

Universal low-energy dynamics for rotating black holes

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Fundamental string theory has been used to show that low-energy excitations of certain black holes are described by a two-dimensional conformal field theory. This picture has been found to be extremely robust. In this paper it is argued that many essential features of the low-energy effective theory can be inferred directly from a semiclassical analysis of the general Kerr-Newman solution of supersymmetric four-dimensional Einstein-Maxwell gravity, without using string theory. We consider the absorption and emission of scalars with orbital angular momentum, which provide a sensitive probe of the black hole. We find that the semiclassical emission rates—including super-radiant emission and greybody factors—for such scalars agree in striking detail with those computed in the effective conformal field theory, in both four and five dimensions. Also the value of the quantum mass gap to the lowest-lying excitation of a charge- Q black hole, $E_{\text{gap}} = 1/8Q^3$ in Planck units, can be derived without knowledge of fundamental string theory. [S0556-2821(97)05320-4]

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I. INTRODUCTION

Recent statistical derivations of the Bekenstein-Hawking entropy have used weakly coupled fundamental string theory as a starting point [1–4]. In full detail, the derivation is not simple and requires a precise understanding of string theory and D -branes. The final answer is, however, much simpler than the derivation: The quantum states of a near-Bogomol'nyi-Prasad-Sommerfield (BPS) black hole are described by a low-energy supersymmetric conformal field theory or effective string whose parameters are functions of the charges. Furthermore, the validity of this effective string picture extends far beyond the domain of validity of its original derivation from fundamental string theory. Indeed, it gives accurate decay rates in the M -theory region where there are no fundamental strings at all.

How did this happen? On general grounds, one expects the near-BPS dynamics of a black hole to be described by some effective field theory, whether or not string theory is weakly coupled. Apparently, we have stumbled upon the effective black hole field theory which is valid, at sufficiently low energies, for all values of the string coupling.

Given this state of affairs, it is natural to ask how much of this effective theory could have been discovered *without* knowledge of fundamental string theory? In this paper we address this question in the simple context of four-dimensional (4D), Einstein-Maxwell gravity. We begin by assuming that, on scales large compared to the Schwarzschild radius, there is some kind of weakly coupled, unitary effective field theory. We then demand consistency of this effective theory with semiclassical, black hole thermodynamics and decay rates. This is a highly overconstrained problem, especially when decay rates into channels with

nonzero angular momentum are considered. The effective superstring theory provides a solution, possibly the only one. Hence we conclude that—had history been a little different—much of the effective string picture of black hole dynamics might have been derived without knowledge of fundamental string theory. Of course, for a complete and systematic picture, string theory remains essential.

The four-dimensional case is considered in Sec. II. In Secs. II A–II J we semiclassically compute the absorption cross section and decay rates for a massless scalar with angular momentum and a near-BPS Kerr-Newman black hole. This depends on five parameters: the mass, charge, and angular momentum of the black hole, as well as the frequency and angular momentum of the scalar. The total emission includes super-radiant emission, which occurs for a rotating black hole even at extremality when the Hawking temperature vanishes. In Sec. II K we argue that the Kerr-Newman entropy formula—including rotation—implies that the black hole degrees of freedom relevant for near-BPS excitations can be described by a (0,4) chiral superconformal field theory with an $SU(2)$ current algebra associated with rotations. We determine the level of the current algebra by equating the bound on L_0 in terms of the $SU(2)$ charge with the bound implied by the absence of a naked singularity. The mass gap is then computed as the energy of the first excited state of this theory. In Sec. II L the proposed conformal field theory is used to compute the decay rates. It turns out that the decay rates are almost completely determined by general properties of the two-dimensional field theory correlators. Comparison with the semiclassical results of Secs. II A–II I reveals detailed agreement.

The five-dimensional case is considered in Sec. III. An important new feature here is that angular momentum can be carried by both left and right movers on the effective string. In particular, an $l=1$ boson can be emitted by the collision of left- and right-moving $l=1/2$ fermions. The rate for this in the effective string picture involves a right and a left *fermi-*

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onic thermal occupation factor. In the semiclassical picture such factors could come only from greybody effects. We will see that such factors indeed arise with exactly the right form. Hence one can directly “see” the fermionic constituents of a black hole in the $l=1$ scalar emission spectrum.

II. FOUR-DIMENSIONAL KERR-NEWMAN BLACK HOLE

A. Classical geometry

The metric for a black hole of charge Q , mass M , and angular momentum $J=Ma$ is

$$ds^2 = - \left(\frac{\Delta - a^2 \sin^2 \theta}{\Sigma} \right) dt^2 - \left(\frac{2a \sin^2 \theta (r^2 + a^2 - \Delta)}{\Sigma} \right) dt d\phi + \left(\frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \right) \sin^2 \theta d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2, \quad (2.1)$$

where

$$\begin{aligned} \Sigma &\equiv r^2 + a^2 \cos^2 \theta, \\ \Delta &\equiv r^2 + a^2 + Q^2 - 2Mr \end{aligned} \quad (2.2)$$

and we are setting the Planck length to 1 and $G_N=1$. The inner and outer horizons are located at the zeros of Δ :

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2 - a^2}. \quad (2.3)$$

The area \mathcal{A} , Hawking temperature T_H , angular velocity Ω , and electric potential Φ at the horizon are

$$\begin{aligned} \mathcal{A} &= 4\pi(2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2 - a^2}) = 4\pi r_+^2, \\ T_H &= \frac{(r_+ - r_-)}{\mathcal{A}}, \\ \Omega &= \frac{4\pi a}{\mathcal{A}}, \\ \Phi &= \frac{4\pi Q r_+}{\mathcal{A}}. \end{aligned} \quad (2.4)$$

These quantities are related by the first law

$$dM = T_H dS + \Omega dJ + \Phi dQ, \quad (2.5)$$

where the entropy is $S = \mathcal{A}/4$.

B. Scalar wave equation

In this section we give the separated form of the wave equation $\square\Phi=0$ for a massless scalar. As for the well-studied case of Kerr [5], the solution separates as

$$\Phi = e^{im\phi - i\omega t} S_A^m(\theta; a\omega) R(r). \quad (2.6)$$

S obeys

$$\left(\frac{1}{\sin\theta} \partial_\theta \sin\theta \partial_\theta - \frac{m^2}{\sin^2\theta} + a^2 \omega^2 \cos^2\theta \right) S_A^m(\theta; a\omega)$$

$$= -A S_A^m(\theta; a\omega). \quad (2.7)$$

For small $a\omega$ (the case of eventual interest to us) the eigenvalues are

$$A = l(l+1) + O(a^2\omega^2). \quad (2.8)$$

R then obeys

$$\Delta \partial_r \Delta \partial_r R + K^2 R - \lambda \Delta R = 0, \quad (2.9)$$

where

$$\begin{aligned} K &\equiv \omega(r^2 + a^2) = ma, \\ \lambda &\equiv A + a^2\omega^2 - 2m\omega a. \end{aligned} \quad (2.10)$$

C. Low-frequency scalar absorption

In this section will calculate the low-energy absorption cross section for the black holes described in Sec. II A. The low-energy condition is $\omega \ll 1/M$, which means that the Compton wavelength of the particle is much larger than the gravitational size of the black hole, defined as the place where the redshift between a static observer and an asymptotic observer becomes of order 1. We also assume that $\Omega \ll 1/M$ for simplicity.

We use a matching procedure, dividing the spacetime outside the horizon, $r_+ \leq r$ into two overlapping regions defined by the near region $r - r_+ \ll 1/\omega$ and the far region $M \ll r - r_+$.

In each region the wave equation can be approximated using the inequalities and then exactly solved. A complete solution can then be obtained by matching. We now discuss each region in turn.

D. Near-region wave equation

In the near region, the coordinate distance $r - r_+$ is small compared with the inverse frequency $1/\omega$. This implies that we can replace the functions $K^2 - \lambda\Delta$ in Eq. (2.9) by

$$K^2 - \lambda\Delta \approx r_+^4 (\omega - m\Omega)^2 - l(l+1)\Delta, \quad (2.11)$$

where the angular velocity Ω of the black hole is given in Eqs. (2.4). We have approximated K by its constant value at small $r \sim r_+$ since the r dependence of the potential is dominated by the term proportional to Δ , and we have also neglected the term involving ωa^2 in Eqs. (2.10). We can approximate the eigenvalues of the angular Laplacian in Eq. (2.7) and λ in Eqs. (2.10) by $l(l+1)$. Equation (2.9) is then, approximately,

$$\Delta \partial_r \Delta \partial_r R + r_+^4 (\omega - m\Omega)^2 R - l(l+1)\Delta R = 0. \quad (2.12)$$

E. Far-region wave equation

In this region we are far from the black hole and its effects disappear. One has simply

$$\frac{1}{r^2} \partial_r r^2 \partial_r R + \omega^2 R - \frac{l(l+1)}{r^2} R = 0, \quad (2.13)$$

the equation for a massless scalar field of frequency ω and angular momentum l in flat spacetime.

F. Near-region solution

In order to solve the near-region equation, we define a new variable

$$z = \frac{r - r_+}{r - r_-}, \quad 0 \leq z \leq 1. \quad (2.14)$$

The horizon is at $z=0$. One finds

$$\Delta \partial_r = (r_+ - r_-) z \partial_z. \quad (2.15)$$

The near-region wave equation (2.12) is then

$$z(1-z)\partial_z^2 R + (1-z)\partial_z R + \left(\frac{\omega - m\Omega}{4\pi T_H}\right)^2 \left(1 + \frac{1}{z}\right) R - \frac{l(l+1)}{1-z} R = 0. \quad (2.16)$$

This can be transformed into the standard hypergeometric form by defining

$$R = A z^{i(\omega - m\Omega)/4\pi T_H} (1-z)^{l+1} F, \quad (2.17)$$

where A is a to-be-determined normalization constant. F then obeys

$$z(1-z)\partial_z^2 F + \left[1 + i \frac{\omega - m\Omega}{2\pi T_H} - \left(1 + 2(l+1) + i \frac{\omega - m\Omega}{2\pi T_H}\right) z\right] \partial_z F - \left((l+1)^2 + i \frac{\omega - m\Omega}{2\pi T_H} (l+1)\right) F = 0. \quad (2.18)$$

Since we are interested in calculating the absorption cross section, we impose the condition that there be only ingoing flux at the horizon $z=0$. This implies that F in Eq. (2.17) is the standard hypergeometric function $F(\alpha, \beta, \gamma; z)$ with

$$\begin{aligned} \alpha &= l+1 + i \frac{\omega - m\Omega}{2\pi T_H}, \\ \beta &= l+1, \\ \gamma &= 1 + i \frac{\omega - m\Omega}{2\pi T_H}. \end{aligned} \quad (2.19)$$

G. Far-region solution

The far-region solution is a linear combination of Bessel functions

$$R = \frac{1}{\sqrt{r}} [\alpha J_{l+1/2}(\omega r) + \beta J_{-l-1/2}(\omega r)]. \quad (2.20)$$

For large R this behaves as

$$R \xrightarrow{r \rightarrow \infty} \frac{1}{r} \sqrt{\frac{2}{\pi \omega}} \left[-\alpha \sin\left(\omega r - \frac{l\pi}{2}\right) + \beta \cos\left(\omega r + \frac{l\pi}{2}\right) \right]. \quad (2.21)$$

H. Matching the far and near solutions

Next we need to match the small- r far-region Bessel functions (2.20) to the large- r ($z \rightarrow 1$) near-region hypergeometric function. At small r Eq. (2.20) behaves as

$$R \simeq \frac{1}{\sqrt{r}} \left[\alpha \left(\frac{\omega r}{2}\right)^{l+1/2} \frac{1}{\Gamma(l+\frac{3}{2})} + \beta \left(\frac{\omega r}{2}\right)^{-l-1/2} \frac{1}{\Gamma(-l+\frac{1}{2})} \right] \quad (2.22)$$

with corrections to both terms suppressed by r^2 . The large- r , $z \rightarrow 1$ behavior of the near-region solution (2.17) follows from the $z \rightarrow 1-z$ transformation law for hypergeometric functions:

$$\begin{aligned} F(\alpha, \beta; \gamma; z) &= \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} \\ &\quad \times F(\alpha, \beta; \alpha + \beta - \gamma + 1; 1-z) \\ &\quad + (1-z)^{\gamma - \alpha - \beta} \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} \\ &\quad \times F(\gamma - \alpha, \gamma - \beta; \gamma - \alpha - \beta + 1; 1-z). \end{aligned} \quad (2.23)$$

Using $1-z \rightarrow (r_+ - r_-)/r$, one finds that for large r Eq. (2.17) is given by

$$\begin{aligned} R &= A \left(\frac{r}{r_+ - r_-}\right)^{-l-1} \Gamma\left(1 + i \frac{\omega - m\Omega}{2\pi T_H}\right) \\ &\quad \times \left(\frac{\Gamma(-2l-1)}{\Gamma(-l)\Gamma(-l + i(\omega - m\Omega)/2\pi T_H)}\right) \\ &\quad + \left(\frac{r}{r_+ - r_-}\right)^{2l+1} \frac{\Gamma(2l+1)}{\Gamma(l+1)\Gamma(l+1 + i(\omega - m\Omega)/2\pi T_H)}, \end{aligned} \quad (2.24)$$

with corrections to both terms suppressed by $1/r^2$. Matching Eq. (2.22) at small r to Eq. (2.24) at large r , one finds $\beta \ll \alpha$ and

$$A = \frac{(r_+ - r_-)^l \omega^{l+1/2} \Gamma(l+1) \Gamma(l+1 + i(\omega - m\Omega)/2\pi T_H)}{2^{l+1/2} \Gamma(l+\frac{3}{2}) \Gamma(2l+1) \Gamma(1 + i(\omega - m\Omega)/2\pi T_H)} \alpha \equiv N\alpha. \quad (2.25)$$

I. Absorption

The conserved flux associated to the radial wave equation (2.9) is

$$f = \frac{2\pi}{i} (R^* \Delta \partial_r R - R \Delta \partial_r R^*). \quad (2.26)$$

Using Eq. (2.21), the incoming flux from infinity is approximately

$$f_{\text{in}} = 2|\alpha|^2. \quad (2.27)$$

The flux across the horizon follows from Eq. (2.17):

$$f_{\text{abs}} = \frac{(\omega - m\Omega)}{T_H} (r_+ - r_-) |N|^2 |\alpha|^2. \quad (2.28)$$

The absorption cross section of the radial problem is

$$\sigma^{l,m} = \frac{f_{\text{abs}}}{f_{\text{in}}} = \frac{(\omega - m\Omega) \mathcal{A}}{2} |N|^2, \quad (2.29)$$

where we used $(r_+ - r_-) = AT_H$. In order to convert the partial wave cross sections to the usual plane wave cross sections, we have to multiply Eq. (2.29) by π/ω^2 . For $l=0$ we find

$$\sigma_{\text{abs}}^0 = \mathcal{A}. \quad (2.30)$$

In fact, for all black holes the low-energy absorption cross section is proportional to the area of the horizon [6]. For $l=1$ we find

$$\sigma_{\text{abs}}^{1,m} = \frac{\omega \mathcal{A}^3}{36} \left(T_H^2 + \frac{(\omega - m\Omega)^2}{4\pi^2} \right) (\omega - m\Omega). \quad (2.31)$$

Note that the absorption cross section is negative for $\omega < m\Omega$, corresponding to super-radiance. This means that the amplitude of the reflected wave is greater than that of the incident wave, and the scattered wave mines energy from the black hole.

The decay rates are related to the scattering amplitudes by

$$\Gamma^l = \frac{\sigma_{\text{abs}}^{l,m}}{e^{(\omega - m\Omega)/T_H} - 1} = \frac{\pi \Gamma(l+1)^2 \omega^{2l-1} T_H^{2l+1} \mathcal{A}^{2l+1}}{2^{2l+2} \Gamma(l+3/2)^2 \Gamma(2l+1)^2} e^{-(\omega - m\Omega)/2T_H} \left| \Gamma \left(l+1 + i \frac{\omega - m\Omega}{2\pi T_H} \right) \right|^2. \quad (2.32)$$

Note that the decay rate, unlike the cross section, is positive for all values of ω . Note also that for a very nonextremal black hole the exponential factors simplify in the low-energy region since $\omega \ll T_H \sim 1/M$. Near extremality, $T_H \rightarrow 0$ and Γ^l reduces to

$$\begin{aligned} \Gamma^l &\rightarrow 0 \quad \text{for } \omega > m\Omega, \\ \Gamma^l &\rightarrow |\sigma_{\text{abs}}^l| \quad \text{for } \omega < m\Omega. \end{aligned} \quad (2.33)$$

In particular, the $l > 0$ decay rate does not vanish for $T_H = 0$ and is dominated by $\omega \leq m\Omega$ emissions.

J. Near-BPS and extremal limits in Einstein-Maxwell gravity

A black hole with generic values of a , M , and Q is quantum mechanically unstable due to both Hawking and super-radiant emissions. The exceptional, stable case is achieved in the limit of BPS saturation:

$$\begin{aligned} M &= Q, \\ a &= 0. \end{aligned} \quad (2.34)$$

In the $N=2$ supersymmetric extension of Einstein-Maxwell gravity, Eqs. (2.34) represent a supersymmetric BPS state of the black hole [7].¹

¹Unlike the $d=5$ case [8], the addition of angular momentum in $d=4$ always breaks supersymmetry [4].

When the angular momentum is included, a black hole can be extremal, but not BPS saturated. Extremality occurs when the horizon is at double zero of Δ ,

$$r_+ = r_-. \quad (2.35)$$

This implies

$$\begin{aligned} M^2 &= Q^2 + a^2, \\ T_H &= 0. \end{aligned} \quad (2.36)$$

In the extremal limit, there is no Hawking radiation. However, if $a > 0$, the black hole still decays through super-radiant emission. The black hole loses its angular momentum much as a $Q=M$ black hole loses its charge in a theory with (mass/charge) < 1 particles: Real pair creation of $l \neq 0$ particles occurs in the Kerr-Newman ergosphere. Actually, in a theory containing particles with (mass/charge) < 1 , like the real world, both extremal limits look very similar.

We wish to analyze low-lying excitations of the black hole about the BPS limit. To do so we expand in the excitation energy

$$E \equiv M - Q, \quad (2.37)$$

taking $E/Q \ll 1$ and $Q \gg 1$ in Planck units. Since a^2 is bounded from above by $M^2 - Q^2$ (greater values give a naked singularity), this implies that a^2 is of order $a^2 \sim EQ$. To leading order in E , we then find

$$r_{\pm} \simeq Q \pm \sqrt{2EQ - a^2},$$

$$\begin{aligned} \mathcal{A} &\simeq 4\pi(Q^2 + 2Q\sqrt{2EQ - a^2}), \\ T_H &\simeq \frac{\sqrt{2EQ - a^2}}{2\pi Q^2}, \\ \Omega &\simeq \frac{a}{Q^2}. \end{aligned} \quad (2.38)$$

The black hole will emit Hawking quanta with frequencies of order T_H and super-radiant quanta with frequencies of order Ω . In both cases this gives

$$\omega \sim \sqrt{\frac{E}{Q^3}}. \quad (2.39)$$

This implies that the greybody factors in Hawking emission will be correctly given by formula (2.32) since the only assumptions that went into that calculation were that $\omega \ll 1/M$ and $\Omega \ll 1/M$ which are certainly obeyed by Eqs. (2.39) and (2.38).

K. Semiclassical derivation of the effective low-energy theory

To begin with, the excess energy $E \equiv M - Q$ of a near-BPS, nonrotating black hole is related to the Hawking temperature by

$$E = 2\pi^2 Q^3 T_H^2. \quad (2.40)$$

In D spacetime dimensions the energy of a weakly interacting field theory scales like T^D . Hence Eq. (2.40) indicates that the quantum states of the black hole are described by a $D=2$ field theory. Since we are taking E to be much less than any other mass scale in the problem² (in particular $1/Q$), this should be a $D=2$ conformal field theory, with associated central charge c . The exact energy-temperature relation for a $D=2$ conformal field theory is

$$E = \frac{\pi}{12} LcT_H^2, \quad (2.41)$$

where L is the volume of the one-dimensional space. This relation is valid if L is large compared to the typical wavelengths of the particles,³

$$L \gg \frac{1}{T_H}. \quad (2.42)$$

The energy E is related to L_0 by

$$E = \frac{2\pi}{L} L_0. \quad (2.43)$$

Comparing Eqs. (2.41) and (2.40), we learn that

$$cL = 24\pi Q^3. \quad (2.44)$$

The group $SU(2)$ of global space rotations is a symmetry of the Hilbert space of black hole states. The Noether currents of this symmetry generate an $SU(2)$ Kac-Moody algebra within the conformal field theory. Let j denote the generator of the $U(1)$ subgroup of rotations about the z axis. Then j obeys

$$[j_m, j_n] = \frac{km}{2} \delta_{m+n,0}, \quad (2.45)$$

where the integer k is the Kac-Moody level and we have chosen the normalization so that j_0 has half integral eigenvalues. The eigenvalues of j_0 are the z components J_z of angular momentum. Standard arguments using the Sugawara construction of the stress tensor imply that L_0 is bounded according to

$$L_0 \geq \frac{J_z^2}{k}. \quad (2.46)$$

This bound is saturated by the extremal states

$$e^{iJ_z \phi} |0\rangle, \quad (2.47)$$

where the $U(1)$ boson $\phi \sim \phi + 2\pi$ is the angle about the z axis defined by $j = (k/2)\partial\phi$. A similar bound can be derived in a completely different manner from the condition $r_+ \geq r_-$; i.e., the absence of a naked singularity in the spacetime solutions. Using Eqs. (2.38), this implies

$$2EQ \geq a^2, \quad (2.48)$$

which, together with Eq. (2.44), yields

$$L_0 \geq \frac{6J_z^2}{c}. \quad (2.49)$$

Consistency of the two dimensional effective field theory with the spacetime analysis requires that Eqs. (2.46) and (2.49) be the same bound. This yields

$$c = 6k. \quad (2.50)$$

This is exactly the relation between k and c encountered in a chiral $(0,4)$ superalgebra. This result is especially striking in that so far we have been discussing the purely bosonic Einstein-Maxwell theory and have not assumed or used supersymmetry in any way.

In a supersymmetric theory the black hole ground state is a soliton preserving four supersymmetries.⁴ We are considering here a chiral theory with only right movers, and so we must have a $(0,4)$ supersymmetry algebra. This algebra contains an $SU(2)_R$ symmetry, which in fact is the same as the $SU(2)$ symmetry of spatial rotations considered above. This implies that $c = 6k$. The black hole mass gap is then the

²Except the black hole mass gap; see below.

³Of course, there are other situations, such as conformal field theories with a small mass gap due to twisted sectors, in which L is not large, but Eq. (2.40) remains valid.

⁴As is the case for extreme Reissner-Nordström solutions of $N = 2$, $N = 4$, and $N = 8$ supergravity.

energy of the lowest-lying excitation. This is the extremal state (2.47) with $J_z = \frac{1}{2}$. Using Eqs. (2.46), (2.50), and (2.44), it has energy

$$E_{\text{gap}} = \frac{1}{8Q^3}. \tag{2.51}$$

The existence and value of the black hole mass gap was first derived using fundamental string theory in [9]. Here we see it can be derived in a purely semiclassical analysis, without any reference to fundamental string theory.

L. Microscopic decay rates

We have seen that the effective string picture correctly reproduces the thermodynamic behavior of a near-BPS Kerr-Newman black hole. In Sec. II I we semiclassically computed the rate at which such a macroscopic black hole emits scalars as a function of emitted energy and angular momentum. In the effective string picture, such decays arise from a coupling of the spacetime scalar ϕ to an operator \mathcal{O} in the conformal field theory. These are of the general form

$$S_{\text{int}} \sim \int dt d\sigma \partial^n \phi(0,t) \mathcal{O}(\sigma+t). \tag{2.52}$$

The spatial argument of ϕ is 0 because we have taken the black hole to be at $x=0$. Here σ is the spatial coordinate in the conformal field theory, and \mathcal{O} depends only on $\sigma+t$ because it is chiral theory.

We know of no principle, without recourse to string theory, which allows us to determine the numerical coefficients in front of the couplings in Eq. (2.52). However, we will be able to determine the energy and angular momentum dependence of the decay rates in great detail. A striking agreement between the macroscopic and microscopic decay rates will be found.

The amplitude \mathcal{M} for the emission of a particle with energy ω has an internal contribution

$$\mathcal{M} \sim \int d\sigma^+ \langle f | \mathcal{O}(\sigma^+) | i \rangle e^{-i\omega\sigma^+}, \tag{2.53}$$

where $\sigma^+ \equiv \sigma+t$. Squaring and summing over final states gives

$$\begin{aligned} \sum_f |\mathcal{M}|^2 &\sim \int d\sigma^+ d\sigma^{+'} \langle i | \mathcal{O}^\dagger(\sigma^+) \mathcal{O}(\sigma^{+'}) | i \rangle \\ &\times e^{-i\omega(\sigma^+ - \sigma^{+'})}, \end{aligned} \tag{2.54}$$

for the rate, where we have used the fact that the sum over final states produces an identity matrix. To compare with the macroscopic decay rate, we should thermally average, at temperature T_H with an angular potential Ω , over the initial state. This potential implies that a state with U(1) charge m is weighted by $\exp[-(\omega - m\Omega)/T_H]$. The rate is then proportional to the thermal correlation function

$$\int d\sigma^+ \langle \mathcal{O}^\dagger(0) \mathcal{O}(\sigma^+) \rangle_{T_H} e^{-i(\omega - m\Omega)\sigma^+}. \tag{2.55}$$

m arises here as the U(1) charge of \mathcal{O} . The shift in ω implements the effects of the angular potential.

This correlation function is completely determined by the conformal weight of the operator \mathcal{O} . If the operator \mathcal{O} has conformal weight Δ , then the correlation function becomes

$$\langle \mathcal{O}^\dagger(0) \mathcal{O}(\sigma^+) \rangle_{T_H} \sim \left[\frac{\pi T_H}{\sinh(\pi T_H \sigma^+)} \right]^{2\Delta}. \tag{2.56}$$

Note that according to Eq. (2.42) we can ignore the periodicity along the spatial direction and take it to be infinite when we calculate Eq. (2.56). The periodicity along Euclidean time translates into

$$\sigma^+ \sim \sigma^+ + i2/T_H. \tag{2.57}$$

Hence we must evaluate the integral

$$\int d\sigma^+ e^{-i(\omega - m\Omega)\sigma^+} \left[\frac{T_H}{\sinh(\pi T_H \sigma^+)} \right]^{2\Delta}. \tag{2.58}$$

This Fourier transform should be performed with an $i\epsilon$ prescription for dealing with the pole at $x=0$. The two different prescriptions correspond to absorption or emission. The one corresponding to emission gives

$$\begin{aligned} &\int d\sigma^+ e^{-i(\omega - m\Omega)(\sigma^+ - i\epsilon)} \left[\frac{T_H}{\sinh(\pi T_H \sigma^+)} \right]^{2\Delta} \\ &\sim (T_H)^{2\Delta-1} e^{-(\omega - m\Omega)/2T_H} \left| \Gamma \left(\Delta + i \frac{(\omega - m\Omega)}{2\pi T_H} \right) \right|^2. \end{aligned} \tag{2.59}$$

The next problem is to determine Δ . The coupling (2.52) and the allowed operators \mathcal{O} are restricted by the symmetries of the theory. The simplest way of satisfying this restrictions is when the integral of \mathcal{O} is invariant under supersymmetry transformations of the conformal field theory and is single valued on the circle (i.e., it is not a twist field).⁵ Under these conditions there is a bound relating the conformal weight of the operator and the U(1) charge:

$$\Delta \geq l + 1. \tag{2.60}$$

Operators which saturate the bound are of the form

$$\mathcal{O} = \{G_{1/2}, \mathcal{C}\}, \tag{2.61}$$

with \mathcal{C} a chiral primary and G a supercharge. The leading contribution in the low-energy expansion comes from the lowest value of Δ , and so we conclude

$$\Delta = l + 1. \tag{2.62}$$

An additional contribution to the energy dependence arises from the external spacetime part of the interaction in Eq. (2.52). For a mode of angular momentum l , the first $l - 1$ spatial derivatives of ϕ vanish at the origin. Hence n

⁵Though this gives the desired answer, the justification of this second assumption is unclear since the states (2.47) themselves are not single valued.

must be at least l in Eq. (2.52). The leading contribution is for $n=l$. Since the field ϕ is massless, derivatives give powers of ω . In other words, we need at least l powers of the spatial momentum to match the SO(3) transformation properties of \mathcal{O} . Hence there is an additional power of ω^{2l} in the rate. Multiplying by a factor of $1/\omega$ for the normalization of the outgoing state and restoring powers of Q with dimensional analysis, the microscopic rate is

$$\Gamma^l \sim \omega^{2l-1} Q^{4l+2} (T_H)^{2l+1} e^{-(\omega-m\Omega)/2\pi T_H} \times \left| \Gamma \left(l+1+i \frac{(\omega-m\Omega)}{2\pi T_H} \right) \right|^2, \quad (2.63)$$

in agreement with Eq. (2.32), noting that $\mathcal{A} \sim Q^2$.

III. FIVE-DIMENSIONAL BLACK HOLES

In this section we consider the interactions of five-dimensional, nonrotating black holes in string theory with a massless scalar, extending the results of [10–13] to include orbital angular momentum for the scalar field. Qualitatively, new features arise in five dimensions because the spatial rotation group is $SU(2) \times SU(2)$, and one $SU(2)$ is carried by right movers, while the other is carried by left movers [8] on the effective string. A single left-moving and a single right-moving fermion can collide and create an outgoing boson. The rate for this process will involve a left and a right fermionic thermal factor. We will find that these factors arise in the greybody calculation. It is fascinating that one can “see” that the black hole is in part made of fermions in such a purely bosonic calculation.

A. Semiclassical scattering

We consider the scattering of scalars from a five-dimensional black hole in the dilute gas approximation considered in [11]. We follow the notation of [11], where further details of the geometry may also be found. The wave equation is

$$\frac{g}{r^3} \frac{d}{dr} r^3 g \frac{d\phi}{dr} + \frac{g}{r^2} \nabla_\theta^2 \phi + \omega^2 f \phi = 0, \quad (3.1)$$

where

$$f = \left(1 + \frac{r_1^2}{r^2} \right) \left(1 + \frac{r_5^2}{r^2} \right) \left(1 + \frac{r_n^2}{r^2} \right), \quad g = 1 - \frac{r_0^2}{r^2}. \quad (3.2)$$

∇_θ^2 is the angular Laplacian which has eigenvalues $l(l+2)$ in five dimensions, and l is an integer which labels the orbital

angular momentum. The rotation group SO(4) can be decomposed as $SO(4) \sim SU(2)_L \times SU(2)_R$. The representations appearing here corresponds to $(l/2, l/2)$ under the two $SU(2)$'s. The degeneracy of this representation is $(l+1)^2$, corresponding to the different allowed J_z values of each $SU(2)$.

We consider low energies satisfying $\omega \ll 1/r_1, 1/r_5$, and we divide space into a far region $r \gg r_1, r_5$ and a near region $r \ll 1/\omega$. We will match the solutions in the overlapping region. In the far region we write $\phi = (1/r)\psi$ and the equation for ψ becomes, with $\rho = \omega r$,

$$\frac{d^2 \psi}{d\rho^2} + \frac{1}{\rho} \frac{d\psi}{d\rho} + \left(1 - \frac{(l+1)^2}{\rho^2} \right) \psi = 0. \quad (3.3)$$

The solutions are the Bessel functions

$$\phi = \frac{1}{\rho} [\alpha J_{l+1}(\rho) + \beta J_{-l-1}(\rho)]. \quad (3.4)$$

From the large- ρ behavior, the incoming flux is found to be

$$f_{\text{in}} = \text{Im}(\phi^* r^3 \partial_r \phi) = \frac{1}{\pi \omega^2} |\alpha e^{i(l+1)\pi/2} + \beta e^{-i(l+1)\pi/2}|^2. \quad (3.5)$$

On the other hand, the small- ρ behavior of the far-region solution is

$$\phi = \frac{1}{\rho} \left[\alpha \left(\frac{\rho}{2} \right)^{l+1} \left(\frac{1}{\Gamma(l+2)} - \mathcal{O}(\rho^2) \right) + \beta \left(\frac{\rho}{2} \right)^{-l-1} \left(\frac{1}{\Gamma(-l)} - \mathcal{O}(\rho^2) \right) \right]. \quad (3.6)$$

Since Eq. (3.6) has a pole for integer l , it is convenient to keep l near an integer value during the calculation and make it integer at the end. Now we turn to the solution in the near region $r \ll 1/\omega$. Defining $v = r_0^2/r^2$, the near-region wave equation is

$$(1-v)^2 \frac{d^2 \phi}{dv^2} - (1-v) \frac{d\phi}{dv} + \left(C + \frac{D}{v} + \frac{E}{v^2} \right) \phi = 0, \quad (3.7)$$

where

$$C = \left(\frac{\omega r_n r_1 r_5}{2r_0^2} \right)^2, \quad D = \frac{\omega^2 r_1^2 r_5^2}{4r_0^2} + \frac{l(l+2)}{4}, \quad E = -\frac{l(l+2)}{4}. \quad (3.8)$$

Defining

$$\phi = v^{-l/2} (1-v)^{-i\omega/4\pi T_H} A F, \quad (3.9)$$

with A a constant, we find that the solution to Eq. (3.7) with only ingoing flux at the horizon is given by Eq. (3.9) with $F = F(\alpha, \beta, \gamma, 1-v)$, a hypergeometric function, with

$$\gamma = 1 + 2q, \quad \alpha = -l/2 + q + i\sqrt{C}, \quad \beta = -l/2 + q - i\sqrt{C}, \quad q = i \frac{\omega}{4\pi T_H}. \quad (3.10)$$

The behavior for small v is

$$\begin{aligned} \phi \sim & Av^{-l/2} \left\{ \frac{\Gamma(1+2q)\Gamma(1+l)}{\Gamma(1+l/2+q-i\sqrt{C})\Gamma(1+l/2+q+i\sqrt{C})} [1+\mathcal{O}(v)] \right. \\ & \left. + v^{1+l} \frac{\Gamma(1+2q)\Gamma(-1-l)}{\Gamma(-l/2+q+i\sqrt{C})\Gamma(-l/2+q-i\sqrt{C})} [1+\mathcal{O}(v)] \right\}. \end{aligned} \tag{3.11}$$

Matching solutions in the overlapping region and anticipating that $\beta \ll \alpha$ in Eq. (3.4), we find

$$\alpha/2 = A(\omega r_0/2)^{-l} \Gamma(1+l)\Gamma(2+l) \left[\frac{\Gamma(1+2q)}{\Gamma(1+l/2+q-i\sqrt{C})\Gamma(1+l/2+q+i\sqrt{C})} \right]. \tag{3.12}$$

The absorbed flux is

$$f_{\text{abs}} = \text{Im}(\phi^* g r^3 \partial_r \phi) = 2r_0^2 \text{Im}(\phi^* g \partial_v \phi) = 4|q|r_0^2|A|^2. \tag{3.13}$$

Hence the absorption cross section for the radial problem is given by the ratio of the two fluxes (3.13) and (3.5). The plane wave cross section is obtained multiplying by $4\pi/\omega^3$, and we obtain

$$\sigma_{\text{abs}}^l = \mathcal{A}_H(r_0\omega)^{2l} \left| \frac{2^l}{\Gamma(l+1)\Gamma(l+2)} \right|^2 \left| \frac{\Gamma((l+2)/2+i\omega/4\pi T_L)\Gamma((l+2)/2+i\omega/4\pi T_R)}{\Gamma(1+i\omega/2\pi T_H)} \right|^2. \tag{3.14}$$

We see the same left-right structure that was found for $l=0$ in [11], but with the difference that the factor of l appears inside the Γ functions. To evaluate this we need the absolute value of $\Gamma(n/2+ib)$. Properties of Γ functions can be used to reduce $\Gamma(n/2+ib)\Gamma(n/2-ib)$ to the form $\Gamma(ib)\Gamma(1-ib)$ or $\Gamma(1/2+ib)\Gamma(1/2-ib)$ (times a polynomial in b). While $\Gamma(ib)\Gamma(1-ib)$ is proportional to $b/\sinh\pi b$, $\Gamma(1/2+ib)\Gamma(1/2-ib)$ is proportional to $1/\cosh\pi b$. This \cosh will eventually translate into a fermionic as opposed to bosonic occupation factor, and this happens for odd l as it should. For example, in the case of $l=1$,

$$\begin{aligned} \sigma_{\text{abs}}^1 &= \frac{\pi^3}{8} (r_1 r_5)^4 \omega [(2\pi T_L)^2 + \omega^2][(2\pi T_R)^2 + \omega^2] \\ &\times \frac{e^{\omega/T_H} - 1}{(e^{\omega/2T_R} + 1)(e^{\omega/2T_L} + 1)}. \end{aligned} \tag{3.15}$$

The corresponding decay rate seems to come from two particles colliding, where both particles are fermions, as one expects from the decomposition of $l=1$ as $(1/2, 1/2)$ under the two $SU(2)$'s. We shall see that this coincides beautifully with the effective string picture. In general, fermions appear for odd l and bosons for even l .

The Hawking rate for emitting particles with angular momentum is obtained by multiplying Eq. (3.14) by the usual Hawking thermal factor, producing

$$\Gamma_H = \frac{2^{4l+4} \pi^{2l+3} (r_1^2 r_5^2 T_L T_R)^{l+1} \omega^{2l-1}}{|\Gamma(l+1)\Gamma(l+2)|^2} e^{-\omega/2T_H}$$

$$\times \left| \Gamma\left(1 + \frac{l}{2} + i \frac{\omega}{4\pi T_L}\right) \Gamma\left(1 + \frac{l}{2} + i \frac{\omega}{4\pi T_R}\right) \right|^2. \tag{3.16}$$

B. Microscopic decay rate

It would be of interest to see how much of the structure of the effective string for the five-dimensional black hole can be deduced, as in the four-dimensional case of the previous sections, without fundamental string theory. However, in this section our main goal is to understand the emission of angular momentum and we shall simply assume the structure implied by string theory. String theory says that the low-energy dynamics is described by a two-dimensional conformal field theory [1] which is a good description of the low-energy dynamics in the black hole region [14]. Some twisted sectors of this superconformal field theory (SCFT) contain excitations which can be viewed as the excitations of a long multiply wound string. Some things are more clear in this picture of a multiply wound string, and some things are more clear thinking about the full conformal field theory; so it is useful to keep both in mind.

In Sec. II L conformal field theoretic arguments were given for determining the microscopic decay rate. The same arguments are applicable here, with a few modifications as follows. Since there are right movers as well as left movers, the rate involves the product of two factors of the form (2.59), one at temperature T_L and conformal weight Δ_L and one at temperature T_R and conformal weight Δ_R . The bounds on Δ give $\Delta_L = \Delta_R = 1 + l/2$. In this section we are considering $\Omega=0$ black holes, and so there is no shift of ω , but it is easy to see how to extend the calculation for general Ω .

It is possible to give the general form of the operator with the correct weights. For a scalar arising from metric deformations of the internal T^4 (an internally polarized graviton), the operator is of the form

$$\epsilon_{IJ}\omega^l \mathcal{O}'_{l/2,L} \mathcal{O}'_{l/2,R}, \quad (3.17)$$

where ϵ_{IJ} indicates the polarization of the graviton in the internal directions and the factors of ω arise as before because the spacetime field must be acted on by l derivatives. The operator $\mathcal{O}_L \mathcal{O}_R$ involves the fields propagating on the effective string and has angular momentum $(l/2, l/2)$ under the $SU(2)_L \times SU(2)_R \sim SO(4)$ symmetry. It must also carry the indices IJ in order to be contracted with the graviton polarization tensor. The I index is carried by the bosonic field ∂X^I living on the brane. The simplest such operators \mathcal{O} are of the form

$$\mathcal{O}'_{l/2} \sim \partial X^I \psi_{i_1} \psi_{i_2} \cdots \psi_{i_l}, \quad (3.18)$$

where ψ_i are different families of fermions propagating on the string. The operator (3.18) has $\Delta = 1 + l/2$, precisely as needed for agreement with the factors in Eq. (3.16).

Putting this all together, our final result for the Hawking emission rate is, up to numerical coefficients,

$$\begin{aligned} \Gamma \sim & (g^2 Q_1 Q_5 T_L T_R)^{l+1} \omega^{2l-1} e^{-\omega/2T_H} \\ & \times \left| \Gamma \left(l/2 + 1 + i \frac{\omega}{4\pi T_L} \right) \right|^2 \left| \Gamma \left(l/2 + 1 + i \frac{\omega}{4\pi T_R} \right) \right|^2. \end{aligned} \quad (3.19)$$

In this expression we have multiplied by a factor of $1/\omega$ from the normalization of the outgoing scalar and a factor of ω^{2l} from squaring the ω factors in the vertex operator (3.17). The factor of g^{2l+2} comes from the fact that in string theory this is a disk amplitude with a closed string plus $2l+2$ open strings ($l+1$ right moving and $l+1$ left moving). Finally, the factor of $(Q_1 Q_5)^{l+1}$ can be understood from the fact that one can create this many different families of open strings.

Once we remember that $g^2 Q_1 Q_5 = r_1^2 r_5^2$, we see that Eq. (3.19) agrees precisely with Eq. (3.16). Once again, we find detailed agreement between the macroscopic and microscopic descriptions of black hole dynamics. It would be interesting to calculate, as was done for the $l=0$ case [10] and the fixed scalar case [13], the precise numerical coefficient in Eq. (3.19) and compare it with Eq. (3.16). As an aside, note that the full energy dependence of the cross sections for the fixed scalars [13] comes from the fact that it couples to an operator with conformal weights $\Delta_L = \Delta_R = 2$.

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