Null dust in canonical gravity

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(Received 18 April 1997)

We present the Lagrangian and Hamiltonian framework which incorporates null dust as a source into canonical gravity. Null dust is a generalized Lagrangian system which is described by six Clebsch potentials of its four-velocity Pfaff form. The Dirac–Arnowitt-Deser-Misner decomposition splits these into three canonical coordinates (the comoving coordinates of the dust) and their conjugate momenta (appropriate projections of four-velocity). Unlike the ordinary dust of massive particles, null dust therefore has three rather than four degrees of freedom per space point. These are evolved by a Hamiltonian which is a linear combination of energy and momentum densities of the dust. The energy density is the norm of the momentum density with respect to the spatial metric. The coupling to geometry is achieved by adding these densities to the gravitational super-Hamiltonian and supermomentum. This leads to appropriate Hamiltonian and momentum constraints in the phase space of the system. The constraints can be rewritten in two alternative forms in which they generate a true Lie algebra. The Dirac constraint quantization of the system is formally accomplished by imposing the new constraints as quantum operator restrictions on state functionals. We compare the canonical schemes for null and ordinary dust and emphasize their differences. [S0556-2821(97)02420-X]

PACS number(s): 04.60.Ds, 04.20.Cv, 04.20.Fy

I. INTRODUCTION

Null dust has been widely used as a simple matter source both in classical and semiclassical gravity. Its equations of motion follow directly from the conservation of the energy-momentum tensor. However, the inclusion of null dust as a source into canonical gravity requires careful identification of its own dynamical degrees of freedom. For this purpose, one needs to construct a spacetime action depending on appropriate Eulerian variables and bring it into canonical form by the Dirac-ADM (Arnowitt-Deser-Misner) procedure. The coupling to gravity, like that of other nonderivative systems, is then entirely straightforward. The ordinary dust of massive particles was treated in this manner by Brown and Kuchař [1]. Our goal is to develop a similar formalism for null dust.

The main application which we have in mind is minisuperspace and midisuperspace quantization of canonical models which include null dust as a source. The specific models based on null dust are both numerous and simple. After the discovery of Vaidya's ''radiating Schwarzschild metric'' [2], there were found many other, more general exact solutions of Einstein's equations with null dust as a matter source. Above all, such models have recently been used to clarify the formation of naked singularities during a spherical gravitational collapse, to describe mass inflation inside black holes, and to model the formation and Hawking evaporation of black holes. We briefly review these topics in Appendix B. Our formalism is designed for studying such issues in quantum rather than in classical or semiclassical contexts.

Null dust is intimately connected with the behavior of zero-rest-mass fields in geometrical optics limit. The energy-momentum tensor of such fields takes in that limit the form of the energy-momentum tensor of null dust. One can then reinterpret some exact solutions of Einstein's equations with null dust as spacetimes produced by zero-rest-mass (in particular, electromagnetic) fields. Careful studies of the high-frequency limit of the gravitational radiation itself revealed that it also can be described by the energy-momentum tensor of null dust. Moreover, which is especially relevant for the present paper, such a connection can be established at the level of a variational principle. All of this indicates that null dust is much more closely related to fundamental fields than ordinary dust formed by phenomenological massive particles. We explain some of these connections in Appendix A.

We start our exposition by reviewing how the dynamics of incoherent dust follows from the conservation law of the energy-momentum tensor (Sec. II). This enables us to pinpoint at the very beginning the main difference between ordinary dust and null dust: The normalization of timelike four-velocity selects its parametrization by proper time, the null normalization of lightlike four-velocity leaves its parametrization arbitrary. This is of paramount importance both for the Lagrangian and Hamiltonian descriptions of null dust.

Since null geodesics are somewhat less intuitive than timelike geodesics, we briefly summarize the basic properties of null congruences in Sec. III. We explain how to obtain an affine parametrization of such congruences, but stress its essential ambiguity which prevents the unique separation of mass distribution of null dust particles from their four-velocity. When it comes to producing the gravitational field, the mass distribution can simply be reabsorbed into four-velocity, which is the reason why it does not naturally occur

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as a separate variable either in the Lagrangian or Hamiltonian frameworks. We formulate a spacetime variational principle from which the null geodesic equations of motion follow in Sec. IV. The variational principle for null dust is quite similar to the variational principle for ordinary dust given by Brown and Kuchař [1], but there are several characteristic differences. The most important one was already mentioned: Null world lines have no natural parametrization and hence the null velocity appears in the variational principle as the Pfaff form of six scalars (the Clebsch potentials) rather than seven scalars which characterize the timelike velocity. Consequences of this distinction can be traced throughout the whole formalism. In Appendix C we illustrate on explicit examples the decomposition of null covector fields into Clebsch potentials which is a prerequisite of our variational principle. In Sec. V we show that any solution of the Euler equations of the variational principle provides enough building blocks to reconstruct an affinely parametrized four-velocity k^α and the associated mass distribution M. We also discuss other special parametrizations. In Sec. VI, we cast our covariant spacetime action into Hamiltonian form by following the Dirac-ADM algorithm. The details of this process are substantially different from the steps which need to be taken for ordinary dust. We express the energy and momentum densities of null dust in terms of appropriate canonical variables. The energy density turns out to be the norm of the momentum density with respect to the spatial metric. It transpires that null dust has only three degrees of freedom per space point, one less than ordinary dust which has four. The missing degree of freedom is a privileged scalar parameter (like proper time) along lightlike geodesics. The missing canonically conjugate momentum is the mass distribution which has been reabsorbed into the four-velocity form. We conclude this section by writing down the standard Hamiltonian and momentum constraints for geometry coupled to null dust. In Sec. VII, we rewrite these constraints in two alternative forms in which they generate a true Lie algebra. In this process, the Hamiltonian constraint is replaced by alternative constraints which contain only geometric variables. This feature of the constraint is related to a Rainich-type ''already unified theory'' for geometry coupled to null dust. In Sec. VIII, we show how one can formally impose the new form of constraints as quantum operator restrictions on state functionals. The outcome of this procedure is a single functional differential equation for physical state functionals $\mathbf{\Psi}[\mathbf{g}]$ which depend solely on the spatial metric $\mathbf{g}$ in the dust frame. In Sec. IX, we compare the canonical formalism and the ensuing quantum theory for null dust with those for ordinary dust and emphasize their differences.

Our conventions follow those of Misner, Thorne, and Wheeler [3], except for our choice of units which are such that $16\pi G=1=c$.

II. DUST AS A SOURCE OF GRAVITY

Incoherent dust is one of the simplest phenomenological sources of gravity in general relativity. Its energy-momentum tensor

$$T^{\alpha\beta}=MU^\alpha U^\beta, \tag{2.1}$$

$$U^\alpha U_\alpha=-1, \tag{2.2}$$

curves the spacetime according to Einstein's law of gravitation

$$G^{\alpha\beta}:=R^{\alpha\beta}-\frac{1}{2}R\gamma^{\alpha\beta}=\frac{1}{2}T^{\alpha\beta}. \tag{2.3}$$

Dust is described by the four-velocity U^α of its particles and the (rest) mass density M of their distribution. The equations of motion of the dust are entirely contained in the energy-momentum conservation law

$$\nabla_\beta T^{\alpha\beta}=0, \tag{2.4}$$

which follows from the Einstein law (2.3) through the Bianchi identities. The structure (2.1) of the energy-momentum tensor allows us to write Eq. (2.4) in the form

$$MU^\beta\nabla_\beta U^\alpha+\nabla_\beta(MU^\beta)U^\alpha=0. \tag{2.5}$$

One sees that

$$MU^\beta\nabla_\beta U^\alpha\propto U^\alpha, \tag{2.6}$$

i.e., that the dust particles move along geodesics. The normalization (2.2) of the four-velocity tells us that the particle world lines are parametrized by proper time. When one multiplies the geodesic equation by U_α, the normalization condition (2.2) implies the rest mass conservation

$$\nabla_\beta(MU^\beta)=0. \tag{2.7}$$

By using Eq. (2.7) back in Eq. (2.5) one learns that proper time is an affine parameter:

$$U^\beta\nabla_\beta U^\alpha=0. \tag{2.8}$$

These facts describe in full detail the motion of ordinary dust of massive particles.

Null dust has the same energy-momentum tensor (2.1) as ordinary dust, but its particles are assumed to follow lightlike world lines:

$$U^\alpha U_\alpha=0. \tag{2.9}$$

The energy-momentum conservation law (2.4) still implies that those world lines are geodesics, Eqs. (2.5) and (2.6). However, the null normalization (2.9) no longer enforces either the conservation law (2.7) or affine parametrization (2.8).

For ordinary dust, the decomposition of the energy-momentum tensor into the mass density M and four-velocity U_α is unique due to the timelike normalization (2.2). For null dust, the lightlike normalization (2.9) is preserved by an arbitrary scaling of U_α:

$$\overline{U^\alpha}=\Lambda U^\alpha,\quad \Lambda>0. \tag{2.10}$$

(The limitation $\Lambda>0$ is needed to preserve the future-pointing orientation of the world lines.) By simultaneously rescaling the scalar M,

$$\overline{M}=\Lambda^{-2}M, \tag{2.11}$$

one preserves the form (2.1) of the energy-momentum tensor. This shows that the decomposition of $T^{\alpha\beta}$ into M and U^α is arbitrary. In particular, by taking $\Lambda = M^{1/2}$, one can eliminate the scalar M altogether and write $T^{\alpha\beta}$ entirely in terms of a single null vector

$$l^\alpha := M^{1/2} U^\alpha \tag{2.12}$$

as

$$T^{\alpha\beta} = l^\alpha l^\beta. \tag{2.13}$$

In terms of l^α, the geodesic equation (2.5) takes the form

$$l^\beta \nabla_\beta l^\alpha + (\nabla_\beta l^\beta) l^\alpha = 0. \tag{2.14}$$

We shall see later that this choice maximally simplifies the form of the null dust action and its canonical decomposition.

III. NULL GEODESIC CONGRUENCES

In a region of spacetime, $\mathcal{M}$, which is filled by dust whose world lines do not intersect, the vector field U^α defines a line congruence $\mathcal{S}$. This congruence can be viewed as an abstract three-dimensional space, the ''dust space,'' whose points are the individual world lines. The world lines $z \in \mathcal{S}$ can be locally labeled by three parameters $z^k(z)$ which introduce a coordinate chart in $\mathcal{S}$. We shall use the indices i,j,k from the middle of the Latin alphabet to denote the components of the objects in $\mathcal{S}$; they take the values 1, 2, 3 . (A global standpoint replacing this local description is discussed in [1].)

Through each event of the region there passes one and only one world line. One can uniquely assign to each event y the labels z^k of that world line:

$$z^k = Z^k(y). \tag{3.1}$$

Our interpretation of the scalar fields $Z^k(y)$ presupposes that their values z^k constitute a good chart in $\mathcal{S}$. Therefore, the three gradients $Z^k{}_{,\alpha}$ must be three linearly independent covectors:[1]

$$U^\alpha \propto \frac{1}{3!} \delta^{\alpha\beta\gamma\delta} Z^i{}_{,\beta} Z^j{}_{,\gamma} Z^k{}_{,\delta} \delta_{ijk} \neq 0. \tag{3.2}$$

Parametrize the curves of $\mathcal{S}$ by a parameter u whose rate of change judged by the size of U^α is unity. In other words, if

$$u = U(y) \tag{3.3}$$

is the value of u on the curve of $\mathcal{S}$ which passes through the event y, it holds that

$$U^\alpha \nabla_\alpha U = 1. \tag{3.4}$$

Equations (3.2) and (3.4) imply that $Z^K = (U, Z^k)$ are four independent functions of spacetime coordinates y^α:

$$\det(Z^K{}_{,\alpha}) = \frac{1}{3!} \delta^{\alpha\beta\gamma\delta} U_{,\alpha} Z^i{}_{,\beta} Z^j{}_{,\gamma} Z^k{}_{,\delta} \delta_{ijk} \neq 0. \tag{3.5}$$

The mapping $Z\colon \mathcal{M} \to \mathbb{R}\times\mathcal{S}$ given locally by Eqs. (3.3) and (3.1) can thus be inverted into the mapping $\Upsilon\colon \mathbb{R}\times\mathcal{S} \to \mathcal{M}$ given locally by

$$y^\alpha = \Upsilon^\alpha(u, z^k). \tag{3.6}$$

Here, z^k distinguish different curves of the congruence $\mathcal{S}$ and u specifies the point on a given curve. The four vectors

$$U^\alpha = \Upsilon^\alpha{}_{,u}, \quad Z^\alpha_k = \Upsilon^\alpha{}_{,k} \tag{3.7}$$

form a basis in $T\mathcal{M}$ dual to the cobasis

$$Z^K{}_{,\alpha} = (U_{,\alpha},\ Z^k{}_{,\alpha}) \tag{3.8}$$

in $T^*\mathcal{M}$. The basis (3.7) and cobasis (3.8) satisfy the standard orthonormality and completeness relations. In particular

$$Z^\alpha_i Z^k{}_{,\alpha} = \delta^k_i. \tag{3.9}$$

So far, everything applies equally well both to timelike and null congruences. Brown and Kuchař [1] specialized the formalism to timelike congruences and applied it to a Lagrangian description of ordinary dust. In this paper, we first briefly recapitulate how to specialize the formalism to null geodesic congruences (see, e.g., [4], [5] for more details) and then use it for Lagrangian description of null dust.

A geodesic null congruence U^α must satisfy the geodesic condition (2.6) and the null condition (2.9). These conditions still hold when the vector field U^α is scaled by an arbitrary factor, Eq. (2.10). Instead of using that scaling for eliminating M from the energy-momentum tensor, Eqs. (2.11)–(2.13), one can use it for enforcing affine parametrization. In terms of l^α, the geodesic equation takes the form (2.14). Unless l^α happens to be divergence free, it is not affinely parametrized. Let us first show that there exists a positive scaling factor

$$\Lambda(y) = e^{\lambda(y)} \tag{3.10}$$

such that

$$\nabla_\beta(\Lambda^{-1} l^\beta) = 0. \tag{3.11}$$

Condition (3.11) amounts to a linear inhomogeneous equation

$$l^\beta \nabla_\beta \lambda = \nabla_\beta l^\beta \tag{3.12}$$

for λ. In the adapted coordinates u, z^k, Eq. (3.12) assumes the form

$$\frac{\partial \lambda(u,z)}{\partial u} = (\nabla_\beta l^\beta)(u,z). \tag{3.13}$$

Its general solution

$$\lambda(u,z) = \int_0^u du\ (\nabla_\beta l^\beta)(u,z) + \lambda_0(z) \tag{3.14}$$

[1]The contravariant tensor density $\delta^{\alpha\beta\gamma\delta}$ of weight 1 is the alternating symbol in $\mathcal{M}$. The covariant tensor density δ_{ijk} of weight -1 is the alternating symbol in $\mathcal{S}$. The Levi-Civita pseudotensor in $\mathcal{M}$ is denoted by $\epsilon^{\alpha\beta\gamma\delta}$.

depends on an arbitrary function $\lambda_0(z)$ of z. By writing Eqs. (2.13), (2.14), and (3.11) in terms of the new variables

$$k^\alpha := \Lambda l^\alpha \quad \text{and} \quad M := \Lambda^{-2} \tag{3.15}$$

one learns that the vector field k^α is affinely parametrized,

$$k^\beta \nabla_\beta k^\alpha = 0, \tag{3.16}$$

the mass distribution M satisfies the continuity equation

$$\nabla_\beta (M k^\beta) = 0, \tag{3.17}$$

and the energy-momentum tensor takes the form

$$T^{\alpha\beta} = M k^\alpha k^\beta. \tag{3.18}$$

The affine parameter v is a monotonically increasing function of the old parameter u:

$$v(u,z) = \int_0^u du\, \Lambda^{-1}(u,z) + v_0(z). \tag{3.19}$$

When we define a new mapping $\Upsilon^\alpha_{\mathrm{AFF}} : \mathbb{R}\times\mathcal{S}\to\mathcal{M}$ by

$$\Upsilon^\alpha_{\mathrm{AFF}}(v,z) := \Upsilon^\alpha(u(v,z),u) \tag{3.20}$$

we obtain

$$k^\alpha = \frac{\partial \Upsilon^\alpha_{\mathrm{AFF}}(v,z)}{\partial v}. \tag{3.21}$$

The affine parametrization (3.19) depends on two arbitrary functions, $\lambda_0(z)$ and $v_0(z)$. This means that along each geodesic the affine parameter is determined only up to a linear transformation

$$\overline{v} = \Lambda_0^{-1}(z) v + \overline{v}_0, \quad \Lambda_0(z) > 0\,. \tag{3.22}$$

When we change the affine parameter by Eq. (3.22), the null vector field $k^\alpha(y)$ is scaled into

$$\overline{k^\alpha}(y) = \Lambda_0(Z(y))\; k^\alpha(y). \tag{3.23}$$

The affinely parametrized null vector field $k^\alpha(y)$ is thus determined only up to an arbitrary positive multiplicative factor Λ_0 which is a function of comoving coordinates $z^k = Z^k(y)$.

A congruence of affinely parametrized null geodesics is characterized by its twist (or rotation) ω, expansion θ, and shear σ. The corresponding scalars are given by

$$\omega = \left(\frac{1}{2} \nabla_{[\alpha} k_{\beta]} \nabla^\alpha k^\beta \right)^{1/2}, \tag{3.24}$$

$$\theta = \frac{1}{2} (\nabla_\alpha k^\alpha), \tag{3.25}$$

$$|\sigma| = \left(\frac{1}{2} \nabla_{(\alpha} k_{\beta)} \nabla^\alpha k^\beta - \theta^2 \right)^{1/2}, \tag{3.26}$$

where the square brackets (parentheses) around indices denote antisymmetrization (symmetrization). The twist can also be determined from the relation

$$\omega k^\alpha = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} k_\beta \nabla_\delta k_\gamma. \tag{3.27}$$

If k_α is proportional to a gradient,

$$k_\alpha(y) = \phi(y)\, \psi_{,\alpha}(y), \tag{3.28}$$

the geodesics of $\mathcal{S}$ form null hypersurfaces $\psi = \text{const}$ to which k^α is orthogonal. A null geodesic congruence is hypersurface orthogonal if and only if it has a vanishing twist: $\omega = 0$.

Under the change (3.23) of affine parametrization, the rotation, expansion, and shear all scale by the same factor:

$$\overline{\omega} = \Lambda_0 \omega, \quad \overline{\theta} = \Lambda_0 \theta, \quad |\overline{\sigma}| = \Lambda_0 |\sigma|. \tag{3.29}$$

Also, by using Eqs. (3.10)–(3.12) and (3.15), one can reexpress them in terms of l^α and its derivatives, and of the undifferentiated scaling factor (3.10) and (3.14):

$$\omega = \Lambda \left(\frac{1}{2} \nabla_{[\alpha} l_{\beta]} \nabla^\alpha l^\beta + \frac{1}{4} (\nabla_\alpha l^\alpha)^2 \right)^{1/2}, \tag{3.30}$$

$$\theta = \Lambda (\nabla_\alpha l^\alpha), \tag{3.31}$$

$$|\sigma| = \Lambda \left(\frac{1}{2} \nabla_{(\alpha} l_{\beta)} \nabla^\alpha l^\beta - \frac{5}{4} (\nabla_\alpha l^\alpha)^2 \right)^{1/2}. \tag{3.32}$$

The scalars (3.30)–(3.32) allow us to introduce other special parametrizations of null geodesic congruences. Considerations about the rate of expansion of a shadow image (see, e.g., [4], [5]) lead to the concept of luminosity distance Λ. This is defined as any solution of the equation

$$\frac{1}{\Lambda} \frac{d\Lambda}{dv} = \theta, \tag{3.33}$$

where v is an affine parameter and θ (assumed to be nonvanishing) is the expansion (3.25). Equations (3.15) and (3.23) ensure that the luminosity distance is identical with the scaling factor (3.10) and (3.12). The luminosity distance played a prominent role in several classical works in radiation theory [6] and in cosmology [7]. The mass distribution M introduced by Eq. (3.15) is the inverse square $M = \Lambda^{-2}$ of the luminosity distance. The parallax distance p,

$$p = \theta^{-1}, \tag{3.34}$$

is also occasionally useful.

IV. SPACETIME ACTION AND THE EULER EQUATIONS

We describe null dust by six spacetime scalars Z^k, W_k. The interpretation of our state variables Z^k, W_k emerges from the form of the action and the resulting equations of motion. We shall see that Z^k are comoving coordinates of null dust particles. By specifying the values z^k of the scalars $Z^k(y)$, we choose a particular null geodesic of the congruence $\mathcal{S}$. The three gradients $Z^k{}_{,\alpha}$ are assumed to be three independent covectors. We shall see later that none of them can be timelike.

The four-velocity covector l_α of a lightlike particle is

given by its components W_k in the cobasis $Z^k{}_{,\alpha}$:

$$l_\alpha = W_k Z^k{}_{,\alpha}. \tag{4.1}$$

This relation expresses the one-form $l = l_\alpha dy^\alpha$ as a Pfaff form

$$l = W_k dZ^k \tag{4.2}$$

of six scalar fields Z^k and W_k. According to Pfaff's theorem [8], four scalar potentials A,B,C,D are sufficient to describe an arbitrary covector in a four-dimensional space:

$$l_\alpha = AB_{,\alpha} + CD_{,\alpha}. \tag{4.3}$$

However, the representation of l_α by six potentials W_k, Z^k is more useful because it has a clear physical interpretation [9].

The null dust action

$$S^{\mathrm{ND}}[Z^k, W_k; \gamma_{\alpha\beta}] = \int d^4y \; L^{\mathrm{ND}}(y) \tag{4.4}$$

is a functional of our six state variables, and of the metric $\gamma_{\alpha\beta}$. The Lagrangian density L^{ND} is taken in the form

$$L^{\mathrm{ND}} = -\frac{1}{2}|\gamma|^{1/2}\gamma^{\alpha\beta} l_\alpha l_\beta, \tag{4.5}$$

where l_α is an abbreviation for expression (4.1).

The equations of motion follow from the variation of the action with respect to W_k and Z^k:

$$0 = \frac{\delta S^{\mathrm{ND}}}{\delta W_k} = -|\gamma|^{-1/2} Z^k{}_{,\alpha} l^\alpha, \tag{4.6}$$

$$0 = \frac{\delta S^{\mathrm{ND}}}{\delta Z^k} = (|\gamma|^{1/2} W_k l^\alpha)_{,\alpha}. \tag{4.7}$$

By multiplying Eq. (4.6) by W_k, we learn that l^α is a null vector field:

$$l^\alpha l_\alpha = 0. \tag{4.8}$$

Equation (4.6) reasserts that Z^k are comoving coordinates. Equation (4.7) tells us that the three currents

$$J^\alpha_k = W_k l^\alpha \tag{4.9}$$

satisfy the continuity equations

$$\nabla_\alpha J^\alpha_k = \nabla_\alpha (W_k l^\alpha) = 0. \tag{4.10}$$

Because each of the three covectors $Z^k{}_{,\alpha}$ is perpendicular to the null vector l^α, none of them can be timelike. If only one of the coefficients W_k in the decomposition (4.1) of l^α does not vanish, the congruence is hypersurface orthogonal [cf. Eq. (3.28)] and thus nontwisting. The covector $Z^k{}_{,\alpha}$ is then null. In the general case of a twisting congruence, all three covectors $Z^k{}_{,\alpha}$ must be spacelike: If any $Z^k{}_{,\alpha}$, k fixed, was null in an open neighborhood $\mathcal{U}$, then $l^\alpha Z^k{}_{,\alpha} = 0$ would imply $l_\alpha \propto Z^k{}_{,\alpha}$ in $\mathcal{U}$, and the congruence would not be twisting in $\mathcal{U}$. Some covectors $Z^k{}_{,\alpha}$ can possibly become null only in lower-dimensional ($d = 0,1,2$) regions of $\mathcal{M}$. A covector $Z^k{}_{,\alpha}$, k fixed, can also become null on a three-dimensional null hypersurface $Z^k = \mathrm{const}$ on which the two remaining coefficients W_i, $i \neq k$, vanish: $W_i = 0$. Then, of course, $l_\alpha \propto Z^k{}_{,\alpha}$ simultaneously lies in this hypersurface and is orthogonal to it.

In Appendix C, we give two examples of twisting null congruences (one of them is the familiar ingoing principal null congruence in the Kerr spacetime), and illustrate the decomposition (4.1) of their tangent null covectors. The spacelike character of the covectors $Z^k{}_{,\alpha}$ is exhibited everywhere except in regions where the twist vanishes.

It now becomes understandable why it would not be useful to represent l_α by more than six potentials. If, say, we wrote $l_\alpha = W_s Z^s{}_{,\alpha}$, $s = 1,2,3,4$, one of the spatial vectors $Z^s{}_{,\alpha}$ could always be written as a linear combination of the remaining three vectors $Z^k{}_{,\alpha}$, $k = 1,2,3$, and the decomposition (4.1) would be regained.

The variation of the action (4.4) and (4.5) with respect to the metric $\gamma_{\alpha\beta}$ yields the energy-momentum tensor

$$T^{\alpha\beta} = 2|\gamma|^{-1/2} \delta S^{\mathrm{ND}} / \delta \gamma_{\alpha\beta}. \tag{4.11}$$

Because l^α is null, Eq. (4.8), this tensor has the structure (2.13). The Pfaff form (4.1) satisfies the identity

$$\nabla_\beta (l_\alpha l^\beta) = -W_{k,\alpha}(Z^k{}_{,\beta} l^\beta) + Z^k{}_{,\alpha} \nabla_\beta (W_k l^\beta) + \frac{1}{2}\nabla_\alpha (l_\beta l^\beta). \tag{4.12}$$

The equations of motion (4.6)–(4.8) then imply the energy-momentum conservation law. In fact, it is well known that the energy-momentum conservation follows from the equations of motion because of the invariance of the action (4.4) and (4.5) under spacetime diffeomorphisms (see, e.g., [1]). We have already seen that the energy-momentum conservation implies that the particles of the null dust move along geodesics, Eq. (2.14). This demonstrates that our action (4.4) and (4.5) correctly reproduces the motion of the null dust on a given background $(\mathcal{M}, \gamma)$.

The null dust is coupled to gravity by adding its action S^{ND} to the Hilbert action

$$S^G[\gamma_{\alpha\beta}] = \int_{\mathcal{M}} d^4y \; L^{\mathrm{G}}, \tag{4.13}$$

$$L^{\mathrm{G}} = |\gamma|^{1/2} R(y;\gamma] \tag{4.14}$$

constructed from the curvature scalar $R(y;\gamma]$.[2] The variation of the total action $S = S^{\mathrm{G}} + S^{\mathrm{ND}}$ with respect to the metric $\gamma_{\alpha\beta}$ yields the Einstein law of gravitation (2.3) with the null dust source (4.11). The conservation law (2.4) then follows independently of the equations of motion directly from Eqs. (2.3) through the Bianchi identities.

[2]The mixed brackets in $R(y;\gamma]$ indicate that the curvature scalar R is a function of y and a functional of $\gamma_{\alpha\beta}(y')$. This convention is used throughout the paper.

V. SPECIAL PARAMETRIZATIONS AND NULL DUST ACTION

The geodesic equation (2.14) which follows from the action (4.4) and (4.5) is not given in affine parametrization. Rather, the vector field $l^\alpha(y)$ is chosen such that it absorbs the mass distribution M of the dust and leads thereby to the energy-momentum tensor (2.13). Let us now show that from any solution $W_k(y)$ and $Z^k(y)$ of the Euler equations (4.6) and (4.7) of the action (4.4), (4.5) one can construct a vector field $k^\alpha(y)$ given in generic affine parametrization.

Start on a spacelike hypersurface Σ transverse to the dust lines l^α. Parametrize Σ by the dust space coordinates z^k of points $z \in \mathcal{S}$. As long as there is any dust on Σ, $W_k(z)$ cannot be a zero covector in $T^*\mathcal{S}$. Choose an arbitrary vector field $\mathbf{\Lambda}^k(z) \in T\mathcal{S}$ such that $\mathbf{\Lambda}^k(z)W_k(z) > 0$. Evolve the fields $Z^k(y), W_k(y)$ from their initial values z^k and $W_k(z)$ on Σ by the Euler equations (4.6), (4.7) and define

$$\mathbf{\Lambda}(y) := [\mathbf{\Lambda}^k(Z(y))W_k(y)]^{-1}. \tag{5.1}$$

The Euler equations imply that

$$\nabla_\alpha(\mathbf{\Lambda}^{-1}l^\alpha) = 0. \tag{5.2}$$

By comparing Eq. (5.2) with Eq. (3.11), one sees that $\mathbf{\Lambda}(y)$ is a scaling factor (3.10) which takes l^α into an affinely parametrized k^α. We already know, Eq. (3.23), that the most general scaling factor $\Lambda(y)$ can differ from our particular scaling factor $\mathbf{\Lambda}(y)$ only by a multiplicative function $\Lambda_0(z)$ of comoving coordinates:

$$\Lambda(y) = \Lambda_0(Z(y))\ \mathbf{\Lambda}(y). \tag{5.3}$$

Equations (5.1) and (5.3) specify an algebraic procedure by which, from any solution $Z^k(y)$, $W_k(y)$ of the Euler equations (4.6) and (4.7), one can construct the most general scaling factor $\Lambda(y)$ which takes the covector field $l_\alpha = W_k Z^k{}_{,\alpha}$ into a covector field

$$k_\alpha(y) = \Lambda(y) l_\alpha(y) \tag{5.4}$$

in affine parametrization. Equation (5.4) simultaneously tells us how to scale the potentials W_k into the corresponding potentials w_k of the Pfaff form of k_α:

$$k_\alpha = w_k Z^k{}_{,\alpha}, \quad \text{with} \quad w_k = \Lambda W_k. \tag{5.5}$$

The mass distribution

$$M = \Lambda^{-2} \tag{5.6}$$

associated with the affine parametrization (5.4) satisfies a continuity equation (3.17). The potentials W_k associated with l_α also satisfy the continuity equation (4.10). However, because in general $\nabla_\alpha l^\alpha \neq 0$, the potentials W_k do not stay the same along the dust lines:

$$l^\alpha \nabla_\alpha W_k \neq 0. \tag{5.7}$$

On the other hand, by virtue of the continuity equations (4.10) and (3.17), the potentials w_k associated with an affinely parametrized k_α of Eq. (5.4) do stay the same along the dust lines:

$$k^\alpha \nabla_\alpha w_k = 0. \tag{5.8}$$

Equations (3.7), (3.9), and (5.5) enable us to interpret w_k geometrically as projections of the null field k_α into the hypersurfaces $y^\alpha = \Upsilon^\alpha_{\mathrm{AFF}}(v,z)$, $v = \text{const}$, of affine foliation:

$$w_k = k_\alpha \frac{\partial \Upsilon^\alpha_{\mathrm{AFF}}(v,z)}{\partial z^k}. \tag{5.9}$$

Notice that while k_α in affine parametrization is built from a solution $Z^k(y)$, $W_k(y)$ of the Euler equations by differentiations (5.5) and algebraic manipulations (5.1) and (5.3), the construction of the affine parameter $v = V(y)$ itself requires solving a differential equation $k^\alpha V_{,\alpha} = 1$, i.e., an integration (3.19).

The other special parameters, the luminosity distance (3.33) and the parallax distance (3.34), can be obtained from $Z^k(y)$, $W_k(y)$ by algebraic operations and differentiations. The luminosity distance $\Lambda(y)$ is simply the scaling factor (5.3). The parallax distance p is the reciprocal value of

$$\begin{aligned}\theta &= \Lambda(\nabla_\alpha l^\alpha) \\ &= \Lambda |\gamma|^{-1/2} (|\gamma|^{1/2} \gamma^{\alpha\beta} W_k Z^k{}_{,\beta})_{,\alpha}.\end{aligned} \tag{5.10}$$

So far, we have shown how to construct the covector field k_α in affine parametrization from a solution $Z^k(y)$, $W_k(y)$ of the Euler equations (4.6) and (4.7) of the action principle (4.4) and (4.5) written in the l_α parametrization. Let us now show how to enforce affine parametrization directly from an action principle. Require one of the potentials W_k in the action (4.4) and (4.5), say $M := W_3$, to be positive, and drop the index from the associated comoving coordinate: $Z := Z^3$. Introduce $\mathrm{w}_A := W_A / W_3$ in place of the remaining two potentials W_A, $A = 1,2$, and write the Lagrangian (4.5) in terms of the new variables M, Z, Z^A, w_A:

$$L^{\mathrm{ND}} = -\frac{1}{2} |\gamma|^{1/2} M \gamma^{\alpha\beta} k_\alpha k_\beta \tag{5.11}$$

with

$$k_\alpha := Z_{,\alpha} + \mathrm{w}_A Z^A{}_{,\alpha}. \tag{5.12}$$

The Pfaff form corresponding to k_α is now constructed only from five potentials Z, Z^A, w_A, though the action (4.4), (5.11), and (5.12) still depends on six scalar variables, due to the presence of M in the Lagrangian (5.11). By varying the action with respect to Z one obtains the continuity equation (3.17). By using the other field equations, one easily derives Eq. (3.16) for affinely parametrized geodesics.

The Lagrangian (4.5) is special because it leads to the simplified form of the energy-momentum tensor, while the Lagrangian (5.11) is special because it leads to an affinely parametrized k^α. By building an additional redundancy into the Lagrangian, one can reach the generic form (2.1) and (2.9) of the energy-momentum source. One simply intro-

duces the seventh scalar M while keeping U_α as the Pfaff form of six scalar fields Z^k, W_k:[3]

$$L^{\mathrm{ND}}=-\frac{1}{2}|\gamma|^{1/2}M\gamma^{\alpha\beta}U_\alpha U_\beta \tag{5.13}$$

with

$$U_\alpha := \mathrm{W}_k Z^k{}_{,\alpha}. \tag{5.14}$$

The new Lagrangian density and all equations of motion are then invariant under the gauge transformation (2.10) and (2.11),

$$\mathrm{W}_k \to \overline{\mathrm{W}}_k = \Lambda \mathrm{W}_k \tag{5.15}$$

and

$$M \to \overline{M} = \Lambda^{-2} M, \tag{5.16}$$

where $\Lambda(y)>0$ is an arbitrary scaling factor.

The canonical form of the action is the same whether one starts from the original Lagrangian (4.5) and (4.1) or the redundant Lagrangian (5.13) and (5.14). The canonical variables recombine the redundant potentials in such a way that the information about the split of l_α into M and U_α gets lost: From the canonical variables one can reconstruct only l_α. It is thus not worth the effort to complicate the spacetime Lagrangian by striving to achieve a superfluous generality. Having learned this lesson, we take the spacetime action (4.4) and (4.5) with l_α given by Eq. (4.1) as our starting point.

VI. CANONICAL DESCRIPTION OF NULL DUST

The familiar ADM algorithm for casting a covariant action into Hamiltonian form works for the null dust in a similar way as for the ordinary dust of massive particles [1]. One foliates the spacetime $\mathcal{M}$ by spacelike hypersurfaces Σ,

$$Y:\mathbb{R}\times\Sigma\to\mathcal{M} \quad \text{by} \quad (t,x)\mapsto y=Y(t,x). \tag{6.1}$$

In local coordinates x^a, $a=1,2,3$, on Σ and y^α, $\alpha=0,1,2,3$, on $\mathcal{M}$, the foliation is represented by

$$(t,x^a)\mapsto y^\alpha = Y^\alpha(t,x^a). \tag{6.2}$$

A transition from one leaf Σ of the foliation to another is described by the deformation vector $\dot{Y}^\alpha := \partial Y^\alpha/\partial t$. Its decomposition into the normal n^α and tangential $Y^\alpha{}_{,a}$ directions to the leaves yields the lapse function $N^\perp$ and the shift vector N^a:

$$\dot{Y}^\alpha = N^\perp n^\alpha + N^a Y^\alpha{}_{,a}. \tag{6.3}$$

On each leaf, the spacetime metric $\gamma_{\alpha\beta}(y)$ induces the intrinsic metric

$$g_{ab}(t,x)=\gamma_{\alpha\beta}(Y(t,x))Y^\alpha{}_{,a}(t,x)Y^\beta{}_{,b}(t,x). \tag{6.4}$$

[3]By comparing Eqs. (5.13) and (5.14) with Eqs. (4.2) and (4.5), we see that $W_k = M^{1/2}\mathrm{W}_k$.

The spacetime metric is reconstructed as

$$\gamma^{\alpha\beta}=-n^\alpha n^\beta + g^{ab}Y^\alpha{}_{,a}Y^\beta{}_{,b}, \tag{6.5}$$

where g^{ab} is the inverse of g_{ab}, and the determinants $|\gamma|$ of $\gamma_{\alpha\beta}$ and $|g|$ of g_{ab} are related by

$$|\gamma|^{1/2}=N^\perp|g|^{1/2}. \tag{6.6}$$

Scalar fields on $\mathcal{M}$, such as the null-dust variables Z^k, W_k can be pulled back to $\mathbb{R}\times\Sigma$ by the mapping (6.1). By using Eq. (6.3) we obtain

$$Z^k{}_{,\alpha}\, n^\alpha = (N^\perp)^{-1}V^k, \tag{6.7}$$

where we have introduced the normal velocities

$$V^k := \dot{Z}^k - Z^k{}_{,a}N^a, \quad Z^k{}_{,a}=Z^k{}_{,\alpha}Y^\alpha{}_{,a}. \tag{6.8}$$

This allows us to write the null-dust action (4.4) and (4.5), with l_α given by (4.1), as an integral over $\mathbb{R}\times\Sigma$, i.e., in the (3+1)-split form:

$$S^{\mathrm{ND}}[Z^k,W_k;g_{ab},N^\perp,N^a]=\int_{\mathbb{R}}dt\int_\Sigma d^3x\, L^{\mathrm{ND}}. \tag{6.9}$$

The Lagrangian density L^{ND} on $\mathbb{R}\times\Sigma$ is a quadratic form of the Lagrange multipliers W_k:

$$L^{\mathrm{ND}}=\frac{1}{2}|g|^{1/2}\,\left((N^\perp)^{-1}V^iV^j-N^\perp g^{ij}\right)\,W_iW_j. \tag{6.10}$$

The metric

$$g^{ij}(t,x)=g^{ab}Z^i{}_{,a}Z^j{}_{,b} \tag{6.11}$$

is the induced metric on Σ expressed in the basis $Z^i{}_{,a}$ of comoving coordinates Z^i.

By varying the action with respect to W_i, we get a system of linear homogeneous equations for W_j:

$$\left(g^{ij}-(N^\perp)^{-2}V^iV^j\right)\,W_j=0. \tag{6.12}$$

This has a nontrivial solution only if the determinant

$$\det\left(g^{ij}-(N^\perp)^{-2}V^iV^j\right)=\left(1-(N^\perp)^{-2}g_{ij}V^iV^j\right)\,\det(g^{ij}) \tag{6.13}$$

vanishes. This imposes the constraint

$$g_{ij}V^iV^j-(N^\perp)^2=0 \tag{6.14}$$

on the velocities $\dot{Z}^i$. (Here, g_{ij} is the inverse of g^{ij}. We can use it for lowering the dust space indices.)

If the constraint (6.14) is satisfied, Eq. (6.12) has a solution $W_j \propto V_j$. Of course, the homogeneous equation (6.12) determines only the direction of W_j, leaving $W=g^{ij}W_iW_j$ undetermined. We write the general solution in the form

$$W_j=\sqrt{W}\;V_j(V_iV^i)^{-1/2}, \tag{6.15}$$

where W is an arbitrary positive factor.

By substituting this solution (6.15) back into the Lagrangian (6.10), we eliminate from the action the multipliers W_j, replacing them by a single multiplier W:

$$L^{\rm ND}=\frac{1}{2}|g|^{1/2}W((N^{\perp})^{-1}g_{ij}V^iV^j-N^{\perp}). \qquad (6.16)$$

The reduced action is a functional of W and Z^k. Its variation with respect to W reproduces the constraint (6.14) which enabled us to express W_j in terms of W and Z^j, $\dot{Z}^j$, Eq. (6.15). Its variation with respect to Z^k gives an equation which, modulo the constraint (6.14) and Eq. (6.15) considered as a definition of W_j, is equivalent to the equations of motion obtained by varying the original action (6.9) and (6.10) with respect to Z^k. The reduced action

$$S^{\rm ND}[Z^k,W;g_{ab},N^{\perp},N^a]=\int_{\mathbb{R}}dt\int_{\Sigma}d^3xL^{\rm ND} \qquad (6.17)$$

with the Lagrangian (6.16) is thus entirely equivalent to the original action (6.9) with the Lagrangian (6.10).

In order to bring the reduced action to canonical form, we perform the Legendre dual transformation from $(Z^k,\ \dot{Z}^k)$ to $(Z^k,\ P_k)$, leaving W as a multiplier. First, we introduce the momenta

$$P_k:=\frac{\partial L^{\rm ND}}{\partial\dot{Z}^k}=|g|^{1/2}W(N^{\perp})^{-1}V_k. \qquad (6.18)$$

To clarify their physical meaning, we return to the definition (6.15) of W_j, the decomposition (4.1) of l_α, and Eqs. (6.7) and (6.8) for the normal velocity. In this way we learn that P_k are normal projections of the currents J^α_k introduced in Eq. (4.9):

$$P_k=|g|^{1/2}J^\alpha_k n_\alpha. \qquad (6.19)$$

Equation (6.18) can be inverted to obtain the velocities

$$\dot{Z}^k=N^{\perp}|g|^{-1/2}W^{-1}g^{kj}P_j+N^aZ^k{}_{,a}. \qquad (6.20)$$

This leads to the Hamiltonian

$$H^{\rm ND}:=P_k\dot{Z}^k-L^{\rm ND}=N^{\perp}H^{\rm ND}_{\perp}+N^aH^{\rm ND}_a \qquad (6.21)$$

which is a linear combination of the momentum density

$$H^{\rm ND}_a=P_kZ^k{}_{,a} \qquad (6.22)$$

and the energy density

$$H^{\rm ND}_{\perp}=\frac{1}{2}W^{-1}|g|^{-1/2}g^{ij}P_iP_j+\frac{1}{2}W|g|^{1/2} \qquad (6.23)$$

$$=\frac{1}{2}W^{-1}|g|^{-1/2}g^{ab}H^{\rm ND}_aH^{\rm ND}_b+\frac{1}{2}W|g|^{1/2} \qquad (6.24)$$

of the dust. The canonical form of the action then reads

$$S^{\rm ND}[Z^k,P_k,W;g_{ab},N^{\perp},N^a]$$

$$=\int_{\mathbb{R}}dt\int_{\Sigma}d^3x\ (P_k\dot{Z}^k-N^{\perp}H^{\rm ND}_{\perp}-N^aH^{\rm ND}_a), \qquad (6.25)$$

where $H^{\rm ND}_a$ and $H^{\rm ND}_{\perp}$ are given by Eqs. (6.22) and (6.24).

At this stage, we are finally able to eliminate the last remaining multiplier W. By varying the action (6.22)–(6.25) with respect to W, we obtain an equation

$$\frac{\delta S^{\rm ND}}{\delta W}=-N^{\perp}\frac{\partial H^{\rm ND}_{\perp}}{\partial W}=0 \qquad (6.26)$$

which determines W in terms of the canonical data:

$$W=|g|^{-1/2}\sqrt{g^{ij}P_iP_j}$$
$$=|g|^{-1/2}\sqrt{g^{ab}H^{\rm ND}_aH^{\rm ND}_b}. \qquad (6.27)$$

By substituting this solution back into $H^{\rm ND}_{\perp}$ we obtain

$$H^{\rm ND}_{\perp}=\sqrt{g^{ab}H^{\rm ND}_aH^{\rm ND}_b}. \qquad (6.28)$$

We see that W is just the scalar form $W=|g|^{-1/2}H^{\rm ND}_{\perp}$ of the Hamiltonian density $H^{\rm ND}_{\perp}$. The final expressions (6.22) for the momentum density and (6.28) for the energy density are simple: The form of the momentum density is dictated by the requirement that it generate the Lie derivative change of the scalars $Z^k(x)$ and scalar densities $P_k(x)$ under spatial diffeomorphisms LDiffΣ [10]. The energy density is the norm of the momentum density with respect to the spatial metric. The resulting reduced canonical action $S^{\rm ND}[Z^k,\ P_k;\ g_{ab},\ N^{\perp},\ N^a]$, with Eqs. (6.22) and (6.28) yields the Hamilton equations for $Z^k(t,x)$ and $P_k(t,x)$. These describe the evolution of the null dust on a given geometrical background $\gamma_{\alpha\beta}\leftrightarrow(N^{\perp}(t,x),N^a(t,x),g_{ab}(t,x))$.[4]

From the solution of the Hamilton equations we can reconstruct the null vector l^α which provides the spacetime description of the dust. It holds that

$$l^\alpha=l^{\perp}n^\alpha+l^aY^\alpha{}_{,a}, \qquad (6.29)$$

where $l^{\perp}$ and l_a are expressed as functions of the canonical variables:

$$l^{\perp}=-l_\alpha n^\alpha=-W^{1/2}, \qquad (6.30)$$

$$l_a=l_\alpha Y^\alpha{}_{,a}=W^{-1/2}|g|^{-1/2}H^{\rm ND}_a. \qquad (6.31)$$

Here, of course, W stands for the scalar form (6.27) of the energy-momentum density. One can check that l^α is a null vector by virtue of its construction (6.29)–(6.31):

$$l^al_\alpha=-(l^{\perp})^2+l^al_a=0. \qquad (6.32)$$

The background variables $N^{\perp}(t,x)$, $N^a(t,x)$, and $g_{ab}(t,x)$ in the dust action $S^{\rm ND}[Z^k,\ P_k;\ g_{ab},\ N^{\perp},\ N^a]$ are not to be varied. The Hamiltonian formalism for null dust on a given

[4]Our Lagrangian and Hamiltonian formalisms can easily be generalized to several mutually noninteracting species (streams) of null dust. This may be useful for the canonical treatment of spherical collapse, in which the ingoing null dust is turned into an outgoing null dust at the center of symmetry, or for the canonical treatment of models involving colliding streams of dust with plane or cylindrical symmetry (see references in [34]).

background is thus entirely unconstrained. To couple null dust to geometry, we must add its action S^{ND} to the gravitational Dirac-ADM action

$$S^{\mathrm{G}}[g_{ab}, p^{ab}; N^{\perp}, N^{a}] = \int_{\mathbb{R}} dt \int_{\Sigma} d^3x \; (p^{ab}\dot{g}_{ab} - N^{\perp}H^{\mathrm{G}}_{\perp} - N^{a}H^{\mathrm{G}}_{a}) \qquad (6.33)$$

with the standard gravitational super-Hamiltonian and supermomentum densities

$$H^{\mathrm{G}}_{\perp}(x; g_{ab}, p^{ab}] = G_{abcd}(x;g)p^{ab}(x)p^{cd}(x) - |g|^{1/2}R(x;g], \qquad (6.34)$$

$$G_{abcd} = \frac{1}{2}|g|^{-1/2}(g_{ac}g_{bd} + g_{ad}g_{bc} - g_{ab}g_{cd}), \qquad (6.35)$$

and

$$H^{\mathrm{G}}_{a}(x; g_{ab}, p^{ab}] = -2D_b p^{b}_{a}(x), \qquad (6.36)$$

where D_b is the spatial covariant derivative.

The variation of the total action with respect to the lapse $N^{\perp}$ and the shift N^{a} then leads to the familiar Hamiltonian and momentum constraints

$$H_{\perp} := H^{\mathrm{G}}_{\perp} + H^{\mathrm{ND}}_{\perp} = 0, \qquad (6.37)$$

$$H_{a} := H^{\mathrm{G}}_{a} + H^{\mathrm{ND}}_{a} = 0 \qquad (6.38)$$

for the coupled system.

VII. NULL DUST CONSTRAINTS THAT GENERATE A LIE ALGEBRA

By using the supermomentum constraint, one can replace the momentum density H^{ND}_{a} of the dust by the gravitational density H^{G}_{a} in the expression (6.28) for the dust energy density $H^{\mathrm{ND}}_{\perp}$. This brings the constraint system (6.38) and (6.37) into an equivalent form (6.38) and

$$H_{\perp} := H^{\mathrm{G}}_{\perp} + \sqrt{g^{ab}H^{\mathrm{G}}_{a}H^{\mathrm{G}}_{b}} = 0. \qquad (7.1)$$

Only the supermomentum constraint (6.38) contains the dust variables. The new Hamiltonian constraint (7.1) is constructed solely from the gravitational variables g_{ab}, p^{ab}. Alternatively, one can get rid of an inconvenient square root by rewriting Eq. (7.1) in the form

$$G := (H^{\mathrm{G}}_{\perp})^2 - g^{ab}H^{\mathrm{G}}_{a}H^{\mathrm{G}}_{b} = 0. \qquad (7.2)$$

Under the positivity condition

$$-H^{\mathrm{ND}}_{\perp} = H^{\mathrm{G}}_{\perp} \leqslant 0, \qquad (7.3)$$

the constraint (7.2) is equivalent to the constraint (7.1).

Brown and Kuchař [1] proved a remarkable fact that the densities (7.2) have strongly vanishing Poisson brackets:

$$\{G(x),\, G(x')\} = 0. \qquad (7.4)$$

By coupling gravity to other simple sources, Kuchař and Romano [11] and Brown and Marolf [12] produced other densitized expressions constructed from the scalar variables $|g|^{-1/2}H^{\mathrm{G}}_{\perp}$ and $|g|^{-1}g^{ab}H^{\mathrm{G}}_{a}H^{\mathrm{G}}_{b}$ which also have strongly vanishing Poisson brackets. Markopoulou [13] posed the question of what is the most general density

$$F = |g|^{w/2}\mathrm{F}(|g|^{-1/2}H^{\mathrm{G}}_{\perp},\; |g|^{-1}g^{ab}H^{\mathrm{G}}_{a}H^{\mathrm{G}}_{b}) \qquad (7.5)$$

of weight w constructed from these variables which has the strongly vanishing Poisson brackets

$$\{F(x),\, F(x')\} = 0. \qquad (7.6)$$

She found an algorithm for generating all such densities. The density (7.2) of weight 2 still seems to be the simplest. Among others, there is the scalar form

$$G_{\surd} := |g|^{-1/2}(H^{\mathrm{G}}_{\perp} + \sqrt{g^{ab}H^{\mathrm{G}}_{a}H^{\mathrm{G}}_{b}}) = 0 \qquad (7.7)$$

of the constraint (7.1) which, as we have just seen, describes null dust.

The constraints $H^{\mathrm{G}}_{a} = 0 = H^{\mathrm{G}}_{\perp}$ of vacuum gravity can be replaced by an alternative system

$$H^{\mathrm{G}}_{a} = 0 = G \quad (\text{or } H^{\mathrm{G}}_{a} = 0 = G_{\surd}). \qquad (7.8)$$

Unlike the original constraints, H^{G}_{a} and G (or H^{G}_{a} and $G_{\surd}$) generate a true Lie algebra. Unfortunately, in vacuum gravity the new constraints (7.2) [and similarly Eq. (7.7)] do not generate the evolution of the geometric data g_{ab}, p^{ab} into a Ricci-flat spacetime. Expression (7.2) is flawed because its Hamiltonian vector field vanishes on the constraint surface (7.8), while expression (7.7) is flawed because its Hamiltonian vector field is ill defined for $H^{\mathrm{G}}_{a} = 0$.

No such difficulty exists for null dust. The momentum constraint (6.38) is different from the vacuum constraint $H^{\mathrm{G}}_{a} = 0$ and, as long as there is any dust at the point in question, H^{ND}_{a} and hence H^{G}_{a} cannot vanish. The Hamiltonian vector fields of the dynamical variables (7.2) or (7.7) then do not vanish on the constraint surface (6.37) and (6.38) of the null dust coupled to geometry. The new constraints (7.2) or (7.7) correctly generate the evolution of geometry produced by null dust. Moreover, as in vacuum spacetime, the constraints (6.38) and (7.2), or (6.38) and (7.7), generate a true Lie algebra. It is thus advantageous to bring the constraints to one of these forms before attempting to quantize the coupled system.

Why is it that the presence of the null dust does not affect Eqs. (7.2) and (7.7) that hold in vacuum gravity? The energy-momentum tensor of the null dust satisfies the condition

$$T^{\alpha}{}_{\gamma}T^{\gamma\beta} = 0. \qquad (7.9)$$

Conversely, any symmetric tensor $T^{\alpha\beta}$ which satisfies Eq. (7.9) must either vanish, or there exists a null vector l^{α} such that

$$T^{\alpha\beta} = l^{\alpha}l^{\beta}. \qquad (7.10)$$

The Einstein law of gravitation (2.3) then implies that l^{α} is a geodesic vector field, i.e., the Euler equations of motion for the null dust. The simple tensor equation

$$G^{\alpha}{}_{\gamma}G^{\gamma\beta}=0 \tag{7.11}$$

imposed on the Einstein tensor thus ensures that the geometry $\gamma_{\alpha\beta}$ is necessarily produced by null dust according to Einstein's law of gravitation.[5]

The $\perp\perp$ projection of Eq. (7.11) gives

$$(G_{\perp\perp})^2-g^{ab}G_{\perp a}G_{\perp b}=0. \tag{7.12}$$

Because the $\perp\perp$ and $\perp\|$ projections of the Einstein tensor yield the gravitational super-Hamiltonian and supermomentum [10],

$$G_{\perp\perp}=-\frac{1}{2}|g|^{-1/2}H^{\mathrm{G}}_{\perp},\quad G_{\perp a}=\frac{1}{2}|g|^{-1/2}H^{\mathrm{G}}_{a}, \tag{7.13}$$

Eq. (7.12) is equivalent to the constraint (7.2). We have already noticed that under the energy positivity condition (7.3) the constraint (7.2) is equivalent to the constraint (7.7). The Rainich-type condition (7.11) thus connects the new form (7.2) or (7.7) of the Hamiltonian constraint with the spacetime picture.

VIII. CONSTRAINT QUANTIZATION OF GEOMETRY COUPLED TO NULL DUST

We have cast the constraint system for geometry coupled to null dust into a form in which it generates a Lie algebra. In this process, the Hamiltonian constraint has been replaced either by the constraint (7.2) or by the constraint (7.7). Either of these constraints have vanishing Poisson brackets (7.4). The momentum constraint is left in its original form (6.38) and (6.22):

$$H_a(x) \coloneqq P_k(x)Z^k{}_{,a}(x)+H^{\mathrm{G}}_a(x)=0. \tag{8.1}$$

The momentum constraints (8.1) close in the way characteristic for the Lie algebra LDiffΣ of the diffeomorphism group DiffΣ. The Poisson brackets of $G(x)$ [or $G_{\sqrt{}}(x)$] with $H_a(x')$ close into $G(x)$ or $[G_{\sqrt{}}(x)]$ in the way which reflects the transformation behavior of $G(x)$ [or $G_{\sqrt{}}(x)$] under spatial diffeomorphisms DiffΣ: $G(x)$ is a density of weight 2, while $G_{\sqrt{}}(x)$ is a scalar.

As for ordinary dust, the constraint system can be vastly simplified by the introduction of an alternative set of canonical variables which reflect the fact that the dust particles define a preferred system of coordinates on Σ. The mapping $Z:\Sigma\to\mathcal{S}$ which, in local coordinates, assumes the form

$$z^k=Z^k(x^a), \tag{8.2}$$

takes the tensorial variables $g_{ab}(x)$ and $p^{ab}(x)$ on Σ into corresponding tensors $\boldsymbol{g}_{ij}(z)$ and $\boldsymbol{p}^{ij}(z)$ on the dust space $\mathcal{S}$:

$$\boldsymbol{g}_{ij}(z) \coloneqq X^a{}_{,i}(z)X^b{}_{,j}(z)g_{ab}(X(z)), \tag{8.3}$$

$$\boldsymbol{p}^{ij}(z) \coloneqq \left|\frac{\partial X(z)}{\partial z}\right| Z^i{}_{,a}(X(z))Z^j{}_{,b}(X(z))p^{ab}(X(z)). \tag{8.4}$$

Here, the t-dependent mapping $X:\mathcal{S}\to\Sigma$ is simply the inverse of Z,

$$X \coloneqq Z^{-1}, \tag{8.5}$$

and $|\partial X(z)/\partial z|$ is the Jacobian for the change of variables $x^a=X^a(z)$.

We rewrite the supermomentum constraint (8.1) in the form

$$\begin{aligned} H_{\uparrow k}(x) &\coloneqq H_a(x)Z^a_k(x) \\ &= P_k(x)+H^{\mathrm{G}}_a(x)Z^a_k(x)=0. \end{aligned} \tag{8.6}$$

Here,

$$Z^a_k(x) \coloneqq X^a{}_{,k}(Z(x)) \tag{8.7}$$

is the inverse matrix to $Z^k{}_{,a}(x)$:

$$Z^j{}_{,a}(x)Z^a_k(x)=\delta^j_k. \tag{8.8}$$

The new supermomentum $H_{\uparrow k}(x)$ smeared by a new shift $N^{\uparrow k}(x)$,

$$H_{\uparrow}[\vec{N}^{\uparrow}] \coloneqq \int_{\Sigma} d^3x\ N^{\uparrow k}(x)H_{\uparrow k}(x), \tag{8.9}$$

generates through the Poisson bracket the change

$$\dot{Z}^k(x) \coloneqq \{Z^k(x),\ H_{\uparrow}[\vec{N}^{\uparrow}]\}=N^{\uparrow k}(x) \tag{8.10}$$

of the dust coordinates $Z^k(x)$ by the amount $N^{\uparrow k}(x)$ [1].

One can prove that the $\mathcal{S}$ variables $\boldsymbol{g}_{ij}(z)$, $\boldsymbol{p}^{ij}(z)$ along with the dust frame variables $Z^k(x)$ and the new supermomentum $H_{\uparrow k}(x)$ form a canonical chart [1]. In particular, this means that the new constraint functions $\mathrm{P}_k(x) \coloneqq H_{\uparrow k}(x)$ have vanishing Poisson brackets among themselves and are the momenta $\mathrm{P}_k(x)$ canonically conjugate to the dust frame variables $Z^k(x)$. Further, because the Poisson brackets of the $\mathcal{S}$ tensors $\boldsymbol{g}_{ij}(z)$, $\boldsymbol{p}^{ij}(z)$ with the smeared supermomentum (8.9) vanish, these $\mathcal{S}$ tensors are invariant under the shifts (8.6).

In terms of the new canonical variables $\boldsymbol{g}_{ij}(z)$, $\boldsymbol{p}^{ij}(z)$, and $Z^k(x)$, $\mathrm{P}_k(x)$ the momentum constraint (8.6) reduces to the condition that the canonical momentum $\mathrm{P}_k(x)$ vanishes:

$$\mathrm{P}_k(x)=0. \tag{8.11}$$

The Hamiltonian constraints (7.2) or (7.7) can then be mapped to the dust space $\mathcal{S}$ according to their weight:

$$\mathbf{G}(z) \coloneqq \left|\frac{\partial X(z)}{\partial z}\right|^2 G(X(z))=0\,, \tag{8.12}$$

[5]Equation (7.11) is perhaps the simplest example of the Rainich-type geometrization of a source field. The general task is to find equations for the Einstein tensor which are equivalent to the Einstein law of gravitation together with the field equations for a given source. The problem was first formulated for the Einstein-Maxwell system by Rainich and solved by him under the assumption that the electromagnetic field is not algebraically special (null) [14], [15]. The Rainich problem for the null electromagnetic field was solved by Hlavatý [16]. The much simpler scalar field case was analyzed by Peres [17] and by Kuchař [18]. The spinor field was treated by Kuchař [19] and the Proca field by Bičák [20].

$$\mathbf{G}_{\surd}(z) \coloneqq G_{\surd}(X(z)) = 0. \tag{8.13}$$

The $\mathcal{S}$ constraints (8.12) and (8.13) are the same functionals of the $\mathcal{S}$ tensors $\boldsymbol{g}_{ij}(z)$ and $\boldsymbol{p}^{ij}(z)$ as the Σ constraints (7.2) or (7.7) were of the Σ tensors $g_{ab}(x)$ and $p^{ab}(x)$. In other words, $\mathbf{G}(z)$ is obtained from $G(x)$ and $\mathbf{G}_{\surd}(z)$ is obtained from $G_{\surd}(x)$ by replacing the Σ tensors $g_{ab}(x)$, $p^{ab}(x)$ by the corresponding $\mathcal{S}$ tensors $\boldsymbol{g}_{ij}(z)$, $\boldsymbol{p}^{ij}(z)$. The $\mathcal{S}$ constraints (8.12) and (8.13) have strongly vanishing Poisson brackets, i.e., they generate an Abelian algebra.

In the Dirac method of quantization, constraints are turned into operators and imposed as restrictions on the state functionals of the system. We choose to work with the dust space variables, so the quantum states of the system are functionals $\boldsymbol{\Psi}[Z,\boldsymbol{g}]$ of the canonical coordinates $Z^k(x)$ and $\boldsymbol{g}_{ij}(z)$, and the constraint operators $\hat{\mathrm{P}}_k(x)$ and $\hat{\boldsymbol{G}}(z)$ [or $\hat{\boldsymbol{G}}_{\surd}(z)$] are obtained by quantizing the classical expressions (8.11)–(8.13). The transition is easy for the momenta (8.11) which are simply replaced by the variational derivatives

$$\hat{\mathrm{P}}_k(x) = -i\frac{\delta}{\delta Z^k(x)}. \tag{8.14}$$

The operators (8.14) automatically commute,

$$[\hat{\mathrm{P}}_i(x),\ \hat{\mathrm{P}}_j(x')] = 0. \tag{8.15}$$

It is far from clear how to replace the remaining classical constraints by operators which not only commute with $\hat{\mathrm{P}}_k(x)$, but also among themselves. We shall proceed under the assumption that there exists a factor ordering and regularization of $\hat{\boldsymbol{G}} \coloneqq \boldsymbol{G}(z;\ \hat{\boldsymbol{g}}_{ij}(z),\ \hat{\boldsymbol{p}}^{ij}(z)]$ and/or $\hat{\boldsymbol{G}}_{\surd} \coloneqq \boldsymbol{G}_{\surd}(z;\ \hat{\boldsymbol{g}}_{ij}(z),\ \hat{\boldsymbol{p}}^{ij}(z)]$ which achieves this goal. If so, the constraint operators can consistently annihilate the physical states. The momentum constraint

$$\hat{\mathrm{P}}_k(x)\boldsymbol{\Psi}[Z,\boldsymbol{g}] = 0, \tag{8.16}$$

where $\hat{\mathrm{P}}_k(x)$ is interpreted as the variational derivative (8.14), means that the state functional $\boldsymbol{\Psi}[Z,\boldsymbol{g}]$ cannot depend on $Z^k(x)$:

$$\boldsymbol{\Psi} = \boldsymbol{\Psi}[\boldsymbol{g}]. \tag{8.17}$$

The constraint system is thereby reduced to a single ∞^3 nontrivial condition that $\hat{\boldsymbol{G}} \coloneqq \boldsymbol{G}(z;\ \hat{\boldsymbol{g}}_{ij}(z),\ \hat{\boldsymbol{p}}^{ij}(z)]$ or ($\hat{\boldsymbol{G}}_{\surd} \coloneqq \boldsymbol{G}_{\surd}(z;\ \hat{\boldsymbol{g}}_{ij}(z),\ \hat{\boldsymbol{p}}^{ij}(z)]$) annihilates the state functional:

$$\boldsymbol{G}(z;\ \hat{\boldsymbol{g}}_{ij}(z),\ \hat{\boldsymbol{p}}^{ij}(z)]\boldsymbol{\Psi}[\boldsymbol{g}] = 0. \tag{8.18}$$

The caveats which need to be born in mind when implementing such a formal procedure for gravity coupled to ordinary dust are carefully spelled out in [1]. An additional difficulty with null dust is that there is no natural variable which would play the role of internal time. As a result, unlike for ordinary dust, the quantum constraint (8.18) does not have the form of a functional Schrödinger equation. It is thus unclear how, even formally, to turn the space of its solutions into a Hilbert space.

IX. COMPARING NULL DUST WITH ORDINARY DUST

Ordinary dust coupled to gravity was turned into a Hamiltonian system and formally quantized by Brown and Kuchař [1]. This scheme turns out to be both similar to and characteristically different from the description of null dust given in this paper. We shall outline the basic similarities and emphasize the differences.

The spacetime action

$$S^{\mathrm{D}}[T,Z^k;\ M,W_k;\gamma_{\alpha\beta}] = \int_{\mathcal{M}} d^4y\ L^{\mathrm{D}}(y) \tag{9.1}$$

of ordinary dust is constructed from eight scalar fields Z^k, W_k and T, M. The Lagrangian density $L^{\mathrm{D}}(y)$ has the form

$$L^{\mathrm{D}} = -\frac{1}{2}|\gamma|^{1/2}M(\gamma^{\alpha\beta}U_\alpha U_\beta + 1). \tag{9.2}$$

The four-velocity U_α is expressed as the Pfaff form

$$U_\alpha = -T_{,\alpha} + W_k Z^k{}_{,\alpha} \tag{9.3}$$

of seven scalar fields W_k, Z^k and T. The matter equations of motion are obtained by varying the dust action (9.1)–(9.3) with respect to the state variables M, W_k, T, and Z^k:

$$0 = \frac{\delta S^{\mathrm{D}}}{\delta M} = -\frac{1}{2}|\gamma|^{1/2}(\gamma^{\alpha\beta}U_\alpha U_\beta + 1), \tag{9.4}$$

$$0 = \frac{\delta S^{\mathrm{D}}}{\delta W_k} = -|\gamma|^{1/2}MZ^k{}_{,\alpha}U^\alpha, \tag{9.5}$$

$$0 = \frac{\delta S^{\mathrm{D}}}{\delta T} = -(|\gamma|^{1/2}MU^\alpha)_{,\alpha}, \tag{9.6}$$

$$0 = \frac{\delta S^{\mathrm{D}}}{\delta Z^k} = (|\gamma|^{1/2}MW_kU^\alpha)_{,\alpha}. \tag{9.7}$$

They lead to the interpretation of the state variables. Equation (9.5) is analogous to Eq. (4.6) for null dust. It ensures that the three vector fields Z^k are constant along the flow lines of U^α and therefore their values z^k can be interpreted as comoving coordinates for the dust. Equation (9.4) ensures that the four-velocity U^α is a unit timelike vector field. It is analogous to Eq. (4.8) which guarantees that the four-velocity l^α of null dust is lightlike. Equation (9.6) allows us to interpret M as the rest mass density of the dust and expresses the law of mass conservation. It is analogous to Eq. (3.17) for the null dust in affine parametrization. Equation (9.7) can be interpreted as the momentum conservation law. It is analogous to Eq. (4.10) for the null dust written again in affine parametrization. By multiplying Eq. (9.3) by U^α and using the field equations (9.4) and (9.5), we learn that

$$T_{,\alpha}U^\alpha = 1, \tag{9.8}$$

i.e., that T is the proper time between a fiducial hypersurface $T=0$ and an arbitrary hypersurface $T=\mathrm{const}$ along the dust world lines. From Eq. (9.3) we see that the W_k variables are the projections of the four-velocity U_α to the hypersurfaces of constant T expressed in the dust space cobasis $Z^k{}_{,\alpha}$. Due

to the conservation laws (9.6) and (9.7), these projections remain the same along a flow line of U^α. In comparison, W_k for the null dust is the component of the null covector l^α in the dust space cobasis $Z^k{}_{,\alpha}$. These components are *not* conserved along the flow lines, Eq. (5.7). However, when one rescales l^α into an affinely parametrized k^α by Eqs. (5.4) and (5.5) and projects k_α into hypersurfaces of constant affine parameter v, one obtains the components w_k of Eq. (5.9) which *are* conserved along the flow lines, Eq. (5.8).

The main difference between the actions S^{D} and S^{ND} is that the dust action depends on eight variables T, M and Z^k, W_k, while the null dust action depends only on six variables Z^k, W_k. The interpretation of the variables Z^k as the comoving coordinates and W_k as the projections of the four-velocities U^α (or l^α) into hypersurfaces of constant T (or U) is analogous. The variables T and M do not appear in the null dust action (4.1), (4.4), and (4.5). This reflects the fact that the mass function M of the null dust is not uniquely determined and it was absorbed into the definition of l^α. Similarly, the affine parameter along the null geodesics is not uniquely determined. If one chooses to enforce the affine parametrization by taking the null dust Lagrangian in the form (5.11), the corresponding M occurs in the action, but the Pfaff form of an affinely parametrized k_α, Eq. (5.12), contains only two independent scalars w_A. One can work in a totally arbitrary parametrization by letting the Lagrangian density to depend on seven variables M, Z^k, W_k instead of six, Eqs. (5.13) and (5.14), but then the action becomes gauge invariant under the scalings (5.15) and (5.16), which makes it effectively dependent only on six of these variables.

These similarities and differences are reflected in the canonical form of the action. For ordinary dust, the energy density $H^{\mathrm{D}}_\perp$ and momentum density H^{D}_a depend on *four* pairs of canonical variables, T and P, and Z^k, P_k. They take the form

$$H^{\mathrm{D}}_a = P T_{,a} + P_k Z^k{}_{,a} \tag{9.9}$$

and

$$H^{\mathrm{D}}_\perp = \sqrt{P^2 + g^{ab} H^{\mathrm{D}}_a H^{\mathrm{D}}_b}. \tag{9.10}$$

On the other hand, similar expressions for null dust, Eqs. (6.22) and (6.28), depend only on *three* pairs of canonical variables, Z^k and P_k. This difference is vital. While ordinary dust has four degrees of freedom per space point $x \in \Sigma$, null dust has only three.

The rest mass density M of ordinary dust is directly related to the momentum P:

$$M = |g|^{-1/2} \frac{P^2}{\sqrt{P^2 + g^{ab} H^{\mathrm{D}}_a H^{\mathrm{D}}_b}}. \tag{9.11}$$

The mass function and affine parametrization of null dust are ambiguous and their only invariant combination is the null vector l^α. This can be reconstructed from the canonical data, Eqs. (6.29)–(6.31), rather than the mass function and the four-velocity separately.

Formally, the momentum and energy densities (6.22) and (6.28) of the null dust are obtained from the corresponding expressions (9.9) and (9.10) for ordinary dust simply by putting $P=0$ and forgetting all about its conjugate variable T. This should not hide the fundamentally different ways in which the Dirac-ADM action is obtained from the spacetime action. The Lagrangian (9.2) and (9.3) for ordinary dust is nondegenerate in the velocities $\dot{T}$, $\dot{Z}^k$. The expressions for the momenta P, P_k can be inverted to yield the velocities. The momenta are in a one-to-one correspondence with the multipliers M and W_k and hence their variation yields equivalent equations. The spacetime action, so to speak, is in an 'already parametrized form.''

To cast the spacetime action (4.4) and (4.5) of the null dust into canonical form requires an entirely different procedure. The null dust Lagrangian (6.8) and (6.10) is singular in the velocities $\dot{Z}^k$. The definition equations for the momenta P_k cannot be inverted. They yield three constraints

$$\delta^{ijk} W_j P_k = 0 \tag{9.12}$$

demanding that the multipliers W_k be parallel to the momenta P_k, which leaves the magnitude of W_k undetermined. The variation of the action with respect to W_k leads to the constraint (6.14) on the velocities $\dot{Z}^k$. If this constraint is satisfied, the multipliers W_k can be replaced by a single multiplier W and the Lagrangian L^{ND} cast into an equivalent form (6.16) which is regular in the velocities. This allows one to perform the Legendre dual transformation to the canonical form of the action. The final elimination of the multiplier W (analogous to the final elimination of the mass multiplier M from the canonical action for ordinary dust) leads to the null dust momentum and energy densities (6.22) and (6.28). To summarize, though these final expressions have similar structure as the densities (9.9) and (9.10) for ordinary dust from which they can be obtained by putting $P=0$, their derivation is fundamentally different.

After the dust is coupled to geometry, the parallels and differences between ordinary and null dust are brought into a new perspective. The momentum and Hamiltonian constraints for ordinary dust can be resolved with respect to the four dust momenta P, P_k which brings them to an equivalent form

$$H_{\uparrow k} := P_k + Z^a_k H^{\mathrm{G}}_a + \sqrt{G} T_{,a} Z^a_k = 0, \tag{9.13}$$

$$H_\uparrow := P - \sqrt{G} = 0, \tag{9.14}$$

where G is given by Eq. (7.2). The new constraint functions $H_{\uparrow K} = (H_\uparrow, H_{\uparrow k})$ have strongly vanishing Poisson brackets:

$$\{H_{\uparrow K}(x),\ H_{\uparrow L}(x')\} = 0. \tag{9.15}$$

The imposition of the constraints (9.13) and (9.14) as operator restrictions on the states $\Psi[T, Z^k; g_{ab}, p^{ab}]$ leads to a functional Schrödinger equation with formally conserved inner product. By mapping the constraints into the dust space, the momentum constraint is eliminated and what remains is a single functional differential Schrödinger equation

$$(\hat{\boldsymbol{P}}(z) - \sqrt{\boldsymbol{G}(z; \hat{\boldsymbol{g}}, \hat{\boldsymbol{p}})})\boldsymbol{\Psi}[\boldsymbol{T}(z), \boldsymbol{g}(z)] = 0. \tag{9.16}$$

The null dust constraints in the form (8.6) and (7.2) can again be obtained from the ordinary dust constraints (9.13)

and (9.14) by disregarding the canonical pair T, P [and squaring Eq. (9.14)]. By mapping them into dust space, the momentum constraint is again eliminated. By imposing the only remaining constraint as an operator restriction on quantum states, one again gets a single functional differential equation (8.18). However, and this is an important difference, Eq. (8.18) is *not* a Schrödinger equation like Eq. (9.16) because there is no internal time $\boldsymbol{T}(z)$. It is thus not clear how to introduce an inner product in the space of its solutions.

Both ordinary dust and null dust provide a standard of space in canonical gravity because the dust particles introduce into spacetime a privileged dust frame $\mathcal{S}$ labeled by comoving coordinates $Z^k(x)$. The crucial difference is that ordinary dust also provides a standard of time: It has an additional degree of freedom $T(x)$ which can be physically interpreted as the proper time along the dust world lines. Null dust does not have any corresponding degree of freedom because affine parametrization of null geodesics is ambiguous. It thus fails to provide a standard of time to the spacetime in which it moves. The story of ordinary dust is that of time regained. The story of null dust is that of time lost again.

ACKNOWLEDGMENTS

We are grateful to Laszlo Gergely and Jorma Louko for their comments. The work on this paper was partially supported by the U.S.–Czech Republic Science and Technology Program Grant No. 92067, by the NSF Grant No. PHY-9507719 to the University of Utah, and by the Czech Republic Grant No. GACR-202/96/0206.

APPENDIX A: NULL DUST AND GEOMETRICAL OPTICS

If at each spacetime point all the energy is transported in one direction with the speed of light, it is appropriate to describe the matter by the energy-momentum tensor of null dust,

$$T^{\alpha\beta}=Mk^{\alpha}k^{\beta}. \tag{A1}$$

The energy-momentum tensor (A1) may be considered as representing an incoherent superposition of waves with random phases and polarizations but moving in a single direction.[6] It is also called the ''geometrical-optics'' or ''pure radiation'' energy-momentum tensor.

As an example, consider the Maxwell theory. (See, in particular, [3], Sec. 22.5, for a detailed exposition of geometrical optics in curved spacetime.) If the electromagnetic waves can *locally* be regarded as plane waves propagating through spacetime of negligible curvature, one can write the electromagnetic vector potential A_{α} in the form

$$A_{\alpha}=\mathrm{Re}(a_{\alpha}\, e^{i\Theta}). \tag{A2}$$

Here, in the first approximation, the complex amplitude $a_{\alpha}(y)$ is independent of the wavelength and is slowly changing as a function of spacetime position y, while the scalar function $\Theta(y)$ is a rapidly changing phase. Following the standard procedure [3], one introduces the wave vector

$$k_{\alpha}=\Theta_{,\alpha}, \tag{A3}$$

the (real) scalar amplitude

$$A=(A_{\alpha}A^{\alpha})^{1/2}=(a^{\alpha}\overline{a_{\alpha}})^{1/2}, \tag{A4}$$

and the (complex) unit polarization vector

$$e_{\alpha}=A^{-1}a_{\alpha}. \tag{A5}$$

As a consequence of the source-free wave equation and the Lorentz gauge condition, both written in the first order of the geometrical optics approximation, the quantities (A3)–(A5) obey the following set of equations:

$$k_{\alpha}k^{\alpha}=0, \tag{A6}$$

$$k^{\beta}\nabla_{\beta}k^{\alpha}=0, \tag{A7}$$

$$\nabla_{\alpha}(A^2k^{\alpha})=0, \tag{A8}$$

and

$$k^{\alpha}e_{\alpha}=0, \quad k^{\beta}\nabla_{\beta}e_{\alpha}=0. \tag{A9}$$

From Eq. (A7) we see that the null vector k^{α} is affinely parametrized. The electromagnetic field tensor is given by

$$F_{\alpha\beta}=2\mathrm{Re}(iAe^{i\Theta}k_{[\alpha}e_{\beta]}). \tag{A10}$$

It represents the electromagnetic field of type N (the null field) since it satisfies the relations

$$(F_{\alpha\beta}+iF^{*}_{\alpha\beta})k^{\beta}=0,$$

$$F_{\alpha\beta}F^{\alpha\beta}=F_{\alpha\beta}F^{*\alpha\beta}=0, \tag{A11}$$

where $F^{*}_{\alpha\beta}$ is dual to $F_{\alpha\beta}$. Equations (A7) and (A8) imply the covariant conservation law for the electromagnetic energy-momentum tensor

$$T^{\alpha\beta}=A^2k^{\alpha}k^{\beta}. \tag{A12}$$

We see that the phenomenological null dust equations (2.9) and (3.16)–(3.18) are the same as Eqs. (A6)–(A8) and (A12) of the high-frequency limit of the Maxwell theory if the null vector field k_{α} is defined by Eq. (A3) and the mass distribution M is identified with the square of the scalar amplitude A:

$$M=A^2. \tag{A13}$$

Null dust thus exhibits all features of the geometrical optics limit of Maxwell's theory except for the polarization properties. However, starting from a solution of the null dust equations one can always construct a polarization vector e_{α} such that Eqs. (A9) are also satisfied. This yields the tensor (A10) which can be regarded as an electromagnetic field tensor in the geometrical optics approximation.

[6]This is different from the energy-momentum tensor of a perfect fluid with the equation of state $p=M/3$, which represents the superposition of waves with random propagation directions.

The laws of geometrical optics can also be interpreted as describing photons that move along null rays with the flux vector which is determined by the amplitude A and the null vector k^α (see [3] for details).

The lightlike particles need not necessarily be photons. It is quite obvious that similar conclusions can be reached for all zero-rest-mass fields in a high-frequency limit. For example, by employing the geometrical optics form (A1) of the energy-momentum tensor, several authors [21] studied the gravitational collapse with escaping neutrinos.

A somewhat special case is the gravitational field itself. Careful studies of the high-frequency limit of the gravitational radiation by Isaacson and others [22] have shown that the energy-momentum tensor (A12) and the null vector field k^α which satisfy Eqs. (A6)–(A8) also describe the behavior of high-frequency gravitational waves. The metric tensor perturbations representing high-frequency waves are given by

$$h_{\alpha\beta}=\mathrm{Re}((a_{\alpha\beta}-\tfrac{1}{2}a\overset{0}{\gamma}_{\alpha\beta})e^{i\Theta}),$$

$$a:=\overset{0}{\gamma}{}^{\alpha\beta}a_{\alpha\beta}, \tag{A14}$$

where $\overset{0}{\gamma}_{\alpha\beta}$ is the background metric (the source of which may be the high-frequency waves themselves). By applying the geometrical optics approximation to the perturbed Einstein's equations, one arrives again at Eqs. (A3), (A6)–(A8), and (A12). Instead of the scalar amplitude (A4) one now gets

$$A=(\tfrac{1}{2}a^{\alpha\beta}\overline{a}_{\alpha\beta})^{1/2}. \tag{A15}$$

One also obtains the equations for the polarization tensor $e_{\alpha\beta}=a_{\alpha\beta}/A$, analogous to Eqs. (A9) (see [3], [22]). The Riemann tensor of the metric (A14) has the Petrov type N. The gravitational field in the high-frequency limit is null, similarly as the electromagnetic field. The well-known peeling-off property of exact radiative (zero-rest-mass) fields in asymptotically flat spacetimes [4] implies that at large distances from the source these fields are null, having the structure of plane waves. In asymptotic regions one can even describe *exact* solutions of the field equations in terms of null dust. In such situations, one can usually find a natural parametrization of null rays, for example, by the proper time of distant observers at rest with respect to an isolated source.

The variational approach of MacCallum and Taub [23] to the high-frequency gravitational waves is especially relevant for the present paper. By applying the ''averaged Lagrangian technique'' of Witham to the second variation Lagrangian for the perturbations of vacuum gravitational field, these authors give a variational principle for approximately periodic gravitational wave described by metric perturbation of the form (A14). Their principle, derived by perturbing and averaging the Hilbert action, implies the geometrical optics equations (A3), (A6)–(A8), and (A12), with A given by Eq. (A15). This principle is closely related to our variational principle for null dust [given in Eqs. (4.4) and (4.5)], in the special case of the hypersurface orthogonal vector field l^α.

APPENDIX B: EXACT SOLUTIONS WITH NULL DUST: EXAMPLES AND SOME RECENT APPLICATIONS

As an illustration, we shall give a few examples of known exact spacetimes with null dust. (A detailed survey of such solutions found before 1980 is given in [24]. The cosmological solutions with null dust were recently reviewed in [25], and the solution representing colliding plane gravitational waves accompanied by null dust in [26].)

Among the simplest solutions directly related to the fields arising in the geometrical optics limit are conformally flat null dust solutions representing special plane waves. They are described by the line element (see, e.g., [24])

$$ds^2=-\frac{1}{4}\Phi^2(u_-)(x^2+y^2)du_-^2-2du_+du_-+dx^2+dy^2, \tag{B1}$$

where Φ is an arbitrary function of a retarded time u_-. The corresponding energy-momentum tensor is

$$T_{\alpha\beta}=\Phi^2k_\alpha k_\beta; \tag{B2}$$

the only nonvanishing component of the null covector k_α is $k_{u_-}=1$. These solutions can always be interpreted as exact solutions of the Einstein-Maxwell equations with the null electromagnetic field given by $F_{\alpha\beta}=2\Phi(u_-)k_{[\alpha}e_{\beta]}$, where $e_\alpha=(0,0,\cos\psi,\sin\psi)$ contains an arbitrary function $\psi=\psi(u_-)$ [cf. Eq. (A10)]. Cylindrical gravitational waves accompanied by null dust are also known [27].

A more complicated class of radiative solutions with ''spherical'' gravitational waves and null dust is formed by the Robinson-Trautman solutions [28]. The energy-momentum tensor again has the form (B2), but the function Φ is now given by $\Phi^2=n^2(\zeta,\overline{\zeta},u_-)/v^2$, where ζ is a complex spatial coordinate, v is an affine parameter along the rays, and u_- is a retarded time. The function n may be arbitrary. If, however, these solutions should represent exact Einstein-Maxwell fields, n must have the form $n^2=2h\overline{h}P^2$, where $h(\zeta,\overline{\zeta},u_-)$ and $P(\zeta,\overline{\zeta},u_-)$ satisfy certain additional conditions [24]. The Robinson-Trautman solutions with null dust include Vaidya's spherically symmetric metric as a special case. In fact, if the evolving null dust is homogeneous, all such Robinson-Trautman spacetimes approach Vaidya's metric as the retarded time goes to infinity [29].

The null vector field k^α in the solutions we have mentioned is hypersurface orthogonal and the corresponding null congruence is thus nontwisting. The twisting null dust solutions are discussed in [30], the best known simple example being the ''radiating Kerr metric.''

Some exact solutions with null dust can also be interpreted as exact solutions of Einstein's equations coupled to a massless scalar field [31]. However, given a conserved energy-momentum tensor in the form (A1), it is not necessarily true that the mass distribution M and the null vector field k^α represent an electromagnetic or a massless scalar field. However, if the null vector field k^α is shear free, a corresponding nontrivial solution of Maxwell's equations can be found by virtue of the Mariot-Robinson theorem [32].

Recently, certain exact solutions with null dust which can be interpreted as ''relativistic rockets'' have been explored in

connection with the properties of gravitational radiation [33]. A number of studies have also been devoted to colliding plane, cylindrical, and spherical systems with null dust [34].

Above all, as we have already stated in the Introduction, the null dust models have been recently used to clarify the formation of naked singularities during a spherical gravitational collapse [35], in the studies of the mass inflation inside black holes [36], and in the models attempting to describe the formation and Hawking evaporation of black holes [37].

APPENDIX C: DESCRIPTION OF TWISTING NULL CONGRUENCES BY PFAFF FORMS: TWO EXAMPLES

Since null congruences are somewhat unusual, we give here two examples of twisting null congruences described by the scalar potentials Z^i and w_i.

(I) In a flat spacetime with Lorentzian coordinates (t,x,y,z), consider a system of lightlike particles which, in each plane perpendicular to the z axis, move in mutually parallel straight lines. As one passes from one plane $z=\text{const}$ to another, the angle α between particle trajectories and the x axis smoothly changes with z: $\alpha=\alpha(z)\in[0,2\pi)$. It is easy to see that the null world lines form a twisting null congruence:

$$\begin{aligned} t&=v+t_0,\\ x&=v\cos\alpha(z)+x_0,\\ y&=v\sin\alpha(z)+y_0,\\ z&=z_0, \end{aligned} \tag{C1}$$

where $v\in\mathbb{R}$ is an affine parameter. The tangent null vectors $k^\alpha=dx^\alpha/dv$ are given by

$$k^\alpha=(1,\cos\alpha(z),\sin\alpha(z),0). \tag{C2}$$

One can readily check that

$$k^\alpha k_\alpha=0, \quad k^\beta\nabla_\beta k_\alpha=0, \tag{C3}$$

confirming that Eq. (C1) is a congruence of null geodesics affinely parametrized by v.

The first comoving coordinate

$$Z^1=z \tag{C4}$$

is trivial: It determines the plane in which the geodesic lies. The second comoving coordinate Z^2 is the coordinate y' of the Cartesian system (x',y',z) obtained from (x,y,z) by the rotation about the z axis by the angle $\alpha(z)$:

$$Z^2=-x\sin\alpha(z)+y\cos\alpha(z). \tag{C5}$$

In the rotated Cartesian systems (x',y',z), the particles move along the x' axes, with $y'=\text{const}$. The third comoving coordinate Z^3 is the retarded time u_- corresponding to that direction:

$$Z^3=u_-=t-x'=t-x\cos\alpha(z)-y\sin\alpha(z). \tag{C6}$$

From Eqs. (C4)–(C6) we obtain the covectors $Z^k{}_{,\alpha}$:

$$\begin{aligned} Z^1{}_{,\alpha}&=(0,0,0,1),\\ Z^2{}_{,\alpha}&=(0,-\sin\alpha(z),\cos\alpha(z),\\ &\quad -x\alpha'(z)\cos\alpha(z)-y\alpha'(z)\sin\alpha(z)),\\ Z^3{}_{,\alpha}&=(1,-\cos\alpha(z),-\sin\alpha(z),\\ &\quad x\alpha'(z)\sin\alpha(z)-y\alpha'(z)\cos\alpha(z)), \end{aligned} \tag{C7}$$

where $\alpha':=d\alpha/dz$. It is easy to see that $Z^k{}_{,\alpha}$ are independent covectors. Since

$$k_\alpha=(-1,\cos\alpha(z),\sin\alpha(z),0), \tag{C8}$$

the decomposition (5.5) is obtained with the coefficients

$$\begin{aligned} w_1&=x\alpha'(z)\sin\alpha(z)-y\alpha'(z)\cos\alpha(z),\\ w_2&=0, \quad w_3=-1. \end{aligned} \tag{C9}$$

One can easily check that w_k are constant along the geodesics, Eq. (5.8). This also follows from Eq. (C1) which allows us to write w_1 in the form $w_1=\alpha'(z_0)(x_0\sin\alpha(z_0)-y_0\cos\alpha(z_0))$. Similarly, one can check that $k^\alpha Z^k{}_{,\alpha}=0$, as given by Eq. (4.6). One can also check the fact mentioned in Sec. IV, that for a twisting congruence all vectors $Z^k{}_{,\alpha}$ are spacelike (except perhaps a set of measure zero). The spacelike character of the vectors $Z^1{}_{,\alpha}$ and $Z^2{}_{,\alpha}$ is evident; for $Z^3{}_{,\alpha}$ we have

$$\eta^{\alpha\beta}Z^3{}_{,\alpha}Z^3{}_{,\beta}=\alpha'(z)^2(x\sin\alpha(z)-y\cos\alpha(z))^2. \tag{C10}$$

The vector $Z^3{}_{,\alpha}$ is thus spacelike unless $\alpha'=0$. Calculating the twist ω of our congruence [see Eq. (3.24)], we find

$$\omega=\frac{1}{2}|\alpha'|. \tag{C11}$$

Hence, if the congruence is twisting, all the three vectors $Z^k{}_{,\alpha}$ are spacelike. When $\alpha'=0$, $k_\alpha=Z^3{}_{,\alpha}$, so that the congruence is hypersurface orthogonal.

Instead of the comoving coordinates Z^k, one can, of course, use other comoving variables $Z^{k'}=Z^{k'}(Z^i)$. Also, one can parametrize the geodesics by a label u different from the affine parameter v. When one changes the parametrization, $v=v(u,Z^k)$, the null vectors are rescaled:

$$k^\alpha\to U^\alpha=\frac{dx^\alpha}{du}=\frac{\partial v}{\partial u}k^\alpha. \tag{C12}$$

This leads to a new decomposition, namely

$$U_\alpha=\mathrm{W}_2Z^2{}_{,\alpha}+\mathrm{W}_3Z^3{}_{,\alpha}, \tag{C13}$$

where $\mathrm{W}_2=(\partial v/\partial u)w_2$, $\mathrm{W}_3=\partial v/\partial u$. As discussed in Sec. V, if the congruence is not affinely parametrized, i.e., if $\partial v/\partial u\neq\text{const}$, the coefficients W_k are not necessarily comoving.

(II) The second example will be described only briefly. It is the familiar ingoing principal null congruence in Kerr spacetime. In the ingoing Kerr coordinates $(\tilde{V},r,\theta,\tilde{\varphi})$ which

generalize the ingoing Eddington-Finkelstein coordinates of the Schwarzschild metric, the Kerr metric reads (our notation follows [3])

$$\begin{aligned} ds^2 = &-(1-2Mr\rho^{-2})d\tilde{V}^2 + 2dr\,d\tilde{V} + \rho^2 d\theta^2 \\ &+ \rho^{-2}[(r^2+a^2)^2 - \Delta a^2 \sin^2\theta]\sin^2\theta\, d\tilde{\varphi}^2 \\ &- 2a\sin^2\theta\, d\tilde{\varphi}\, dr - 4aMr\rho^{-2}\sin^2\theta\, d\tilde{\varphi}\, d\tilde{V}. \end{aligned} \tag{C14}$$

Here, the constant parameters M and a are the mass and angular momentum per unit mass, and the functions Δ and ρ have the form

$$\Delta = r^2 - 2Mr + a^2, \quad \rho^2 = r^2 + a^2\cos^2\theta. \tag{C15}$$

The ingoing null Kerr congruence is given by

$$\tilde{V} = \text{const}, \quad r = -v, \quad \theta = \text{const}, \quad \tilde{\varphi} = \text{const}, \tag{C16}$$

where we have absorbed a constant energy parameter into the affine parameter v (cf. [3]). The coordinates $Z^1 := \tilde{V}$, $Z^2 := \theta$, $Z^3 := \tilde{\varphi}$ are clearly comoving. We can easily form the basis vectors $Z^k{}_{,\alpha}$:

$$\begin{aligned} Z^1{}_{,\alpha} &= (1, 0, 0, 0), \\ Z^2{}_{,\alpha} &= (0, 0, 1, 0), \\ Z^3{}_{,\alpha} &= (0, 0, 0, 1). \end{aligned} \tag{C17}$$

The covariant components of the tangent null vector $k^\alpha = dx^\alpha/dv$ are

$$k_\alpha = (-1, 0, 0, a\sin^2\theta). \tag{C18}$$

Its decomposition into the three covectors $Z^k{}_{,\alpha}$ yields the coefficients

$$w_1 = -1, \quad w_2 = 0, \quad w_3 = a\sin^2\theta, \tag{C19}$$

which are constant along the null geodesics (C16), in accordance with Eq. (5.8). The covariant metric can be read off from Eq. (C14). The norms of the vectors $Z^k{}_{,\alpha}$ are

$$\begin{aligned} g^{\alpha\beta} Z^1{}_{,\alpha}\, Z^1{}_{,\beta} &= \rho^{-2} a^2 \sin^2\theta, \\ g^{\alpha\beta} Z^2{}_{,\alpha}\, Z^2{}_{,\beta} &= \rho^{-2}, \\ g^{\alpha\beta} Z^3{}_{,\alpha}\, Z^3{}_{,\beta} &= (\rho\sin\theta)^{-2}, \end{aligned} \tag{C20}$$

where ρ^2 is given by Eq. (C15). We see that all the vectors $Z^k{}_{,\alpha}$ are spacelike as long as $a \neq 0$, i.e., when the congruence is twisting. The twist ω, given by Eq. (3.24), is

$$\omega = |a\cos\theta|\rho^{-2}. \tag{C21}$$

The congruence (C16) is twisting even in the flat-space limit of the Kerr metric, obtained by putting $M = 0$. In fact, Eqs. (C20) and (C21) are independent of M. With $a = 0$, the vector $Z^1{}_{,\alpha}$ becomes null and the congruence is hypersurface orthogonal, $k_\alpha = -Z^1{}_{,\alpha}$, i.e., nontwisting.

The comoving coordinates $Z^2 = \theta$, $Z^3 = \tilde{\varphi}$ are simple, but they become singular at the axis $\theta = 0$ and $\theta = \pi$, the magnitude of the vector $Z^3{}_{,\alpha}$ becoming infinite, Eq. (C20). It is easy, however, to cure this defect by going over to another pair of comoving coordinates, $Z^{2'}$ and $Z^{3'}$, e.g.,

$$Z^{2'} = \sin\tilde{\varphi}\,\sin\theta, \quad Z^{3'} = \cos\tilde{\varphi}\,\sin\theta. \tag{C22}$$

Then

$$k_\alpha = -Z^1{}_{,\alpha} + a\sin\theta\,\cos\tilde{\varphi}\, Z^{2'}{}_{,\alpha} - a\sin\theta\,\sin\tilde{\varphi}\, Z^{3'}{}_{,\alpha}, \tag{C23}$$

and

$$\begin{aligned} g^{\alpha\beta} Z^{2'}{}_{,\alpha}\, Z^{2'}{}_{,\beta} &= \rho^{-2}(1 - \sin^2\tilde{\varphi}\,\sin^2\theta), \\ g^{\alpha\beta} Z^{3'}{}_{,\alpha}\, Z^{3'}{}_{,\beta} &= \rho^{-2}(1 - \cos^2\tilde{\varphi}\,\sin^2\theta) \end{aligned} \tag{C24}$$

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