta_H\omega/2\pi)\Gamma(1-2\xi)}{\Gamma(1-\xi-i\,\beta_L\omega/4\pi)\Gamma(1-\xi-i\,\beta_R\omega/4\pi)}+x^{\xi-1}\frac{\Gamma(1-i\,\beta_H\omega/2\pi)\Gamma(2\xi-1)}{\Gamma(\xi-i\,\beta_L\omega/4\pi)\Gamma(\xi-i\,\beta_R\omega/4\pi)}. \quad (74)$$

Here each term admits corrections for large x that are subleading in $1/x$.

Similarly a basis adapted to the behavior at the inner horizon can be chosen. The wave function that has only an ingoing component at the outer horizon has both an outgoing and an ingoing component at the inner horizon. In physical terms the scattering off the background invariably mixes the components. The basis adapted to the inner horizon will play no role in the present investigation.

The angular Laplacian. The angular Laplacian $\vec{K}^2$ of a flat five-dimensional background [Eq. (41)] is the quadratic Casimir of the group $SO(4)\simeq SU(2)_L\times SU(2)_R$. It has eigenvalues $\vec{K}^2=\frac{1}{4}K(K+2)$ where K is an integer. The presence of the curved background modifies the angular Laplacian to [Eq. (40)]

$$\hat{\Lambda}=4\vec{K}^2+(l_1^2+l_2^2)\omega^2+(l_2^2-l_1^2)\omega^2\cos 2\theta. \quad (75)$$

The solutions $e^{i(m_\phi\phi+m_\psi\psi)}\chi(\theta)$ to the corresponding eigenvalue problem cannot in general be found in closed form.[8] As a qualitative result we note that the contributions from the rotation of the black hole are always positive. In the special case $l_1=l_2$ the eigenfunctions $\chi(\theta)$ are hypergeometric functions and the eigenvalues are very simple:

$$\Lambda=K(K+2)+(l_1^2+l_2^2)\omega^2. \quad (76)$$

Corrections can be calculated perturbatively. The leading term is of second order in $(l_2^2-l_1^2)\omega^2$ because $\cos 2\theta$ vanishes when averaged over all angles. We can use Eq. (76) as approximate eigenvalues for large classes of problems, including those relevant for low-energy perturbations or for black holes with nearly coincident rotation parameters.

VI. ABSORPTION CROSS SECTIONS

The calculation of absorption cross sections follows much previous work (including [26,27,11,23,24,13,14,28]). In this section we find the necessary generalizations due to angular momentum and sharpen the ranges of validity previously established for nonrotating black holes. We first carry out the algebraic manipulations, and then consider their ranges of validity.

In the absorption geometry the wave function close to the horizon has only an incoming component. We normalize the wave function as $A_0\Phi_0^{\rm in}$. Then Eq. (71) gives the flux at the horizon as

$$\text{flux}=\frac{1}{2i}(\overline{\Phi}r^3\partial_r\Phi-\text{c.c.})=|A_0|^2\frac{\beta_H\omega\Delta}{4\pi}. \quad (77)$$

Similarly, we write the wave function in the asymptotic region as $A_\infty^+\Phi_\infty^+$ and expand at very large distances:

$$A_\infty^+\Phi_\infty^+\sim A_\infty^+\sqrt{\frac{2}{\pi x^{3/4}\Delta^{1/2}\omega}}\cos\left(\omega\sqrt{x\Delta}-\xi\pi+\frac{1}{4}\pi\right), \quad (78)$$

and so the flux becomes

$$\text{flux}=\frac{1}{2i}(\overline{\Phi}r^3\partial_r\Phi-\text{c.c.})=|A_\infty^+|^2\frac{\Delta}{4\pi}. \quad (79)$$

The effective two-dimensional transmission coefficient $|T_K|^2$ is the ratio of these fluxes. Using a geometric relation derived in [29] the absorption cross section of the Kth partial wave becomes

$$\sigma_{\rm abs}^{(K)}(\omega)=\frac{4\pi(K+1)^2}{\omega^3}|T_K|^2=\frac{4\pi\beta_H}{\omega^2}(K+1)^2\left|\frac{A_0}{A_\infty^+}\right|^2. \quad (80)$$

To find the ratio $|A_0/A_\infty^+|$ we consider a general wave function in the asymptotic region,

$$\Phi_\infty=A_\infty^+\frac{1}{x^{1/2}}J_{2\xi-1}(\omega\sqrt{x\Delta})+A_\infty^-\frac{1}{x^{1/2}}J_{-(2\xi-1)}(\omega\sqrt{x\Delta}), \quad (81)$$

and expand for small arguments of the Bessel function:

$$\Phi_\infty\sim A_\infty^+x^{\xi-1}\frac{1}{\Gamma(2\xi)}\left(\frac{\sqrt{\Delta}\omega}{2}\right)^{2\xi-1}+A_\infty^-x^{-\xi}\frac{1}{\Gamma(2-2\xi)}\left(\frac{\sqrt{\Delta}\omega}{2}\right)^{1-2\xi}. \quad (82)$$

This should be compared with the near-horizon wave function $A_0\Phi_0^{\rm in}$ for large x [Eq. (74)]. Assuming that these limiting forms have an overlapping regime of validity we find

$$\left|\frac{A_\infty^+}{A_0}\right|=\left(\frac{\sqrt{\Delta}\omega}{2}\right)^{1-2\xi}\Gamma(2\xi)\Gamma(2\xi-1)\times\frac{\Gamma(1-i\,\beta_H\omega/2\pi)}{\Gamma(\xi-i\,\beta_L\omega/4\pi)\Gamma(\xi-i\,\beta_R\omega/4\pi)}, \quad (83)$$

Note that the "matching region" of overlapping validity is necessarily at large x, and so, for $\xi>\frac{1}{2}$, the $x^{\xi-1}$ terms

[8] In fact the differential equation is the analytical continuation of the radial equation (50): The constant term is analogous to the mass term and the $\cos 2\theta$ term corresponds to the energy at infinity [omitted in Eq. (50)].

dominate and the $x^{-\xi}$ terms can be neglected. This fact was anticipated already in the derivation of the flux [Eq. (79)], where A_{∞}^{-} was ignored.[9]

Collecting the results the partial absorption cross sections become

$$\sigma_{\rm abs}^{(K)}(\omega)=\frac{4\pi(K+1)^2\beta_H}{\omega^2}\left(\frac{\sqrt{\Delta}\omega}{2}\right)^{4\xi-2}\left|\times\frac{\Gamma(\xi-i\,\beta_L\omega/4\pi)\Gamma(\xi-i\,\beta_R\omega/4\pi)}{\Gamma(2\xi-1)\Gamma(2\xi)\Gamma(1-i\,\beta_H\omega/2\pi)}\right|^2. \quad (84)$$

Recall the definition $\xi=\frac{1}{2}(1+\sqrt{1+\Lambda-M\omega^2})$ where Λ was given in Eq. (40).

We turn next to the range of validity for the matching procedure that leads to this cross section. It is most transparent to derive the conditions directly from the general wave equation [Eq. (36)] written as

$$\frac{\partial}{\partial x}\left(x^2-\frac{1}{4}\right)\frac{\partial}{\partial x}\Phi_0+\frac{1}{4}\left[x\Delta\omega^2-\Lambda+M\omega^2+\frac{1}{x-\frac{1}{2}}\left(\frac{(\beta_R+\beta_L)}{4\pi}\right)^2\omega^2-\frac{1}{x+\frac{1}{2}}\left(\frac{(\beta_R-\beta_L)}{4\pi}\right)^2\omega^2\right]\Phi_0=0. \quad (85)$$

[We assume $m_R=m_L=0$ for convenience, but generality could be restored using Eqs. (64) and (65).] The Bessel function is valid when we can ignore the horizon terms and the $\frac{1}{4}$ in the derivative terms, and the hypergeometric function requires that the energy at infinity $\frac{1}{4}x\Delta\omega^2$ is negligible. We must show that there is an intermediate matching region where both approximations are valid. We consider two useful strategies in the following subsections.

A. Matching on a vanishing potential

The first possibility is that *all potential terms are small* in the matching region. Then only the kinetic term remains, and the equation integrates to a constant solution. This constant value of the wave function is the coincident amplitude of the Bessel function at small argument and the hypergeometric function at large x.[10]

Matching on a vanishing potential requires a range of x that satisfies

$$x\gg 1,\quad \Delta x\omega^2\ll 1,\quad \frac{1}{x}\beta_R\beta_L\omega^2\ll 1,\quad |-\Lambda+M\omega^2|\ll 1. \quad (86)$$

The necessary and sufficient conditions for the existence of such x are

$$\Delta\omega^2\ll 1,\quad \beta_R\beta_L\Delta\omega^4\ll 1,\quad |-\Lambda+M\omega^2|\ll 1. \quad (87)$$

For higher partial waves a positive integer contributes to Λ [Eq. (40)], and so the last condition can only be satisfied in rather special circumstances. In this subsection we only consider the S wave. The last condition automatically implies $\xi\simeq 1$, and so the coincident wave functions in the matching region [Eq. (74) or (82)] indeed reduce to constants, as expected. Moreover, the absorption cross section takes a particularly simple and suggestive form

$$\sigma_{\rm abs}^{(0)}(\omega)=A\left|\frac{\Gamma(1-i\,\beta_L\omega/4\pi)\Gamma(1-i\,\beta_R\omega/4\pi)}{\Gamma(1-i\,\beta_H\omega/2\pi)}\right|^2=A\frac{\beta_L(\omega/2)\,\beta_R(\omega/2)}{\beta_H\omega}\frac{(e^{\beta_H\omega}-1)}{(e^{\beta_L\,\omega/2}-1)(e^{\beta_R\,\omega/2}-1)}, \quad (88)$$

where A denotes the area of the black hole. [In rewriting Eq. (84) we used $\Delta=\beta_H^{-1}S$, $S=(1/4G_N)A$, and $G_N=\frac{1}{4}\pi$.] This cross section can be interpreted microscopically in terms of a two-body process of the effective string theory that parametrizes the collective excitations of the black hole [12,11].

Note that we have not assumed $\beta_R\omega\sim\beta_L\omega\sim 1$, and so there are regimes where either one or both of the Bose distribution factors simplify to either the Maxwell distribution or to the Bose degenerate state. The classical calculation is still reliable in these cases.

Next we consider some specific examples.

Low-energy limit. In the S wave the angular operator $\Lambda\propto\omega^2$, and so for an arbitrary black hole all conditions in Eq. (87) can be satisfied by taking the energy ω sufficiently small. In this case Eq. (88) applies and the cross section becomes

$$\sigma_{\rm abs}^{(0)}(\omega\to 0)=A. \quad (89)$$

This relation is well known for scattering off nonrotating black holes (see [30] and references therein), but the result here also applies to nonrotating ones.

Two large charges. Assume that two of the boost parameters are large, say, $\delta\equiv\delta_1\sim\delta_2\gg 1$, and treat the last one as order unity. We generalize this ''dilute gas'' region of Maldacena and Strominger [11] by including also large angular momenta with $J_R\sim J_L\sim\mu^{3/2}e^{2\delta}$ or, equivalently, $l_1\sim l_2\sim\mu^{1/2}$. [$l_{1,2}$ were defined in Eq. (6).] In this case $\Delta\sim\mu$, $M\sim\mu e^{2\delta}$, $\Lambda\sim\mu\omega^2$, and $\beta_R\sim\beta_L\sim\mu^{1/2}e^{2\delta}$. According to Eq. (87) the cross section, Eq. (88), is reliable for frequencies that satisfy $e^{\delta}\mu^{1/2}\omega\ll 1$. This includes (but is not limited to) the interesting range $\omega\sim\beta_{R,L}^{-1}\sim\mu^{-1/2}e^{-2\delta}$. The thermodynamic parameters of the absorption cross section, Eq. (88), have a nontrivial dependence on angular momenta, and the inferred distribution functions agree in detail with those expected from counting arguments [20,31].

Rapidly spinning black holes. The freedom provided by the angular momenta also allows for a new kind of limit: All the boosts are arbitrary but a dilute-gas-type region can nevertheless be reached by tuning the angular momentum pa-

[9]The case where ξ becomes a complex number corresponds to large frequencies. Here both A_{∞}^{-} and A_{∞}^{+} must be taken into account. In this case the appropriate modifications are given in an appendix of [13].

[10]The coefficient of the linearly independent solution, proportional to x^{-1}, can be determined by matching derivatives. This term contributes a flux that is suppressed by $(\Delta\omega^2)^2\ll 1$, due to the large matching x.

rameters so that both inverse temperatures are large. This is accomplished by taking $l_2=0$ and tuning $\mu-l_1^2=\mu\epsilon^2\ll\mu$ [$l_{1,2}$ were defined in Eq. (6)]. Then $\Delta\sim\mu\epsilon^2$, $M\sim\mu$, $\Lambda\sim\mu\omega^2$, and $\beta_R\sim\beta_L\sim\mu^{1/2}\epsilon^{-1}$. The matching conditions, Eq. (87), require $\mu\omega^2\ll 1$. This range of frequencies includes the interesting ones with $\omega\sim\beta_{R,L}^{-1}\sim\mu^{-1/2}\epsilon$. Note that in this example *no hierarchy in the charges is necessary*, and so we capture the entire functional dependence of the temperatures on the boost parameters. It is also interesting that in this case the black hole is not even approximately supersymmetric.

Near-BPS limit. We generalize the nonrotating near-BPS black hole (considered in [23,24]) by including angular momenta $l_1\sim l_2\sim\mu^{1/2}$. (This implies $J_R\sim\mu^{3/2}e^{\delta}$ and $J_L\sim\mu^{3/2}e^{3\delta}$, and so there is a hierarchy in the angular momenta.) Close to extremality all the boosts are large $\delta_i\gg 1$ and we expand systematically in e^{δ} (where $\delta\sim\delta_i$). Then $\Delta\sim\mu$, $M\sim\mu e^{2\delta}$, $\Lambda\sim\mu\omega^2$, $\beta_R\sim\mu^{1/2}e^{3\delta}$, and $\beta_L\sim\mu^{1/2}e^{\delta}$. The conditions, Eq. (87), are satisfied for frequencies in the range $\mu^{1/2}\omega e^{\delta}\ll 1$. There is a hierarchy of the temperatures ($\beta_R\gg\beta_L$) in this case, and so there is no regime where both Bose factors are significant simultaneously. The applicable range of frequencies is $\omega\sim\beta_R^{-1}\sim\mu^{-1/2}e^{-3\delta}$ but not $\omega\sim\beta_L^{-1}\sim\mu^{-1/2}e^{-\delta}$, and so only the β_R can be reliable probed.

Near-extreme Kerr-Newman limit. As the final example we consider the near-extreme Kerr-Newman limit defined by $\mu-(l_1-l_2)^2=\mu\epsilon^2\ll\mu$ (with $l_2\neq 0$). Here $\Delta\sim\mu\epsilon$, $M\sim\Lambda/\omega^2\sim\mu$, $\beta_L\sim\mu^{1/2}$, and $\beta_R\sim\mu^{1/2}\epsilon^{-1}$, and so the condition on the frequency becomes $\mu\omega^2\ll 1$. As in the near-BPS case we can probe β_R, but not β_L.

It is interesting that in the limit $\epsilon\to 0$ the entropies approach $S_R=0$ and

$$S=S_L=2\pi\sqrt{Q_1Q_2Q_3+J_R^2-J_L^2}=2\pi\sqrt{n_1n_2n_3+J_R^2-J_L^2},\tag{90}$$

where the n_i are quantized charges. The near-extreme Kerr-Newman limit is not supersymmetric, but the form of the entropy is nevertheless reminiscent of the BPS case: The entropy does not depend on moduli, and the counting arguments can be made notably less heuristic.

B. Matching on a constant potential

In this case *the screening term* dominates in the matching region. Then the wave equation is solved by the polynomials $x^{\xi-1}$ and $x^{-\xi}$. The coincident wave functions [Eq. (74) or (82)] indeed reduce to precisely these polynomials.

Matching on a constant potential requires a range of x so that

$$x\gg 1,\quad x\Delta\omega^2\ll|-\Lambda+M\omega^2|,\quad \frac{1}{x}\beta_R\beta_L\omega^2\ll|-\Lambda+M\omega^2|.\tag{91}$$

If $|-\Lambda+M\omega^2|\ll 1$, the present procedure corresponds to matching on a vanishing potential, but in this case the conditions, Eq. (91), are nevertheless stronger than Eq. (87), because here we insist that the screening term dominates even though it is small when $|-\Lambda+M\omega^2|\ll 1$. Therefore the two matching procedures must be considered separately to find the most generous ranges of validity.

The necessary and sufficient conditions for the existence of x satisfying Eq. (91) are

$$\Delta\omega^2\ll|-\Lambda+M\omega^2|,\quad \beta_R\beta_L\Delta\omega^4\ll|-\Lambda+M\omega^2|^2.\tag{92}$$

In the S wave, $\Lambda\propto\omega^2$, and so in this case there are *no* assumptions about the frequency of the radiation.[11] Indeed, in the S wave the entire potential in Eq. (85) is proportional to ω^2, and so conditions on the relative size of potential terms must be frequency independent.

We consider a few specific examples.

Higher angular momentum modes. The simplest example of matching on a constant potential concerns a particular partial wave K, but otherwise the same restrictions as in the case of matching on a vanishing potential. This is consistent with Eq. (91) [but not Eq. (87)]. In this case $\Lambda\simeq K(K+2)$ and $M\omega^2\ll 1$, and so the absorption spectrum is Eq. (84) with $\xi=K/2+1$. The process can be modeled microscopically as an impinging closed string that is absorbed by bound state of D-branes, with $2K$ fermions being excited in the process [14,29,32].

One large charge. We consider the S wave and take $\delta\equiv\delta_3\gg 1$ and $\delta_{1,2}$ of order 1 [13]. Angular momenta $l_{1,2}\sim\mu^{1/2}$ can be included. Then $\Delta\sim\mu$, $M\sim\mu e^{2\delta}$, $\Lambda\sim\mu\omega^2$, and $\beta_R\sim\beta_L\sim\mu^{1/2}e^{\delta}$. This is sufficient to satisfy the conditions $\Delta\ll M$ and $\beta_R\beta_L\Delta\ll M^2$ required by Eq. (91), and so the absorption cross section is given by Eq. (84) with a general value of ξ.

VII. DISCUSSION

We would like to conclude the paper with remarks on the microscopic interpretation of our results. As a starting point for the discussion we consider the Hawking emission rate

$$\Gamma_{\rm em}=\sigma_{\rm abs}(\omega)\frac{1}{e^{\beta_H\omega}-1}\frac{d^4k}{(2\pi)^4}.\tag{93}$$

In the regime where matching on a vanishing potential is justified [Eq. (87)] we use Eq. (88) for the cross section and find

$$\Gamma_{\rm em}=A\frac{\beta_L(\omega/2)\beta_R(\omega/2)}{\beta_H\omega}\frac{1}{(e^{\beta_L\omega/2}-1)(e^{\beta_R\omega/2}-1)}\frac{d^4k}{(2\pi)^4}\tag{94}$$

$$=8\pi G_NL\frac{1}{\omega}\left(\frac{\omega}{2}\right)^2\frac{1}{(e^{\beta_L\omega/2}-1)(e^{\beta_R\omega/2}-1)}\frac{d^4k}{(2\pi)^4},\tag{95}$$

where the intermediate step used relations given in Sec. II, and we defined L as

[11]Note, however, that we only give the final result for $\xi>\frac{1}{2}$, but the argument shows that the analogous calculation for ξ complex is reliable as well.

$$L=2\pi\mu^2\left(\prod_i \cosh^2\delta_i-\prod_i \sinh^2\delta_i\right). \tag{96}$$

It was shown by Das and Mathur that the emission rate, Eq. (95), is identical, including the coefficient, to the two-body annihilation rate for small amplitude waves propagating on an effective string of length L [12]. In this model of the emission process the length of the effective string parametrizes the strength of the interactions. It is satisfying that in our case the length L is both U-duality invariant and independent of angular momenta.

For large black holes L is much larger that the naive string length. The importance of this kind of ''tension renormalization'' was recognized already in the early countings of nonperturbative string states [33–35], and it is now understood from D-brane properties how this may come about [36,37]. The near-BPS black holes related to momentum-carrying bound states of $D1$- and $D5$-branes [10,38] are special cases of the general formula, Eq. (96): Here two boosts are large $\delta_1\sim\delta_2\gg 1$ and the length reduces to $L=2\pi Q_1Q_2=2\pi n_1n_2R$, where $n_{1,2}$ are the quantized $D1$- and $D5$-brane charges and R is the length of the dimension that the $D1$-brane wraps around [12]. However, the general expression for L accounts for emission from a larger class of black holes than has previously been considered. For example, the full dependence on boost parameters is needed in the case of rapidly spinning black holes even though the thermodynamic properties of this case are analogous to the ''dilute gas'' regime of [11].

In the microscopic interpretation the colliding quanta have Bose distributions with inverse temperatures [Eqs. (15), (16)]:

$$\beta_{R,L}=\frac{2\pi\mu(\Pi_i\cosh\delta_i\pm\Pi_i\sinh\delta_i)}{\sqrt{\mu-(l_1\pm l_2)^2}}. \tag{97}$$

The dynamical considerations therefore give direct information about properties of the microscopic theory. In particular, this gives a concrete physical meaning to the temperatures derived at each event horizon. However, the two-body form of the emission rate is a low-energy approximation, and so only the cases where the precise requirement [Eq. (87)] on the frequency is consistent with the interesting ranges $\omega\sim\beta_R^{-1}$ and $\omega\sim\beta_L^{-1}$ can be probed in detail [11]. Despite this restriction we can verify the dependence of the inverse temperatures on all boost parameters by considering rapidly rotating black holes. Our expressions for the U(1) potentials [Eqs. (19) and (20)] can similarly be checked in some regimes, by considering emission of charged particles, and the angular potentials [Eqs. (17) and (18)] can be probed by considering the emission of higher partial waves.[12] Hence the microscopic model based on the thermodynamics of two horizons provides an economical summary of a large class of special cases, including some that have not been considered before.

The Hawking emission process can be described as a two-body process in the entire regime where matching on a constant potential is justified [Eq. (87)]. For generic nonextremal black holes this implies $\beta_{L,R}\omega\ll 1$, and so the agreement between the microscopic model and the macroscopic calculation reduces to a single number, namely, the universal low-energy absorption cross section. This is nevertheless nontrivial because we consider that the most general black holes and the model capture the full functional dependence on all parameters. It has previously been argued (along somewhat different lines) that the universal low-energy scattering off Schwarzschild [5] and Reissner-Nordström [39] black holes can be accounted for by an effective string model. Our result includes these observations as special cases as well as the D-brane-inspired string models for near-BPS black holes. Let us summarize the argument: From the horizon structure we identify distribution functions for right- and left-moving string excitations, from rapidly spinning black holes we infer the coupling between the two sectors, and then a calculation gives the universal low-energy cross section for *all* black holes. In this sense the version of the effective string model presented in this paper has some applicability even for generic nonextremal black holes.

The remaining problem becomes one of interactions, rather than that of state counting. Here it is concerning that in general the typical Hawking particle is too energetic to result from a simple two-body process. This may simply indicate that interactions are more involved at larger energies, at least in the range of parameters where matching on a constant potential is justified [Eq. (91)] [13]. Here the absorption cross section [Eq. (84)] depends on the parameter $\xi=\frac{1}{2}(1+\sqrt{1+\Lambda-M\omega^2})$. The angular momentum eigenvalue Λ [Eq. (76)] depends on the angular momentum of the particle as well as that of the background. When the main contribution to ξ is from particle angular momentum the ξ is integer or half-integer and the spectrum can be understood qualitatively from many-body kinematics [14,29,32]. In general the background mass and angular momenta contribute to ξ but the emission spectrum retains its qualitative character. It is therefore reasonable to suspect that further understanding of many-body effects might account also for this case.

As we saw in Sec. IV the geometry of the region in the vicinity of the horizons immediately suggests an effective description in string theory. The matching on a vanishing potential corresponds to the situation where this suggestive near-horizon region can be unambiguously distinguished from the surrounding space. In the case of matching on a constant potential the long-range fields make the distinction less clear, but presumably still valid, as we argued in the previous paragraph. However, in the most general problem the distinction seems ambiguous, and it is the processes that are sensitive to this coupling between the near-horizon region and the asymptotic space that we are presently unable to account for even classically.[13] This seems to be a barrier that will remain difficult to surmount in the string theory descrip-

[12]This calculation uses matching on a constant potential, not a vanishing one.

[13]It is possible that investigations involving particles with non-minimal coupling (initiated in [40]) might help, for example, by being less sensitive to the term at infinity.

tion. It is not yet clear whether this represents an obstacle of purely technical nature or a more profound crisis.

ACKNOWLEDGMENTS

We would like to thank V. Balasubramanian, C. Callan, S. Mathur, H. Verlinde, A. Peet, and F. Wilczek for discussions, and D. Kastor and J. Maldacena for correspondence. This work was supported in part by DOE Grant No. DOE-EY-76-02-3071, NSF Grant No. PHY95-12732 (M.C.), and NATO collaborative Grant No. CGR 949870 (MC).

APPENDIX: THE BLACK HOLE SOLUTION

The Einstein metric of the black holes is [15]

$$
\begin{aligned}
\bar{\Delta}^{-1/3}ds_E^2 = {} & -\frac{(r^2+l_1^2\cos^2\theta+l_2^2\sin^2\theta)(r^2+l_1^2\cos^2\theta+l_2^2\sin^2\theta-2m)}{\bar{\Delta}}dt^2+\frac{r^2}{(r^2+l_1^2)(r^2+l_2^2)-2mr^2}dr^2+d\theta^2\\
& +\frac{4m\cos^2\theta\sin^2\theta}{\bar{\Delta}}\bigg[l_1l_2\bigg\{(r^2+l_1^2\cos^2\theta+l_2^2\sin^2\theta)-2m\prod_{i<j}\sinh^2\delta_i\sinh^2\delta_j\bigg\}+2m(l_1^2+l_2^2)\prod_i\cosh\delta_i\\
& \times\sinh\delta_i-4ml_1l_2\prod_i\sinh^2\delta_i\bigg]d\phi\, d\psi-\frac{4m\sin^2\theta}{\bar{\Delta}}\bigg[(r^2+l_1^2\cos^2\theta+l_2^2\sin^2\theta)\bigg(l_1\prod_i\cosh\delta_i-l_2\prod_i\sinh\delta_i\bigg)\\
& +2ml_2\prod_i\sinh\delta_i\bigg]d\phi\, dt-\frac{4m\cos^2\theta}{\bar{\Delta}}\bigg[(r^2+l_1^2\cos^2\theta+l_2^2\sin^2\theta)\bigg(l_2\prod_i\cosh\delta_i-l_1\prod_i\sinh\delta_i\bigg)\\
& +2ml_1\prod_i\sinh\delta_i\bigg]d\psi\, dt+\frac{\sin^2\theta}{\bar{\Delta}}\bigg[(r^2+2m\sinh^2\delta_3+l_1^2)(r^2+2m\sinh^2\delta_1+l_1^2\cos^2\theta+l_2^2\sin^2\theta)(r^2+2m\sinh^2\delta_2\\
& +l_1^2\cos^2\theta+l_2^2\sin^2\theta)+2m\sin^2\theta\bigg\{(l_1^2\cosh^2\delta_3-l_2^2\sinh^2\delta_3)(r^2+l_1^2\cos^2\theta+l_2^2\sin^2\theta)+4ml_1l_2\prod_{i<j}\cosh\delta_i\sinh\delta_j\\
& -2m\sinh^2\delta_1\sinh^2\delta_2(l_1^2\cosh^2\delta_3+l_2^2\sinh^2\delta_3)-2ml_2^2\sinh^2\delta_3(\sinh^2\delta_1+\sinh^2\delta_2)\bigg\}\bigg]d\phi^2\\
& +\frac{\cos^2\theta}{\bar{\Delta}}\bigg[(r^2+2m\sinh^2\delta_3+l_2^2)(r^2+2m\sinh^2\delta_1+l_1^2\cos^2\theta+l_2^2\sin^2\theta)(r^2+2m\sinh^2\delta_2+l_1^2\cos^2\theta+l_2^2\sin^2\theta)\\
& +2m\cos^2\theta\bigg\{(l_2^2\cosh^2\delta_3-l_1^2\sinh^2\delta_3)(r^2+l_1^2\cos^2\theta+l_2^2\sin^2\theta)+4ml_1l_2\prod_{i<j}\cosh\delta_i\sinh\delta_j\\
& -2m\sinh^2\delta_1\sinh^2\delta_2(l_2^2\cosh^2\delta_3+l_1^2\sinh^2\delta_3)-2ml_1^2\sinh^2\delta_3(\sinh^2\delta_1+\sinh^2\delta_2)\bigg\}\bigg]d\psi^2,
\end{aligned}
\tag{A1}
$$

where

$$
\bar{\Delta}=\prod_i\,(r^2+2m\sinh^2\delta_i+l_1^2\cos^2\theta+l_2^2\sin^2\theta). \tag{A2}
$$

The notation follows [15], except that the indices on the boosts δ have been redefined $(e1,e2,e)\to(1,2,3)$. The μ of the main text is related to m through $\mu=2m$. Note that the complete solution also includes gauge fields and other matter fields (of considerable complexity). They are given in [15].

It is possible that the metric can be written in a more compact and symmetrical form, but we are not aware of any substantial simplifications. One helpful identity (that is non-trivial to verify) is

$$
\sqrt{-g}=r\bar{\Delta}^{1/3}\sin\theta\cos\theta. \tag{A3}
$$

We inverted the metric using this relation repeatedly and, after lengthy manipulation of the resulting formulas, found certain complete squares in the resulting wave equation. These are the terms that are recognized as the horizon terms in the general equation [Eq. (36)], after the linear change of radial variable [Eq. (35)].

[1] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[2] J. Maldacena, Ph.D. thesis, Princeton University (unpublished).
[3] G. Horowitz, ''Quantum states of black holes,'' gr-qc/9704072.
[4] G. T. Horowitz, J. M. Maldacena, and A. Strominger, Phys. Lett. B **383**, 151 (1996).
[5] E. Halyo, B. Kol, A. Rajaraman, and L. Susskind, Phys. Lett. B **401**, 15 (1997).
[6] G. T. Horowitz and J. Polchinski, Phys. Rev. D **55**, 6189 (1997).
[7] M. Cvetič and D. Youm, Phys. Rev. D **54**, 2612 (1996).
[8] M. Cvetič, Nucl. Phys. B (Proc. Suppl.) **56**, 1 (1997).
[9] F. Larsen, Phys. Rev. D **56**, 1005 (1997).
[10] C. Callan and J. Maldacena, Nucl. Phys. **B472**, 591 (1996).
[11] J. Maldacena and A. Strominger, Phys. Rev. D **55**, 861 (1996).
[12] S. Das and S. Mathur, Nucl. Phys. **B478**, 561 (1996).
[13] I. R. Klebanov and S. Mathur, ''Black hole grey body factors and absorption of scalars by effective strings,'' hep-th/9701187.
[14] J. Maldacena and A. Strominger, this issue, Phys. Rev. D **56**, 4975 (1997).
[15] M. Cvetič and D. Youm, Nucl. Phys. **B476**, 118 (1996).
[16] M. Cvetič and I. Gaida, ''Duality invariant nonextreme black holes in toroidally compactified string theory,'' hep-th/9703134.
[17] M. Cvetič and C. Hull, Nucl. Phys. **B480**, 296 (1996).
[18] A. Strominger and C. Vafa, Phys. Lett. B **379**, 99 (1996).
[19] R. Dijkgraaf, E. Verlinde, and H. Verlinde, Nucl. Phys. **B486**, 77 (1996).
[20] J. C. Breckenridge, R. C. Myers, A. W. Peet, and C. Vafa, Phys. Lett. B **391**, 93 (1997).
[21] M. Cvetič and F. Larsen, Nucl. Phys. B (to be published).
[22] S. S.Gubser and I. R. Klebanov, Nucl. Phys. **B482**, 173 (1996).
[23] S. W. Hawking and M. M. Taylor-Robinson, Phys. Rev. D **55**, 7680 (1997).
[24] F. Dowker, D. Kastor, and J. Traschen, ''U-duality, D-branes and black hole emission rates: agreements and disagreements,'' hep-th/9702109.
[25] E. L. Ince, *Ordinary Differential Equations* (Dover, New York, 1956).
[26] D. Page, Phys. Rev. D **13**, 3260 (1976).
[27] W. G. Unruh, Phys. Rev. D **14**, 3251 (1976).
[28] R. Emparan, Phys. Rev. D **56**, 3591 (1997).
[29] S. Mathur, ''Absorption of angular momentum by black holes and D-branes,'' hep-th/9704156.
[30] S. Das, G. Gibbons, and S. Mathur, Phys. Rev. Lett. **78**, 417 (1997).
[31] J. Breckenridge, D. Lowe, R. Myers, A. Peet, A. Strominger, and C. Vafa, Phys. Lett. B **381**, 423 (1996).
[32] S. Gubser, Phys. Rev. D **56**, 4984 (1997).
[33] L. Susskind, ''Some speculations about black hole entropy in string theory,'' hep-th/9309145.
[34] F. Larsen and F. Wilczek, Phys. Lett. B **375**, 37 (1996).
[35] M. Cvetič and A. Tseytlin, Phys. Rev. D **53**, 5619 (1996).
[36] J. Maldacena and L. Susskind, Nucl. Phys. **B475**, 679 (1996).
[37] R. Dijkgraaf, G. Moore, E. Verlinde, and H. Verlinde, Commun. Math. Phys. **185**, 197 (1997).
[38] G. T. Horowitz and A. Strominger, Phys. Rev. Lett. **77**, 2369 (1996).
[39] E. Halyo, ''Reissner-Nordström black holes and strings with rescaled tension,'' hep-th/9610068.
[40] C. G. Callan, S. S. Gubser, I. R. Klebanov, and A. A. Tseytlin, Nucl. Phys. **B489**, 65 (1997).