Scattering off the extreme Reissner-Nordström black hole in $N=2$ supergravity

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The scattering amplitudes for the perturbed fields of $N=2$ supergravity about an extreme Reissner-Nordström black hole are examined. Owing to the fact that the extreme hole is a BPS state of the theory and preserves an unbroken global supersymmetry ($N=1$), the scattering amplitudes of the component fields should be related to each other. In this paper, we derive the formula of the transformation of the scattering amplitudes. [S0556-2821(97)05720-2]

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I. INTRODUCTION

Solitons that are nonperturbative configurations play an important role for studying nonperturbative aspects of quantum field theories. A soliton is a classical solution which is a stationary, regular, and classically and quantum-mechanically stable configuration with finite localized energy. Solitons often have some conserved charges. From the stability of the configurations in classical and quantum theory, we may think of a soliton as the lowest energy state whose energy is given by the charges. Further we may expect the inequality between the mass and charges of a soliton. In fact, we have the inequality in supersymmetric theories and call saturated states Bogomol'nyi-Prasad-Sommerfield (BPS) states [1]. Although BPS states are massive and break supersymmetry, they still have some unbroken supersymmetry.

In the Einstein-Maxwell theory, extreme Reissner-Nordström solutions behave as gravitational solitons [2,3]. The Einstein and Einstein-Maxwell systems can be embedded in supergravity theories. In asymptotically flat spacetime, we can obtain global charges that generate rigid supersymmetry [4]. Therefore we can follow the argument in rigid supersymmetry formally. For $N=1$ supergravity positivity of the energy is suggested [4,5]. Subsequently Witten established the positivity for general relativity using the trick of a ''Witten spinor'' [6]. Further, using the ''Witten spinor'' motivated by the transformation law of gravitini in $N=2$ supergravity, Gibbons and Hull [7] established the inequality between the mass and the electromagnetic charges. Further they showed that the saturated configurations are Majumdar-Papapetrou (MP) solutions, which are assemblages of the extreme Reissner-Nordström holes, and that the MP solutions have unbroken supersymmetries. More generalizations of their results are available in Refs. [8–10].

In addition, the nonrenormalization theorem of the on-shell effective action for the MP solutions was established [9,11]. Although supergravity has better ultraviolet behavior than general relativity, it is known that supergravity is non-renormalizable at the perturbative level and is not regarded as the final theory. However, we may expect that the final theory should include supergravity and may think of the non-renormalization for the MP solutions as a guiding principle to the final theory.

Thus, it is very significant to investigate extreme black holes in the classical and semiclassical frameworks through general relativity and supergravity. To better understand extreme holes that are the lowest energy states for given charges, it is important to study the fluctuation (excitation) about them.

Originally, the study of the perturbation about the extreme Reissner-Nordström hole was motivated by the interest in the no hair conjecture in supergravity [12] and by the interpretation problem of the paradoxical thermal properties of extreme-dilaton black holes [13].

Recently in the study of the method of calculating quasi-normal frequencies of the extreme Reissner-Nordstöm hole, Onozawa *et al.* numerically [14] found that the quasinormal frequencies of gravitational waves and electromagnetic waves about it coincide by a suitable shift of the angular momentum indices. Because of the fact that the quasinormal frequency is the resonance pole of scattering waves, we may expect that gravitational and electromagnetic waves have the same reflection and transmition amplitudes. Subsequently, they established the coincidence between S matrices of perturbations of gravitational, electromagnetic, and spin-3/2 fields (gravitini) about the extreme Reissner-Nordström background in the $N=2$ supergravity by finding relation between the Regge-Wheeler potentials of perturbations [15]. From the fact that the extreme Reissner-Nordström hole is a BPS state in extended supergravity, we expect that the coincidence is related to the fact that BPS states preserve unbroken supersymmetries. The purpose of this paper is to derive the relation between scattering matrices of graviton, gravitini, and photon using supersymmetric transformation.

The paper is organized as follows: In Sec. II we briefly review the perturbation equations through the Newman-Penrose formalism and the scattering problem. In Sec. III we give the supersymmetric transformation law between the curvatures of perturbed fields. In Sec. IV we seek the correspondence between the radial parts of the perturbations with the suitable total angular momentum and the relations of the reflection and transmition coefficients for them. Section V is devoted to a summary.

II. PERTURBATION EQUATIONS

By linearizing $N=2$ supergravity [16] about a purely bosonic background, we have the perturbation equations for

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the linearized Einstein-Maxwell system and for the linearized O(2) doublet of spin-3/2 fields, from which they are decoupled. Here we follow Chandrasekhar [17] and Torres del Castillo and Silva-Ortigoza [18] for bosonic perturbations and fermionic ones, respectively.

The line element of the Reissner-Nordström solution is given as

$$ds^2=\frac{\Delta(r)}{r^2}dt^2-\frac{r^2}{\Delta(r)}dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2), \tag{2.1}$$

$$\Delta(r)=r^2-2r+Q^2, \tag{2.2}$$

where we adopt the unit $M=1$, M being the mass of the background black hole, and Q is the electric charge of the black hole.

On the background of the Reissner-Nordström solution, the bosonic perturbations are described by the Regge-Wheeler equation and the fermionic ones by a similar equation through the Newmann-Penrose formalism. Since this background is static and spherically symmetric, the perturbation equations are separable and nontrivial equations are radial part ones.

For the perturbations with helicity $(+1,+\frac{3}{2},+2)$, their equations of radial parts of the perturbations, $Y_{+s}(r)(s=1,\frac{3}{2},2)$, in the phantom gauge [19], are given by

$$\Lambda^2 Y_{+s}+P_s(r)\ \ \Lambda_- Y_{+s}-Q_s(r)Y_{+s}=0, \tag{2.3}$$

$$\Lambda_\pm\equiv\frac{d}{dr_*}\pm i\Omega,\quad \Lambda^2\equiv\Lambda_+\Lambda_-, \tag{2.4}$$

$$\frac{d}{dr_*}=\frac{\Delta}{r^2}\frac{d}{dr}, \tag{2.5}$$

where r_* is the tortoise coordinate and Ω is the frequency of perturbations. We omit the index of distinguishing two gravitini because they follow the same equation as expected from the O(2) symmetry between them.

Each $P_s(r)$ and $Q_s(r)$ is given by, for $s=1,2$,

$$P_s(r)\equiv\frac{d}{dr_*}\ln\left(\frac{r^8}{D_s}\right),\quad D_s\equiv\Delta^2\left(1+\frac{2q_s}{\mu^2 r}\right), \tag{2.6a}$$

$$Q_s(r)\equiv\mu^2\frac{\Delta}{r^4}\left(1+\frac{2q_s}{\mu^2 r}\right)\left(1+\frac{q_{s'}}{\mu^2 r}\right)\quad (s,s'=1,2;\ \ s\neq s'), \tag{2.6b}$$

where $q_{1,2}$ are defined by

$$q_1=3+\sqrt{9+4Q^2\mu^2},\quad q_2=3-\sqrt{9+4Q^2\mu^2}. \tag{2.7}$$

And for $s=3/2$,

$$P_{3/2}(r)\equiv\frac{3}{r^3}(r^2-3r+2Q^2), \tag{2.8a}$$

$$Q_{3/2}(r)\equiv\frac{\Delta}{r^6}(\lambda r^2+2r-2Q^2). \tag{2.8b}$$

The radial parts $Y_{+s}(r)$ of perturbations are constructed in two ways. One is by the perturbed Weyl scalar Ψ_0 and the perturbed spin connection κ as

$$Y_{+s}(r)=\frac{\Delta^2}{r^3}F_{+s}(r),\quad F_{+s}=R_{+2}(r)+\frac{q_s k(r)}{\mu}, \tag{2.9}$$

$$\Psi_0=R_{+2}(r)S_{+2}(\theta)e^{i(\Omega t+m\phi)},$$

$$\kappa=\sqrt{2}r^2k(r)S_{+1}(\theta)e^{i(\Omega t+m\phi)}, \tag{2.10}$$

where the constant μ is an eigenvalue of the spin-weighted spherical harmonics,

$$\mathcal{L}_{-1}^\dagger\mathcal{L}_2 S_{+2}=-\mu^2 S_{+2},\quad \mathcal{L}_2\mathcal{L}_{-1}^\dagger S_{+1}=-\mu^2 S_{+1}, \tag{2.11}$$

$$\mu=\sqrt{(J-1)(J+2)}. \tag{2.12}$$

The operators $\mathcal{L}_n$ and $\mathcal{L}_n^\dagger$ are defined by

$$\mathcal{L}_n\equiv\partial_\theta+\frac{m}{\sin\theta}+n\cot\theta, \tag{2.13a}$$

$$\mathcal{L}_n^\dagger\equiv\partial_\theta-\frac{m}{\sin\theta}+n\cot\theta. \tag{2.13b}$$

Besides, the functions S_{+1} and S_{+2} are related in the manner

$$\mathcal{L}_2 S_{+2}=\mu S_{+1},\quad \mathcal{L}_{-1}^\dagger S_{+1}=-\mu S_{+2}. \tag{2.14}$$

Another is by the Weyl scalar Ψ_1 and the spin connection σ. Using G_{+s}, which are defined as

$$G_{+s}(r)=R_{+1}(r)+\frac{q_s}{\mu}\ s(r), \tag{2.15}$$

$$\Psi_1=\frac{1}{r\sqrt{2}}R_{+1}(r)S_{+1}(\theta)e^{i(\Omega t+m\phi)},$$

$$\sigma=rs(r)S_{+2}(\theta)e^{i(\Omega t+m\phi)}, \tag{2.16}$$

and the relations

$$\Delta\left(D_2^\dagger-\frac{3}{r}\right)F_{+s}=\mu\left(1+\frac{2q_s}{\mu^2 r}\right)G_{+s'}\quad (s,s'=1,2;\ \ s\neq s'), \tag{2.17}$$

$$D_n\equiv\partial_r+\frac{ir^2\Omega}{\Delta}+2n\frac{r-1}{\Delta},\quad D_n^\dagger=(D_n)^*, \tag{2.18}$$

G_{+s} are related to Y_{+s}.

In order to obtain $Y_{+3/2}(r)$ for the helicity-$(+3/2)$ perturbations, we first construct the supersymmetric gauge-invariant quantities H_0^i from the supercovariant curvature $\Psi^{i\,A}{}_{BC}$ of the spin-3/2 fields ψ_μ^i $(i=1,2)$ as

$$H_0^i=\Psi^i_{(ABC)}o^A o^B o^C, \tag{2.19}$$

$$\Psi^{i\,A}{}_{BC}=\tfrac{1}{2}\left[\boldsymbol{D}_{(B|A'|}\psi^{iA}{}_{C)}{}^{A'}+\frac{i}{M_p}\epsilon^{ij}\ \ \boldsymbol{F}^A{}_{(B}\overline{\psi^{jA'}}{}_{|A'|C)}\right], \tag{2.20}$$

where the boldface letters indicate the background quantities and o^A is a principal spinor of $\boldsymbol{F}_{AB}$ that is a two-spinor representation of the self-dual part of electromagnetic field strength $\boldsymbol{F}_{\mu\nu}$. $\boldsymbol{D}_\mu$ is covariant derivative with respect to the spin connection $\boldsymbol{\omega}_{\mu ab}=\boldsymbol{\omega}_{\mu AB}\epsilon_{A'B'}+\overline{\boldsymbol{\omega}}_{\mu A'B'}\epsilon_{AB}$, for example,

$$\boldsymbol{D}_\mu \eta_A=\partial_\mu \eta_A+\boldsymbol{\omega}_{\mu A}{}^B\eta_B. \tag{2.21}$$

From H_0^i, we obtain the radial parts of the helicity-$(+3/2)$ perturbations,

$$Y^j_{+3/2}(r)=\frac{\Delta^{3/2}}{r^2}R^j_{+3/2}, \tag{2.22}$$

$$H_0^j=R^j_{+3/2}(r)S_{+3/2}(\theta)\quad e^{i(\Omega t+m\phi)}, \tag{2.23}$$

where m is $+1/2$ or $-1/2$ and the spin-weight $+3/2$ spherical harmonics $S_{+3/2}$ satisfies

$$\mathcal{L}^\dagger_{-1/2}\mathcal{L}_{3/2}S_{+3/2}=-\lambda S_{+3/2}, \tag{2.24}$$

$$\lambda=(J_s-\tfrac{1}{2})(J_s+\tfrac{3}{2})\quad (J_s=\tfrac{3}{2},\tfrac{5}{2},\ldots). \tag{2.25}$$

Equation (2.3), which is derived from the above procedures, is a ''master'' equation for the physical process of linear perturbations about black holes, for instance, the scattering problem. The scattering problem of the linear perturbations about black holes is closely related to that of the one-dimensional scattering problem of Eq. (2.3) [17]. The scattering problem as a one-dimensional system is easily formulated by converting Eq. (2.3) into a Regge-Wheeler-type equation without the first order derivative term [17,18]. Here we only show the results in terms of Y_{+s} variables.

To set the scattering problem, we need the normalized in(out)going wave forms for Y_{+s} at the asymptotic regions $(r_*\to\pm\infty)$. At $r_*\to\infty$, its asymptotic form of each normalized perturbation $Y_{+s}(s=1,\frac{3}{2},2)$ becomes

$$Y^{(+\infty,\mathrm{in})}_{+s}\sim-4\Omega^2 e^{+i\Omega r_*}\quad\text{and}$$

$$Y^{(+\infty,\mathrm{out})}_{+s}\sim-\frac{K_s}{4\Omega^2 r^{2\|s\|}}e^{-i\Omega r_*}, \tag{2.26}$$

where $\|s\|=2$ for $s=1,2$ and $\|s\|=\frac{3}{2}$ for $s=\frac{3}{2}$. Here K_s are defined by

$$K_s\equiv\mu^2(\mu^2+2)+2i\Omega\beta_s,\quad \beta_s^2\equiv q_{s'}^2,\quad (s,s'=1,2;s\neq s'), \tag{2.27}$$

$$K_{3/2}\equiv 2i\Omega(\kappa_{3/2}+2i\Omega\beta_{3/2}), \tag{2.28}$$

$$\kappa^2_{3/2}\equiv[(J_s-\tfrac{1}{2})(J_s+\tfrac{1}{2})(J_s+\tfrac{3}{2})]^2,\quad \beta^2_{3/2}\equiv 4. \tag{2.29}$$

Similarly, at $r_*\to-\infty$, for $s=1,2$,

$$Y^{(-\infty,\mathrm{out})}_{+s}\sim 4i\Omega\left(i\Omega-\frac{r_+-Q^2}{r_+^3}\right)\exp(+i\Omega r_*),$$

$$Y^{(-\infty,\mathrm{in})}_{+s}\sim\frac{K_s(1+2q_s/\mu^2 r_+)\Delta^{\|s\|}}{4r_+^{4\|s\|}[i\Omega-2(r_+-1)/r_+^4][i\Omega-(r_+-Q^2)/r_+^3]}\times\exp(-i\Omega r_*), \tag{2.30}$$

and, for $s=3/2$,

$$Y^{(-\infty,\mathrm{out})}_{+3/2}\sim 4i\Omega\left(i\Omega-\frac{r_+-1}{2r_+^2}\right)\exp(+i\Omega r_*),$$

$$Y^{(-\infty,\mathrm{in})}_{+3/2}\sim\frac{K_s\Delta^{\|s\|}}{4r_+^{4\|s\|}[i\Omega-(r_+-1)/2r_+^2][i\Omega-\frac{3}{4}(r_+-1)/r_+^2]}\times\exp(-i\Omega r_*). \tag{2.31}$$

Using the above basis, the scattering problems of the perturbations are set as

$$\begin{aligned}Y_{+s}&\sim Y^{(+\infty,\mathrm{in})}_{+s}+R_s(\Omega)Y^{(+\infty,\mathrm{out})}_{+s}\quad (r_*\to\infty)\\&\sim T_s(\Omega)Y^{(-\infty,\mathrm{out})}_{+s}\quad (r_*\to-\infty),\end{aligned} \tag{2.32}$$

where R_s and T_s are the reflection and transmition coefficients, respectively.

III. THE TRANSFORMATION LAW OF THE CURVATURES

In the previous section, we summarized the perturbation equations governing the physical modes. On the extreme Reissner-Nordström background, the quasinormal frequencies of the perturbations with different helicity coincide by a suitable shift of the total angular momentum [14,15]. This fact suggests that the reflection and transmition amplitudes are equivalent among the perturbations with different helicity.

It is well known that the extreme Reissner-Nordström background has an unbroken global supersymmetry in $N=2$ supergravity [7]. This implies that the perturbations with the different helicity are related to each other.

In this section, we obtain the supersymmetric transformation laws between the curvatures of the perturbed fields through $N=2$ supergravity. The actions of $N=2$ supergravity are represented by

$$\begin{aligned}\mathcal{L}=&-\frac{M_p{}^2 e}{2}R-\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\psi^i_{A\mu}e^{AA'}_\nu D_\rho\overline{\psi}^{\bar{i}}_{A'\sigma}\\&-\overline{\psi}^{\bar{i}}_{A'\mu}e^{AA'}_\nu D_\rho\psi^i_{A\sigma})\\&-\frac{i}{32M_p{}^2}\epsilon^{\mu\nu\rho\sigma}\;[(\epsilon^{ij}\psi^{iA}_\mu\psi^j_{A\nu})\;(\epsilon^{kl}\psi^{kB}_\rho\psi^l_{B\sigma})\\&-(\epsilon^{ij}\overline{\psi}^{\bar{i}A'}_\mu\overline{\psi}^{\bar{j}}_{A'\nu})\;(\epsilon^{kl}\overline{\psi}^{\bar{k}B'}_\rho\overline{\psi}^{\bar{l}}_{B'\sigma})]-\frac{e}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\\&+\frac{i}{8M_p}\epsilon^{\mu\nu\rho\sigma}\hat{F}_{\rho\sigma}\epsilon^{ij}\;(\psi^{iA}_\mu\psi^j_{A\nu}+\overline{\psi}^{\bar{i}A'}_\mu\overline{\psi}^{\bar{j}}_{A'\nu}),\end{aligned} \tag{3.1}$$

$$\hat{F}_{\mu\nu}=F_{\mu\nu}+\frac{1}{2M_p}\epsilon^{ij}(\psi^{iA}_{\mu}\psi^{j}_{A\nu}-\overline{\psi^{iA'}_{\mu}}\,\overline{\psi^{j}_{A'\nu}})\quad (i,j,k,l=1,2), \tag{3.2}$$

where $M_p=(8\pi G)^{-1/2}$ is the Planck mass, $e^{AA'}_{\mu}=e^{a}_{\mu}\sigma^{AA'}_{a}$, and D_μ is a covariant derivative with respect to $\omega_\mu{}^{ab}$. The connection $\omega_{\mu ab}$ including the torsion is given by a tetrad and gravitini through varying the action with respect to $\omega_{\mu ab}$,

$$\omega_\mu{}^{ab}=\omega^{(0)ab}_{\mu}+K_\mu{}^{ab}, \tag{3.3a}$$

$$\omega^{(0)ab}_{\mu}=e^{a\nu}\partial_{[\mu}e^{b}_{\nu]}-\frac{1}{2}e_{c\mu}e^{a\nu}e^{b\lambda}\partial_\nu e^{c}_{\lambda}-(a\leftrightarrow b), \tag{3.3b}$$

$$K_\mu{}^{ab}=\frac{i}{2M_p{}^2}\left(e^{a\nu}\overline{\psi^{i}_{A'[\mu}}\sigma^{bAA'}\psi^{i}_{|A|\nu]}-\frac{1}{2}e^{a\nu}e^{b\lambda}\overline{\psi^{i}_{A'\nu}}e^{cAA'}_{\mu}\psi^{i}_{A\lambda}\right)-(a\leftrightarrow b), \tag{3.3c}$$

and the curvature is given by

$$R_{\mu\nu}{}^{ab}=2\partial_{[\mu}\omega_{\nu]}{}^{ab}+2\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^{b}, \tag{3.4a}$$

$$R_{ab}=e^{\mu}_{a}e^{\nu}_{c}R_{\mu\nu b}{}^{c}. \tag{3.4b}$$

The action is invariant under the supersymmetric transformations as

$$\delta e_{a\mu}=-\frac{i}{2M_p}(\alpha^{i}_{A}\sigma^{AA'}_{a}\overline{\psi^{i}_{A'\mu}}+\overline{\alpha^{i}_{A'}}\sigma^{AA'}_{a}\psi^{i}_{A\mu}), \tag{3.5a}$$

$$\delta A_\mu=-\tfrac{1}{2}\epsilon^{ij}(\alpha^{iA}\psi^{j}_{A\mu}-\overline{\alpha^{iA'}}\,\overline{\psi^{j}_{A'\mu}}), \tag{3.5b}$$

$$\delta\psi^{i}_{A\mu}=M_pD_\mu\alpha^{i}_{A}-i\epsilon^{ij}\hat{F}_{A}{}^{B}e_{\mu BA'}\overline{\alpha^{jA'}}, \tag{3.5c}$$

where α^{i}_{A} are Grassmann odd transformation parameters and $\hat{F}_{AB}$ is a two-spinor representation of the self-dual part of $\hat{F}_{\mu\nu}$.

We can check that $\omega_{\mu ab}$ and $\hat{F}_{\mu\nu}$ are supercovariant, i.e., that their transformations have no derivative of transformation parameters. For the spin-3/2 fields, we introduce the supercovariant curvatures of $\psi^{i}_{ABA'}=e^{\mu}_{BA'}\psi^{i}_{A\mu}$ in the two-spinor representation:

$$\Psi^{iA}{}_{BC}=\frac{1}{2}\left[D_{(B|A'|}\psi^{iA}{}_{C)}{}^{A'}+\frac{i}{M_p}\epsilon^{ij}\hat{F}^{A}{}_{(B}\overline{\psi^{jA'}}{}_{|A'|C)}\right], \tag{3.6a}$$

$$\Psi^{iA}{}_{B'C'}=\frac{1}{2}\left[D_{B(B'}\psi^{iAB}{}_{C')}-\frac{i}{M_p}\epsilon^{ij}\hat{F}^{A}{}_{B}\overline{\psi^{j}_{(B'C')}}{}^{B}\right]. \tag{3.6b}$$

They are transformed according to

$$\delta\Psi^{iA}{}_{BC}=\frac{M_p}{2}R_{BC}{}^{AD}\alpha^{i}_{D}+\frac{i}{2}\epsilon^{ij}[D_{(B}{}^{A'}\hat{F}_{C)}{}^{A}]\overline{\alpha^{j}_{A'}}+O(\psi^2), \tag{3.7a}$$

$$\delta\Psi^{iA}{}_{B'C'}=\frac{M_p}{2}R_{B'C'}{}^{AD}\alpha^{i}_{D}-\frac{i}{2}\epsilon^{ij}[D_{B(B'}\hat{F})^{AB}]\overline{\alpha^{j}_{C)'}}+\frac{1}{2M_p}\hat{F}^{AD}\overline{\hat{F}}_{B'C'}\alpha^{i}_{D}+O(\psi^2). \tag{3.7b}$$

Because we will analyze the perturbations about a purely bosonic background, it is sufficient to obtain the transformation laws at linear order of $\psi^{i}_{A\mu}$.

We introduce an expansion parameter λ and replace the fundamental fields, tetrad, connection, gravitini, and electromagnetic potential about a background as, for example, $\psi^{i}_{A\mu}\rightarrow\boldsymbol{\psi}^{i}_{A\mu}+\lambda\psi^{i}_{A\mu}$, where we use boldface for the background quantities and standard letters for the perturbed quantities, respectively. Various equations and relations for perturbed fields are given by expanding with respect to λ.

Let us consider supersymmetric transformation laws. Because of $\boldsymbol{\psi}^{j}_{A\mu}=0$, the bosonic background quantities are invariant under a supersymmetric transformation. On the other hand, fermionic quantities generally change due to a nontrivial bosonic background. For example, the background gravitini transform under the supersymmetric transformation into

$$\delta\boldsymbol{\psi}^{j}_{A\mu}=M_p\boldsymbol{D}_\mu\alpha^{i}_{A}-i\epsilon^{ij}\boldsymbol{F}_{A}{}^{B}e_{\mu BA'}\overline{\alpha^{jA'}}. \tag{3.8}$$

Therefore, if there are some supercovariantly constant spinors (SCCS's)

$$\boldsymbol{D}_\mu\zeta^{i}_{A}-\frac{i}{M_p}\epsilon^{ij}\boldsymbol{F}_{A}{}^{B}\;e_{\mu BA'}\;\overline{\zeta^{jA'}}=0, \tag{3.9}$$

the background configurations are invariant under the supersymmetric transformations that are induced by SCCS's. And then, unbroken supersymmetry persists on the system consisting of the perturbed fields.

Next, the perturbed supercovariant curvatures of gravitini transform into

$$\delta\Psi^{i}_{ABC}=\frac{M_p}{2}R_{BCA}{}^{D}\alpha^{i}_{D}+\frac{i}{2}\epsilon^{ij}[\boldsymbol{D}_{(B}{}^{A'}F_{C)A}]\;\overline{\alpha^{j}_{A'}}+\frac{i}{2}\epsilon^{ij}\sigma_{a(B}{}^{A'}[\omega^{a}{}_{C)}{}^{D}\boldsymbol{F}_{DA}+\omega^{a}{}_{|A|}{}^{D}\boldsymbol{F}_{C)D}]\;\overline{\alpha^{j}_{A'}}, \tag{3.10}$$

where we omit the transformation laws of $\Psi^{iA}{}_{B'C'}$ because they vanish due to the equations of motion.

Since the physical modes of gravitini are given by H^{i}_{0}, we are interested in the transformation laws of H^{i}_{0} generated by SCCS's, ζ^{i}_{A},

$$\delta H_0^i=\frac{M_p}{2}[\zeta^i_{(0)}\Psi_1-\zeta^i_{(1)}\Psi_0]$$
$$+\frac{i}{2}\epsilon^{ij}(\boldsymbol{D}_a\phi_0)(\overline{\zeta^J_{(0')}}m^a-\overline{\zeta^J_{(1')}}l^a)$$
$$+i\epsilon^{ij}\phi_0(\boldsymbol{\beta}\overline{\zeta^J_{(0')}}-\boldsymbol{\epsilon}\overline{\zeta^J_{(1')}})-i\epsilon^{ij}\boldsymbol{\phi}_1(\sigma\overline{\zeta^J_{(0')}}-\kappa\overline{\zeta^J_{(1')}}),\tag{3.11}$$

where $\zeta^i_{(0)}=o^A\zeta^i_A$ and $\zeta^i_{(1)}=\iota^A\zeta^i_A$ and they have the spin weights $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively. And then, Ψ_0, Ψ_1, ϕ_0, σ, and κ are perturbed Weyl scalars, Maxwell scalar, and complex spin coefficients, respectively. Further $\boldsymbol{\epsilon}$ and $\boldsymbol{\beta}$ are background spin coefficients.

IV. RELATIONS OF THE REFLECTION AND TRANSMISSION COEFFICIENTS

In the previous section, we obtained the transformation law between the perturbed curvatures of gravitini. Using it, we can relate the decoupled modes Y_{+s} of perturbations about the extreme Reissner-Nordström black hole.

On the extreme hole, there exist the supercovariantly constant spinors

$$\zeta^i_{(0)}=\sqrt{2}\quad\eta^i_{(0)}(\theta)\exp(im'\phi),\tag{4.1a}$$

$$\zeta^i_{(1)}=\frac{\Delta^{1/2}}{r}\ \eta^i_{(1)}(\theta)\exp(im'\phi),\tag{4.1b}$$

where m' is $+\frac{1}{2}$ or $-\frac{1}{2}$ and η^i_A satisfy

$$\mathcal{L}^{m'\dagger}_{-1/2}\eta^i_{(0)}=\mathcal{L}^{m'}_{-1/2}\eta^i_{(1)}=0,\tag{4.2a}$$

$$\mathcal{L}^{m'}_{+1/2}\eta^i_{(0)}=\eta^i_{(1)},\tag{4.2b}$$

$$\mathcal{L}^{m'\dagger}_{+1/2}\eta^i_{(1)}=-\eta^i_{(0)},\tag{4.2c}$$

where the operators $\mathcal{L}^m_n$ and $\mathcal{L}^{m\dagger}_n$ are the same operators as defined in Eqs. (2.13) in the previous section and we manifest azimuthal angular momentum dependence with index m. The supercovariantly constant spinors satisfy the relation

$$\frac{i}{M_p}\epsilon^{ij}\boldsymbol{\phi}_1\overline{\zeta^J_{A'}}=-\boldsymbol{\gamma}\left(\frac{2r^2}{\Delta}\overline{\iota}_{A'}\iota^A+\overline{o}_{A'}o^A\right)\zeta^i_A.\tag{4.3}$$

From Eq. (4.3), the transformation of H^i_0, Eq. (3.11) becomes

$$\delta H_0^i=\frac{M_p}{2}\left[\frac{\Delta^{1/2}}{r}\left(\Psi_0+4\boldsymbol{\gamma}\frac{r^2}{\Delta}\sigma\right)\eta^i_{(1)}\right.$$
$$\left.-\sqrt{2}(\Psi_1-2\boldsymbol{\gamma}\kappa)\quad\eta^i_{(0)}\right]e^{im'\phi},\tag{4.4}$$

where we, of course, adopt the phantom gauge $\phi_0=\phi_2=0$.

According to Sec. II, we decompose Ψ_0, Ψ_1, κ, and σ by spin-weighted spherical harmonics, and we manifest angular momentum dependence. For example,

$$\Psi_0=R^J_{+2}(r)S^J_{+2}(\theta)e^{i(\Omega t+m\phi)},\tag{4.5}$$

$$\mathcal{L}^{m\dagger}_{-1}\mathcal{L}^m_2S^J_{+2}=-\mu_J^2S^J_{+2},\tag{4.6}$$

$$\mu_J=\sqrt{(J-1)(J+2)}.\tag{4.7}$$

And then

$$\delta H_0^i=\frac{M_p}{2}\left\{\frac{\Delta^{1/2}}{r}\left[R^J_{+2}+r\frac{d}{dr}\ln\left(\frac{\Delta}{r^2}\right)s^J\right](\eta^i_{(1)}S^J_{+2})\right.$$
$$\left.-\frac{1}{r}\left[R^J_{+1}-r^3\frac{d}{dr}\left(\frac{\Delta}{r^2}\right)k^J\right](\eta^i_{(0)}S^J_{+1})\right\}e^{i(\Omega t+M\phi)},\tag{4.8}$$

where $M=m+m'$.

Each quantity $\eta^i_{(1)}S^J_{+2}$ and $\eta^i_{(0)}S^J_{+1}$ has spin weight $+3/2$, but not an eigenstate of total angular momentum, respectively. Hence we need decompose them into $(S^{J+1/2}_{+3/2})$ and $(S^{J-1/2}_{+3/2})$, which are eigenstates of the total angular momentum. It is easy to check the equations

$$\mathcal{L}^{M\dagger}_{-1/2}\mathcal{L}^M_{3/2}(\eta^i_{(1)}S^J_{+2})=-\mu_J(\eta^i_{(0)}S^J_{+1})-\mu_J^2(\eta^i_{(1)}S^J_{+2}),\tag{4.9a}$$

$$\mathcal{L}^{M\dagger}_{-1/2}\mathcal{L}^M_{3/2}(\eta^i_{(0)}S^J_{+1})=-\mu_J(\eta^i_{(1)}S^J_{+2})-(\mu_J^2+3)(\eta^i_{(0)}S^J_{+1}).\tag{4.9b}$$

From these equations, we can decompose $(\eta^i_{(1)}S^J_{+2})$ and $(\eta^i_{(0)}S^J_{+1})$ as

$$\eta^i_{(1)}S^J_{+2}=\xi^i\frac{S^{J+1/2}_{+3/2}-S^{J-1/2}_{+3/2}}{q_2-q_1},\tag{4.10a}$$

$$\eta^i_{(0)}S^J_{+1}=\xi^i\frac{q_1\ \ S^{J+1/2}_{+3/2}-q_2\ \ S^{J-1/2}_{+3/2}}{2\mu_J(q_2-q_1)},\tag{4.10b}$$

where ξ^i are arbitrary Grassmann odd constants and $q_{1,2}$ are the extremal limit $(Q^2=1)$ values of Eq. (2.7),

$$q_1=3+\sqrt{9+4\mu_J^2}=2(J+2),$$

$$q_2=3-\sqrt{9+4\mu_J^2}=-2(J-1).\tag{4.11}$$

Because $\Delta=(r-1)^2$ in the extreme case, Eq. (4.8) is rewritten as

$$\delta H_0^i=\frac{M_p\Delta^{1/2}\xi^i}{2r(q_2-q_1)}\Bigg[S_{+3/2}^{J+1/2}\left(F_{+1}^J-\frac{q_1}{2\Delta^{1/2}\mu_J}G_{+2}^J\right)-S_{+3/2}^{J-1/2}\left(F_{+2}^J-\frac{q_2}{2\Delta^{1/2}\mu_J}G_{+1}^J\right)\Bigg]e^{i(\Omega t+M\phi)},$$

$$=\frac{M_p\Delta^{1/2}\xi^i}{2r(q_2-q_1)}\Bigg[S_{+3/2}^{J+1/2}\left\{F_{+1}^J-\frac{q_1}{2}\frac{r\Delta^{1/2}}{\mu_J^2r+2q_1}\times\left(D_2^\dagger-\frac{3}{r}\right)F_{+1}^J\right\}-S_{+3/2}^{J-1/2}\left\{F_{+2}^J-\frac{q_2}{2}\frac{r\Delta^{1/2}}{\mu_J^2r+2q_2}\times\left(D_2^\dagger-\frac{3}{r}\right)F_{+2}^J\right\}\Bigg]e^{i(\Omega t+M\phi)}, \tag{4.12}$$

where we use the relation (2.17). Equation (4.12) shows that the helicity-$(+3/2)$ modes with $J+1/2$ are generated by unbroken supersymmetry from the helicity-$(+1)$ mode F_{+1}^J with total angular momentum J or the helicity-$(+2)$ mode F_{+2}^{J+1} with $J+1$.

From Eq. (4.12), the radial parts $Y_{+3/2}^{J_s}$ of perturbed curvatures of gravitini are generated from Y_{+s}^J as

$$Y_{+3/2}^{kJ_s}=\xi^k[Y_{+s}^J-C_s^J(r)\ \ \Lambda_-Y_{+s}^J] \tag{4.13a}$$

or, equivalently,

$$\xi^k\left[2i\Omega+\frac{1}{C_s^J}-C_s^JQ_s+\frac{1}{2}\ P_{3/2}\right]Y_{+s}^J=\left[2i\Omega+\frac{1}{C_s^J}+\frac{1}{2}\ P_{3/2}\right]Y_{+3/2}^{kJ_s}+\Lambda_-Y_{+3/2}^{kJ_s}, \tag{4.13b}$$

where

$$C_s^J(r)\equiv\frac{q_sr^2\Delta^{3/2}}{2\mu_J^2D_s},\quad D_s\equiv\Delta^2\left(1+\frac{2q_s}{\mu_J^2r}\right), \tag{4.14}$$

and J_s is $J+\frac{1}{2}$ for $s=1$ and $J-\frac{1}{2}$ for $s=2$. Equations (4.13) are our main result, and in principle, we can also obtain the relations between potentials of perturbations with different helicities. Hereafter we omit the index k which distinguishes two gravitini.

From Eqs. (4.13), we can obtain the relation between reflection and transmission amplitudes. From them, it follows that $Y_{+3/2}^{J_s}$ derived from $Y_{+s}^{J(+\infty,\mathrm{in})}$ and $Y_{+s}^{J(+\infty,\mathrm{out})}(s=1,2)$ have, respectively, the asymptotic behaviors, at $r_*\to\infty$,

$$Y_{+3/2}^{J_s}\sim Y_{+3/2}^{J_s(+\infty,\mathrm{in})}\quad\text{and}\quad Y_{+3/2}^{J_s}\sim\gamma_s\ \ Y_{+3/2}^{J_s(+\infty,\mathrm{out})}, \tag{4.15}$$

where $\gamma_s\equiv(i\Omega q_s/\mu_J^2)K_s/K_{3/2}$ and $|\gamma_s|=1$.

Similarly, it follows that $Y_{+3/2}^{J_s}$ derived from $Y_{+s}^{J(-\infty,\mathrm{in})}$ and $Y_{+s}^{J(-\infty,\mathrm{out})}(s=1,2)$ have, respectively, the asymptotic behaviors, at $r_*\to-\infty$,

$$Y_{+3/2}^{J_s}\sim Y_{+3/2}^{J_s(-\infty,\mathrm{out})}\quad\text{and}\quad Y_{+3/2}^{J_s}\sim\gamma_s\ \ Y_{+3/2}^{J_s(-\infty,\mathrm{in})}. \tag{4.16}$$

Therefore the asymptotic form of $Y_{+3/2}^{J_s}$ derived from the solution for Y_{+s}^J $(s=1,2)$ having the asymptotic behavior

$$Y_{+s}^J\sim Y_{+s}^{J(+\infty,\mathrm{in})}+R_s^J(\Omega)Y_{+s}^{J(+\infty,\mathrm{out})}\quad(r_*\to\infty)$$
$$\sim T_s^J(\Omega)Y_{+s}^{J(-\infty,\mathrm{out})}\quad(r_*\to-\infty), \tag{4.17}$$

has the asymptotic behavior

$$Y_{+3/2}^{J_s}\sim Y_{+3/2}^{J_s(+\infty,\mathrm{in})}+R_s^J(\Omega)\gamma_sY_{+3/2}^{J_s(+\infty,\mathrm{out})}\quad(r_*\to\infty)$$
$$\sim T_s^J(\Omega)Y_{+3/2}^{J_s(-\infty,\mathrm{out})}\quad(r_*\to-\infty). \tag{4.18}$$

Accordingly, we obtain the relations of reflection and transmission coefficients,

$$R_{3/2}^{J_s}(\Omega)=\gamma_sR_s^J(\Omega)\quad\text{and}\quad T_{3/2}^{J_s}(\Omega)=T_s^J(\Omega)\quad(s=1,2). \tag{4.19}$$

Thus, under a suitable shift of angular momenta, while the amplitudes of the transmitted waves are identically the same for three perturbed fields, the reflected amplitudes differ only in their phases.

V. SUMMARY

In the previous section, using the unbroken supersymmetry that remains on the extreme Reissner-Nordström black hole, we obtained the relation between the reflection and transmission coefficients of decoupled modes with (helicity, total angular momentum) $=(1,J)$, $(\frac{3}{2},J+\frac{1}{2})$, $(2,J+1)$.

These relations are also expected for the perturbations about the superpartners of the extreme Reissner-Nordström black hole [20] and for matter multiplets about them.

In a previous paper [15], we observed that the Regge-Wheeler potential of gravitational perturbation coincides with the one of electromagnetic perturbation by inversion of the tortoise coordinate, that is, exchange of the horizon for infinity and vice versa. It is interesting to understand the above correspondence by using the relations of the perturbations obtained in the previous section.

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