

Quantum inequalities in two-dimensional Minkowski spacetime

Éanna É. Flanagan

Cornell University, Newman Laboratory, Ithaca, New York 14853-5001

(Received 3 June 1997)

We generalize some results of Ford and Roman constraining the possible behaviors of the renormalized expected stress-energy tensor of a free massless scalar field in two-dimensional Minkowski spacetime. Ford and Roman showed that the energy density measured by an inertial observer, when averaged with respect to the observers proper time by integrating against some weighting function, is bounded below by a negative lower bound proportional to the reciprocal of the square of the averaging time scale. However, the proof required a particular choice for the weighting function. We extend the Ford-Roman result in two ways. (i) We calculate the optimum (maximum possible) lower bound and characterize the state which achieves this lower bound; the optimum lower bound differs by a factor of six from the bound derived by Ford and Roman for their choice of smearing function. (ii) We calculate the lower bound for arbitrary, smooth positive weighting functions. We also derive similar lower bounds on the spatial average of energy density at a fixed moment of time. [S0556-2821(97)07420-1]

PACS number(s): 04.62.+v, 03.70.+k, 42.50.Dv

I. INTRODUCTION AND SUMMARY

In classical physics, the energy densities measured by all observers are non-negative, so that the matter stress-energy tensor T_{ab} obeys $T_{ab}u^a u^b \geq 0$ for all timelike vectors u^a . This ‘‘weak energy condition’’ strongly constrains the behavior of solutions of Einstein’s field equation: once gravitational collapse has reached a certain critical stage, the formation of singularities becomes inevitable [1]; traversable wormholes are forbidden [2]; and the asymptotic gravitational mass of isolated objects must be positive [3].

However, as is well known, in quantum field theory the energy density measured by an observer at a point in spacetime can be unboundedly negative [4]. Examples of situations where observers measure negative energy densities include the Casimir effect [5] and squeezed states of light [6], both of which have been probed experimentally. In addition, the theoretical prediction of black hole evaporation [7] depends in a crucial way on negative energy densities. If nature were to place no restrictions on negative energies, it might be possible to violate cosmic censorship [8,9], or to produce traversable wormholes or closed timelike curves [10]. As a consequence, in recent years there has been considerable interest in constraints on negative energy density that follow from quantum field theory. For reviews of recent results and their ramifications see, e.g., Refs. [11–14].

In this paper we shall be concerned with so-called ‘‘quantum inequalities,’’ which are constraints on the magnitude and duration of negative energy fluxes and densities measured by inertial observers, first introduced by Ford [15] and extensively explored by Ford and Roman [9,11,12,16,17].

A. Quantum inequalities

Consider a free, massless scalar field Φ in two dimensional Minkowski spacetime. We consider the following three different spacetime-averaged observables. Fix a smooth, strictly positive function $\rho = \rho(\xi)$ with

$$\int_{-\infty}^{\infty} \rho(\xi) d\xi = 1, \tag{1.1}$$

which we will call the smearing function. Let \hat{T}_{ab} be the stress tensor, and let (x, t) be coordinates such that the metric is $ds^2 = -dt^2 + dx^2$. Define

$$\hat{\mathcal{E}}_S[\rho] = \int_{-\infty}^{\infty} dx \rho(x) \hat{T}_{tt}(x, 0), \tag{1.2}$$

$$\hat{\mathcal{E}}_T[\rho] = \int_{-\infty}^{\infty} dt \rho(t) \hat{T}_{tt}(0, t), \tag{1.3}$$

and

$$\hat{\mathcal{E}}_F[\rho] = \int_{-\infty}^{\infty} dt \rho(t) \hat{T}^{xt}(0, t). \tag{1.4}$$

The quantity $\hat{\mathcal{E}}_S[\rho]$ is the spatial average of the energy density over the spacelike hypersurface $t=0$, while $\hat{\mathcal{E}}_T[\rho]$ is the time average with respect to proper time of the energy density measured by an inertial observer, and $\hat{\mathcal{E}}_F[\rho]$ is the time average with respect to proper time of the energy flux measured by an inertial observer. Of these three observables, $\hat{\mathcal{E}}_S$ and $\hat{\mathcal{E}}_T$ are classically positive, while $\hat{\mathcal{E}}_F$ is classically positive when only the right-moving sector of the theory contains excitations.

In the quantum theory, let $\mathcal{E}_{S,\min}[\rho]$ and $\mathcal{E}_{T,\min}[\rho]$ denote the minimum over all states of the expected value of the observables $\hat{\mathcal{E}}_S[\rho]$ and $\hat{\mathcal{E}}_T[\rho]$, respectively. Similarly, let $\mathcal{E}_{F,\min}[\rho]$ denote the minimum over all states in the right moving sector of the expected value of $\hat{\mathcal{E}}_F[\rho]$. Ford and Roman have previously derived lower bounds on $\mathcal{E}_{T,\min}[\rho]$ and $\mathcal{E}_{F,\min}[\rho]$, for a particular choice of the smearing function ρ . Specifically, they showed that [11,17]

$$\mathcal{E}_{T,\min}[\rho_0] \geq -\frac{1}{8\pi\tau^2} \quad (1.5)$$

and [15]

$$\mathcal{E}_{F,\min}[\rho_0] \geq -\frac{1}{16\pi\tau^2}, \quad (1.6)$$

where

$$\rho_0(t) \equiv \frac{\tau}{\pi} \frac{1}{t^2 + \tau^2}. \quad (1.7)$$

The main result of this paper is that

$$\mathcal{E}_{T,\min}[\rho] = \mathcal{E}_{S,\min}[\rho] = 2\mathcal{E}_{F,\min}[\rho] = -\frac{1}{24\pi} \int_{-\infty}^{\infty} dv \frac{\rho'(v)^2}{\rho(v)}, \quad (1.8)$$

for arbitrary smearing functions $\rho(v)$. Equation (1.8) generalizes the Ford-Roman results and shows that the qualitative nature of those results does not depend on their specific choice of smearing function (which was chosen to facilitate the proofs of the inequalities), as one would expect. Equation (1.8) also gives the optimum, maximum possible lower bound on the averaged energy density, in contrast to the lower bounds (1.5) and (1.6). For the particular choice (1.7) of smearing function, Eq. (1.8) shows that the optimum lower bounds are a factor of six smaller in absolute value than the bounds (1.5) and (1.6).

Equation (1.8) holds not just for smearing functions $\rho(v)$ which are strictly positive [as is the Ford-Roman smearing function (1.7)], but also for smearing functions which are strictly positive only in an open interval $v_1 < v < v_2$ (with v_1, v_2 finite) and zero elsewhere, as long as $\rho(v)$ is smooth on $-\infty < v < \infty$. For such smearing functions, the quantity $\rho'(v)^2/\rho(v)$ appearing in Eq. (1.8) should be interpreted to be zero when $\rho(v) = 0$.

Equation (1.8) also shows that the lower bounds on the temporal averages and spatial averages of energy are identical, which is not surprising in a two-dimensional theory.

We derive the result (1.8) in Sec. II below. In Sec. III we discuss some of its implications: we show that the total amount of negative energy that can be contained in a finite region $0 \leq x \leq L$ at a fixed moment of time is infinite, but that if $\alpha > 0$ is a number such that, for some state, $\langle \hat{T}_{tt}(x, 0) \rangle \leq -\alpha$ for all x with $0 \leq x \leq L$, then α cannot be arbitrarily large.

II. DERIVATION OF THE QUANTUM INEQUALITY

We start by showing that the minimum values of the three observables $\hat{\mathcal{E}}_S$, $\hat{\mathcal{E}}_T$, and $\hat{\mathcal{E}}_F$ that we have defined are not independent of each other, cf., the first part of Eq. (1.8) above. To see this, introduce null coordinates $u = t + x$, $v = t - x$, so that the field operator can be decomposed as

$$\hat{\Phi}(x, t) = \hat{\Phi}_R(v) + \hat{\Phi}_L(u). \quad (2.1)$$

Here $\hat{\Phi}_R(v)$ acts on the right-moving sector and $\hat{\Phi}_L(u)$ on the left-moving sector of the theory. The nonzero components of the stress tensor in the (u, v) coordinates are $\hat{T}_{uu}(u) = :(\partial_u \hat{\Phi}_L)^2:$ and

$$\hat{T}_{vv}(v) = :(\partial_v \hat{\Phi}_R)^2:, \quad (2.2)$$

where the colons denote normal ordering. Define the right-moving and left-moving energy flux observables

$$\hat{\mathcal{E}}^{(R)}[\rho] \equiv \int dv \rho(v) \hat{T}_{vv}(v) \quad (2.3)$$

and

$$\hat{\mathcal{E}}^{(L)}[\rho] \equiv \int du \rho(u) \hat{T}_{uu}(u). \quad (2.4)$$

Then we have $\hat{\mathcal{E}}_S[\rho] = \hat{\mathcal{E}}_T[\rho] = \hat{\mathcal{E}}^{(R)}[\rho] + \hat{\mathcal{E}}^{(L)}[\rho]$, while $\hat{\mathcal{E}}_F[\rho] = \hat{\mathcal{E}}^{(R)}[\rho] - \hat{\mathcal{E}}^{(L)}[\rho]$. It follows that $\mathcal{E}_{T,\min}[\rho] = \mathcal{E}_{S,\min}[\rho] = 2\mathcal{E}_{F,\min}[\rho] = 2\mathcal{E}_{\min}^{(R)}[\rho]$, where

$$\mathcal{E}_{\min}^{(R)}[\rho] \equiv \min_{\text{states}} \langle \hat{\mathcal{E}}^{(R)}[\rho] \rangle, \quad (2.5)$$

from which the first part of Eq. (1.8) follows.

Thus, to establish Eq. (1.8) it is sufficient to consider the right-moving sector of the theory and to show that

$$\mathcal{E}_{\min}^{(R)}[\rho] = -\frac{1}{48\pi} \int_{-\infty}^{\infty} dv \frac{\rho'(v)^2}{\rho(v)}. \quad (2.6)$$

We derive the result (2.6) in this section in two stages. First, in Sec. II A, we give a simple derivation which is valid only for smearing functions which are strictly positive and for which the minimum over states in Eq. (2.5) is achieved by a state in the usual Hilbert space [c.f. Eq. (2.23) below]. Then, in Sec. II B, we use the algebraic formulation of quantum field theory to extend the proof to more general smearing functions.

A. Bogolubov transformation

The key idea in our proof is to make a Bogolubov transformation which transforms the quadratic form (2.3) into a simple form. In general spacetimes such a Bogolubov transformation is difficult to obtain, but in flat, two-dimensional spacetimes it can be obtained very simply by using a coordinate transformation, as we now explain.

We can write the mode expansion of the right-moving field operator as

$$\hat{\Phi}_R(v) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} d\omega \frac{1}{\sqrt{2\omega}} [e^{-i\omega v} \hat{a}_\omega + \text{H.c.}], \quad (2.7)$$

where H.c. means Hermitian conjugate. The Hamiltonian of the right-moving sector is

$$\hat{H}_R = \int_0^{\infty} d\omega \omega \hat{a}_\omega^\dagger \hat{a}_\omega. \quad (2.8)$$

Consider now a new coordinate V which is a monotonic increasing function of v ,

$$V=f(v), \quad (2.9)$$

say, where the function f is a bijection from the real line to itself. We define a mode expansion with respect to the V coordinate [18]:

$$\hat{\Phi}_R(v)=\hat{\Phi}_R[f^{-1}(V)]=\frac{1}{\sqrt{2\pi}}\int_0^\infty d\omega\frac{1}{\sqrt{2\omega}}[e^{-i\omega V}\hat{b}_\omega+\text{H.c.}]. \quad (2.10)$$

Since the function f is a bijection, the algebra spanned by the operators \hat{a}_ω coincides with the algebra spanned by the operators \hat{b}_ω . [In Sec. II B below we will consider the case where f is a bijection from a finite open interval (v_1, v_2) to the real line, and where correspondingly the operators \hat{a}_ω and \hat{b}_ω span different algebras.] Thus, the operators \hat{b}_ω can be expressed as linear combinations of the \hat{a}_ω 's and \hat{a}_ω^\dagger 's, and conversely.

We now assume that there exists a unitary operator \hat{S} such that

$$\hat{S}\hat{a}_\omega\hat{S}^\dagger=\hat{b}_\omega. \quad (2.11)$$

Such an operator will not always exist, as we discuss in Sec. II B below, but for the remainder of this subsection we will restrict attention to coordinate transformations $v\rightarrow V(v)$ for which the operator \hat{S} does exist. It follows from Eq. (2.11) that

$$\hat{S}^\dagger\hat{\Phi}_R(v)\hat{S}=\hat{\Phi}_R[f(v)]. \quad (2.12)$$

Consider now the transform $\hat{S}^\dagger\hat{T}_{vv}(v)\hat{S}$ of the operator $\hat{T}_{vv}(v)$. Using Eq. (2.2) this can be written as

$$\hat{S}^\dagger\hat{T}_{vv}(v)\hat{S}=\lim_{\bar{v}\rightarrow v}\hat{S}^\dagger\partial_{\bar{v}}\partial_v[\hat{\Phi}_R(\bar{v})\hat{\Phi}_R(v)-H(v-\bar{v})]\hat{S}, \quad (2.13)$$

where

$$H(\Delta v)=-\frac{1}{4\pi}[\ln|\Delta v|+\pi i\Theta(-\Delta v)] \quad (2.14)$$

is the distribution that the normal ordering procedure effectively subtracts off. Here Θ is the step function. Equations (2.2), (2.9), (2.12), and (2.13) now yield

$$\begin{aligned} \hat{S}^\dagger\hat{T}_{vv}(v)\hat{S} &= \lim_{\bar{v}\rightarrow v}\partial_{\bar{v}}\partial_v[\hat{\Phi}_R[f(\bar{v})]\hat{\Phi}_R[f(v)]-H(v-\bar{v})] \\ &= \lim_{\bar{v}\rightarrow v}V'(v)^2\partial_{\bar{V}}\hat{\Phi}_R(\bar{V})\partial_V\hat{\Phi}_R(V)-\partial_{\bar{v}}\partial_vH(v-\bar{v}) \\ &= V'(v)^2\cdot[\partial_V\hat{\Phi}_R(V)]^2-\Delta(v), \\ &= V'(v)^2\hat{T}_{vv}(V)-\Delta(v), \end{aligned} \quad (2.15)$$

where primes denote derivatives with respect to v and

$$\Delta(v)=\lim_{\bar{v}\rightarrow v}\partial_{\bar{v}}\partial_v\{H(v-\bar{v})-H[f(v)-f(\bar{v})]\}. \quad (2.16)$$

Using Eq. (2.14) we find

$$\begin{aligned} \Delta(v) &= \frac{1}{4\pi}\left[\frac{V'''(v)}{6V'(v)}-\frac{V''(v)^2}{4V'(v)^2}\right] \\ &= -\frac{1}{12\pi}\sqrt{V'(v)}\left(\frac{1}{\sqrt{V'(v)}}\right)'' \end{aligned} \quad (2.17)$$

The relation (2.15) is the key result that we shall use. Note that taking the expected value of Eq. (2.15) in the vacuum state yields

$$\langle\psi|\hat{T}_{vv}(v)|\psi\rangle=-\Delta(v), \quad (2.18)$$

where $|\psi\rangle=\hat{S}|0\rangle$ is the natural vacuum state associated with the V coordinate, which satisfies $\hat{b}_\omega|\psi\rangle=0$. This reproduces the standard formula for the expected stress tensor in the vacuum state associated with a given null coordinate; see, e.g., Ref. [19].

Now integrate Eq. (2.15) against the smearing function $\rho(v)$. From Eq. (2.3) this yields

$$\begin{aligned} \hat{S}^\dagger\hat{\mathcal{E}}^{(R)}[\rho]\hat{S} &= \int dv\rho(v)V'(v)^2\hat{T}_{vv}[V(v)] \\ &\quad - \int dv\rho(v)\Delta(v). \end{aligned} \quad (2.19)$$

We now choose the coordinate V to be such that $\rho(v)V'(v)=1$; note that this prescription yields a bijection $v\rightarrow V(v)$ since $\rho(v)>0$. The first term on the right-hand side of Eq. (2.19) now becomes $\int dV\hat{T}_{vv}(V)$, which is just the Hamiltonian \hat{H}_R ; cf. Eq. (2.8) above. Inserting the relation $V'(v)=1/\rho(v)$ into Eqs. (2.17) and (2.19) gives

$$\hat{S}^\dagger\hat{\mathcal{E}}^{(R)}[\rho]\hat{S}=\hat{H}_R-\Delta, \quad (2.20)$$

where

$$\Delta=-\frac{1}{12\pi}\int dv\sqrt{\rho(v)}[\sqrt{\rho(v)}]''=\frac{1}{48\pi}\int dv\frac{\rho'(v)^2}{\rho(v)}. \quad (2.21)$$

On the second line we have integrated by parts, and have assumed that $\rho'(v)\rightarrow 0$ as $v\rightarrow\pm\infty$.

It is clear from Eq. (2.20) that $\hat{\mathcal{E}}_{\min}^{(R)}[\rho]=-\Delta$, since \hat{H}_R is a positive operator with minimum eigenvalue zero. Equation (2.6) then follows from Eq. (2.21). Also, the state which achieves the minimum value $-\Delta$ of $\hat{\mathcal{E}}^{(R)}[\rho]$ is just the vacuum state $|\psi\rangle=\hat{S}|0\rangle$ associated with the V coordinate; this is a generalized (multimode) squeezed state. The V coordinate is given in terms of $\rho(v)$ by

$$V(v)=\int\frac{dv}{\rho(v)}. \quad (2.22)$$

B. Algebraic reformulation

The derivation just described suffers from the limitation that in certain cases the ‘‘scattering matrix’’ \hat{S} will fail to exist. This operator \hat{S} will exist when [20]

$$\int_0^\infty d\omega \int_0^\infty d\omega' |\beta_{\omega\omega'}|^2 < \infty, \quad (2.23)$$

where

$$\beta_{\omega\omega'} = \int_{-\infty}^\infty dv \frac{[\omega - \omega' V'(v)]}{\sqrt{\omega\omega'}} e^{-i\omega v} e^{-i\omega' V(v)}. \quad (2.24)$$

The condition (2.23) will be violated unless $|V'(v) - 1| < 1$ everywhere, i.e., unless

$$\rho(v) > 1/2 \quad (2.25)$$

everywhere. Therefore, for smearing functions which satisfy the normalization condition (1.1), the Bogolubov transformation to the mode basis associated with the new coordinate (2.22) does not yield a well defined scattering operator \hat{S} . The proof outlined in Sec. II A above is valid only for non-normalizable smearing functions satisfying Eq. (2.25).

However, it is straightforward to generalize the proof to smearing functions for which the condition (2.23) is violated using the algebraic formulation of quantum field theory [20], as we now outline. The following proof also applies to smearing functions which are strictly positive in an open region $v_1 < v < v_2$ (with v_1 and/or v_2 finite) and which vanish outside that open region. For any algebraic state η on Minkowski spacetime, let

$$\mathcal{F}_{g,\eta}(v) = \langle T_{vv}(v) \rangle_\eta \quad (2.26)$$

denote the expected value of the vv component of the stress tensor in the state η . Here $g = g_{ab}$ denotes the flat Minkowski metric

$$g_{ab} dx^a dx^b = -dt^2 + dx^2 = -dudv, \quad (2.27)$$

where $x^a = (x, t)$. Now suppose that V is a coordinate on the open interval (v_1, v_2) [which may be $(-\infty, \infty)$] given by $V = f(v)$, where f is a monotonically increasing bijection from (v_1, v_2) to $(-\infty, \infty)$. Consider the metric \bar{g}_{ab} which is conformally related to g_{ab} given by

$$\bar{g}_{ab} dx^a dx^b = -dudV = -V'(v) dudv. \quad (2.28)$$

This metric is defined on the submanifold \bar{M} of the original spacetime defined by the inequality $v_1 < v < v_2$; the pair (\bar{M}, \bar{g}_{ab}) is itself a two-dimensional Minkowski spacetime.

We can naturally associate with the state η on Minkowski spacetime (M, g_{ab}) a state $\bar{\eta}$ on the spacetime (\bar{M}, \bar{g}_{ab}) which has the same n point distributions $\langle \hat{\Phi}_R(v_1) \dots \hat{\Phi}_R(v_n) \rangle$. It can be checked that the resulting algebraic state $\bar{\eta}$ obeys the Hadamard and positivity conditions on the spacetime (\bar{M}, \bar{g}_{ab}) and so is a well defined state. If we define

$$\mathcal{F}_{\bar{g},\bar{\eta}}(v) = \langle T_{vv}(v) \rangle_{\bar{\eta}}, \quad (2.29)$$

then a straightforward point-splitting computation exactly analogous to that outlined in Sec. II A above yields

$$\mathcal{F}_{g,\eta}(v) = V'(v)^2 \mathcal{F}_{\bar{g},\bar{\eta}}[V(v)] - \Delta(v), \quad (2.30)$$

where $\Delta(v)$ is the quantity defined by Eq. (2.17) above. Now choosing $V'(v) = 1/\rho(v)$ yields, in obvious notation,

$$\langle \hat{\mathcal{E}}^{(R)}[\rho] \rangle_\eta = \langle \hat{H}_R \rangle_{\bar{\eta}} - \Delta, \quad (2.31)$$

where Δ is given by Eq. (2.21) but with the domain of integration being (v_1, v_2) . Finally we use the fact that the quadratic form \hat{H}_R is positive indefinite for all algebraic states $\bar{\eta}$ (not just for states in the folium of the vacuum state). The remainder of the proof now follows just as before.

III. IMPLICATIONS

In this section we discuss some of the implications of our result (1.8). First, it is possible to deduce from Eq. (1.8) constraints on the *maximum* energy density rather than the averaged energy density in a region of space. Specifically, the quantity

$$\min_{\text{states}} \max_{0 \leq x \leq L} \langle \hat{T}_{tt}(x, 0) \rangle \quad (3.1)$$

is bounded below for any $L > 0$, which confirms in this context a conjecture made in Ref. [14]. To see that the quantity (3.1) is bounded below, note that $\hat{T}_{tt}(x, t=0) = \hat{T}_{uu}(u=x) + \hat{T}_{vv}(v=-x)$, so that

$$\max_{0 \leq x \leq L} \langle \hat{T}_{tt}(x, 0) \rangle \leq \max_{0 \leq u \leq L} \langle \hat{T}_{uu}(u) \rangle + \max_{-L \leq v \leq 0} \langle \hat{T}_{vv}(v) \rangle. \quad (3.2)$$

Thus, it is sufficient to bound each term on the right-hand side of Eq. (3.2). Next, for any state, and for any smearing function $\rho(v)$ with support in $[-L, 0]$ and normalized according to Eq. (1.1), we have

$$\langle \hat{\mathcal{E}}_R[\rho] \rangle \leq \max_{-L \leq v \leq 0} \langle T_{vv}(v) \rangle. \quad (3.3)$$

One can write down a similar inequality for the other term on the right-hand side of Eq. (3.2). Taking the minimum over states and using Eqs. (2.6), (3.2), and (3.3) now yields

$$\min_{\text{states}} \max_{0 \leq x \leq L} \langle \hat{T}_{tt}(x, 0) \rangle \geq 2 \max_{\rho} \mathcal{E}_{R,\min}[\rho], \quad (3.4)$$

where the maximum is taken over all smooth normalizable smearing functions ρ with support in $[0, L]$. It is clear on dimensional grounds that the right-hand side of Eq. (3.4) is proportional to $-\hbar/L^2$, and hence we obtain

$$\min_{\text{states}} \max_{0 \leq x \leq L} \langle \hat{T}_{tt}(x, 0) \rangle \geq -k \frac{\hbar}{L^2}, \quad (3.5)$$

for some constant k .

The second implication of our result is that the *total* amount of negative energy that can be contained in a finite region $0 \leq x \leq L$ of space in two dimensions is infinite. This

can be seen from our result applied to the observable $\hat{\mathcal{E}}_S[\rho]$, by taking the limit where the smearing function $\rho(x)$ approaches the function

$$\rho_{\text{box}}(x) = \begin{cases} 1, & 0 \leq x \leq L, \\ 0, & \text{otherwise.} \end{cases} \quad (3.6)$$

In this limit the quantity $\mathcal{E}_{S,\min}[\rho]$ diverges. However, this divergence is merely an ultraviolet edge effect, in the sense that states which have large total negative energies inside the finite region will have most of the energy density concentrated near the edges at $x=0$ and $x=L$ [this can be seen from Eq. (3.5)], and furthermore such states will have compensating large positive energy densities just outside the finite region.

IV. CONCLUSION

We have derived a very general constraint on the behavior of renormalized expected stress tensors in free field theory in

two dimensions, generalizing earlier results of Ford and Roman [21]. Our result confirms the generality of the Ford-Roman time-energy uncertainty-principle-type relation [11]: that the amount ΔE of energy measured over a time Δt is constrained by

$$\Delta E \geq -\frac{\hbar}{\Delta t}. \quad (4.1)$$

We also showed that the total energy in a one-dimensional box is unbounded below, but that the maximum energy density in such a box is bounded below.

ACKNOWLEDGMENTS

The author thanks Robert Wald, Tom Roman, and Larry Ford for helpful discussions. This research was supported in part by NSF Grants No. PHY 9514726 and No. PHY 9408378.

-
- [1] R. Penrose, Phys. Rev. Lett. **14**, 57 (1965); S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, London, 1973).
- [2] J. Friedman, K. Schleich, and D. Witt, Phys. Rev. Lett. **71**, 1486 (1993).
- [3] R. Schoen and S.T. Yau, Phys. Rev. Lett. **43**, 1457 (1979).
- [4] H. Epstein, V. Glaser, and A. Jaffe, Nuovo Cimento **36**, 1016 (1965).
- [5] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. **B51**, 793 (1948); L. S. Brown and G. J. Maclay, Phys. Rev. **184**, 1272 (1969).
- [6] L.-A. Wu, H. J. Kimble, J. L. Hall, and H. Wu, Phys. Rev. Lett. **57**, 2520 (1986).
- [7] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
- [8] L. H. Ford and T. A. Roman, Phys. Rev. D **41**, 3662 (1990).
- [9] L. H. Ford and T. A. Roman, Phys. Rev. D **46**, 1328 (1992).
- [10] M. Morris and K. Thorne, Am. J. Phys. **56**, 395 (1988); M. Morris, K. Thorne, and Y. Yurtsever, Phys. Rev. Lett. **61**, 1446 (1988).
- [11] L. H. Ford and T. A. Roman, Phys. Rev. D **51**, 4277 (1995).
- [12] L. H. Ford and T. A. Roman, Phys. Rev. D **53**, 1988 (1996).
- [13] U. Yurtsever, Phys. Rev. D **51**, 5797 (1995).
- [14] É. É. Flanagan and R. M. Wald, Phys. Rev. D **54**, 6233 (1996).
- [15] L. H. Ford, Proc. R. Soc. London **A364**, 227 (1978); L. H. Ford, Phys. Rev. D **43**, 3972 (1991).
- [16] L. H. Ford and T. A. Roman, Phys. Rev. D **53**, 5496 (1996).
- [17] L. H. Ford and T. A. Roman, Phys. Rev. D **55**, 2082 (1997).
- [18] Note that we are using a notation where the argument of $\hat{\Phi}_R$ is always the v coordinate of the spacetime point, never the V coordinate; in other words we shall *not* use the notation $\hat{\Phi}_R(V)$ to denote the field operator at the spacetime point with V -coordinate value V and v coordinate value $f^{-1}(V)$.
- [19] S. A. Fulling and P. C. W. Davies, Proc. R. Soc. London **A348**, 393 (1976).
- [20] R. M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (University of Chicago Press, Chicago, 1994).
- [21] Roman has recently derived constraints on averages over both space and time of energy density [T. A. Roman (unpublished)]. Similar constraints can also be obtained by an application of our result (2.6): we can obtain lower bounds on very general space-time smearings of the stress-energy tensor using the fact that for any smooth tensor $f^{ab}(x,t)$,
- $$\int dx dt f^{ab}(x,t) \hat{T}_{ab}(x,t) = \int du \hat{T}_{vv}(v) f(v) + \int dv \hat{T}_{uu}(u) g(u),$$
- where $f(v) \equiv \int f^{vv}(u,v) du$ and $g(u) \equiv \int f^{uu}(u,v) dv$. The expected value of two terms on the right-hand side can be bounded below using Eq. (2.6).