Tensor perturbations in high-curvature string backgrounds

M. Gasperini
Dipartimento di Fisica Teorica, Università di Torino, Via P. Giuria 1, 10125 Turin, Italy
and Istituto Nazionale di Fisica Nucleare, Sezione di Torino, Turin, Italy
(Received 15 April 1997)

We derive a generalized equation for the evolution of tensor perturbations in a cosmological background, taking into account higher-curvature contributions and a tree-level coupling to the dilaton in the string frame. The equation is obtained by perturbing the gravidilaton string effective action, expanded up to first order in α'. The α' corrections can modify the low-energy perturbation spectrum, but the modifications are shown to be small when the background curvature keeps constant in the string frame. [S0556-2821(97)04520-7]

PACS number(s): 04.30.Db, 98.80.Cq

I. INTRODUCTION

The generation of a primordial perturbation spectrum, able to explain the observed large scale anisotropy, is one of the most celebrated aspects of the present inflationary cosmological models [1].

Such a spectrum is obtained through the amplification of the quantum fluctuations of the metric and of the background sources around their initial configuration [2,3], and is usually computed in the context of a lowest-order, scalar-tensor effective theory of gravity. Higher-derivative (i.e., higher-curvature) corrections are usually neglected or, when included, are parametrized by arbitrary functions of the scalar curvature [2,4] (see [5] for a possible exception), without consequence for the primordial spectrum since the effective perturbation equations are left unchanged.

The absence of higher-curvature contributions, however, is hard to swallow for a perturbation spectrum that originates just in the primordial cosmological phase when the Universe, according to the standard scenario, is highly curved and is expected to approach the Planck scale and the quantum gravity regime. It would seem more appropriate, in such a context, to include all possible high-curvature corrections into the effective action, and ask whether their inclusion can in general modify the primordial spectrum obtained from the lowest-order perturbation equations.

The answer to this question is, unfortunately, model dependent, being subordinate to the explicit form of the higher-order terms added to the action. The aim of this paper is to discuss higher-curvature corrections to the tensor perturbation spectrum, in the particular case of the gravidilaton effective action of string theory [6]. In the so-called ''pre-Big Bang'' cosmological models [7] obtained in that context, the occurrence of a high-curvature string phase in which higher-derivative corrections cannot be neglected is indeed an unavoidable consequence of the initial conditions, chosen to approach the string perturbative vacuum [8]. More generally, high-curvature corrections should appear in any complete cosmological scenario based on string theory. Fortunately, in that case, the corrections cannot be added arbitrarily, but are rigidly prescribed by the ''α' expansion'' of the string effective action [9].

We shall limit, in this paper, to the first order in α', namely to the four-derivative terms corresponding to quadratic curvature corrections. Also, we shall work in a homogeneous and isotropic background, with flat $d=3$ spatial dimensions (see [10] for the possible influence of the higher-dimensional dynamics on the scalar, tensor and axion [11] perturbation spectrum). Already at this level, as we will show, the amplification of the tensor fluctuations of the vacuum is governed by a modified perturbation equation. In a background with constant curvature and linearly evolving dilaton, like in the string phase [12–14] typical of the pre-Big Bang scenario, the differences between the corrected higher-order equation and the approximated low-energy equation are small, however, and can be neglected for an order-of-magnitude estimate of the graviton spectrum. In other backgrounds, with running curvature and long enough duration of the high-curvature phase, the modified equation found in this paper may lead instead to important differences in the perturbation spectrum.

The paper is organized as follows. In Sec. II we present the background equations and we perturb the gravidilaton effective action, up to second order in the tensor perturbation variable, including the quadratic curvature terms prescribed by string theory. The diagonalization of the perturbed action defines a new canonical variable, for the normalization of the spectrum to the quantum fluctuations of the vacuum. In Sec. III we discuss the time evolution of the normalized canonical variable in a high-curvature string background, and we estimate the resulting tensor perturbation spectrum. Section IV contains a brief summary and our concluding remarks. The details of the perturbative computation are presented in the Appendix.

II. BACKGROUND AND PERTURBATION EQUATIONS

In the string frame, and to first order in the high-derivative α' expansion, the effective action that reproduces the gravidilaton sector of the tree-level string S matrix can be written in the form [9]

$$S=\int d^4x\sqrt{-g}e^{-\phi}\left\{-R-\partial_\mu\phi\partial^\mu\phi+\frac{k\alpha'}{4}\times[R_{\rm GB}^2-(\partial_\mu\phi\partial^\mu\phi)^2]\right\}. \quad (1)$$

Here ϕ is the dilaton field, $\alpha'=\lambda_s^2$ is the fundamental stringlength parameter governing the importance of the higher-curvature corrections, $k=1,2$ for the bosonic and heterotic string, respectively, and $R^2_{\rm GB}\equiv R^2_{\mu\nu\alpha\beta}-4R^2_{\mu\nu}+R^2$ is the usual Gauss-Bonnet invariant [conventions: $g_{\mu\nu}=(+---)$, $R_{\nu\alpha}=R_{\mu\nu\alpha}{}^{\mu}$, and $R_{\mu\nu\alpha}{}^{\beta}=\partial_\mu\Gamma_{\nu\alpha}{}^{\beta}-\cdots$]. Note that we have chosen a field redefinition that removes terms with higher-than-second derivatives from the effective equations [9,15]. Also, we are considering the case of critical (super-)strings, in which no effective cosmological term appears in the action. This means, in $d=3$ spatial dimensions, that some ''passive'' sector is assumed to be present to cancel the central charge deficit $(3-d_{\rm crit})/\alpha'$. All the computations of this paper can be easily extended, however, to include a nonvanishing cosmological constant.

We shall consider perturbations around a $d=3$, spatially flat background, parametrized by

$$ds^2=N^2(t)dt^2-a^2(t)dx_i^2, \quad \phi=\phi(t), \quad a(t)=e^{\beta(t)} \tag{2}$$

$(i,j=1,2,3)$. For such a background the action (1) becomes, after integration by parts (see the Appendix),

$$S=\int dte^{3\beta-\phi}\left[\frac{1}{N}(6\dot\beta\dot\phi-6\dot\beta^2-\dot\phi^2)+\frac{\alpha'}{4N^3}(8\dot\phi\dot\beta^3-\dot\phi^4)\right]. \tag{3}$$

A dot denotes differentiation with respect to t, and we have put $k=1$, for simplicity. By varying the action with respect to N, β, and ϕ, and imposing the cosmic time gauge $N=1$, we get, respectively, the equations

$$6H^2+\dot\phi^2-6H\dot\phi-\frac{3}{4}\alpha'(8\dot\phi H^3-\dot\phi^4)=0, \tag{4}$$

$$-18H^2-3\dot\phi^2+12H\dot\phi+6\ddot\phi-12\dot H+\alpha'(12\dot\phi H^3+\tfrac{3}{4}\dot\phi^4 -6H^2\dot\phi^2+6H^2\ddot\phi+12\dot\phi H\dot H)=0, \tag{5}$$

$$12H^2+\dot\phi^2-6H\dot\phi-2\ddot\phi+6\dot H+\alpha'(6H^4+\tfrac{3}{4}\dot\phi^4-3H\dot\phi^3 +6H^2\dot H-3\dot\phi^2\ddot\phi)=0 \tag{6}$$

(we have adopted the usual notation for the Hubble parameter, $H=\dot a/a\equiv\dot\beta$, $\dot H=\ddot\beta$). Their solutions provide the gravidilaton background for the propagation of metric perturbations. Note that only two of these equations are independent [14] (unless $\dot\phi=3\dot\beta$), and that the first equation, following from the variation of the lapse function, can be used as a ''non-dynamical'' constraint on the set of initial conditions.

In this paper we shall restrict our attention to tensor metric perturbations, parametrized by the transverse, trace-free variable $h_{\mu\nu}$:

$$g_{\mu\nu}\to g_{\mu\nu}+\delta g_{\mu\nu}, \quad \delta g_{\mu\nu}=h_{\mu\nu}, \quad \nabla_\nu h_\mu{}^\nu=0=h_\mu{}^\nu, \tag{7}$$

where ∇_μ denotes covariant differentiation with respect to $g_{\mu\nu}$, and the indices of $h_{\mu\nu}$ are raised and lowered with the unperturbed metric, $h_\mu{}^\nu=g^{\nu\alpha}h_{\mu\alpha}$. By expanding, up to second order in h, the controvariant components of the metric tensor,

$$\delta^{(1)}g^{\mu\nu}=-h^{\mu\nu}, \quad \delta^{(2)}g^{\mu\nu}=h^{\mu\alpha}h_\alpha{}^\nu \tag{8}$$

($\delta^{(k)}A$ denotes the kth term in the expansion of a variable A in powers of h), of the volume density,

$$\delta^{(1)}\sqrt{-g}=0, \quad \delta^{(2)}\sqrt{-g}=-\frac{1}{4}\sqrt{-g}h_{\mu\nu}h^{\mu\nu}, \tag{9}$$

and of the components of the Riemann and Ricci tensor (see the Appendix), we will obtain from Eq. (1) an action quadratic in the perturbation variable $h_\mu{}^\nu$, governing the dynamic of tensor perturbations in the corresponding gravidilaton background. The method is exactly the same as the one first used in [16] for determining the effective action of tensor perturbations in a cosmological background (see also [2,17]), with the only difference that in the present case the perturbed action includes higher-curvature terms, and a nonminimal coupling of the metric to the dilaton field.

It is convenient to work in the synchronous gauge, where

$$g_{00}=1, \quad g_{0i}=0, \quad g_{ij}=-a^2\delta_{ij},$$

$$h_{00}=0, \quad h_{0i}=0, \quad g^{ij}h_{ij}=0, \quad \partial_j h_i{}^j=0. \tag{10}$$

After repeated use of the identities (see the Appendix)

$$g^{jk}\dot h_{ik}=\dot h_i{}^j+2Hh_i{}^j,$$

$$g^{jk}\ddot h_{ik}=\ddot h_i{}^j+2\dot H h_i{}^j+4H\dot h_i{}^j+4H^2h_i{}^j, \tag{11}$$

we can express the second-order variation of the action (1) as a quadratic form depending on the first and second derivatives of the symmetric, trace-free matrix $h\equiv h_i{}^j$, with time-dependent coefficients fixed by the background fields $a(t)$, $\phi(t)$:

$$
\begin{aligned}
\delta^{(2)}S \equiv& \int d^4x e^{-\phi}\left[-\delta^{(2)}(\sqrt{-g}R)-\delta^{(2)}(\sqrt{-g}\partial_\mu\phi\partial^\mu\phi)+\frac{\alpha'}{4}\delta^{(2)}(\sqrt{-g}R^2_{GB})-\frac{\alpha'}{4}\delta^{(2)}(\sqrt{-g}(\partial_\mu\phi\partial^\mu\phi)^2)\right]\\
=& \int d^4x e^{-\phi}a^3 \,\mathrm{Tr}\left[h^2\left(\frac{1}{4}\dot\phi^2-\frac{3}{2}\dot H-3H^2\right)-h\ddot h-4Hh\dot h-\frac{3}{4}\dot h^2+\frac{1}{4}h\frac{\nabla^2}{a^2}h\right]\\
&+\frac{\alpha'}{4}\int d^4x e^{-\phi}a^3\mathrm{Tr}\left[h^2\left(\frac{1}{4}\dot\phi^4-6\dot H H^2-6H^4\right)-4H^2h\ddot h-h\dot h(8H\dot H+16H^3)-\dot h^2(\dot H+7H^2)\right.\\
&\left.-2H\dot h\ddot h+(2\ddot h+\dot H h+4H\dot h+H^2h)\frac{\nabla^2}{a^2}h+2\dot h\frac{\nabla^2}{a^2}\dot h\right].
\end{aligned}
\tag{12}
$$

Here $\nabla^2=\delta^{ij}\partial_i\partial_j$ is the flat-space Laplace operator, $\mathrm{Tr}h^2\equiv h_i{}^j h_j{}^i$, and so on (the explicit computation of the various terms is reported in the Appendix). Integrating by parts the terms with more than two partial derivatives acting on h, as well as the terms in $h\dot h$ and $h\ddot h$, we can put the action in the convenient form

$$
\begin{aligned}
\delta^{(2)}S\equiv&\int d^4x e^{-\phi}a^3\mathrm{Tr}\left\{\frac{1}{4}\dot h^2(1-\alpha' H\dot\phi)+\frac{1}{4}h\frac{\nabla^2}{a^2}h[1+\alpha'(\dot\phi^2-\ddot\phi)]\right.\\
&\left.+h^2\left[\frac{1}{2}\ddot\phi+H\dot\phi-\frac{1}{4}\dot\phi^2-\dot H-\frac{3}{2}H^2+\frac{\alpha'}{4}\left(\frac{1}{4}\dot\phi^4+2H^2\ddot\phi-2H^2\dot\phi^2+4\dot\phi\dot H H+4\dot\phi H^3\right)\right]\right\}.
\end{aligned}
\tag{13}
$$

The absence of terms with more than two derivatives follows from the fact that the higher-curvature corrections appear in the action as an Euler form (the Gauss-Bonnet invariant). Note also that all α' corrections disappear in the limit $\phi=$const, since in that case the higher-curvature part of the action (1) reduces (in $d=3$) to a total derivative that does not contribute to the variation.

The coefficient of the h^2 term in Eq. (13) is identically vanishing, thanks to the background equation (5). By decomposing the matrix $h_i{}^j$ into the two physical polarization modes of tensor perturbations, h_+ and $h_\times$,

$$
\mathrm{Tr}h^2\equiv h_i{}^j h_j{}^i=2(h_+^2+h_\times^2),
\tag{14}
$$

we can finally write the action, for each polarization mode $h(x,t)$, as

$$
\delta^{(2)}S_h=\frac{1}{2}\int d^4x e^{-\phi}a^3\left\{\dot h^2(1-\alpha' H\dot\phi)+h\frac{\nabla^2}{a^2}h[1+\alpha'(\dot\phi^2-\ddot\phi)]\right\},
\tag{15}
$$

where h is now a scalar variable standing for either one of the two polarization amplitudes h_+, $h_\times$. The variation of the action with respect to h gives then the modified perturbation equation:

$$
\ddot h(1-\alpha' H\dot\phi)+\dot h[3H-\dot\phi-\alpha'(3H^2\dot\phi-H\dot\phi^2+\dot H\dot\phi+H\ddot\phi)]-\frac{\nabla^2}{a^2}h[1+\alpha'(\dot\phi^2-\ddot\phi)]=0.
\tag{16}
$$

In the absence of α' corrections, and for a constant dilaton background, we recover the well-known result $\Box h=0$, describing the propagation of a massless scalar degrees of freedom minimally coupled to the background metric [2,16–18] When $\alpha'=0$, and $\dot\phi\neq 0$, we recover instead the perturbation equation in a Brans-Dicke background [19], $\Box h-\dot\phi\dot h=0$, describing the propagation of gravity waves in the string frame according to the lowest-order string effective action.

Equation (16) controls the time evolution of the Fourier components h_k of the two polarization modes. In order to normalize the spectrum to the quantum fluctuations of the vacuum, however, we need the canonical variable that diagonalizes the perturbed action [2,20], and that represents in this case the normal modes of tensor oscillations of our gravi-dilaton background. We note, to this purpose, that introducing the conformal time coordinate η, defined by $a=dt/d\eta$, the action (15) can be written in the form

$$
\delta^{(2)}S_h=\frac{1}{2}\int d^3x d\eta[z^2(\eta)h'^2+y^2(\eta)h\nabla^2 h],
\tag{17}
$$

where a prime denotes differentiation with respect to η, and

$$
z^2(\eta)=e^{-\phi}\left(a^2-\alpha'\frac{a'}{a}\phi'\right),
$$

$$
y^2(\eta)=e^{-\phi}\left[a^2+\alpha'\left(\phi'^2-\phi''+\frac{a'}{a}\phi'\right)\right].
\tag{18}
$$

By setting $\psi=zh$ the action becomes

$$
\delta^{(2)}S_h=\frac{1}{2}\int d^3x d\eta\left(\psi'^2+\frac{z''}{z}\psi^2+\frac{y^2}{z^2}\psi\nabla^2\psi\right).
\tag{19}
$$

For each Fourier mode we can thus define a canonical variable, $\psi_k = z h_k$, that diagonalizes the kinetic part of the action, and that satisfies an evolution equation of the usual form [2,18],

$$\psi_k'' + [k^2 - V_k(\eta)]\psi_k = 0,$$

$$V_k(\eta) = \frac{z''}{z} - \frac{k^2}{z^2}(y^2 - z^2), \tag{20}$$

with the only difference that the effective potential $V_k(\eta)$ is, in general, k dependent.

This equation, that encodes into the effective potential the higher-curvature corrections [through Eq. (18)], represents the main result of this paper, and will be used in Sec. III to discuss the amplification of tensor fluctuations in the gravidilaton background of a typical string cosmology model.

III. HIGHER-CURVATURE CONTRIBUTIONS TO THE GRAVITON SPECTRUM

In the context of the pre-Big Bang scenario [7,8], typical of string cosmology, the background equations (4)–(6) describe the evolution of the Universe from an asymptotic initial state with $H=0$ and $\dot{\phi}=0$ (the string perturbative vacuum).

As long as the space-time curvature and the dilaton kinetic energy are small in string units, $\alpha' H^2 \ll 1$, $\alpha' \dot{\phi}^2 \ll 1$, the higher-order α' corrections can be neglected, and the low-energy solutions [21] of the background equations describe an accelerated growth of the curvature and of the string coupling, with $H>0$, $\dot{H}>0$, $\dot{\phi}>0$. As soon as the curvature reaches the string scale, however, the effect of the α' corrections tends to stabilize the background in a phase of constant curvature and linearly evolving dilaton, $H=$const, $\dot{\phi}=$const, as recently discussed in [14]. The final transition to the standard, decelerated evolution eventually occurs when the radiation backreaction becomes important, and generates quantum loop corrections to the effective action [22] (the duration of the constant curvature phase, however, is presently unknown and appears into the equations as a phenomenological parameter). The typical accelerated evolution of H and $\dot{\phi}$, obtained through a numerical integration of Eqs. (4)–(6) with the perturbative vacuum as initial condition at $t \to -\infty$, is shown in Fig. 1.

As clearly shown in the figure, the inflationary evolution of this class of backgrounds can be sharply divided into two distinct regimes: an initial dilaton-driven phase, in which the α' corrections are negligible, and a high curvature string phase, in which the α' corrections are dominant and stabilize the background curvature at the string scale. In the first phase Eq. (20) reduces to the usual perturbation equation (including a tree-level coupling to the dilaton [19])

$$\psi_k'' + \left(k^2 - \frac{\xi''}{\xi}\right)\psi_k = 0, \quad \xi = a e^{-\phi/2}. \tag{21}$$

In the second phase the α' corrections cannot be neglected, but the background curvature is constant, $H = a'/a^2 = c_1/\sqrt{\alpha'}$ and $\dot{\phi} = \phi'/a = c_2/\sqrt{\alpha'}$, so that

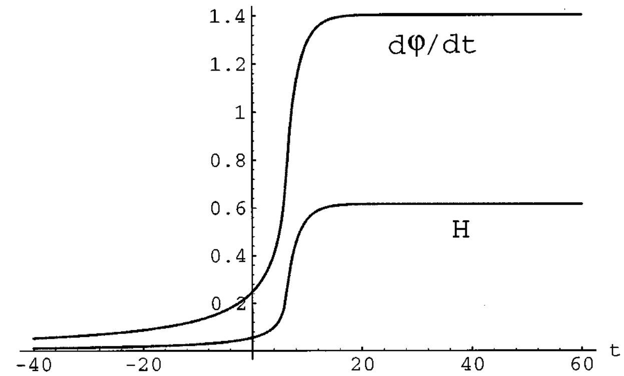

FIG. 1. Time evolution of the gravidilaton background, with the perturbative vacuum as initial condition at $t \to -\infty$ (in units $\alpha' = 1$). The plot shows the results of a numerical integration of the string cosmology equations (4)–(6).

$$z^2 = \xi^2(1 - c_1 c_2), \quad y^2 = \xi^2(1 + c_2^2). \tag{22}$$

The only change of Eq. (21) is thus an effective shift of the comoving frequency:

$$\psi_k'' + \left[k^2(1+c) - \frac{\xi''}{\xi}\right]\psi_k = 0, \quad \xi = a e^{-\phi/2}, \tag{23}$$

$$c = \alpha' \frac{H\dot{\phi} + \dot{\phi}^2}{1 - \alpha' H \dot{\phi}} = \frac{c_1 c_2 + c_2^2}{1 - c_1 c_2} = \text{const.}$$

The important consequence of the above equations is that, for the whole class of background that we are considering, no modification is induced by the high-curvature terms on the evolution of ψ_k outside the horizon (i.e., for $|k\eta| \ll 1$). Such an evolution is uniquely determined, both in the low- and high-curvature regime, by the time behavior of the background variable $\xi(\eta)$, according to the asymptotic solutions of Eqs. (21) and (23):

$$\psi_k = A_k \xi(\eta) + B_k \xi(\eta) \int^{\eta} d\eta' \, \xi^{-2}(\eta') \tag{24}$$

(A_k, B_k are integration constants). Hence, no modification is induced in the perturbation spectrum (both for modes crossing the horizon in the dilaton-driven and in the string phase), compared with the spectrum determined without the α' corrections in the perturbation equation.

Let us consider, in particular, the string phase, assuming that $\psi(\eta) \sim \xi(\eta) \sim (-\eta)^{\alpha}$, $\alpha \leqslant 1/2$, is the dominant term in the asymptotic solution (24) for $\eta \to 0_-$. By using the correct normalization of the canonical variable [2] at horizon crossing (hc), $|\psi_k|_{\rm hc} = 1/\sqrt{k}$, we find indeed, for $\eta \to 0_-$, the power spectrum

$$k^{3/2}|\psi_k| = k \frac{\xi}{\xi_{\rm hc}} = k|k\eta\sqrt{1+c}|^{\alpha} \sim k^{1+\alpha}, \tag{25}$$

which is exactly the same as that provided by the low-energy perturbation equation. The only effect of the high-curvature terms is the shift $\Delta\psi_k/\psi_k$ of the asymptotic amplitude, due to a shift of the horizon crossing scale,

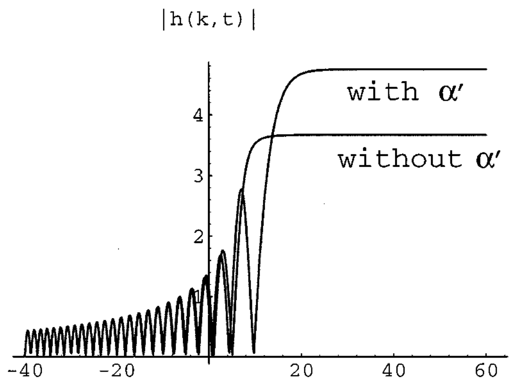

FIG. 2. Time evolution of $|h_k(t)|$ for a mode of comoving frequency $k=1$ (in units $\alpha'=1$), in the background solution corresponding to Fig. 1. The evolution is obtained through a numerical integration of the perturbation equation (16), with and without α' corrections.

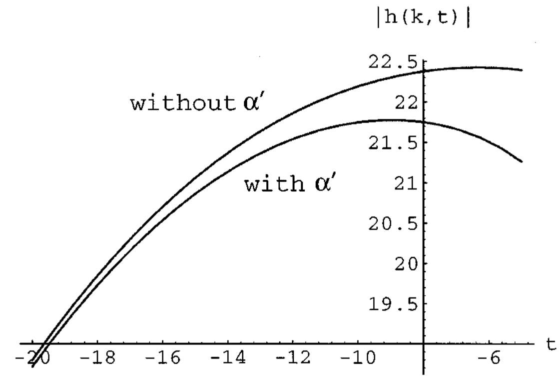

FIG. 3. Time evolution of $|h_k(t)|$ outside the horizon, for a mode of comoving frequency $k^2=0.005$ (in units $\alpha'=1$). The evolution is obtained through a numerical integration of Eq. (16), with and without α' corrections, in the background $a=(-t)^{-1/2}$, $\phi=-3\ln(-t)$.

$$\frac{|\Delta\psi_k|}{|\psi_k|}=|(\sqrt{1+c})^{\alpha}-1|. \qquad (26)$$

Such a shift is the same for all modes, and is of order one if, as expected, the constant values of H and $\dot{\phi}$ are of order one in string units [unless c_1 and c_2 are both exactly equal to one, in which case $c\to\infty$, see Eq. (23)].

The above conclusions are confirmed by a numerical integration of the system formed by the background equations (4)–(6) and by the exact perturbation equation (16). The results are shown in Fig. 2 where we have compared, for a mode crossing the horizon in the string phase, the evolution in cosmic time of the amplitude $|h_k(t)|$ obtained from the exact perturbation equation (16), with the amplitude that one would obtain (for the *same* mode k and in the *same* background) neglecting the α' corrections in the perturbation equation. In both cases the amplitude oscillates outside the horizon, and the oscillations are damped outside the horizon, as expected. The effect of the α' corrections, when they become important, is to induce an effective shift of the comoving frequency, with a resulting shift of the final asymptotic amplitude, as clearly shown by the figure. Since the shift is typically of order one, the previous computations [13] concerning graviton production during the string phase (see also [8,23,24]), performed without α' corrections, remain valid as an order of magnitude estimate.

It is important to stress that this conclusion is valid provided H and $\dot{\phi}$ stay constant for the whole duration of the high-curvature string phase. In the opposite case the α' corrections may affect the time-evolution outside the horizon and, as a consequence, the perturbation spectrum. Of course, in a general background in which the growth of the curvature scale is unbounded, our modified perturbation equation can only be applied for $\alpha' H\dot{\phi}<1$ (at higher scales, higher powers of the curvature should be added). The importance of the α' corrections can be consistently checked, however, even inside the range of validity of our equations, as illustrated in Fig. 3.

The figure shows the result of a numerical integration of Eq. (16), with and without α' corrections, for the typical power-law background $a=(-t)^{-1/2}$, $\phi=-3\ln(-t)$ (which corresponds, in the low-energy limit, to a background dominated by a perfect gas of unstable, stretched strings [7,21], with equation of state $p=-\rho/3$). In the absence of α' corrections the solution of Eq. (16) gives an amplitude $h_k(t)$ which grows, asymptotically, like $(-t)^{-1/2}$. With the α' terms included, the asymptotic behavior of the amplitude is significantly different, and the differences are k dependent, leading to a final modified spectrum. Note that the induced shift is amplified in time when the comoving amplitude is not asymptotically constant, like for the case illustrated in Fig. 3.

We come back, finally, to the case in which the high-curvature phase is frozen at the string scale, and the corrected perturbation equation takes the form (23). The perturbation spectrum, for modes crossing the horizon in the string phase, is determined by the corresponding evolution of $\xi(\eta)$. By using the asymptotic values of H and $\dot{\phi}$ given by the model of background discussed in this paper,

$$c_1=\sqrt{\alpha'}H=0.616\ldots, \quad c_2=\sqrt{\alpha'}\dot{\phi}=1.40\ldots, \qquad (27)$$

(see Fig. 1, and also [14]), we find

$$a(\eta)=-\frac{1}{H\eta}=-\frac{\sqrt{\alpha'}}{c_1\eta},$$

$$\xi(\eta)=ae^{-\phi/2}\sim(-\eta)^{c_2/2c_1-1}\sim(-\eta)^{0.136}. \qquad (28)$$

The resulting slope of the string branch of the spectrum,

$$k^{3/2}|\psi_k|\sim k^{1.36}, \qquad (29)$$

is thus flatter than the slope of the low-energy dilatonic branch ($\sim k^{3/2}$, see [7,25]), as anticipated in [13,26].

The particular power (29) obtained in this simple example should not be taken, however, as a firm prediction for the string branch of the graviton spectrum. The constant asymptotic values of H and $\dot{\phi}$, parametrizing in phase space the fixed points of the σ-model β functions associated to the string effective action [14], may indeed differ from those given in Eq. (27) when a more ''realistic'' model of background (including dilaton potential, quantum loop corrections) is considered. It may be interesting to point out, in particular, that for $H=\text{const}$ and $\dot{\phi}\to 0$ we have $\xi\to(-\eta)^{-1}$, and the spectrum tends to be flat, i.e., scale invariant. The fact that a flat perturbation spectrum may emerge from a high-curvature string phase provides an important counterexample to the expectation that, in string cosmology, the perturbation spectra grow in general too fast [25,27] to have observable effects at large angular scales.

IV. CONCLUSION

In this paper we have perturbed the gravidilaton string effective action around a spatially flat gravidilaton background, and we have found the higher-derivative corrections, up to first order in α', to the low-energy equation of tensor perturbations.

Applying the results to the pre-Big Bang cosmological models we have shown that the high-curvature terms, which have a crucial influence on the evolution in time of the background, do not affect in a qualitative way the evolution in time of the perturbations, both inside and outside the horizon (modulo a constant shift of the final amplitude).

The low-energy perturbation equation thus remains valid for an order-of-magnitude estimate of the spectrum. This result, however, is a direct consequence of having a background with a high-curvature phase in which H and $\dot{\phi}$ stay frozen at the string scale. In more general backgrounds the time evolution of perturbations, and the final spectrum, may be significantly affected by the high-curvature corrections.

The perturbative computations of this paper have been explicitly performed in the string frame, and with an appropriate representation of the background fields that eliminates higher-than-second derivatives from the effective equations. The results about the perturbation spectrum, however, should be frame independent, and should remain invariant under arbitrary field redefinitions, even when such redefinitions modify the explicit form of the action. To a modification of the background should correspond indeed a modification of the perturbation equation, in such a way that the solution of the new perturbation equation, in the new background, should be the same as the solution of the old equation in the old background (as explicitly checked in [7] for the transformation from the string to the Einstein frame, in the case of the lowest order effective action).

Finally, the results reported in this paper about tensor perturbations are expected to be qualitatively valid also in the case of scalar perturbations. For modes crossing the horizon during the string phase, in particular, the scalar perturbation spectrum should be determined by the constant values of H and $\dot{\phi}$, and should tend to a scale-invariant spectrum in the limit $\dot{\phi}\to 0$, as discussed in Sec. III for the graviton case.

ACKNOWLEDGMENTS

This work was supported in part by EC Contract No. ERBCHRX-CT94-0488. I am grateful to Gabriele Veneziano for many useful discussions. Special thanks are due to Edward W. Kolb for raising stimulating questions about string corrections to the graviton spectrum, that motivated in part the work presented in this paper.

APPENDIX: SECOND-ORDER PERTURBATION OF THE ACTION

Consider the homogeneous and isotropic gravidilaton background, parametrized in $d=3$ by

$$ds^2=N^2(t)dt^2-a^2(t)dx_i^2, \quad \phi=\phi(t) \tag{A1}$$

$(i,j=1,2,3)$. We define, as usual, $H=\dot{a}/a$ and $F=\dot{N}/N$. With the conventions $g_{\mu\nu}=(+---)$, $R_{\mu\nu\alpha}{}^{\beta}=\partial_\mu\Gamma_{\nu\alpha}{}^{\beta}-\cdots$, and $R_{\nu\alpha}=R_{\mu\nu\alpha}{}^{\mu}$, we compute the scalar curvature

$$R=\frac{1}{N^2}(6HF-6\dot{H}-12H^2), \tag{A2}$$

and the Gauss-Bonnet invariant

$$R_{GB}^2=R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}-4R_{\mu\nu}R^{\mu\nu}+R^2 = \frac{24}{N^4}(\dot{H}H^2+H^4-FH^3). \tag{A3}$$

Summing up all contributions, and setting $a=e^{\beta}$, $H=\dot{\beta}$, $\dot{H}=\ddot{\beta}$, the string effective action (1) becomes, in this background,

$$S=\int dt e^{3\beta-\phi}\left[\frac{1}{N}(6\ddot{\beta}+12\dot{\beta}^2-6\dot{\beta}F-\dot{\phi}^2)+\frac{k\alpha'}{4N^3}\times(24\ddot{\beta}\dot{\beta}^2+24\dot{\beta}^4-24\dot{\beta}^3F-\dot{\phi}^4)\right]. \tag{A4}$$

The derivatives of the lapse function can be eliminated, by noting that

$$\frac{e^{3\beta-\phi}}{N}(6\ddot{\beta}-6\dot{\beta}F)=\frac{e^{3\beta-\phi}}{N}(6\dot{\beta}\dot{\phi}-18\dot{\beta}^2)+\frac{d}{dt}\left(6\dot{\beta}\frac{e^{3\beta-\phi}}{N}\right) \tag{A5}$$

and

$$\frac{e^{3\beta-\phi}}{N^3}(24\ddot{\beta}\dot{\beta}^2+24\dot{\beta}^4-24\dot{\beta}^3F) = 8\dot{\phi}\dot{\beta}^3\frac{e^{3\beta-\phi}}{N^3}+\frac{d}{dt}\left(8\dot{\beta}^3\frac{e^{3\beta-\phi}}{N^3}\right). \tag{A6}$$

Using these results into Eq. (A4) we recover the action (3), modulo a total derivative that does not contribute to the equations of motion.

Let us now compute the second-order variation of the action (1), for a transverse and traceless metric perturbation $\delta g_{\mu\nu}=h_{\mu\nu}(x,t)$, with $h_\mu{}^{\mu}=0=\nabla_\nu h_\mu{}^{\nu}$ (the indices of $h_{\mu\nu}$ are raised with the unperturbed metric $g^{\mu\nu}$). We work in the synchronous gauge, where

$$g_{00}=1, \quad g_{0i}=0, \quad g_{ij}=-a^2\delta_{ij},$$

$$h_{00}=0, \quad h_{0i}=0, \quad g^{ij}h_{ij}=0, \quad \partial_j h_i{}^{j}=0. \tag{A7}$$

The nonvanishing components of the Christoffel connection and of the Ricci and Riemann tensor, for the unperturbed metric, are given by

$$\Gamma_{0i}{}^{j}=H\delta_i^j, \quad \Gamma_{ij}{}^{0}=-Hg_{ij},$$
$$R_{00}=-3(\dot H+H^2), \quad R_{ij}=-g_{ij}(\dot H+3H^2),$$
$$R_{0i0j}=g_{ij}(\dot H+H^2), \quad R_{ikjl}=H^2(g_{ij}g_{kl}-g_{kj}g_{il}). \tag{A8}$$

In this gauge,

$$\nabla_0 h_i{}^j=\dot h_i{}^j, \quad \nabla_0 h_{ij}=\dot h_{ij}-2Hh_{ij}, \quad \nabla_0\nabla_0 h_i{}^j=\ddot h_i{}^j,$$
$$\nabla_0\nabla_0 h_{ij}=\ddot h_{ij}-2\dot H h_{ij}-4H\dot h_{ij}+4H^2h_{ij} \tag{A9}$$

(∇_0 is the covariant time derivative, while a dot denotes partial time derivative). From these equations one easily derives the identities (11), useful to express all perturbed quantities in powers of the convenient variable $h_i{}^j$ and of its derivatives.

The nonvanishing controvariant components of the metric tensor, at first and second order in h, are given by

$$\delta^{(1)}g^{ij}=-h^{ij}, \quad \delta^{(2)}g^{ij}=h^{ik}h_k{}^j. \tag{A10}$$

For the determinant of the metric tensor we thus obtain

$$\delta^{(1)}\sqrt{-g}=0, \quad \delta^{(2)}\sqrt{-g}=-\frac{1}{4}a^3h_i{}^jh_j{}^i. \tag{A11}$$

The nonvanishing components of the perturbed connection are, at first order,

$$\delta^{(1)}\Gamma_{0i}{}^{j}=\frac{1}{2}\dot h_i{}^j, \quad \delta^{(1)}\Gamma_{ij}{}^{0}=-\frac{1}{2}\dot h_{ij},$$
$$\delta^{(1)}\Gamma_{ij}{}^{k}=\frac{1}{2}(\partial_i h_j{}^k+\partial_j h_i{}^k-\partial^k h_{ij}), \tag{A12}$$

and, at second order,

$$\delta^{(2)}\Gamma_{0i}{}^{j}=-\frac{1}{2}\dot h_i{}^k h_k{}^j,$$
$$\delta^{(2)}\Gamma_{ij}{}^{k}=-\frac{1}{2}h^k{}_l(\partial_i h_j{}^l+\partial_j h_i{}^l-\partial^l h_{ij}). \tag{A13}$$

At first order, and using the identities (11), the nonvanishing components of the Ricci tensor can be given in terms of $h_i{}^j$ as

$$\delta^{(1)}R_i{}^j=-\frac{1}{2}\left(\ddot h_i{}^j+3H\dot h_i{}^j-\frac{\nabla^2}{a^2}h_i{}^j\right)\equiv-\frac{1}{2}\Box h_i{}^j, \tag{A14}$$

$$\delta^{(1)}R_{ij}=-\frac{1}{2}g_{ik}\left(\ddot h_j{}^k+2\dot H h_j{}^k+3H\dot h_j{}^k+6H^2h_j{}^k-\frac{\nabla^2}{a^2}h_j{}^k\right)$$

(where $\nabla^2=\delta^{ij}\partial_i\partial_j$). At second order,

$$\delta^{(2)}R_{00}=\delta^{(2)}R_0{}^0=\frac{1}{2}\left(\ddot h_i{}^jh_j{}^i+\frac{1}{2}\dot h_i{}^j\dot h_j{}^i+2\dot h_i{}^jh_j{}^i\right),$$

$$\delta^{(2)}R_{ij}=\frac{H}{2}g_{ij}h_k{}^l\dot h_l{}^k+\frac{1}{4}g_{ik}\dot h_j{}^l\dot h_l{}^k+\frac{1}{4}g_{jk}\dot h_i{}^l\dot h_l{}^k-\frac{1}{4}\partial_i h_k{}^l\partial_j h_l{}^k+\frac{1}{2}\partial_k h_i{}^l\partial^k h_{lj},$$

$$\delta^{(2)}R_i{}^j=\frac{1}{2}\left(h_i{}^k\ddot h_k{}^j+3Hh_i{}^k\dot h_k{}^j+H\delta_i^jh_k{}^l\dot h_l{}^k+\dot h_i{}^k\dot h_k{}^j-\frac{1}{2}\partial_i h_l{}^k\partial^j h_k{}^l\right) \tag{A15}$$

(in the variation of R_{ij} we have neglected all terms that after integration by parts do not contribute to the perturbed action, because of the gauge condition $\partial_i h_j{}^i=0$).

We can now compute the second-order perturbation of the various terms appearing in the string effective action. We adopt, for simplicity, a matrix notation for $h_i{}^j$, setting $h_i{}^jh_j{}^i=\mathrm{Tr}h^2$, $h_i{}^j\dot h_j{}^i=\mathrm{Tr}(h\dot h)$, and so on. From Eqs. (A10)–(A15) we obtain

$$\delta^{(2)}(\sqrt{-g}\,\partial_\mu\phi\partial^\mu\phi)=-\frac{1}{4}a^3\dot\phi^2\mathrm{Tr}h^2, \tag{A16}$$

$$\delta^{(2)}[\sqrt{-g}(\partial_\mu\phi\partial^\mu\phi)^2]=-\frac{1}{4}a^3\dot\phi^4\mathrm{Tr}h^2, \tag{A17}$$

$$\delta^{(2)}(\sqrt{-g}R)=R\,\delta^{(2)}\sqrt{-g}+\sqrt{-g}(\delta^{(1)}g^{\mu\nu}\delta^{(1)}R_{\mu\nu}+R_{\mu\nu}\delta^{(2)}g^{\mu\nu}+g^{\mu\nu}\delta^{(2)}R_{\mu\nu})$$
$$=a^3\mathrm{Tr}\left[h^2\left(\frac{3}{2}\dot{H}+3H^2\right)+h\ddot{h}+4Hh\dot{h}+\frac{3}{4}\dot{h}^2-\frac{h}{4}\frac{\nabla^2}{a^2}h\right], \tag{A18}$$

$$\delta^{(2)}(\sqrt{-g}R^2)=R^2\,\delta^{(2)}\sqrt{-g}+2\sqrt{-g}R(\delta^{(1)}g^{\mu\nu}\delta^{(1)}R_{\mu\nu}+R_{\mu\nu}\delta^{(2)}g^{\mu\nu}+g^{\mu\nu}\delta^{(2)}R_{\mu\nu})$$
$$=-6a^3(\dot{H}+2H^2)\mathrm{Tr}\left[h^2\left(\frac{3}{2}\dot{H}+3H^2\right)+2h\ddot{h}+8Hh\dot{h}+\frac{3}{2}\dot{h}^2-\frac{h}{2}\frac{\nabla^2}{a^2}h\right], \tag{A19}$$

$$\delta^{(2)}(\sqrt{-g}R_{\mu\nu}R^{\mu\nu})=(R_{\mu\nu})^2\delta^{(2)}\sqrt{-g}+\sqrt{-g}(\delta^{(1)}R_\mu{}^\nu\delta^{(1)}R_\nu{}^\mu+2R_\mu{}^\nu\delta^{(2)}R_\nu{}^\mu)$$
$$=a^3\ \mathrm{Tr}\left[-h\dot{h}(12\dot{H}H+24H^3)-\frac{\dot{h}^2}{2}\left(5\dot{H}+\frac{18}{4}H^2\right)-h^2(3\dot{H}^2+9\dot{H}H^2+9H^4)-h\ddot{h}(4\dot{H}+6H^2)+\frac{\ddot{h}^2}{4}\right.$$
$$\left.+\frac{3}{2}H\dot{h}\ddot{h}+\frac{1}{4}\left(\frac{\nabla^2}{a^2}h\right)^2-\frac{1}{2}(\ddot{h}+3H\dot{h}-\dot{H}h-3H^2h)\frac{\nabla^2}{a^2}h\right]. \tag{A20}$$

Finally, to complete the perturbation of the Gauss-Bonnet invariant, we need the perturbations of the Riemann tensor. We find, to first order,

$$\delta^{(1)}R_{0i}{}^{0j}=\frac{1}{2}(\ddot{h}_i{}^j+2H\dot{h}_i{}^j),$$

$$\delta^{(1)}R_{0i}{}^{jk}=\frac{1}{2}(\partial^j\dot{h}_i{}^k-\partial^k\dot{h}_i{}^j),$$

$$\delta^{(1)}R_{ik}{}^{0j}=\frac{1}{2}(\partial_i\dot{h}_k{}^j-\partial_k\dot{h}_i{}^j),$$

$$\delta^{(1)}R_{ik}{}^{jl}=\frac{1}{2}(\partial_i\partial^jh_k{}^l-\partial_k\partial^jh_i{}^l+\partial_k\partial^lh_i{}^j-\partial_i\partial^lh_k{}^j)+\frac{H}{2}(\delta_i^j\dot{h}_k{}^l-\delta_k^j\dot{h}_i{}^l+\delta_k^l\dot{h}_i{}^j-\delta_i^l\dot{h}_k{}^j). \tag{A21}$$

At second order, what we need for the perturbation of $R^2_{\mu\nu\alpha\beta}$ are the mixed components

$$\delta^{(2)}R_{0i}{}^{0j}=-\frac{1}{2}\left(\ddot{h}_i{}^kh_k{}^j+\frac{1}{2}\dot{h}_i{}^k\dot{h}_k{}^j+2H\dot{h}_i{}^kh_k{}^j\right), \tag{A22}$$

and the contraction

$$R_{jl}{}^{ik}\delta^{(2)}R_{ik}{}^{jl}=-\frac{H^2}{2}\left(\dot{h}_i{}^j\dot{h}_j{}^i+8Hh_i{}^j\dot{h}_j{}^i-h_i{}^j\frac{\nabla^2}{a^2}h_j{}^i\right) \tag{A23}$$

(we have neglected terms that, after integration by parts, vanish because of the trasversality condition). Therefore

$$\delta^{(2)}(\sqrt{-g}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta})=(R_{\mu\nu\alpha\beta})^2\delta^{(2)}\sqrt{-g}+\sqrt{-g}(\delta^{(1)}R_{\mu\nu}{}^{\alpha\beta}\delta^{(1)}R_{\alpha\beta}{}^{\mu\nu}+2R_{\mu\nu}{}^{\alpha\beta}\delta^{(2)}R_{\alpha\beta}{}^{\mu\nu})$$
$$=(R_{\mu\nu\alpha\beta})^2\delta^{(2)}\sqrt{-g}+\sqrt{-g}(4\,\delta^{(1)}R_{0i}{}^{0j}\delta^{(1)}R_{0j}{}^{0i}+2\,\delta^{(1)}R_{0i}{}^{jk}\delta^{(1)}R_{jk}{}^{0i}+2\,\delta^{(1)}R_{ik}{}^{0j}\delta^{(1)}R_{0j}{}^{ik}$$
$$+\delta^{(1)}R_{ik}{}^{jl}\delta^{(1)}R_{jl}{}^{ik}+8R_{0i}{}^{0j}\delta^{(2)}R_{0j}{}^{0i}+2R_{jl}{}^{ik}\delta^{(2)}R_{ik}{}^{jl})$$
$$=a^3\ \mathrm{Tr}\left[\dot{h}^2(2H^2-2\dot{H})-h\ddot{h}(4H^2+4\dot{H})-h^2(3\dot{H}^2+6\dot{H}H^2+6H^4)-h\dot{h}(8H\dot{H}+16H^3)+\ddot{h}^2\right.$$
$$\left.+4H\dot{h}\ddot{h}+\left(\frac{\nabla^2}{a^2}h\right)^2+2\dot{h}\frac{\nabla^2}{a^2}\dot{h}+(H^2h-2H\dot{h})\frac{\nabla^2}{a^2}h\right], \tag{A24}$$

modulo a total derivative that does not contribute to the action.

Summing up the results (A16)–(A20) and (A24) we finally obtain the action (12) reported in Sec. II.

[1] E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, Redwood City, CA, 1990); A. D. Linde, *Particle Physics and Inflationary Cosmology* (Harwood, Chur, Switzerland, 1990). See also, for a recent review, M. S. Turner, ''Ten things everyone should know about inflation,'' astro-ph/9704062.
[2] V. F. Mukhanov, A. H. Feldman, and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992).
[3] A. R. Liddle and D. H. Lyth, Phys. Rep. **231**, 1 (1993), and references therein.
[4] V. F. Mukhanov, L. A. Kofman, and D. Y. Pogosyan, Phys. Lett. B **193**, 427 (1987); V. F. Mukhanov, *ibid.* **218**, 17 (1989); L. Amendola, M. Litterio, and F. Occhionero, *ibid.* **231**, 43 (1989); **237**, 348 (1990); J. Hwang, Phys. Rev. D **42**, 2601 (1990); Class. Quantum Grav. **7**, 1613 (1990).
[5] H. Noh and J. Hwang, Phys. Rev. D **55**, 5222 (1997); A. A. Starobinski, Pis'ma Zh. Eksp. Teor. Fiz. **34**, 460 (1981) [JETP Lett. **34**, 438 (1981)].
[6] C. Lovelace, Phys. Lett. **135B**, 75 (1984); E. S. Fradkin and A. A. Tseytlin, Nucl. Phys. **B261**, 1 (1985); C. G. Callan *et al.*, *ibid.* **B262**, 593 (1985); A. Sen, Phys. Rev. Lett. **55**, 1846 (1985).
[7] M. Gasperini and G. Veneziano, Astropart. Phys. **1**, 317 (1993); Mod. Phys. Lett. A **8**, 3701 (1993); Phys. Rev. D **50**, 2519 (1994). An updated collection of papers on the pre-Big Bang scenario is available at http://www.to.infn.it/teorici/gasperini/.
[8] G. Veneziano, in *Proceedings of the 4th Course of the International School of Astrophysics D. Chalonge*, Erice, 1995, edited by N. Sanchez and A. Zichichi (Kluwer Academic, Dordrecht, 1996), p. 285; M. Gasperini, *ibid.*, p. 305.
[9] R. R. Metsaev and A. A. Tseytlin, Nucl. Phys. **B293**, 385 (1987).
[10] M. Gasperini and M. Giovannini, Class. Quantum Grav. **14**, 735 (1997).
[11] E. J. Copeland, R. Easther and D. Wands, Phys. Rev. D **56**, 874 (1997).
[12] M. Gasperini, M. Giovannini, and G. Veneziano, Phys. Rev. Lett. **75**, 3796 (1995).
[13] R. Brustein, M. Gasperini, M. Giovannini, and G. Veneziano, Phys. Lett. B **361**, 45 (1995).
[14] M. Gasperini, M. Maggiore, and G. Veneziano, Nucl. Phys. **B494**, 315 (1997).
[15] B. Zwiebach, Phys. Lett. **156B**, 315 (1985).
[16] L. H. Ford and L. Parker, Phys. Rev. D **16**, 1601 (1977).
[17] B. Allen and S. Koranda, Phys. Rev. D **50**, 3713 (1994).
[18] L. P. Grishchuk, Sov. Phys. JETP **40**, 409 (1975); A. A. Starobinski, JETP Lett. **30**, 682 (1979).
[19] M. Gasperini and M. Giovannini, Phys. Rev. D **47**, 1519 (1993); J. D. Barrow, J. P. Mimoso, and M. R. de Garcia Maia, *ibid.* **48**, 3630 (1993).
[20] N. Deruelle, C. Gundlach, and D. Polarski, Class. Quantum Grav. **9**, 137 (1992).
[21] G. Veneziano, Phys. Lett. B **265**, 287 (1991).
[22] I. Antoniadis, J. Rizos, and K. Tamvakis, Nucl. Phys. **B415**, (1994); R. Easther and K. Maeda, Phys. Rev. D **54**, 7252 (1996); S. J. Rey, Phys. Rev. Lett. **77**, 1929 (1996); M. Gasperini and G. Veneziano, Phys. Lett. B **387**, 715 (1996).
[23] M. Gasperini, in *New development in String Gravity and Physics at the Planck Energy Scale*, edited by N. Sanchez (World Scientific, Singapore, in press), hep-th/9607146.
[24] A. Buonanno, M. Maggiore, and C. Ungarelli, Phys. Rev. D **55**, 3330 (1997).
[25] R. Brustein, M. Gasperini, M. Giovannini, F. Mukhanov, and G. Veneziano, Phys. Rev. D **51**, 6744 (1995).
[26] R. Brustein, M. Gasperini, and G. Veneziano, Phys. Rev. D **55**, 3882 (1997).
[27] J. Hwang and H. Noh, ''Density spectrum from kinetic inflation,'' gr-qc/9612005.