Casimir energies for massive scalar fields in a spherical geometry

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The Casimir energy corresponding to a massive scalar field with Dirichlet boundary conditions on a spherical surface is obtained. The field is considered, separately, inside and outside the surface. The renormalization procedure that is necessary to apply in each situation is studied in detail, in particular, the differences occurring with respect to the case when the field occupies the whole space. The final result contains several constants that experience renormalization and can be determined only experimentally. The nontrivial finite parts that appear in the massive case are found exactly, providing a precise determination of the complete, renormalized zero-point energy for the first time. [S0556-2821(97)04220-3]

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I. INTRODUCTION

Calculations of Casimir energy in spherically symmetric situations have attracted the interest of physicists for well over forty years now. This is not strange, since in many contexts the inclusion of quantum fluctuations about semiclassical configurations turns out to be essential. Historically a first far reaching idea involving vacuum energies originated with Casimir himself. He proposed that the force stabilizing a classical electron model arises from the zero-point energy of the electromagnetic field within and without a perfectly conducting spherical shell [1]. Having found an attractive force between parallel plates due to the vacuum energy [2], the hope was that the same would occur for the spherically symmetric situation.[1] Unfortunately, as Boyer [3] first showed, for this geometry the stress is repulsive [4,5]. Nowadays it is known that the Casimir energy depends strongly on the geometry of the space-time and on the boundary conditions imposed. This is still a very active field of research because a satisfactory understanding of the behavior has not yet been found. For a number of results obtained in the last ten years see for instance [6–8].

Actual interest in the Casimir effect results from a considerable improvement of the experimental verification [9] as well as from its possible relevance to sonoluminescence (see, however, [10].) A different source of interest results from the MIT bag model in QCD [11–14]. In this connection there had been a number of calculations for the spherical geometry (see [15–18]), but they are still not completed, for instance for massive quark fields.

The Casimir effect was considered up to now mostly for massless fields (e.g., in [19,20]). In the case of massive fields a number of new features as additional divergences [21] occurs which call for a systematic renormalization procedure [22].

For massless fermions the zero-point energy was considered in [19,20]. The massless fermionic field inside and outside the spherical surface was analyzed in [20]. In the last case, a cancellation of divergences between the inner and outer spaces occurs and finite zero-point energies are found. Considering only the inner space, divergences appear and it is necessary to introduce contact terms and perform a renormalization of their coupling. The case of a D-dimensional sphere has been investigated in [23]. Results for the massive fermionic fields contain new ultraviolet-divergent terms in addition to those occurring in the massless case, as has been discussed in [21]. Further considerations, especially on the renormalization procedure necessary in order to carry out these calculations, and on its precise interpretation, can be found in [22].

In nearly all of the mentioned works the authors have used a Green's function approach in order to calculate the zero-point energy. An exception is Ref. [22], where, in the general setting of an ultrastatic spacetime with or without boundaries, a systematic procedure which makes use of zeta function regularization was developed (see also [24]). In this approach, a knowledge of the zeta function of the operator associated with the field equation together with (eventually) some appropriate boundary conditions is needed. Recently, a detailed description of how to obtain the zeta function for a massive scalar field inside a ball satisfying Dirichlet or Robin boundary conditions has been given by some of the authors [25,26] of the present work. An analytical continuation to the whole complex plane has been obtained and then applied to find an arbitrary number of heat-kernel coefficients. In the ensuing Refs. [27,28] the functional determinant has been considered too and, furthermore, the method has been also applied to spinors [29,30] and p-forms [31,32]. For an alternative approach involving scalars and spinors see also the developments in [33]. All the above considerations yield purely analytical and quite explicit formulas. In order to actually obtain values for the Casimir energy, however, a

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[1]This could be expected because the Casimir force between plates was shown to be the same as the force resulting from the retarded (always attractive) van der Waals forces between the atoms of the plates.

numerical integration had to be performed. This has been done in different cases, in particular for the massless scalar and electromagnetic fields [34], partly reobtaining previous results.

Here we will consider the massive scalar field inside and outside the spherical surface, separately. We will discuss in detail the renormalization procedure which is necessary to apply in this situation and, specifically, the differences occurring with respect to the case when one assumes that the field occupies the whole space. The final result for the zero-point energy contains several constants which experience renormalization and whose physical values can be determined only experimentally. However, for the massive field—as is clear from dimensional grounds—nontrivial finite parts which depend on a dimensionless variable involving the mass are present, and we will, in this case, be able to find here for the first time the complete, renormalized zero-point energy.

The aim of this paper is to perform the first complete (including numbers, respectively, plots) calculation of the Casimir energy for a *massive* field (whereby we restrict ourselves to a scalar field obeying Dirichlet boundary conditions) in the spherical geometry. As it turns out this allows for a number of conclusions concerning the normalization conditions and the interpretation of the effect itself [especially backwards for the massless case in model (i) below] as well as it adds a new piece to the puzzle on the sign. In addition the present calculation should serve as a guideline for work on more realistic models.

The organization of the paper is as follows. In Sec. II we describe in detail the models considered and the regularization and renormalization procedure employed. In Sec. III we summarize briefly the formulas that are needed in the subsequent study of the zeta function of the problem at hand. We shall start with the scalar field inside the ball. Some additional considerations are necessary because the representation given in the previous articles [25,26] was only applicable when the mass of the field is $m \leqslant 1$. Here we will derive formulas which are valid for arbitrary mass and very useful for numerical evaluations. All divergences and finite parts of the zero-point energy are calculated and its renormalization is performed with some care. The explicit dependence of the finite part in the mass is determined numerically. Afterwards, the exterior space is considered and a corresponding analysis is performed for this situation. Adding up both contributions, we see clearly how the divergences cancel among themselves as well as the influence of this cancellation on the compulsory renormalization process. Section IV is devoted to conclusions. The appendix contains some hints and technical details that are used in the derivation of the zeta function for the nonzero mass case.

II. DESCRIPTION OF THE MODEL AND ITS RENORMALIZATION

The physical system that we will consider consists of two parts.

(1) A classical system consisting of a spherical surface (''bag'') of radius R. Its energy reads

$$E_{\text{class}} = pV + \sigma S + FR + k + \frac{h}{R}, \tag{1}$$

where $V = \frac{4}{3}\pi R^3$ and $S = 4\pi R^2$ are the volume and surface, respectively. This energy is determined by the parameters p = pressure, σ = surface tension, and F, k, and h which do not have special names.

(2) A quantized field $\hat{\varphi}(x)$ whose classical counterpart obeys the Klein-Gordon equation

$$(\Box + m^2)\varphi(x) = 0, \tag{2}$$

as well as suitable boundary conditions on the surface ensuring self-adjointness of the corresponding elliptic operator on perturbations. We choose Dirichlet boundary conditions as the easiest to handle. The quantum field has a ground state energy

$$E_0 = \frac{1}{2} \sum_{(k)} \sqrt{\lambda_{(k)}}, \tag{3}$$

where the $\sqrt{\lambda_{(k)}}$'s are the one-particle energies with the quantum number k.

For this system we shall consider three models, which will behave in a different way. These models consist of the classical part given by the surface and (i) the quantized field in the interior of the surface, (ii) the quantized field in the exterior of the surface, and (iii) the quantized field in both regions together, respectively.

The ground state energy is divergent and we shall regularize it by

$$E_0 = \frac{1}{2} \sum_{(k)} (\lambda_{(k)})^{1/2-s} \mu^{2s}, \quad \mathcal{R}s > 2, \tag{4}$$

where μ is the known arbitrary parameter with the dimension of mass prescribed by the regularization and introduced in order to get the correct dimension for the energy. The one particle energies $\sqrt{\lambda_{(k)}}$ are determined by the eigenvalue equation

$$(-\Delta + m^2)\varphi_{(k)}(x) = \lambda_{(k)}\varphi_{(k)}(x), \tag{5}$$

together with Dirichlet boundary conditions on the surface

$$\varphi_{(k)}(x)\big|_{|x|=R} = 0. \tag{6}$$

For the field in the interior, the meaning of $\lambda_{(k)}$ is obvious: $(k) = (l,m,n)$, $\lambda_{(l,m,n)} = \sqrt{j^2_{l+1/2,n}/R^2 + m^2}$, $J_{l+1/2}(j_{l+1/2,n}) = 0$.

For the calculations we use the corresponding zeta function:

$$\zeta(s) = \sum_{(k)} \lambda_{(k)}^{-s}. \tag{7}$$

In the interior region, we have

$$\zeta_{(\text{int})}(s) = \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} (2l+1)\lambda_{(l,m,n)}^{-s}. \tag{8}$$

For the exterior zeta function $\zeta_{(\text{ext})}(s)$ we must take into account that the radial quantum number is continuous. We have to subtract the Minkowski space contribution. This procedure is well known (see, for example, [35]) and need not be repeated here. In the case of the third model we take the spectrum as the union of the spectra of the first two models and we simply add the interior and the exterior zeta functions: namely,

$$\zeta_{(\mathrm{total})}(s)=\zeta_{(\mathrm{int})}(s)+\zeta_{(\mathrm{ext})}(s). \tag{9}$$

This means that the interior and exterior region are independent one from the other.

In any case the regularized ground state energy is given by

$$E_0^{(\mathrm{mod})}=\tfrac{1}{2}\zeta_{(\mathrm{mod})}(s-\tfrac{1}{2})\mu^{2s}, \tag{10}$$

where ''mod'' means the model: ''int,'' ''ext'', or ''total.'' Let us note that the *regularized* energy of the third model is just the sum of the regularized energy of the first two models.

The divergent contributions of the ground state energy can be found most easily using standard heat kernel expansion

$$K(t)=\sum_{(k)} e^{-\lambda_{(k)}t}\sim\left(\frac{1}{4\pi t}\right)^{3/2} e^{-tm^2}\sum_{j=0,1/2,1,\dots}^{\infty} B_j t^j, \quad t\to 0^+, \tag{11}$$

by means of

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1}K(t), \tag{12}$$

where we have to take into account that, in the presence of a boundary, coefficients with half integer numbers are nonzero. As is well known and can be easily found from the above formulas, in order to obtain the contribution of the first five coefficients B_i $(i=0,\frac{1}{2},1,\frac{3}{2},2)$, one cannot simply put $s=-\frac{1}{2}$ under the sign of the integral in $\zeta(s)$, respectively, $s=0$ in the energy, because this leads to a divergent integral coming from the lower integration bound. An appropriate analytical continuation is required. We will call the corresponding contribution to the energy $E_{0(\mathrm{div})}$. It reads

$$\begin{aligned}E_{0(\mathrm{div})}=&-\frac{m^4}{64\pi^2}B_0\left[\frac{1}{s}-\frac{1}{2}+\ln\left(\frac{4\mu^2}{m^2}\right)\right]-\frac{m^3}{24\pi^{3/2}}B_{1/2}\\&+\frac{m^2}{32\pi^2}B_1\left[\frac{1}{s}-1+\ln\left(\frac{4\mu^2}{m^2}\right)\right]+\frac{m}{16\pi^{3/2}}B_{3/2}\\&-\frac{1}{32\pi^2}B_2\left[\frac{1}{s}-2+\ln\left(\frac{4\mu^2}{m^2}\right)\right].\end{aligned} \tag{13}$$

A remark is here in order. The zeta functional regularization used leaves the contributions of the coefficients with half integer index finite in the limit $s\to 0$. This is a specific feature of the regularization, often much appreciated. However, in other regularizations, as for example the proper time cutoff [22], these contributions are divergent. Equation (13) then contains all contributions of the regularized zero point energy which will experience renormalization.

The heat kernel coefficients needed in Eq. (13) are well known (see for instance [36]). For the interior region,

$$B_0^{(\mathrm{int})}=\tfrac{4}{3}\pi R^3,\quad B_{1/2}^{(\mathrm{int})}=-2\pi^{3/2}R^2,\quad B_1^{(\mathrm{int})}=\tfrac{8}{3}\pi R,$$

$$B_{3/2}^{(\mathrm{int})}=-\tfrac{1}{6}\pi^{3/2},\quad B_2^{(\mathrm{int})}=\frac{16}{315}\frac{\pi}{R}.$$

In the exterior region,

$$B_i^{(\mathrm{ext})}=B_i^{(\mathrm{int})},\quad i=\tfrac{1}{2},\tfrac{3}{2},\dots, \tag{14}$$

$$B_i^{(\mathrm{ext})}=-B_i^{(\mathrm{int})},\quad i=0,1,2,\dots . \tag{15}$$

In order to perform the renormalization we choose as general scheme the following: all contributions of the heat kernel coefficients which can lead to divergences in some regularization have to be subtracted by means of a renormalization of the corresponding parameters in the classical part of the system. Thus, we have in each of the first two models five divergent contributions. In the third model we note that, in accordance with Eq. (9),

$$E_{0(\mathrm{total})}^{(\mathrm{div})}=E_{(\mathrm{div})}^{(\mathrm{int})}+E_{(\mathrm{div})}^{(\mathrm{ext})},$$

and owing to the known cancellation of divergent contributions, which is in fact due to Eq. (15), only two of them remain.

As physical system we will consider the classical part and also the ground state energy of the quantum field together, and write, for the complete energy,

$$E=E_{(\mathrm{class})}+E_0. \tag{16}$$

In this context the renormalization can be achieved by shifting the parameters in $E_{(\mathrm{class})}$ by an amount which cancels the divergent contributions and removes completely the contribution of the corresponding heat kernel coefficients. In the first two models, we have

$$\begin{aligned}&p\to p\mp\frac{m^4}{64\pi^2}\left[\frac{1}{s}-\frac{1}{2}+\ln\left(\frac{4\mu^2}{m^2}\right)\right],\quad \sigma\to\sigma+\frac{m^3}{48\pi},\\&F\to F\pm\frac{m^2}{12\pi}\left[\frac{1}{s}-1+\ln\left(\frac{4\mu^2}{m^2}\right)\right],\quad k\to k-\frac{m}{96},\\&h\to h\pm\frac{1}{630\pi}\left[\frac{1}{s}-2+\ln\left(\frac{4\mu^2}{m^2}\right)\right],\end{aligned} \tag{17}$$

where the upper sign corresponds to the first model and the lower sign to the second. In the third model there are only two contributions, which are divergent in some regularizations. The renormalization reads

$$\sigma\to\sigma+\frac{m^3}{24\pi},\quad k\to k-\frac{m}{48}. \tag{18}$$

Within the method of zeta function regularization this is a finite renormalization.

After the subtraction of these contributions from E_0 we denote it by $E_0^{(\mathrm{ren})}=E_0-E_0^{(\mathrm{div})}$ and the complete energy becomes

$$E=E_{(\mathrm{class})}+E_0^{(\mathrm{ren})}. \tag{19}$$

In our renormalization scheme, we have defined a unique renormalized ground state energy $E_0^{(\mathrm{ren})}$. Often, the renormalization arbitrariness is removed by imposing some normalization conditions. In this case a natural candidate would be the requirement, that $E_0^{(\mathrm{ren})}$ vanishes for $R\rightarrow\infty$. This is certainly fulfilled in our case. However, such a requirement does not fix completely the renormalization in the first two models, since a finite renormalization of h is still possible.

With respect to this renormalization there is a qualitative difference between the first two and the third model. In the latter one some divergences have been canceled when adding up the contributions from the interior and exterior regions. The corresponding terms in the classical energy do not need to be renormalized. In general, as mentioned in [22] and much earlier in connection with the renormalization of QED, those contributions which need renormalization are not of quantum nature, but have to be present in the classical part of the system and are to be determined from outside (like the electron mass and charge in QED) or by the dynamics of the classical part. For example, in the bag model one has to look for a minimum of the complete energy while varying these parameters. The contributions which do not need renormalization—like the one resulting from B_0, B_1, or B_2 in the third model—may be absent in the classical part and can be considered as pure quantum contributions. In this sense the third model contains only two classical parameters (σ and k).

Having in mind that only the complete energy has physical meaning, a change of the normalization conditions—respectively, a finite renormalization—would be equivalent to a change of the classical parameters and should not influence issues like that of finding a minimum of the complete energy by varying the parameters of the classical part.

Special remarks are in order for the case of a massless quantum field. There, only contributions from B_2 remain (one performs the limit $m\rightarrow 0$ in the regularized expressions, the B_i's are proportional to m^{4-2i}). In the third model these contributions are finite and can be considered as pure quantum corrections. They yield, together with the finite contributions (cf. the quantity N in the next section), the known $1/R$ contributions to the Casimir energy for a sphere and a massless field with various boundary conditions. However, in the models (i) and (ii) the B_2 contribution is divergent and the corresponding $1/R$ term in the energy must be considered as a classical contribution. Thus, in these cases the ground state energy for a massless field can be removed by a finite renormalization and the energy of the system remains formally the classical one. In this sense there is no Casimir effect. The same is true in the presence of a thick spherical shell (interior and exterior regions are separated by a finite distance, with no quantum field), because here no cancellation between interior and exterior modes occurs.

Similar remarks hold not only for a spherical surface, but also for an arbitrarily shaped one. For the infinitely thin surface a cancellation between interior and exterior modes does occur, while if it has a finite thickness this is not true anymore.

III. CALCULATION OF THE GROUND STATE ENERGY

First we consider the interior case. As it is easily seen from Eq. (4), the task that remains for the evaluation of the zero-point energy is to perform a convenient analytical treatment of the zeta function (8). A precise way to obtain an analytical continuation of $\zeta(s)$ to $s=-1/2$ has been described in [25,26] in detail, what allows us to be brief here. We may write the zeta function for the interior space in the form

$$\zeta_{(\mathrm{int})}(s)=N_{(\mathrm{int})}(s)+\sum_{i=-1}^{3} A_i(s), \qquad (20)$$

where the A_i's are the contributions of the first five terms of the uniform asymptotic expansion of the modified Bessel functions as $\nu\rightarrow\infty$ and $k\rightarrow\infty$, with ν/k fixed. It is sufficient to subtract these five contributions in order to absorb all possible divergent contributions. A higher number of subtractions is possible in order to speed up the convergence of the remaining numerical expressions. We have called N the zeta function where all these asymptotic terms have been subtracted:

$$\begin{aligned}N_{(\mathrm{int})}(s)=&\frac{\sin(\pi s)}{\pi}\sum_{l=0}^{\infty}(2l+1)\int_{mR/\nu}^{\infty}dx\left[\left(\frac{x\nu}{R}\right)^2-m^2\right]^{-s}\\&\times\frac{\partial}{\partial x}\left[\ln I_\nu(\nu x)-\ln\left(\frac{e^{\nu\eta}}{\sqrt{2\pi\nu}(1+x^2)^{1/4}}\right)\right.\\&\left.-\frac{1}{\nu}D_1(t)-\frac{1}{\nu^2}D_2(t)-\frac{1}{\nu^3}D_3(t)\right], \qquad (21)\end{aligned}$$

being $t=1/\sqrt{1+x^2}$ and $\eta=\sqrt{1+x^2}+\ln[x/(1+\sqrt{1+x^2})]$. In this formula the parameter s can be put equal to $-1/2$ under the integration and summation signs. The evaluation of $N(1/2)$ is the remaining numerical task. The polynomials D_i are

$$D_1(t)=\sum_{a=0}^{1} x_{1,a}t^{1+2a}\equiv\frac{1}{8}t-\frac{5}{24}t^3,$$

$$D_2(t)=\sum_{a=0}^{2} x_{2,a}t^{2+2a}\equiv\frac{1}{16}t^2-\frac{3}{8}t^4+\frac{5}{16}t^6, \qquad (22)$$

$$D_3(t)=\sum_{a=0}^{3} x_{3,a}t^{3+2a}\equiv\frac{25}{384}t^3-\frac{531}{640}t^5+\frac{221}{128}t^7-\frac{1105}{1152}t^9,$$

and, in terms of their coefficients, $x_{i,a}$, the functions $A_i(s)$ are given by

$$\begin{aligned}A_{-1}(s)=&\frac{R^{2s}}{2\sqrt{\pi}\Gamma(s)}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}(mR)^{2j}\frac{\Gamma[j+s-(1/2)]}{s+j}\\&\times\zeta_H(2j+2s-2;1/2),\end{aligned}$$

$$\begin{aligned}A_0(s)=&-\frac{R^{2s}}{2\Gamma(s)}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}(mR)^{2j}\Gamma(s+j)\\&\times\zeta_H(2j+2s-1;1/2), \qquad (23)\end{aligned}$$

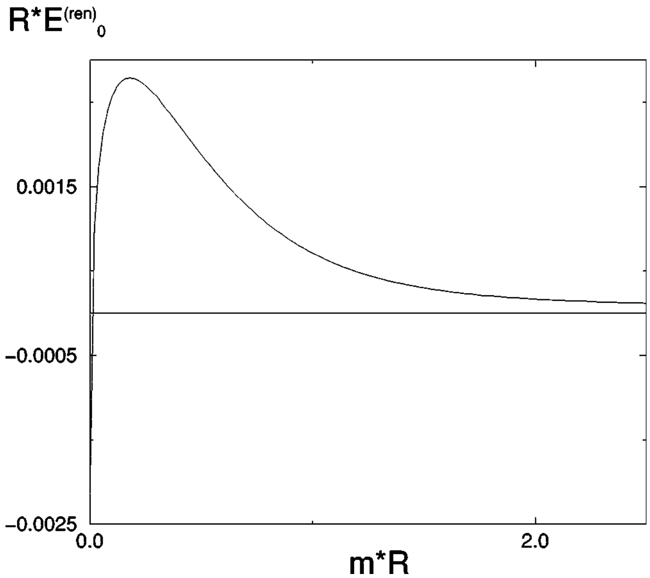

FIG. 1. Plot of the renormalized vacuum energy $E_0^{\rm ren}$ measured in units of the inverse of the radius.

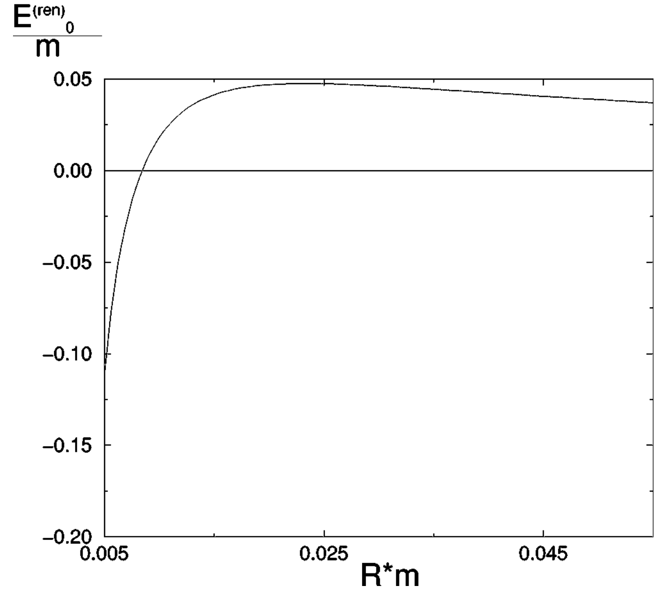

FIG. 2. Plot of the renormalized vacuum energy $E_0^{\rm ren}$ measured in units of the mass. The plot has been restricted to a domain around the maximum value.

$$A_i(s)=-\frac{2R^{2s}}{\Gamma(s)}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}(mR)^{2j}\zeta_H(-1+i+2j+2s;1/2)\sum_{a=0}^{i}x_{i,a}\frac{\Gamma[s+a+j+(i/2)]}{\Gamma[a+(i/2)]}.$$

It is easy to see that the above series are convergent for $|mR|\leqslant 1$. Alternative representations valid for arbitrary values of mR are derived in the Appendix. Using the above formulas, or alternatively, the ones given in the Appendix, we can perform the renormalization and calculate the renormalized ground state energy numerically. The result for the ground state energy is shown in Fig. 1 for $R=1$ as a function of m. For very small values of the argument, mR, the function goes to a finite, negative value (remember that we are plotting $RE_0^{\rm (ren)}$, not $E_0^{\rm (ren)}$), whereas for large values of the argument the function goes to zero.

The dependence of $E_0^{\rm ren}$ on the radius for fixed mass is depicted in Fig. 2. This plot also exhibits a maximum for $mR\simeq 0.023$, and here we have restricted the domain to a region around it.

The zero-point energy in the exterior of the spherical surface can be calculated in a much similar manner. Indeed, only a few changes are necessary. Subtracting the Minkowski space zeta function from the zeta function associated with the field outside the surface, the starting point here reads

$$\zeta_{\rm ext}(s)=N_{\rm ext}(s)+\sum_{i=-1}^{3}(-1)^i A_i(s), \qquad (24)$$

with

$$N_{\rm ext}(s)=\frac{\sin(\pi s)}{\pi}\sum_{l=0}^{\infty}(2l+1)\int_{mR/\nu}^{\infty}dx\left[\left(\frac{x\nu}{R}\right)^2-m^2\right]^{-s}\times\frac{\partial}{\partial x}\left[\ln K_\nu(\nu x)-\ln\left(\frac{\sqrt{\pi}e^{-\nu\eta}}{\sqrt{2\nu}(1+x^2)^{1/4}}\right)+\frac{1}{\nu}D_1(t)-\frac{1}{\nu^2}D_2(t)+\frac{1}{\nu^3}D_3(t)\right]. \qquad (25)$$

As is clear, one just needs to substitute the Bessel function K_ν for I_ν. The asymptotic contributions, as compared with those for the interior space, get an alternating sign coming from the asymptotics of the Bessel function K_ν—which exhibit this sign when compared with those of the function I_ν (see Ref. [37]). By construction, $N_{\rm ext}(s)$ is finite at $s=-1/2$. The results for $A_i(s=-1/2)$ are given in the Appendix. They are the same as in the previous case. Again, the renormalized ground state energy can be calculated. The result is shown in Fig. 3. It is apparent that the slope is always negative and that the plot always gives positive values. It is clear that, had we plotted the same quantity in units of the mass, a curve with both these properties would have been obtained too. In particular, it would not develop a maximum as the one observed for the interior case.

In the case of the third model, i.e., for the quantum field extending to both regions altogether, we just have to add the two results above. As shown in Fig. 4, there is an interval where the slope is positive. This would seem to leave open the possibility, that a plot in units of the mass could exhibit a maximum. We have carefully investigated this possibility, but the answer is negative. In other words, such alternative plot is always monotonically decreasing.

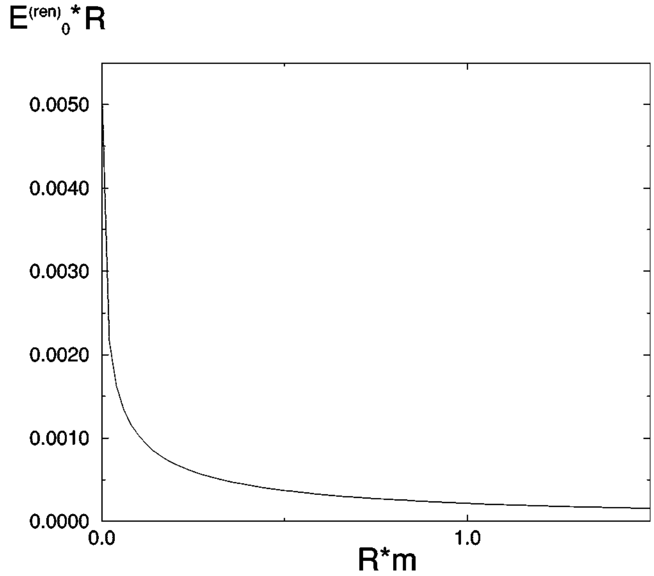

FIG. 3. Plot of the renormalized vacuum energy in units of the inverse of the radius.

IV. CONCLUSIONS

In this paper we have developed a systematic approach to the calculation of the Casimir energy of a massive field obeying Dirichlet boundary conditions on a spherical surface. The models corresponding to the quantum field confined to the interior and to the exterior regions of the surface, respectively, have been discussed separately, and the differences in the renormalization of these models with respect to the case where the field is present in the whole space have been investigated in detail.

Figures 1–4 show the quantum contribution to the renormalized ground state energy. This quantum ground state energy of the interior region exhibits a maximum for variable radius and fixed mass, as is clear from Fig. 2. Thus, we may say that if the surface is small enough, the quantum part of the vacuum energy induces an attractive force. We would like to remark that the appearance of this attractive force is sensitive to the normalization conditions chosen. For instance, it can be removed by performing a suitable finite renormalization of h [cf. formula (17)]. Here the consideration of a massive field allows for a physical justification of the normalization prescription chosen. The point is that the complete subtraction of the contributions resulting from the corresponding heat kernel coefficients [as it is done by means of (17)] automatically removes all contributions which do not vanish in the infinite mass limit. But the latter is a physically reasonable requirement—a field with infinite mass should not produce quantum effects.

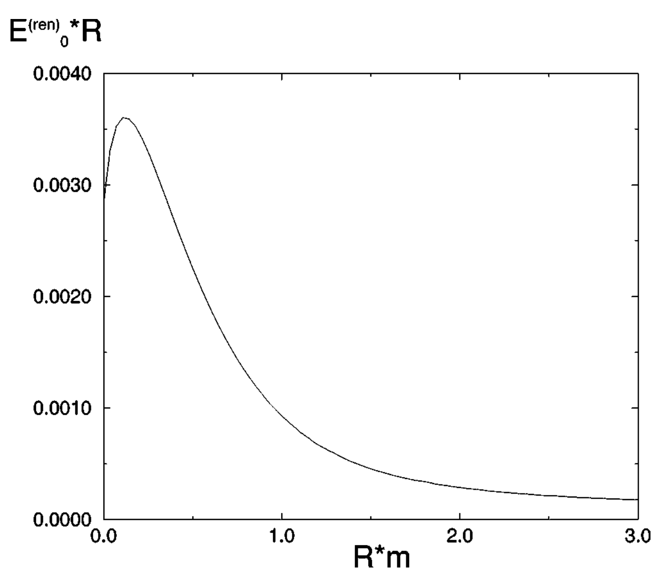

FIG. 4. The renormalized vacuum energy represented in units of the inverse of the radius.

The appearance of attractive forces in this kind of situation is not new, however, and has been found also when dealing with spinors [19] and with the electromagnetic field on a dielectric cylinder [38].

Our analysis presents the first complete treatment of the massive Casimir effect in the presence of nonplanar boundaries and it shows specific properties not encountered before. If one considers planar boundaries, the only case completely solved up to now (see, i.e., [8]), for the range $mR \gg 1$ the Casimir energy is exponentially small and thus of very short range. This is due to the vanishing extrinsic curvature of planar boundaries.

In our case the above remarks do not hold anymore. Using Eq. (11) in Eq. (12) one easily obtains the asymptotic series for $m \to \infty$ of $\zeta(s)$ in the form

$$\zeta(s) = \frac{1}{(4\pi)^{3/2}\Gamma(s)} \sum_{j=0,1/2,1,\ldots}^{\infty} \Gamma(s+j-3/2) B_j m^{3-2j-2s}. \tag{26}$$

The contributions coming from $j=0,1/2,1,3/2,2$ were involved in the renormalization procedure (see Sec. II), the remaining finite pieces behave like $1/m$. These pieces are present due to the nonvanishing of the higher coefficients B_j, $j=5/2,3,\ldots$, which is a result of the nonvanishing extrinsic curvature in our example. For that reason, in general one cannot say that the Casimir force for the massive case is of very short range or that the contributions due to the mass are negligibly small compared to the massless case. These comments may all be realized in Figs. 1 to 4 of our paper.

A remark must be added. Having in mind a Greens function treatment of the considered problem and methods like the multireflection expansion of the Greens function, one would expect that the Casimir energy is exponentially small for large masses. How this is related to the global calculation presented here should be clarified by an investigation of the local energy density.

Robin boundary conditions can be treated in complete analogy, as has been described in detail in Refs. [26–28]. Also the interior and exterior regions can be considered separately, and adding up the contributions coming from each region the same cancellation of divergences appear. The ground state energy of the electromagnetic field subject to superconductor boundary conditions (i.e., vanishing normal component of the magnetic field and vanishing tangential component of the electric field) is the sum of the ground state energy of two scalar fields satisfying, respectively, Dirichlet boundary conditions and Robin boundary conditions (TE and TM modes). The $B_{1/2}$ heat kernel coefficient has opposite sign for Dirichlet and Robin boundary conditions, what leads to a partial cancellation of divergences between the TE and TM modes (in the massive case). Doing the same kind of calculation as the one presented here and taking the massless limit, previous results are reobtained [4,5], what provides a further check of the procedure.

Along the same lines, it would be interesting to perform the calculations for higher spin fields and to apply the results to realistic physical models, as the MIT bag model for instance. Furthermore, in complete analogy, the case of two concentric spherical shells can be treated with our method too.

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APPENDIX: REPRESENTATIONS FOR THE ASYMPTOTIC CONTRIBUTIONS INSIDE THE SURFACE

In this appendix we derive explicit representations of the $A_i(s)$, $i=-1,\ldots,3$ [see Eq. (23)], which are valid for arbitrary mR. Let us start with $A_{-1}(s)$, which is actually the most difficult piece to treat. Instead of Eq. (23), one may use the representation [25,26]

$$A_{-1}(s)=2\frac{\sin(\pi s)}{\pi}\sum_{l=0}^{\infty}\nu^2\int_{mR/\nu}^{\infty}\left[\left(\frac{x\nu}{R}\right)^2-m^2\right]^{-s}\frac{\sqrt{1+x^2}-1}{x}, \tag{A1}$$

of which we need the analytical continuation to $s=-1/2$. With the substitution $t=(x\nu/R)^2-m^2$, this expression results in the following one:

$$\begin{aligned}A_{-1}(s)&=\frac{\sin(\pi s)}{\pi}\sum_{l=0}^{\infty}\nu\int_0^{\infty}dt\frac{t^{-s}}{t+m^2}\{\sqrt{\nu^2+R^2(t+m^2)}-\nu\}\\&=-\frac{1}{2\sqrt{\pi}}\frac{\sin(\pi s)}{\pi}\sum_{l=0}^{\infty}\nu\int_0^{\infty}dtt^{-s}\int d\alpha e^{-\alpha(t+m^2)}\\&\quad\times\int_0^{\infty}d\beta\beta^{-3/2}\{e^{-\beta(\nu^2+R^2[t+m^2])}-e^{-\beta\nu^2}\},\end{aligned} \tag{A2}$$

where the Mellin integral representation for the single factors has been used. As we see, the β integral is well defined. Introducing a regularization parameter δ, $A_{-1}(s)$ can then be written as

$$A_{-1}(s)=\lim_{\delta\to 0}[A_{-1}^1(s,\delta)+A_{-1}^2(s,\delta)], \tag{A3}$$

with

$$\begin{aligned}A_{-1}^1(s,\delta)=&-\frac{1}{2\sqrt{\pi}}\frac{\sin(\pi s)}{\pi}\sum_{l=0}^{\infty}\nu\int d\alpha e^{-\alpha m^2}\\&\times\int_0^{\infty}d\beta\beta^{-3/2+\delta}e^{-\beta(\nu^2+R^2m^2)}\\&\times\int_0^{\infty}dtt^{-s}e^{-t(\alpha+\beta R^2)}\end{aligned}$$

and

$$\begin{aligned}A_{-1}^2(s,\delta)=&\frac{1}{2\sqrt{\pi}}\Gamma(1-s)\frac{\sin(\pi s)}{\pi}\sum_{l=0}^{\infty}\nu\int d\alpha e^{-\alpha m^2}\alpha^{s-1}\\&\times\int_0^{\infty}d\beta\beta^{-3/2+\delta}e^{-\beta\nu^2}.\end{aligned}$$

Let us proceed with the remaining pieces. In $A_{-1}^1(s,\delta)$ one of the integrations can be done, yielding

$$\begin{aligned}A_{-1}^1(s,\delta)=&-\frac{R^{1-2\delta}}{2\sqrt{\pi}\Gamma(s)}\Gamma(s+\delta-1/2)\\&\times\sum_{l=0}^{\infty}\nu\int_0^{\infty}dyy^{\delta-3/2}\left[m^2+y\left(\frac{\nu}{R}\right)^2\right]^{1/2-s-\delta}.\end{aligned} \tag{A4}$$

For $A_{-1}^2(s,\delta)$, one gets

$$\begin{aligned}A_{-1}^2(s,\delta)&=\frac{m^{-2s}}{2\sqrt{\pi}}\Gamma(\delta-1/2)\sum_{l=0}^{\infty}\nu^{2-2\delta}\\&=\frac{R^{1-2\delta}}{2\sqrt{\pi}\Gamma(s)}\Gamma(s+\delta-1/2)\\&\quad\times\sum_{l=0}^{\infty}\nu\int_0^{\infty}dxx^{s-1}\left[m^2x+\left(\frac{\nu}{R}\right)^2\right]^{1/2-s-\delta}.\end{aligned} \tag{A5}$$

And adding up Eqs. (A4) and (A5) yields

$$\begin{aligned}A_{-1}(s)=&\frac{R}{2\sqrt{\pi}\Gamma(s)}\Gamma(s-1/2)\\&\times\sum_{l=0}^{\infty}\nu\int_0^{1}dxx^{s-1}\left[m^2x+\left(\frac{\nu}{R}\right)^2\right]^{1/2-s},\end{aligned} \tag{A6}$$

a form suited for the treatment of the angular momentum sum.

To perform the summation over l, we will use

$$\sum_{\nu=1/2}^{\infty}f(\nu)=\int_0^{\infty}d\nu f(\nu)-i\int_0^{\infty}d\nu\frac{f(i\nu+\epsilon)-f(-i\nu+\epsilon)}{1+e^{2\pi\nu}}, \tag{A7}$$

where $\epsilon\to 0$ is understood and appropriate analytic properties of the function $f(\nu)$ are assumed. When expanding the function $f(\nu)$ in a Taylor series, one arrives at the well known Euler-Maclaurin summation formula (a thorough treatment of the Euler-Maclaurin summation formula can be found in Ref. [39]).

In order to get the Casimir energy we will need only the expansion of $A_{-1}(s)$ around $s=-1/2$. Using Eq. (A7) one finds, after a lengthy calculation,

$$
\begin{aligned}
A_{-1}(s)=&\left(\frac{1}{s+1/2}+\ln R^2\right)\left(\frac{7}{1920\pi R}+\frac{m^2R}{48\pi}-\frac{m^4R^3}{24\pi}\right)\\
&+\ln 4\left(\frac{7}{1920\pi R}+\frac{m^2R}{48\pi}-\frac{m^4R^3}{24\pi}\right)+\frac{7}{1920\pi R}\\
&-\frac{m^2R}{48\pi}+\frac{m^4R^3}{48\pi}[1+4\ln(mR)]\\
&-\frac{1}{\pi R}\int_0^\infty d\nu\frac{\nu}{1+e^{2\pi\nu}}(\nu^2-m^2R^2)\ln|\nu^2-m^2R^2|\\
&-\frac{2m^2R}{\pi}\int_0^\infty d\nu\frac{\nu}{1+e^{2\pi\nu}}\Bigg(\ln|\nu^2-m^2R^2|\\
&+\frac{\nu}{mR}\ln\left|\frac{mR+\nu}{mR-\nu}\right|\Bigg).
\end{aligned}
\tag{A8}
$$

All other $A_i(s)$ can be treated in a much easier way. As a starting point for $A_0(s)$ we choose [25,26]

$$
A_0(s)=-\frac{m^{-2s}}{2}\sum_{l=0}^{\infty}\nu\left[1+\left(\frac{\nu}{mR}\right)^2\right]^{-2s}. \tag{A9}
$$

Using Eq. (A7), this yields immediately

$$
A_0(s)=\frac{1}{6}R^2m^3-m\int_0^{mR}d\nu\frac{\nu}{1+e^{2\pi\nu}}\sqrt{1-\left(\frac{\nu}{mR}\right)^2}. \tag{A10}
$$

For the remaining A_i's we proceed in a different way. Let us explain the method using one of the contributions of $A_1(s)$, say

$$
\begin{aligned}
\sum_{l=0}^{\infty}&\nu^{-2s}\left[1+\left(\frac{\nu}{mR}\right)^2\right]^{-s-1/2}\\
&=\zeta_H(2s;1/2)-(s+1/2)(mR)^2\zeta_H(2s+2;1/2)\\
&+\sum_{l=0}^{\infty}\nu^{-2s}\Bigg\{\left[1+\left(\frac{\nu}{mR}\right)^2\right]^{-s-1/2}-1\\
&+(s+1/2)\left(\frac{mR}{\nu}\right)^2\Bigg\}.
\end{aligned}
$$

This provides the immediate continuation of the sums to $s=-1/2$. In fact, the infinite sum is convergent and in the Hurwitz zeta function the analytical continuation to $s=-1/2$ is easily performed. All pieces in A_i, $i=1,2,3$ have a similar aspect and may be treated in the same way. Thus (we just write down the results)

$$
\begin{aligned}
A_1(s)=&\left(\frac{1}{s+1/2}+\ln R^2\right)\left(\frac{7}{48\pi}m^2R+\frac{1}{192\pi R}\right)\\
&-\frac{1}{72\pi R}[2+9\zeta_R'(-1)]\\
&+\frac{1}{24\pi}m^2R(-2+7\gamma+21\ln 2)\\
&+\frac{1}{8\pi R}\sum_{l=0}^{\infty}\nu\left\{\left(\frac{mR}{\nu}\right)^2-\ln\left[1+\left(\frac{mR}{\nu}\right)^2\right]\right\}\\
&-\frac{5}{12\pi R}\sum_{l=0}^{\infty}\nu\left\{\left[1+\left(\frac{mR}{\nu}\right)^2\right]^{-1}-1+\left(\frac{mR}{\nu}\right)^2\right\},
\end{aligned}
$$

$$
\begin{aligned}
A_2(s)=&\frac{1}{16R}\sum_{l=0}^{\infty}\left\{\left[1+\left(\frac{mR}{\nu}\right)^2\right]^{-1/2}-1\right\}\\
&-\frac{3}{16R}\sum_{l=0}^{\infty}\left\{\left[1+\left(\frac{mR}{\nu}\right)^2\right]^{-3/2}-1\right\}\\
&+\frac{15}{128R}\sum_{l=0}^{\infty}\left\{\left[1+\left(\frac{mR}{\nu}\right)^2\right]^{-5/2}-1\right\},
\end{aligned}
\tag{A11}
$$

$$
\begin{aligned}
A_3(s)=&-\frac{229}{40320\pi R}\left(\frac{1}{s+1/2}+\ln R^2\right)\\
&+\frac{2152-687\gamma-2061\ln 2}{60480\pi R}+\frac{25}{192\pi R}\\
&\times\sum_{l=0}^{\infty}\frac{1}{\nu}\left\{\left[1+\left(\frac{mR}{\nu}\right)^2\right]^{-1}-1\right\}\\
&-\frac{177}{160\pi R}\sum_{l=0}^{\infty}\frac{1}{\nu}\left\{\left[1+\left(\frac{mR}{\nu}\right)^2\right]^{-2}-1\right\}\\
&+\frac{221}{120\pi R}\sum_{l=0}^{\infty}\frac{1}{\nu}\left\{\left[1+\left(\frac{mR}{\nu}\right)^2\right]^{-3}-1\right\}\\
&-\frac{221}{252\pi R}\sum_{l=0}^{\infty}\frac{1}{\nu}\left\{\left[1+\left(\frac{mR}{\nu}\right)^2\right]^{-4}-1\right\}.
\end{aligned}
$$

This completes the list of expressions necessary for the analysis of the massive scalar field inside the surface with Dirichlet boundary conditions.

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