Gravitational geons revisited

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A careful analysis of the gravitational geon solution found by Brill and Hartle is made. The gravitational wave expansion they used is shown to be consistent and to result in a gauge-invariant wave equation. It also results in a gauge-invariant effective stress-energy tensor for the gravitational waves provided that a generalized definition of a gauge transformation is used. To leading order this gauge transformation is the same as the usual one for gravitational waves. It is shown that the geon solution is a self-consistent solution to Einstein's equations and that, to leading order, the equations describing the geometry of the gravitational geon are identical to those derived by Wheeler for the electromagnetic geon. An appendix provides an existence proof for geon solutions to these equations. [S0556-2821(97)02220-0]

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I. INTRODUCTION

Brill and Hartle [1] (BH) developed a very useful method for finding approximate solutions to Einstein's equations that correspond to high frequency gravitational waves propagating in a background geometry, which is created by the average stress energy of the waves themselves. In their paper they applied this method to the case of a static spherically symmetric background geometry and found that gravitational waves can remain confined in a region for a time much longer than the region's light-crossing time. This so-called *gravitational geon* is generated by a large number of high frequency, small amplitude gravitational waves. The time average of the curvature due to these waves creates the background geometry of the geon, and this background geometry traps the waves for a long time in a region of space called the ''active'' region.[1] The BH solution is important because it serves as an example in which the gravitational field both creates and responds to its own effective stress energy. It is also an example of a nontrivial (approximate) solution to the vacuum Einstein equations that has no curvature singularities.

There are three reasons for analyzing the solution found by BH in more detail than was done in their original paper. The first is that BH did not investigate the question, is the stress-energy tensor they used conserved or gauge invariant? To have a self-consistent set of equations it is necessary that the effective stress-energy tensor have these properties. Second, the validity of the geon solution found by BH has been questioned by Cooperstock, Faraoni, and Perry [4,5]. They claim that the thin shell approximation used by BH to describe the active region is invalid, and that gravitational geons due to high frequency, large angular momentum waves do not exist. The third reason for analyzing the BH solution in more detail is that, because they used a thin shell approximation, BH did not determine the form of the geometry within the active region. The active region is the only region where the gravitational waves have a significant size, and it is the region where spacetime is most strongly curved. It is clearly important to know the geometry of the active region, if one wishes to learn anything about the details of the geon solution.

In this paper a thorough analysis of the geon solution found by BH is presented. A self-consistent expansion of the metric and curvature tensors is found. Using results recently obtained regarding effective stress-energy tensors for gravitational waves [6], it is argued that, to leading order, the wave equation and stress-energy tensor are gauge invariant. It is also shown that the stress-energy tensor is conserved to leading order with respect to the background geometry. The wave and backreaction equations are explicitly derived in the high frequency and large angular momentum limits. It is shown that in and near the active region these equations can be cast in a form which is mathematically identical to the equations derived by Wheeler [2] for the electromagnetic geon. Thus, to leading order, the background geometry of the gravitational geon found by BH is identical to that of the electromagnetic geon found by Wheeler.[2] This means that the details of the active region as discussed by Wheeler [2] are identical (mutatis mutandis) to those of the gravitational geon found by BH. There is of course a difference in the wave equations. However, the effective time averaged stress-energy of the gravitational waves is, to leading order, identical to that of the electromagnetic waves.

In Sec. II a review of the BH solution is given. In Sec. III

[1]Gravitational geons are analogous to the original electromagnetic geons of Wheeler [2], which are virtual gravitationally bound states of electromagnetic energy. As such, geons have a finite lifetime and are not true nonradiative solutions. Gibbons and Stewart [3] have shown that nonradiative geons, or other exactly periodic solutions, cannot exist in Einstein's theory.

[2]It is already clear from their paper that the geometry BH found in the regions exterior to the active region is exactly the same as that found by Ernst [7] for the electromagnetic geon.

a self-consistent expansion of the wave equation and the effective stress-energy tensor for the gravitational waves is presented. Explicit leading order equations are derived in Sec. IV, and it is shown that in the active region the equations to leading order are those found by Wheeler for the electromagnetic geon. Extension of the calculation beyond the leading order is also discussed. Our conclusions are summarized in Sec. V.

II. THE BRILL-HARTLE SOLUTION

The gravitational geon found by BH is a solution to Einstein's equations that consists of gravitational waves propagating on a static spherically symmetric background, which is created by the waves. There is a large number of waves, each with a small amplitude, a high frequency, and a large angular momentum. The waves have different angular orientations and somewhat different frequencies. The stress energy of the waves is significant in and near a spherical shell called the active region, and insignificant elsewhere. The solution is self-consistent in that the geometry produced by the stress energy of the waves traps them in the active region for a long time.

We begin by reviewing the BH gravitational wave expansion and the BH definition of an effective stress-energy tensor for the gravitational waves. Consider a separation of the metric into a background part $\gamma_{\mu\nu}$ and a perturbation $h_{\mu\nu}$:

$$g_{\mu\nu}=\gamma_{\mu\nu}+h_{\mu\nu}. \tag{1}$$

The Einstein tensor can also be divided into a part describing the curvature due to the background geometry and that due to the perturbation:

$$G_{\mu\nu}(g_{\alpha\beta})=G_{\mu\nu}(\gamma_{\alpha\beta})+\triangle G_{\mu\nu}(\gamma_{\alpha\beta},h_{\alpha\beta}). \tag{2}$$

It is important to note that $\triangle G_{\mu\nu}$ is defined by this equation.

One way to specify the separation (1) is to use some smoothing or averaging procedure $\langle\ \rangle$ acting on $G_{\mu\nu}$, and demand that this procedure not affect the value of $G_{\mu\nu}(\gamma_{\alpha\beta})$. In addition, one demands both the exact and the averaged Einstein equations for vacuum. This leads to

$$\triangle G_{\mu\nu}(\gamma,h)=\langle\triangle G_{\mu\nu}(\gamma,h)\rangle, \tag{3a}$$

$$G_{\mu\nu}(\gamma)=-\langle\triangle G_{\mu\nu}(\gamma,h)\rangle. \tag{3b}$$

(Here and hereafter the indices of the arguments of $G_{\mu\nu}$ and $\triangle G_{\mu\nu}$ are suppressed for notational simplicity.) The first equation is regarded as a wave equation, which gives the behavior of the perturbed geometry. The second equation is the backreaction equation, which describes how the background geometry is affected by the perturbed geometry. From the backreaction equation it is natural to define an effective stress-energy tensor for the gravitational waves as

$$\langle T_{\mu\nu}\rangle=-\frac{1}{8\pi}\langle\triangle G_{\mu\nu}(\gamma,h)\rangle. \tag{4}$$

For any valid perturbation expansion $\triangle G_{\mu\nu}$ can be written in the form

$$\triangle G_{\mu\nu}=\triangle_1 G_{\mu\nu}+\triangle_2 G_{\mu\nu}+\cdots. \tag{5}$$

BH obtained an expansion of the form (5) by first expanding $\triangle G$ in powers of h and its derivatives. They then considered the high frequency, large angular momentum limit of this expansion. Thus their leading order term $\triangle_1 G$ consists of the high frequency large angular momentum limit of the terms in the original expansion which are linear in h. Their second-order term $\triangle_2 G$ consists of the appropriate high frequency, large angular momentum limits of the terms in the original expansion which are quadratic in h.[3] BH used a time averaging for their stress-energy tensor. This plus the high frequency of the waves resulted in a stress-energy tensor consisting of the time average of the high frequency large angular momentum limits of the quadratic terms in the original expansion of $\triangle G$ in powers of h and its derivatives. To completely fix the value of the stress-energy tensor they implicitly made the choice[4] $\langle\triangle_1 G_{\mu\nu}\rangle=0$. The resulting wave and backreaction equations are

$$\triangle_1 G_{\mu\nu}(\gamma,h)=0, \tag{6a}$$

$$G_{\mu\nu}(\gamma)=-\langle\triangle_2 G_{\mu\nu}(\gamma,h)\rangle. \tag{6b}$$

The background metric for the geon solution is static and spherically symmetric, so it can be written in the form

$$\gamma_{\mu\nu}=\mathrm{diag}\,(-e^{\nu},e^{\lambda},r^2,r^2\sin^2\theta). \tag{7}$$

BH used a variational approach to find their solution, the essence of which is to show that the effective averaged stress-energy tensor has vanishing trace to leading order. This suffices to define the metric outside the active region, which is near $r=a$. The value of a depends on the ratio of the angular momentum to the frequency of the waves. In the infinite frequency and angular momentum limits the active region shrinks to an infinitely thin shell at $r=a$. BH's self-consistent treatment found in this limit that

$$e^{\nu}=\frac{1}{9},\quad e^{-\lambda}=1,\quad r<a,$$

$$e^{\nu}=e^{-\lambda}=1-\frac{2M}{r},r>a, \tag{8}$$

and that the mass of the geon is $M=(4/9)a$. Examination of Eq. (8) suggests that, in the active region, $\partial\lambda/\partial r$ is large while $\partial\nu/\partial r$ is of order unity.[5]

[3]It also contains certain terms from the original expansion that are linear in h. However these vanish when a time average is taken.

[4]It is shown in [6] that Eq. (4) does not completely specify the definition of the stress-energy tensor. An extra condition must be imposed to determine uniquely the value of $\langle\triangle G\rangle$. Although not presented as such in their paper, the condition $\langle\triangle_1 G\rangle=0$ imposed by BH served to fix the value of $\langle\triangle G\rangle$ in the gauge they worked in. See Sec. III A for further discussion.

[5]We use the term ''unity'' to mean the power a^n of the geon radius a, with n adjusted so ''unity'' has the same dimension as the quantity with which it is being compared.

III. CONSISTENCY OF THE GRAVITATIONAL WAVE EXPANSION

Several consistency questions must be answered to validate the BH solution to Eqs. (6a) and (6b). First, the effective stress-energy tensor of the gravitational waves, Eq. (4), must be conserved with respect to the background geometry γ in order to be a source of this background geometry. Second, the solution ansatz for the wave equation must be sufficiently general so that this set of equations can be simultaneously solved. Third it is necessary to investigate gauge invariance. For the wave and backreaction equations to be consistent, the effective stress-energy tensor must be at least approximately gauge invariant if the wave equation is. Also BH used a gauge transformation to reduce the number of components of $h_{\mu\nu}$. Finally it is necessary to show that the expansion of $\triangle G$ used by BH is a valid one for the geon solution.

It is appropriate to note that we are here constructing one particular type of geon, as described in the beginning of Sec. II. No attempt is made to discuss other possible gravitational geons, such as thick-shell ones [9]. To show existence of geon solutions it is only necessary to establish that one particular, carefully selected, set of waves does form a geon.

A. Conservation and gauge invariance

Conservation of the effective stress-energy tensor for gravitational waves can be established in a straightforward manner. If we have a solution of Eq. (3b), then the Bianchi identity satisfied by $G_{\mu\nu}(\gamma)$ implies that the exact stress-energy tensor (4) is conserved with respect to the background geometry $\gamma_{\mu\nu}$. Thus the approximate stress-energy tensor is similarly conserved to leading order so long as a self-consistent expansion is used for the gravitational waves.

The question of whether the wave equation and the effective stress-energy tensor are gauge invariant is much more difficult to answer. It is useful in this regard to discuss some general results, first for the wave equation and then for the stress-energy tensor.

If the background geometry γ is a solution to the vacuum Einstein equations then it can be shown that if an expansion of the form (5) is used with $\triangle_n G$ consisting of terms of nth order in h and its derivatives, then the wave equation (6a) is exactly invariant under gauge transformations of the form

$$\overline{h}_{\mu\nu}(x)=h_{\mu\nu}(x)-\gamma_{\mu\alpha}(x)\xi^{\alpha}{}_{,\nu}-\gamma_{\alpha\nu}(x)\xi^{\alpha}{}_{,\mu}-\gamma_{\mu\nu,\alpha}\xi^{\alpha}. \tag{9}$$

Here ξ results from an arbitrary coordinate transformation of the form

$$\overline{x}^{\mu}=x^{\mu}+\xi^{\mu}. \tag{10}$$

If the same type of expansion is used but the background geometry is a solution of Eq. (6b) and if the transformations are restricted so that ξ and its derivatives are no larger in magnitude than h and its derivatives, then it can be shown that the wave equation is invariant to $O(h^2)$ under the above type of gauge transformation. For high frequency, large momentum gravitational waves and the same restrictions on ξ, Isaacson [10] showed that the leading order wave equation is invariant under gauge transformations of the form (9) to second order in his expansion. It is not hard, using an argument similar to that used by Isaacson, to show that the wave equation for the gravitational geon solution is also invariant to second order under gauge transformations of the form (9) so long as ξ and its derivatives are no larger in magnitude than h and its derivatives. Here and hereafter by nth order we mean of the same order as $\triangle_n G(\gamma,h)$ in a self-consistent expansion of the form (5).

Gauge invariance of the effective stress-energy tensor is a different matter. For Eqs. (6a) and (6b) to be consistent, the stress-energy tensor must at least be gauge invariant to second order. However, Isaacson showed that $\triangle_2 G$ is not invariant under transformations of the form (9). The only case he found in which the stress-energy tensor and thus the back reaction equation (6b) are gauge invariant to second order is the case of high frequency, large momentum waves when the averaging is over a region of spacetime which is large compared to the wavelengths of the waves but small compared to the scale on which the background geometry varies.

One might hope that in the geon case, where high frequency, large angular momentum waves are used, that the stress-energy tensor would similarly be gauge invariant to second order. However, an explicit calculation using the gauge transformation (17) below shows that this is not the case. In fact, even when the background geometry is a solution to the vacuum Einstein equations (including the case of the flat space solution), the stress-energy tensor is not gauge invariant if time averaging rather than spacetime averaging is used. Thus it appears that Eqs. (6a) and (6b), which were implicitly solved by BH to obtain the geon solution, are inconsistent when time averaging is used.

The resolution to this very serious problem is given in Ref. [6]. It is as follows: First Eqs. (6a) and (6b) must be replaced by the equations that result from substituting Eq. (5) into Eqs.(3a) and (3b). The result to nth order is

$$\triangle_1 G+\cdots+\triangle_n G=\langle\triangle_1 G+\cdots+\triangle_n G\rangle, \tag{11a}$$

$$G(\gamma)=-\langle\triangle_1 G+\cdots+\triangle_n G\rangle. \tag{11b}$$

Then a generalized gauge transformation is used. It is arrived at by using the coordinate transformation (10) and not allowing the functional form of the background geometry to change under this coordinate transformation, that is $\overline{g}_{\mu\nu}=\gamma_{\mu\nu}+\overline{h}_{\mu\nu}$. Then $\overline{h}_{\mu\nu}$ is given implicitly by the equation

$$\begin{aligned}\gamma_{\mu\nu}(x)+h_{\mu\nu}(x)=&\ \gamma_{\mu\nu}(\overline{x})+\overline{h}_{\mu\nu}(\overline{x})\\&+[\gamma_{\mu\alpha}(\overline{x})+\overline{h}_{\mu\alpha}(\overline{x})]\xi^{\alpha}{}_{,\nu}\\&+[\gamma_{\alpha\nu}(\overline{x})+\overline{h}_{\alpha\nu}(\overline{x})]\xi^{\alpha}{}_{,\mu}\\&+[\gamma_{\alpha\beta}(\overline{x})+\overline{h}_{\alpha\beta}(\overline{x})]\xi^{\alpha}{}_{,\mu}\xi^{\beta}{}_{,\nu}.\end{aligned} \tag{12}$$

Here derivatives of ξ are with respect to x not $\overline{x}$. The generalized gauge transformation is defined as one in which the quantity $\overline{h}(x)$ is substituted for $h(x)$ into the expression of interest. If h, ξ and their derivatives are small enough, then to leading order this gauge transformation reduces to the usual transformation (9).

It is proven in Ref. [6] that, when the vacuum field equations are satisfied, $\triangle G_{\mu\nu}$ is invariant under this generalized gauge transformation. It is also argued in that paper that the quantity

$$\triangle_1 G_{\mu\nu} + \cdots + \triangle_n G_{\mu\nu}$$

is gauge invariant to nth order. This implies that Eqs. (11a) and (11b) are invariant to nth order under generalized gauge transformations. Thus there is no problem with gauge invariance to any order so long as generalized gauge transformations are used.

In the same paper it was shown that there is a large amount of freedom available in choosing the form of the effective stress-energy tensor for gravitational waves. This freedom is related to the freedom one has in choosing in the split between the background and the perturbed geometry. Thus, as noted above, the BH ansatz

$$\langle \triangle_1 G_{\mu\nu} \rangle = 0 \tag{13}$$

is simultaneously a choice of the form of the stress-energy tensor and a definition of the separation between the background and perturbed geometry that is to be used. From the above discussion it is clear that the condition (13) is, in general, only gauge invariant to first order. The lack of exact gauge invariance in this condition is a reflection of the fact that the splitting between the background and the perturbed metric is inherently gauge dependent.

As mentioned above, an explicit calculation which we have made using the gauge transformation (17) below shows that, even after time averaging, $\langle \triangle_2 G_{\mu\nu} \rangle$ is not separately gauge invariant. The problem is simply that the condition (13) was used along with the expansion (5) in the derivation of Eqs. (6a) and (6b). However, this condition can only be imposed in a particular gauge since it is not exactly gauge invariant. Since BH solved Eqs. (6a) and (6b) and since we also use those equations as the starting point of the derivations in Sec. IV, it is necessary to show that these equations are consistent with the second-order version of Eqs. (11a) and (11b).

To begin note that to actually solve Eqs. (11a) and (11b) in a particular gauge one can follow Isaacson [10] and make the following expansion for h:

$$h = h^{(1)} + h^{(2)} + \cdots, \tag{14}$$

where $h^{(n)}$ is defined such that $\triangle_1 G(\gamma, h^{(n)})$ is of the same order as $\triangle_n G(\gamma, h^{(1)})$. Then to second order Eqs. (11a) and (11b) can be written

$$\triangle_1 G(\gamma, h^{(1)}) = \langle \triangle_1 G(\gamma, h^{(1)}) \rangle, \tag{15a}$$

$$\triangle_1 G(\gamma, h^{(2)}) + \triangle_2 G(\gamma, h^{(1)}) = \langle \triangle_1 G(\gamma, h^{(2)}) + \triangle_2 G(\gamma, h^{(1)}) \rangle, \tag{15b}$$

$$G(\gamma) = -\langle \triangle_1 G(\gamma, h^{(1)}) + \triangle_1 G(\gamma, h^{(2)}) + \triangle_2 G(\gamma, h^{(1)}) \rangle. \tag{15c}$$

Condition (13) can be imposed by requiring that $\langle \triangle_1 G(\gamma, h^{(n)}) \rangle = 0$ for all n. Then to lowest order Eqs. (15a)–(15c) are equivalent to Eqs.(6a) and (6b) with the substitution $h \rightarrow h^{(1)}$.

It is important to emphasize that it is the original equations (11a) and (11b) that are gauge invariant to nth order and that Eqs. (15a)–(15c) and thus Eqs. (6a) and (6b) are in general only correct in a particular gauge. The procedure being followed here is the usual one for finding solutions to Einstein's equations. First the equations are written down in a particular coordinate system and then they are solved (here in an approximate manner) in that coordinate system.

B. A valid expansion of the Einstein tensor

Since the gravitational waves making up the BH geon are high frequency, large angular momentum waves, it is necessary to find an expansion of the Einstein tensor that is appropriate for these waves. The key point for a thin-shell geon is, that while the waves' amplitudes are much smaller than unity, their derivatives are large compared to unity. Further, while the background metric is of order unity, some, but not all, of its derivatives are much larger than unity in the active region (although they are of order unity or smaller well outside of the active region). Thus, whether a term is of leading order will be determined not only by the power of h but also by the power of the frequency ω or of the harmonic order l^* that it contains.

What must be done is to find a self-consistent expansion of the Einstein tensor with the above constraints on the background metric and the perturbations. Since some derivatives of the background metric are large in the active region but of order unity outside of it, this is a complicated task. It clearly depends on the solutions to both the wave equation and back reaction equation which in turn depend on the expansion used.

Fortunately, as is shown by direct calculation in Sec. IV, to second order the method used by BH and discussed in Sec. II works. However for the reasons discussed above, it would be significantly more difficult to compute the third- and higher-order terms in the correct expansion of $\triangle G$.

IV. THE WAVE AND BACKREACTION EQUATIONS

Derivations of the wave and backreaction equations have been attempted at the time of the BH paper [8] and subsequently by us and independently by [5]. Our version provides a correct, independent treatment of these important equations. We begin with the wave equation.

A. The wave equation

The geon is composed of a large number of gravitational waves and the equations for each are the same. BH used the equations $\triangle_1 R_{\mu\nu} = 0$, with $R_{\mu\nu}$ the Ricci tensor. BH showed that to leading order these equations are equivalent to Eq. (6a). We first expand $\triangle R_{\mu\nu}$ in powers of $h_{\mu\nu}$ and its derivatives, so the wave equation is the high frequency and large angular momentum limit of

$$\gamma^{\alpha\beta}[h_{\mu\nu;\alpha\beta}+h_{\alpha\beta;\mu\nu}-h_{\mu\alpha;\nu\beta}-h_{\nu\alpha;\mu\beta}]=0. \tag{16}$$

Here indices as usual are raised and lowered by the background metric $\gamma_{\alpha\beta}$.

Following BH we analyze the angular behavior of solutions to these equations for static spherically symmetric spacetimes by the method of Regge and Wheeler [11], and limit ourselves to azimuthally symmetric, odd-parity functions. In the gauge of Regge and Wheeler the perturbation h is obtained from the general h via a gauge transformation of the form

$$\xi^t=\xi^r=0,$$

$$\xi^\mu=e^{-i\omega t}\Lambda(r)\epsilon^{\mu\nu}\frac{\partial}{\partial x^\nu}Y_l{}^0(\theta) \quad (\mu,\nu=\theta,\phi). \tag{17}$$

Here $\Lambda(r)$ is some function of r, $\epsilon^{\mu\nu}$ is an antisymmetric tensor with $\epsilon^{32}=(r^2\sin\theta)^{-1}$, and $Y_l{}^0(\theta)$ is a spherical harmonic. In their gauge the "odd" solutions are then written in the form

$$h_{\mu\nu}=r_{\mu\nu}e^{-i\omega t}\sin\theta\ \frac{dY_l{}^0(\theta)}{d\theta}+\text{c.c.},$$

$$r_{\mu\nu}(r)=h_0(r)[\delta_\mu{}^0\delta_\nu{}^3+\delta_\nu{}^0\delta_\mu{}^3]+h_1(r)[\delta_\mu{}^1\delta_\nu{}^3+\delta_\nu{}^1\delta_\mu{}^3]. \tag{18}$$

When these equations are substituted into Eq. (16), and when the background metric is of the form (7), there result the following three equations for h_0 and h_1:

$$i\omega e^{-\nu}\left(\frac{dh_0}{dr}-\frac{2h_0}{r}\right)+h_1\left[\frac{l^{*2}}{r^2}-\omega^2e^{-\nu}+\frac{e^{-\lambda}}{r}\left(\lambda_r-\nu_r-\frac{2}{r}\right)\right]=0, \tag{19a}$$

$$i\omega h_0e^{-\nu}+e^{-\lambda}\left[\frac{1}{2}(\nu_r-\lambda_r)h_1+\frac{dh_1}{dr}\right]=0, \tag{19b}$$

$$\frac{d^2h_0}{dr^2}+i\omega\left[\frac{dh_1}{dr}+h_1\left(\frac{2}{r}-\frac{1}{2}(\lambda_r+\nu_r)\right)\right]-\frac{1}{2}(\lambda_r+\nu_r)\frac{dh_0}{dr}-\left[\frac{e^{\lambda}l^{*2}}{r^2}-\frac{2\nu_r}{r}\right]h_0=0. \tag{19c}$$

Here we have put $l^{*2}\equiv l(l+1)$, and denoted radial derivatives of the background metric by subscripts. From Eq. (19b) it is clear that if we take h_1 to be real then h_0 is imaginary. Taking note of the complex conjugate term in Eq. (18) it is then seen that the part of the disturbance associated with h_0 has a time dependence of the form $\sin(\omega t)$ while the part of the disturbance associated with h_1 has a time dependence of the form $\cos(\omega t)$.

By using Eq. (19b) the function h_0 can be eliminated from Eqs. (19a) and (19c). Then the change of variables

$$dr^*=e^{(1/2)(\lambda-\nu)}dr, \tag{20a}$$

$$Q=e^{(1/2)(\nu-\lambda)}\frac{h_1}{r} \tag{20b}$$

results in the two equations[6]

$$\frac{d^2Q}{dr^{*2}}+\left[\omega^2-e^{\nu}\ \frac{l^{*2}}{r^2}-e^{-\lambda/2+\nu/2}\ \frac{3\lambda_{r*}}{2r}+e^{-\lambda/2+\nu/2}\ \frac{3\nu_{r*}}{2r}\right]Q=0, \tag{21a}$$

$$\frac{d^3Q}{dr^{*3}}+\left[-\nu_{r*}+e^{-\lambda/2+\nu/2}\ \frac{3}{r}\right]\frac{d^2Q}{dr^{*2}}+\left[\omega^2-e^{\nu}\ \frac{l^{*2}}{r^2}-e^{-\lambda/2+\nu/2}\ \frac{3\lambda_{r*}}{2r}+e^{-\lambda/2+\nu/2}\ \frac{3\nu_{r*}}{2r}\right]\frac{dQ}{dr^*}+\left[-\omega^2\ \nu_{r*}-e^{-\lambda/2+3\nu/2}\ \frac{l^{*2}}{r^3}+e^{-\lambda+\nu}\ \frac{2\nu_{r*}}{r^2}+e^{-\lambda/2+\nu/2}\ \frac{\lambda_{r*}{}^2}{4r}-e^{-\lambda/2+\nu/2}\ \frac{\lambda_{r*r*}}{2r}-e^{-\lambda/2+\nu/2}\ \frac{\nu_{r*}{}^2}{4r}+e^{-\lambda/2+\nu/2}\frac{\nu_{r*r*}}{2r}+e^{-\lambda/2+\nu/2}\ 3\ \omega^2/r\right]Q=0. \tag{21b}$$

[6]By taking the derivative of Eq. (21a) with respect to r^* it is easy to show that the two equations are not consistent unless the background metric $\gamma_{\mu\nu}$ is an exact solution to the vacuum Einstein equations. This is because for a general background the lack of exact gauge invariance does not allow the simplifying gauge transformation (17), as explained above. However, the approximate gauge invariance of the wave equation results in the equations being consistent to leading order in l^* if the background geometry satisfies the backreaction equations to leading order.

To ascertain the correct limiting form of these equations we take a hint from the EM geon's background and put

$$\omega \sim l^*$$

,

$$\frac{dQ}{dr^*} \sim l^* Q,$$

$$\frac{d\lambda}{dr^*} \ll O(l^*),$$

$$\frac{d^2\nu}{dr^{*2}} \ll O(l^*). \tag{22}$$

(In the following subsections we show that this indeed allows self-consistent solutions.) To leading order (l^{*2}), Eqs. (21a) and (21b) are

$$\frac{d^2Q}{dr^{*2}} + \left[\omega^2 - \frac{e^{\nu} l^{*2}}{r^2}\right] Q = 0, \tag{23a}$$

$$\frac{d^3Q}{dr^{*3}} + \left[\omega^2 - \frac{e^{\nu} l^{*2}}{r^2}\right] \frac{dQ}{dr^*} = 0. \tag{23b}$$

Note that these equations are consistent to leading order in l^*. Note also that it is not being assumed here that $d\lambda/dr^*$ and $d^2\nu/dr^{*2}$ are of order unity or less, simply that they are not of order l^*. It turns out that for the geon solution both of these quantities are of order unity outside of the active region, but they are of order $l^{*2/3}$ inside the active region. It also turns out that dQ/dr^* is of order l^*Q outside the active region, but is of order $l^{*2/3}Q$ inside the active region. Thus Eqs. (23a) and (23b) describe the leading order behavior of the gravitational waves, both inside and outside of the active region.

B. The backreaction equations

The next step in deriving the equations that result in a gravitational geon solution is to consider the effective stress-energy tensor, Eq. (4), for the gravitational waves. As with the wave equation, the easiest way to derive the stress-energy tensor to leading order is to consider the linear and quadratic terms in an expansion of the Einstein tensor in powers of $h_{\mu\nu}$. The large l^* limit of these terms is to be taken in the same way as it was for the wave equation in the previous subsection. The result is averaged over times which are long compared with the gravitational wave period.

The background metric for the geon solution is static and spherically symmetric, so from the backreaction equation (6b) it is clear that the stress-energy tensor should have these same symmetries. Time averaging automatically makes it static over the geon lifetime. Spherical symmetry is a more difficult problem. Arranging the waves so that the stress-energy tensor is spherically symmetric has been discussed in detail by Wheeler [2] in the case of the electromagnetic geon. The arrangements are identical for the gravitational geon. They are briefly summarized here.

First, as can be seen from Eq. (18), to have spherical symmetry in the large l^* limit it is necessary to have many gravitational waves with different angular orientations. Thus we put

$$h_{\mu\nu} = \sum_i (h_{\mu\nu})_i \tag{24}$$

with each $(h_{\mu\nu})_i$ a solution to Eq. (6a) with a different value of ω, a different phase, and a different polar axis. If the phases are random and the values of ω, while different, are all approximately equal and large, then after the time averaging is done, to leading order in ω the cross terms in $\langle \triangle_2 G_{\mu\nu} \rangle$ vanish and

$$\langle \triangle_2 G_{\mu\nu} \rangle \approx \sum_i \langle (\triangle_2 G_{\mu\nu})_i \rangle. \tag{25}$$

It is useful to denote $(\triangle_2 G_{\mu\nu})_I$ as the wave which has a pole at $\theta = 0$. Wheeler [2] has shown that if the distribution of the polar axes of the waves is uniform, then when the waves are all added together one has

$$\langle \triangle_2 G_t{}^t \rangle = \frac{1}{2}\int_0^{\pi} \langle (\triangle_2 G_t{}^t)_I \rangle \sin\theta\, d\theta,$$

$$\langle \triangle_2 G_r{}^r \rangle = \frac{1}{2}\int_0^{\pi} \langle (\triangle_2 G_r{}^r)_I \rangle \sin\theta\, d\theta,$$

$$\langle \triangle_2 G_\theta{}^\theta \rangle = \frac{1}{4}\int_0^{\pi} [\langle (\triangle_2 G_\theta{}^\theta)_I \rangle + \langle (\triangle_2 G_\phi{}^\phi)_I \rangle] \sin\theta\, d\theta. \tag{26}$$

Finally there is the question of conservation. From the discussion of Sec. III A it is clear that $\langle T_{\mu\nu} \rangle$ should be automatically conserved to leading order. Since the geon is here treated to leading order, that is in the large l^* limit, one expects conservation to hold. Direct calculation shows that the stress-energy tensor is conserved to leading order in this limit.

In the large l^* limit the backreaction equations coming from the (t,t), (r,r), and (θ,θ) components of Eq. (6b) in an orthonormal frame are

$$-\frac{1}{r^2} + e^{-\lambda}\frac{1}{r^2} - e^{-\lambda/2-\nu/2}\frac{\lambda_{r^*}}{r} = \frac{-l^{*4}}{8(2l+1)\,\omega^2\, r^4}\left[l^{*2} Q^2 + e^{-\nu}\omega^2 r^2 Q^2 + e^{-\nu} r^2 \left(\frac{dQ}{dr^*}\right)^2\right], \tag{27a}$$

$$-\frac{1}{r^2}+e^{-\lambda}\frac{1}{r^2}+\frac{e^{-\lambda/2-\nu/2}\ \nu_{r*}}{r}=\frac{l^{*4}}{8\,(2\,l+1)\ \omega^2\ r^4}\Bigg[-l^{*2}\ Q^2+e^{-\nu}\omega^2\ r^2\ Q^2-e^{-\lambda/2-\nu/2}6rQ\frac{dQ}{dr^*}+e^{-\lambda/2-3\nu/2}\frac{8\omega^2r^3}{l^{*2}}\ Q\ \frac{dQ}{dr^*}+e^{-\nu}r^2\left(\frac{dQ}{dr^*}\right)^2\Bigg], \tag{27b}$$

$$e^{-(\lambda+\nu)/2}\left(\frac{\nu_{r*}-\lambda_{r*}}{2r}\right)+e^{-\nu}\frac{\nu_{r*r*}}{2}=\frac{l^{*4}}{8(2l+1)\omega^2r^4}\Bigg[-2l^{*2}Q^2+e^{-\nu}7\omega^2r^2Q^2-e^{-2\nu}\frac{4\ \omega^4\ r^4}{l^{*2}}\ Q^2-e^{-\nu}3r^2\left(\frac{dQ}{dr^*}\right)^2+e^{-2\nu}\frac{4\ \omega^2\ r^4}{l^{*2}}\left(\frac{dQ}{dr^*}\right)^2\Bigg]. \tag{27c}$$

In arriving at these equations the following identities were used:

$$\frac{1}{2}\int_{-1}^{1}[P_l(x)]^2dx=\frac{1}{2l+1},$$

$$\frac{1}{2}\int_{-1}^{1}[xP_l(x)P_l{}'(x)]dx=\frac{l}{2l+1},$$

$$\frac{1}{2}\int_{-1}^{1}[P_l{}'(x)]^2dx=\frac{l(l+1)}{2},$$

$$\frac{1}{2}\int_{-1}^{1}(1-x^2)[P_l{}'(x)]^2dx=\frac{l(l+1)}{(2l+1)}. \tag{28}$$

Here $P_l(x)$ is a Legendre polynomial.

The combined set of Eqs. (23a), (23b), (27a), (27b), and (27c) completely specifies the background geometry and the gravitational waves in the high frequency and angular momentum limit. In the next subsection Wheeler's expansion is used to show that in and near the active region it is possible to arrive at a set of equations which are independent of both ω and l^*.

C. Wheeler's expansion for the active region

The active region is defined as the region in which the gravitational waves undergo radial oscillations[7] rather than radial damping, and where their amplitudes are significant. As can be seen from Eq. (23a), the waves will radially oscillate whenever $\omega^2 > e^{\nu} l^{*2}/r^2$. This happens at large r, but in order to construct a geon that lasts for a long time, we must use waves whose amplitudes are completely negligible there. Radial oscillation also happens near $r=a$, where from Eqs. (8) and (23a) one finds[8]

$$a \sim \frac{l^*}{\omega}. \tag{29}$$

By expanding all of the relevant quantities in powers of $l^{*1/3}$, Wheeler was able to arrive at a set of equations which describe the electromagnetic geon in and near the active region and which do not explicitly depend on the value of l^*. A similar derivation is presented here. Before making the expansions it is useful first to change to new variables,

$$r=\frac{\rho}{\omega}, \tag{30a}$$

$$e^{-\lambda}=1-\frac{2L(\rho)}{\rho}, \tag{30b}$$

$$e^{\nu}=\left(1-\frac{2L(\rho)}{\rho}\right)S^2(\rho), \tag{30c}$$

$$Q=\frac{[8(2l+1)]^{1/2}}{l^{*2}}f(\rho). \tag{30d}$$

To remove the dependence on l^* a new radial variable x is defined such that

$$x=(\rho^*-l^*)l^{*-1/3}, \tag{31a}$$

$$dx=l^{*-1/3}d\rho^*. \tag{31b}$$

Then the following expansions are made:

$$\rho=l^*+l^{*1/3}r_0(x)+\cdots, \tag{32a}$$

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[7] By radial oscillations we mean here that the waves oscillate as a function of the radius, not that they oscillate in the radial direction.

[8] Until the equations are solved, the location of the active region is unknown. Wheeler [2] initially expanded about $\rho=l^*$. After solving the equations he found the active region was near $\rho=l^*/3$. He used a scale invariance present in the original equations for the EM geon [and also present in Eqs. (23a), (23b), and (27a)–(27c)] to rescale the solutions accordingly. To facilitate comparison with the equations derived by Wheeler we also expand about $\rho=l^*$.

$$L = l^*\lambda_0(x) + l^{*2/3}\lambda_1(x) + l^{*1/3}\lambda_2(x) + \cdots, \tag{32b}$$

$$S = \frac{1}{k(x)} + l^{*-1/3} q_1(x) + l^{*-2/3} q_2(x) + \cdots, \tag{32c}$$

$$f = l^{*1/3}\phi(x) + \phi_1(x) + l^{*-1/3}\phi_2(x) + \cdots, \tag{32d}$$

$$\left[1 - \left(\frac{l^*S}{\rho}\right)^2 \left(1 - \frac{2L}{\rho}\right)\right] = l^{*-2/3} j(x)k(x) + \ \ldots . \tag{32e}$$

It is next useful to derive several intermediate identities. Using Eqs. (32a)–(32c) in Eq. (32e) one finds

$$\lambda_0 = \frac{1}{2}(1 - k^2), \tag{33a}$$

$$\lambda_1 = q_1 k^3. \tag{33b}$$

Combining Eqs. (20a), (30a), (31b), (32a), and (33a) one finds

$$\frac{dr_0}{dx} = k \tag{33c}$$

Finally Eqs. (23a), (27a), (27b), (30a)–(30d), (31a), (31b), (32a)–(32e), (33a)–(33c) can be combined to obtain, to leading order in l^*, the equations

$$\frac{d^2\phi}{dx^2} + jk\phi = 0, \tag{34a}$$

$$\frac{dk}{dx} + \phi^2 = 0, \tag{34b}$$

$$\frac{dj}{dx} - 3 + \frac{1}{k^2}\left[1 + \left(\frac{d\phi}{dx}\right)^2\right] = 0. \tag{34c}$$

These equations are exactly the same equations that Wheeler [2] found for the electromagnetic geon.[9] Thus we see that, to leading order, the geometry of the gravitational geon is exactly the same as that for the electromagnetic geon. This was already found to be the case by BH for the spacetime outside of the active region. It has now been shown to be true for the geometry inside of the active region.

This result was hinted at when BH noted that in the large frequency and angular momentum limit the equation of motion for the gravitational waves is identical to that for electromagnetic waves. It is also consistent with the result of Isaacson [10] that when the effective stress-energy tensor for high frequency, large momentum gravitational waves is averaged over a small region of spacetime, it is of the same form as that for electromagnetic waves.

For a thin-shell geon it is necessary that the solution to the wave equation (34a) fall off rapidly on both sides of the active region. Equations (34) are therefore similar to an eigenvalue problem in which the amplitude of ϕ (as measured, for example, by its peak value) plays the role of the eigenvalue: through Eqs. (34b) and (34c) the amplitude determines the strength of the ''potential'' jk, which must be just right to have ϕ as a zero energy eigenfunction. For the case of the lowest eigenfunction these equations have been discussed numerically by Wheeler [2], and more recently by Cooperstock *et al.* [5]. The geon metric outside the active region, Eq. (8), was first derived from the numerical solution.[10] Work is currently in progress to investigate the properties of the other solutions. The results will be presented elsewhere.

Although the numerical results constitute strong evidence for the existence of geon solutions, without an existence proof of exact solutions to the basic equations (34) the viability of any type of geon is open to doubt [5]. We provide the existence proof in the Appendix.

A few words are in order about extensions to higher orders in the expansion. In the present approximation the waves are a kind of ''null fluid'' that can be isotropically distributed in the active region. If the expansion of the Einstein tensor is carried out to higher order in l^* the ''graininess'' of the stress-energy due to non-negligible cross terms of finite wavelength in the stress-energy tensor becomes apparent. This invalidates Eq. (25) and makes the stress-energy tensor much more difficult to compute. The background geometry could however remain spherically symmetric by a different choice of splitting between waves and background.

V. CONCLUSIONS

We have obtained a correct self-consistent set of equations for the gravitational geon, which describes the gravitational waves and the background geometry. These equations are accurate in the high frequency, large angular momentum limit. In and near the active region they have been shown to be the same set of equations as those found by Wheeler for the electromagnetic geon. Thus, to leading order, the geometry both inside and outside of the active region of the gravitational geon is identical to that of the electromagnetic geon.

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[9] Cooperstock *et al.* [5] found a different set of equations. Part of the difference is related to the fact that they used a different normalization for the variable f in Eq. (30d). This does not lead to qualitatively different solutions. However, sign errors in their equations do result in qualitatively different solutions. The right-hand sides of their Eqs. (4.22) and (4.23) have the wrong signs given the sign conventions used in their paper. These sign errors result in overall sign errors on the right hand sides of their Eqs. (4.30), (4.31) and (4.46). They also result in a sign error in the last term on the right-hand side of Eq. (4.47).

[10] Wheeler found $e^\nu \approx 0.11$, and Ernst [7] showed $e^\nu = 1/9$ exactly. The method of BH yields the same exact result for all fields that form geons and whose (effective) stress-energy tensor has vanishing trace. Our development in the Appendix yields yet another way to derive the exact result.

APPENDIX: DISCUSSION OF THE GEON EQUATION

Our geons are taken to be governed by Eqs. (34). These equations focus on the active region, and neglect the other oscillating region [beyond $r \sim 2.67M$, where $\omega^2 > e^{\nu} l^{*2}/r^2$, according to Eq (8)]. We can therefore solve Eq. (34a) as a true bound state problem in the effective potential jk, rather than a virtual bound state. The appropriate boundary condition on $\phi(x)$ is therefore

$$\phi(-\infty) = \phi(+\infty) = 0, \tag{A1}$$

From Eq. (34b) it is seen that we can follow Wheeler [2] and choose the boundary condition on k to be

$$k(-\infty) = 1. \tag{A2}$$

[Since $e^{\nu}|_{r=0} = k^2|_{x=-\infty}$ this contradicts Eq. (8), and is due to the use of Wheeler's expansion before rescaling; see footnote 10. Nevertheless, the ratio $e^{\nu}|_{r=0}/e^{\nu}|_{r=\infty} = k^2|_{x=-\infty}/k^2|_{x=\infty}$ will be correctly given.] A third condition follows from Eq. (34c),

$$j(-\infty) = -\infty. \tag{A3}$$

We cannot expect to solve Eqs. (34) with any more conditions than these three; the integration of the equation itself will tell us whether solutions of the geon type are possible, and what their remaining properties are.

We reduce the system (34) to a single equation by considering k as the independent variable and using Eq. (34b) to convert derivatives, $d/dx = -\phi^2 d/dk$. We will denote d/dk by a prime ($'$). The unknown function $j(k)$ can then be eliminated between the two remaining equations. The resulting third-order equation is conveniently written in terms of a new function

$$H = F^2 F'' - 2 - 6k^2,$$

where

$$F = \phi^2, \tag{A4}$$

with the result

$$-H + kH' = 0. \tag{A5}$$

This can be integrated to yield $H = Ak$, with A a constant of integration. Thus the equation to be solved is

$$F^2 F'' = Ak + 2 + 6k^2, \tag{A6}$$

To evaluate the constant A we use the boundary conditions (A1) and (A2) at $x = -\infty$. Because $F^2 F'' = d^2F/dx^2 - F^{-1}(dF/dx)^2$, Eq. (A6) must yield zero at $k = 1$, hence $A = -8$. Next we use Eq. (A1) at $x = +\infty$ to conclude that Eq. (A6) also vanishes for $k(\infty)$, so that

$$k(\infty) = \frac{1}{3}. \tag{A7}$$

In view of Eqs. (30c) and (32c) and the rescaling this yields the geon metric (8) outside the active region (with M not yet determined in terms of ϕ). It is appropriate to rewrite Eq. (A6) in terms of a shifted variable, $u = k - 2/3$ that exhibits the symmetry of this equation,

$$F^2 F'' = 6u^2 - \frac{2}{3}. \tag{A8}$$

The boundary conditions (A1) and (34a) now show that ϕ, F, and jk are even functions of u, and by Eq. (34b) we can choose x to be an odd function of u. Thus our four boundary conditions can be replaced by the more convenient form

$$F' = 0 \quad \text{at} \quad u = 0, \tag{A9a}$$

$$F \to 0 \quad \text{when} \quad u \to 1/3, \tag{A9b}$$

$$x = 0 \quad \text{at} \quad u = 0, \tag{A9c}$$

$$2jF = 1 \quad \text{at} \quad u = 0. \tag{A9d}$$

Of course only Eqs. (A9a) and (A9b) are needed to solve Eq. (A8).

To show existence of solutions it is enough to concentrate on the interval $I: 0 \leqslant u \leqslant 1/3$. Starting with some positive initial value $F(0)$ and condition (A9a), we can always integrate Eq. (A8) to larger and larger u as long as $F(u)$ remains bounded away from zero. If $F(0)$ is sufficiently large, the solution stays positive to $u = 1/3$. For example, the estimate $F^2 F'' = 6u^2 - 2/3 > -2/3$ shows that $F(0) > 1/3(16/\pi^2)^{1/3} \approx 0.392$ is sufficient. We consider all solutions that are positive in the interval I, and call them "solutions in I" for brevity. Because the right side of Eq. (A8) is negative in I, these solutions are decreasing functions in I. Also, if F_1 and F_2 are two solutions in I with $F_2(0) > F_1(0)$, then the difference $F_2 - F_1$ is a finite, increasing function in I because it satisfies

$$(F_2 - F_1)'' = \left(\frac{2}{3} - 6u^2\right)\left(\frac{F_2^2 - F_1^2}{F_1^2 F_2^2}\right) > 0.$$

Therefore solution curves do not cross in I, and the "final value" $F(1/3)$ specifies a unique solution in I [as does the initial value, $F(0)$]. At each $u \in I$ the solutions in I depend continuously (and monotonically) on initial and on final values. Now consider the greatest lower bound (glb) of the final values. If this were positive it could be lowered, for example, by integrating from a smaller initial value, for the integration will run to $u = 1/3$ unless $F(u)$ approaches zero at some $u < 1/3$; but this cannot happen for finite final values because $F(u)$ is a decreasing function. Hence the glb of final values is zero. The corresponding limit of solutions $F(u)$ must therefore be a solution that is positive in $0 \leqslant u < 1/3$ and approaches zero at $u = 1/3$. This is the desired "eigenfunction" that satisfies the boundary condition (A9b).[11]

It remains to be verified that the solution satisfying the boundary conditions in u also satisfies the boundary condi-

[11]The corresponding "eigenvalue," the glb of initial values $F(0) = \phi^2(0)$, can easily be found numerically to be 0.3556 . . . .

tion in x. This follows from Eq. (34b), $du/dx = -F(u)$. The asymptotic form of $F(u)$ near $u = \pm 1/3$,

$$F(u) \to (12)^{1/3}\left(\frac{1}{3} - |u|\right)\left[-\ln\left(\frac{1}{3} - |u|\right)\right]^{1/3}$$

can be integrated to yield

$$x \to \mp \frac{3}{(96)^{1/3}}\left[-\ln\left(\frac{1}{3} - |u|\right)\right]^{2/3} + \text{const.}$$

Thus $u = \mp 1/3$ does correspond to $x = \pm\infty$. (By deriving integral relationships from the differential equations the same conclusion, as well as the fulfillment of the boundary conditions by all the unknown functions, can be established without using asymptotic forms. We refrain from displaying these somewhat involved relationships.)

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