Radiative multipole moments of integer-spin fields in curved spacetime

Stephen W. Leonard and Eric Poisson
Department of Physics, University of Guelph, Guelph, Ontario, Canada N1G 2W1
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Radiative multipole moments of scalar, electromagnetic, and linearized gravitational fields in Schwarzschild spacetime are computed to third order in v in a weak-field, slow-motion approximation, where v is a characteristic velocity associated with the motion of the source. These moments are defined for all three types of radiation by relations of the form $\Psi(t,\vec{x})=r^{-1}\Sigma_{lm}\mathcal{M}_{lm}(u)Y_{lm}(\theta,\phi)$, where Ψ is the radiation field at infinity and $\mathcal{M}_{lm}$ are the radiative moments, functions of retarded time $u=t-r-2M\ln(r/2M-1)$; M is the mass parameter of the Schwarzschild spacetime and $(t,\vec{x})=(t,r,\theta,\phi)$ are the usual Schwarzschild coordinates. For all three types of radiation the moments share the same mathematical structure: To zeroth order in v, the radiative moments are given by relations of the form $\mathcal{M}_{lm}(u)\propto(d/du)^l\int\rho(u,\vec{x})r^l\overline{Y_{lm}}(\theta,\phi)d\vec{x}$, where ρ is the source of the radiation. A radiative moment of order l is therefore given by the corresponding source moment differentiated l times with respect to retarded time. To second order in v, additional terms appear inside the spatial integrals, and the radiative moments become $\mathcal{M}_{lm}(u)\propto(d/du)^l\int[1+O(r^2\partial_u^2)+O(M/r)]\rho r^l\overline{Y_{lm}}\,d\vec{x}$. The term involving $r^2\partial_u^2$ can be interpreted as a special-relativistic correction to the wave-generation problem. The term involving M/r comes from general relativity. These correction terms of order v^2 are *near-zone* corrections which depend on the detailed behavior of the source. Furthermore, the radiative multipole moments are still *local* functions of u, as they depend on the state of the source at retarded time u only. To third order in v, the radiative moments become $\mathcal{M}_{lm}(u)\rightarrow\mathcal{M}_{lm}(u)+2M\int_{-\infty}^{u}[\ln(u-u')+\text{const}]\ddot{\mathcal{M}}_{lm}(u')\,du'$, where overdots indicate differentiation with respect to u'. This expression shows that the $O(v^3)$ correction terms occur outside the spatial integrals, so that they do not depend on the detailed behavior of the source. Furthermore, the radiative multipole moments now display a nonlocality in time, as they depend on the state of the source at *all times* prior to the retarded time u, with the factor $\ln(u-u')$ assigning most of the weight to the source's recent past. (The term involving the constant is actually local.) The correction terms of order v^3 are *wave-propagation* corrections which are heuristically understood as arising from the scattering of the radiation by the spacetime curvature surrounding the source. The radiative multipole moments are computed explicitly for all three types of radiation by taking advantage of the symmetries of the Schwarzschild metric to separate the variables in the wave equations. Our calculations show that the truly nonlocal wave-propagation correction — the term involving $\ln(u-u')$ — takes a universal form which is independent of multipole order and field type. We also show that in general relativity, temporal and spatial curvatures contribute *equally* to the wave-propagation corrections. Finally, we produce an alternative derivation of the radiative moments of a scalar field based on the retarded Green's function of DeWitt and Brehme. This calculation shows that the tail part of the Green's function is entirely responsible for the wave-propagation corrections in the radiative moments. [S0556-2821(97)03820-4]

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I. INTRODUCTION AND SUMMARY

A. Tails in waves

It has long been known that, in general, the propagation of massless fields in curved spacetime does not proceed along characteristics only, but is accompanied by wave tails. Much attention has been devoted to this topic since the groundbreaking work by Hadamard [1]. Here are some of the highlights.

In 1952, Choquet-Bruhat [2] studied the initial value problem of general relativity and showed that the gravitational field at some event P depends not only on the data put on the intersection of P's past light cone with the initial surface, but also on the data put inside this region. This result indicates that in general relativity, field propagation proceeds at all speeds less than, or equal to, the speed of light.

In 1960, DeWitt and Brehme [3] constructed Green's functions for the scalar and electromagnetic wave equations in curved spacetime, and showed that these split naturally into a direct part, with support on, and only on, the light cone, and a tail part, with support within the light cone. A similar Green's function was also constructed for the Einstein field equations [4].

In 1968, Kundt and Newman [5] established that for hyperbolic partial differential equations in two dimensions, the presence of wave tails is the rule rather than the exception. This conclusion was extended by McLenaghan and coworkers [6–9] to the case of conformally invariant wave equations in four dimensions.

Wave tails are known to have important physical consequences. For example, DeWitt and Brehme [3] have shown that the tail part of the electromagnetic field is of paramount importance in deriving the equations of motion for charged particles in curved spacetime. Similarly, Mino, Sasaki, and Tanaka [10], as well as Quinn and Wald [11], have recently shown that tails are entirely responsible for the gravitational radiation reaction force. And as a final example, Price [12] has shown that gravitational-wave tails play an integral part in the physical process by which a recently formed black hole relaxes to a stationary state, as is demanded by the no-hair theorems.

The presence of tails in the gravitational waves produced by an isolated source was first demonstrated in 1965 by Bonnor and Rotenberg [13]. In 1968, this work was extended by Couch *et al.* [14], who showed that an initially outgoing wave will be partly backscattered by the spacetime curvature surrounding the source, thereby creating an incoming wave. A further extension of this work has appeared very recently [15].

In 1992, Blanchet and Damour [16] considered for the first time the effect of tails on the behavior of gravitational waves at infinity, thereby concentrating on effects that could potentially be observed directly. They found that gravitational waves at time t depend not only on the state of the source at the corresponding retarded time u [essentially $u = t - r/c$, where r is the distance to the source, but see Eq. (1.2) for a more precise definition], but also on the state of the source at *all times* prior to the retarded time. (Once again, this indicates that wave propagation proceeds at all speeds less than, or equal to, the speed of light.) Subsequently [17–20], it was shown that tails play an important role in the generation of gravitational waves by the orbital inspiral of a compact binary system. These waves are among the most promising for detection by future kilometer-scale interferometers such as the American Laser Interferometer Gravitational-wave Observatory [21] (LIGO) and the French-Italian VIRGO [22].

Given the physical relevance of tails in the propagation of radiation in curved spacetime, it appeared to us worthwhile to seek a deeper understanding of this phenomenon by asking how it depends on the type of radiation being considered and by digging further into the nature of its physical origin. This paper reports on the results of such an investigation, in which we study the influence of tails on those properties of massless fields that are directly measurable to an observer at infinity: the radiative multipole moments. We consider the cases of scalar, electromagnetic, and gravitational radiation generated by an isolated source and propagating to infinity in a spacetime curved by a nonrotating central mass M. All of our results are derived on the basis of a weak-field, slow-motion approximation. Throughout this paper we use units such that $G=c=1$, and we employ the definitions and conventions of Misner, Thorne, and Wheeler [23].

B. Scalar radiation

We begin the summary of our results with the simplest case, that of a scalar field $\Phi(\boldsymbol{x})$ obeying the wave equation $\Box\Phi = -4\pi\rho$, where $\Box = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$, $g_{\alpha\beta}$ is the Schwarzschild metric and $\rho(\boldsymbol{x})$ a given source. The symbol $\boldsymbol{x}$ collectively designates all Schwarzschild coordinates $\{t,r,\theta,\phi\}$. As is shown in Sec. III, the radiative part of the scalar field, which dominates at infinity, can be written as

$$\Phi_{\rm rad}(t,\vec{x}) = \frac{1}{r}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \mathcal{Z}_{lm}(u) Y_{lm}(\theta,\phi), \tag{1.1}$$

where $\mathcal{Z}_{lm}(u)$ are radiative multipole moments, depending on retarded time

$$u = t - r - 2M\ln(r/2M - 1), \tag{1.2}$$

and $Y_{lm}(\theta,\phi)$ are the usual spherical harmonics. The symbol $\vec{x}$ collectively designates all spatial coordinates $\{r,\theta,\phi\}$.

In a leading-order calculation in a weak-field, slow-motion approximation, the radiative multipole moments are found to be given by

$$\mathcal{Z}^{(0)}_{lm}(u) = \frac{4\pi}{(2l+1)!!}\left(\frac{d}{du}\right)^l \int \rho(u,\vec{x}) r^l \overline{Y_{lm}}(\theta,\phi) d\vec{x}, \tag{1.3}$$

where $d\vec{x} = r^2\, dr\, d\cos\theta\, d\phi$ and the integration is over the region of space occupied by the source; we assume that this region is bounded. Equation (1.3) shows that the radiative moments are obtained from the source moments $\int \rho r^l \overline{Y_{lm}} d\vec{x}$ by taking a number of time derivatives equal to the multipole order.

In a more accurate calculation, incorporating corrections of order v^2 with respect to the leading-order results (with $v \ll 1$ a characteristic velocity associated with the motion of the source), we find

$$\mathcal{Z}^{(2)}_{lm}(u) = \frac{4\pi}{(2l+1)!!}\left(\frac{d}{du}\right)^l \int \left[1 + \frac{(r\partial_u)^2}{2(2l+3)} - l\frac{M}{r}\right] \times \rho(u,\vec{x}) r^l \overline{Y_{lm}}(\theta,\phi) d\vec{x}. \tag{1.4}$$

It is easy to see that the correction terms are indeed of order v^2. The term involving $(r\partial_u)^2$ is of order $(r_c/t_c)^2$, where r_c is a characteristic radius within the source and t_c a characteristic time scale associated with its motion; the ratio r_c/t_c *defines* the characteristic velocity v. This term can be understood as arising from special-relativistic corrections to the wave-generation problem. On the other hand, the term proportional to M/r comes from general relativity, and is also of order v^2 by virtue of the virial theorem for bound motion in a gravitational field. It should be noted that in Eq. (1.4), the correction terms occur *inside* the spatial integrals, so that they depend on the detailed behavior of the source. Furthermore, these corrections are purely *local* in time, as $\mathcal{Z}^{(2)}_{lm}(u)$ depends on the state of the source at the time u only. The terms of order v^2 are therefore *near-zone* corrections that have nothing to do with the tail effect discussed previously.

A calculation carried out to order v^3 in a weak-field, slow-motion approximation does reveal the influence of the tails. Indeed, the radiative multipole moments are now given by

$$\mathcal{Z}^{(3)}_{lm}(u) = \mathcal{Z}^{(2)}_{lm}(u) + 2M\int_{-\infty}^{u}\left[\ln\left(\frac{u-u'}{4M}\right) + \beta_l^{\rm scalar} + \gamma\right] \times \ddot{\mathcal{Z}}^{(0)}_{lm}(u')du', \tag{1.5}$$

which clearly displays a nonlocality in time. Here, overdots indicate differentiation with respect to u', and

$$\beta_l^{\rm scalar} = \psi(l+1) + \frac{1}{2}, \tag{1.6}$$

where $\psi(l+1)=-\gamma+\sum_{k=1}^{l}k^{-1}$ is the digamma function ($\gamma\simeq 0.577\,21$ is Euler's number). We notice that the correction terms of order v^3 occur *outside* the spatial integrals, so that they do not depend on the detailed behavior of the source. These are *wave-propagation* corrections, which are readily associated with the occurrence of wave tails in curved spacetime. We must point out that of the three $O(v^3)$ terms in Eq. (1.5), two are actually *instantaneous*, as they are equal to $2M(\beta_l^{\rm scalar}+\gamma)\dot{\mathcal{Z}}_{lm}^{(0)}(u)$. The remaining term is truly nonlocal, and the factor $\ln(u-u')$ assigns most of the weight to the source's recent past.

Equations (1.1)–(1.6) were derived by integrating the wave equation for a scalar field in Schwarzschild spacetime, for which the variables can conveniently be separated. (The actual work of integrating the radial equation is carried out in Sec. II.) But because our calculations use only the weak-field behavior of the Schwarzschild solution, our results are insensitive to the detailed form of the metric. Although we rely heavily on the symmetries of the Schwarzschild metric to separate the variables in the wave equation, our results rely only on the fact that the field is spherically symmetric at large distances. Our results would therefore hold also in more general, nonrotating spacetimes, with spherical symmetry holding only approximately at large distances. Staticity, however, is a crucial assumption, and our results would not be valid if the spacetime were rotating. Although the wave-propagation corrections to the radiative multipole moments would take the same form as in Eq. (1.5), the spacetime's rotation would create additional terms of order v^3. These would occur inside the spatial integrals, and would describe near-zone corrections of the spin-orbit type [24–27].

C. Electromagnetic radiation

In Sec. IV we turn to the case of electromagnetic radiation produced by a given current density $J^\alpha(\boldsymbol{x})$ in Schwarzschild spacetime. (The remarks of the preceding paragraph, regarding the generality of our results, apply equally well here.) The radiative part of the vector potential is given by

$$A_\alpha^{\rm rad}(t,\vec{x})=\frac{1}{r}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}[\mathcal{I}_{lm}(u)Y_\alpha^{E,lm}(\theta,\phi)+\mathcal{S}_{lm}(u)Y_\alpha^{B,lm}(\theta,\phi)],\tag{1.7}$$

where $\mathcal{I}_{lm}(u)$ and $\mathcal{S}_{lm}(u)$ are charge and current multipole moments, respectively, while $Y_\alpha^{E,lm}(\theta,\phi)$ and $Y_\alpha^{B,lm}(\theta,\phi)$ are the vectorial spherical harmonics described in Appendix A. In a calculation accurate to order v^3 in a weak-field, slow-motion approximation, we find

$$\mathcal{I}_{lm}^{(3)}(u)=\mathcal{I}_{lm}^{(2)}(u)+2M\int_{-\infty}^{u}\left[\ln\left(\frac{u-u'}{4M}\right)+\beta_l^{\rm em}+\gamma\right]\times\dddot{\mathcal{I}}_{lm}^{(0)}(u')du'\tag{1.8}$$

and

$$\mathcal{S}_{lm}^{(3)}(u)=\mathcal{S}_{lm}^{(2)}(u)+2M\int_{-\infty}^{u}\left[\ln\left(\frac{u-u'}{4M}\right)+\beta_l^{\rm em}+\gamma\right]\times\dddot{\mathcal{S}}_{lm}^{(0)}(u')du',\tag{1.9}$$

where

$$\beta_l^{\rm em}=\psi(l+1)+\frac{1}{2}-\frac{1}{2l(l+1)}.\tag{1.10}$$

These relations are analogous to Eq. (1.5) and have the same physical interpretation. The second-order multipole moments are given by

$$\mathcal{I}_{lm}^{(2)}(u)=\frac{4\pi}{(2l+1)!!}\sqrt{\frac{l+1}{l}}\left(\frac{d}{du}\right)^l\times\int\left[1+\frac{l+3}{2(l+1)(2l+3)}(r\partial_u)^2-(l-1)\frac{M}{r}\right]\sigma(u,\vec{x})r^l\overline{Y_{lm}}(\theta,\phi)d\vec{x}\tag{1.11}$$

and

$$\mathcal{S}_{lm}^{(2)}(u)=-\frac{4\pi}{(2l+1)!!}\left(\frac{d}{du}\right)^l\int\left[1+\frac{(r\partial_u)^2}{2(2l+3)}-\frac{l^2-1}{l}\frac{M}{r}\right]J^\alpha(u,\vec{x})r^l\overline{Y_\alpha^{B,lm}}(\theta,\phi)d\vec{x},\tag{1.12}$$

where $\sigma\equiv J^t-r\partial_u J^r/(l+1)$. These relations are analogous to Eq. (1.4) and have the same physical interpretation. Finally, the zeroth-order moments $\mathcal{I}_{lm}^{(0)}(u)$ and $\mathcal{S}_{lm}^{(0)}(u)$ are obtained from Eqs. (1.11) and (1.12) by discarding all $O(v^2)$ terms; in this limit, $\sigma=J^t$.

D. Gravitational radiation

The case of gravitational radiation is conceptually very different from the previous cases, because of the fact that the spacetime metric is now dynamical. However, if we assume that $T^{\alpha\beta}(\boldsymbol{x})$, the given stress-energy tensor responsible for the radiation, is small, then the Einstein field equations may be linearized in the small deviation of the metric with respect to the Schwarzschild form. This results in a wave equation for the metric perturbation [28], and mathematically, the gravitational-radiation problem ends up resembling closely the scalar and electromagnetic analogues. This is the problem considered in Sec. V.

The traceless-transverse gravitational-wave field is given by [29]

$$h_{\alpha\beta}^{\rm rad}(t,\vec{x})=\frac{1}{r}\sum_{l=2}^{\infty}\sum_{m=-l}^{l}[\mathcal{I}_{lm}(u)T_{\alpha\beta}^{E2,lm}(\theta,\phi)+\mathcal{S}_{lm}(u)T_{\alpha\beta}^{B2,lm}(\theta,\phi)],\tag{1.13}$$

where $\mathcal{I}_{lm}(u)$ and $\mathcal{S}_{lm}(u)$ are mass and current multipole moments, respectively, while $T_{\alpha\beta}^{E2,lm}(\theta,\phi)$ and $T_{\alpha\beta}^{B2,lm}(\theta,\phi)$

are the tensorial spherical harmonics described in Appendix A. The weak-field, slow-motion approximations to the multipole moments are

$$\mathcal{I}^{(3)}_{lm}(u)=\mathcal{I}^{(2)}_{lm}(u)+2M\int_{-\infty}^{u}\left[\ln\left(\frac{u-u'}{4M}\right)+\beta_l^{\rm grav,mass}+\gamma\right]\times\ddot{\mathcal{I}}^{(0)}_{lm}(u')du' \tag{1.14}$$

and

$$\mathcal{S}^{(3)}_{lm}(u)=\mathcal{S}^{(2)}_{lm}(u)+2M\int_{-\infty}^{u}\left[\ln\left(\frac{u-u'}{4M}\right)+\beta_l^{\rm grav,current}+\gamma\right]\times\ddot{\mathcal{S}}^{(0)}_{lm}(u')du', \tag{1.15}$$

where

$$\beta_l^{\rm grav,current}=\psi(l+1)+\frac{1}{2}-\frac{2}{l(l+1)} \tag{1.16}$$

and

$$\beta_l^{\rm grav,mass}=\beta_l^{\rm grav,current}-\frac{6}{(l-1)l(l+1)(l+2)}. \tag{1.17}$$

The second-order moments are given by

$$\mathcal{I}^{(2)}_{lm}(u)=\frac{16\pi}{(2l+1)!!}\sqrt{\frac{(l+1)(l+2)}{2(l-1)l}}\left(\frac{d}{du}\right)^{l}\times\int\left[1+\frac{l+9}{2(l+1)(2l+3)}(r\partial_u)^2-(l+2)\frac{M}{r}\right]\sigma(u,\vec{x})r^l\overline{Y_{lm}}(\theta,\phi)d\vec{x} \tag{1.18}$$

and

$$\mathcal{S}^{(2)}_{lm}(u)=-\frac{16\pi}{(2l+1)!!}\sqrt{\frac{2(l+2)}{l-1}}\left(\frac{d}{du}\right)^{l}\times\int\left[1+\frac{l+4}{2(l+2)(2l+3)}(r\partial_u)^2-\frac{(l-1)(l+2)}{l}\frac{M}{r}\right]J^{\alpha}(u,\vec{x})r^l\overline{Y_{\alpha}^{B,lm}}(\theta,\phi)d\vec{x}, \tag{1.19}$$

where $\sigma\equiv T^{tt}+T^{rr}+r^2T^{\theta\theta}+r^2\sin^2\theta T^{\phi\phi}-4r\partial_u T^{tr}/(l+1)$, $J^{\alpha}\equiv T^{t\alpha}-r\partial_u T^{r\alpha}/(l+2)$. Finally, the zeroth-order expressions are recovered by discarding all $O(v^2)$ terms from Eqs. (1.18) and (1.19); in this limit, $\sigma=T^{tt}$ and $J^{\alpha}=T^{t\alpha}$.

Equations (1.14)–(1.19) were previously derived (in a different representation involving symmetric tracefree tensors) by Blanchet [30] (see also Refs. [16] and [19]) on the basis of post-Newtonian theory. The physical interpretation of these results is the same as in the previous cases, and the results share the same degree of generality as the previous ones (see the concluding paragraph of Sec. I B).

E. Universality of the tail correction

A survey of the preceding subsections reveals that the multipole moments of scalar, electromagnetic, and gravitational radiation fields all share the same mathematical structure, with terms of order v^2 near-zone corrections depending on the detailed behavior of the source and with terms of order v^3 wave-propagation corrections independent of the detailed behavior of the source. And while the near-zone corrections are local in time, the wave-propagation corrections introduce a nonlocality.

We also observe that the tail corrections depend on the multipole order l and on the field's type only through the terms involving the various β_l's. These terms are actually *instantaneous*, because after integration over du', they are found to be proportional to the first derivative of the zeroth-order moments evaluated at u. The truly nonlocal tail corrections, which involve the weighting function $\ln(u-u')$, are *independent* of multipole order and field type. This remarkable result, that the tail correction has a *universal* form, is one of the main new contributions of this paper.

F. Physical origin of the tail term

The nonlocality (in time) of the radiative multipole moments is heuristically understood as arising from the scattering of the radiation field by the spacetime curvature surrounding the mass M, and a survey of our previous results indeed reveals that the tail terms are proportional to M. Now, the mass parameter enters twice in the metric of an asymptotically flat spacetime: Assuming that the weak-field metric is expressed in Schwarzschild-like coordinates, we have $g_{tt}\sim-1+2M/r$ and $g_{rr}\sim1+2M/r$ at large distances. Because of this degeneracy, it is impossible to tell whether it is ''g_{tt}'s mass'' which is ''mostly responsible'' for the tail effect or whether it is ''g_{rr}'s mass,'' or whether both are ''equally responsible.'' In other words, we cannot tell how the temporal and spatial curvatures separately contribute to the tail effect.

We examine this question in Sec. VI for the specific case of scalar radiation. To lift the degeneracy, we artificially introduce an additional mass parameter $\hat{\gamma}M$ in the description of our spacetime. This is defined so that the metric functions are now given by $g_{tt}\sim-1+2M/r$ and $g_{rr}\sim1+2\hat{\gamma}M/r$ in the weak-field limit. General relativity is recovered by setting $\hat{\gamma}=1$.

Integrating the scalar wave equation for the modified spacetime yields Eq. (1.1) with

$$\mathcal{Z}^{(3)}_{lm}(u)=\mathcal{Z}^{(2)}_{lm}(u)+(1+\hat{\gamma})M\int_{-\infty}^{u}\left[\ln\left(\frac{u-u'}{4M}\right)+\beta_l^{\rm scalar}+\gamma\right]\times\ddot{\mathcal{Z}}^{(0)}_{lm}(u')du', \tag{1.20}$$

where $\beta_l^{\rm scalar}=\psi(l+1)+\hat{\gamma}/(1+\hat{\gamma})$,

$$\mathcal{Z}_{lm}^{(2)}(u)=\frac{4\pi}{(2l+1)!!}\left(\frac{d}{du}\right)^l\int\left[1+\frac{(r\partial_u)^2}{2(2l+3)}-\frac{(2l+1)\hat{\gamma}-1}{2}\frac{M}{r}\right]\rho(u,\vec{x})r^l\overline{Y_{lm}}(\theta,\phi)d\vec{x},\tag{1.21}$$

and $\mathcal{Z}_{lm}^{(0)}(u)$ is given by Eq. (1.3). These expressions reduce to Eqs. (1.5) and (1.4), respectively, when $\hat{\gamma}=1$. We see that in the modified spacetime, the tail terms are proportional to $(1+\hat{\gamma})$. This allows us to conclude that in general relativity, the temporal and spatial curvatures contribute *equally* to the tail effect. This intriguing observation is another main contribution of this paper.

G. Spacetime approach

The final section of the paper, Sec. VII, contains an alternative derivation of Eqs. (1.1)–(1.6) based on the spacetime approach of DeWitt and Brehme [3]. For simplicity, we again restrict attention to the case of scalar radiation.

The mathematical methods employed in Secs. III, IV, and V to derive expressions for the radiative multipole moments of integer-spin fields are based upon a separation of variables approach made possible by the symmetries of the Schwarzschild solution. These methods, though convenient for practical computations, do not reflect closely the physical picture of wave propagation in curved spacetime. In particular, the distinction between direct terms (which are local in time) and tail terms (which are nonlocal) emerges only at the very end of the calculation.

The spacetime approach of Sec. VII is based instead on $G(\boldsymbol{x},\boldsymbol{x}')$, the DeWitt-Brehme retarded Green's function for the scalar wave equation [3]. As was mentioned in Sec. I A, $G(\boldsymbol{x},\boldsymbol{x}')$ is expressed as a sum of two parts. The first part has support on, and only on, the past light cone of the field point $\boldsymbol{x}$, and gives rise to the direct terms in the waves. The second part has support inside the past light cone of $\boldsymbol{x}$, and gives rise to the tail terms. In the spacetime approach of DeWitt and Brehme, the mathematics reflects the physical picture quite closely.

The radiative multipole moments calculated in Sec. VII agree precisely with those calculated in Sec. III. Therefore, the calculation based on the spacetime approach tells us nothing new in terms of the final answer. Nevertheless, this alternative derivation is very instructive, because of the fact that the mathematical origin of the tail correction is clear from the outset. We regard this as another important contribution of this paper.

H. Organization of this paper

The remaining sections of the paper contain the detailed derivations of the results summarized above. After laying some preliminary ground work in Sec. II, we integrate the wave equations for scalar, electromagnetic, and gravitational radiation in Schwarzschild spacetime in Secs. III, IV, and V, respectively. All calculations are carried out in a weak-field, slow-motion approximation. In Sec. VI we integrate the scalar wave equation for the artificially modified spacetime. And, finally, in Sec. VII we integrate the scalar wave equation using the spacetime approach of DeWitt and Brehme [3]. Various technical details are relegated to five Appendixes.

II. GENERALIZED REGGE-WHEELER EQUATION

The generalized Regge-Wheeler equation [31]

$$\left\{\frac{d^2}{dr^{*2}}+\omega^2-f\left[\frac{l(l+1)}{r^2}-\frac{2(s^2-1)M}{r^3}\right]\right\}X_l(\omega;r)=0\tag{2.1}$$

has long been known to govern the evolution of integer-spin fields in Schwarzschild spacetime. Here, r is the usual Schwarzschild coordinate, $f=1-2M/r$ (with M denoting the mass of the spacetime), and $d/dr^*=f d/dr$. Also, ω denotes the frequency of the field, l its multipole order, and $s=\{0,1,2\}$ its spin. The precise relation between the mode functions $X_l(\omega,r)$ and the corresponding scalar, electromagnetic, and gravitational fields will be described in Secs. III, IV, and V, respectively. In this section we consider the purely mathematical problem of integrating Eq. (2.1) in the low-frequency limit.

We first examine the question of boundary conditions. It is easy to check that $X_l(\omega;r)$ must behave as $e^{\pm i\omega r^*}$, where

$$r^*=r+2M\ln(r/2M-1),\tag{2.2}$$

in the asymptotic limits $r\to 2M$, $r\to\infty$. It will become clear in the following sections that the desired solution is the one which describes purely incoming waves at the black-hole event horizon. We therefore select the function $X_l^H(\omega;r)$, such that

$$X_l^H(\omega;r\to 2M)\sim(\text{const})e^{-i\omega r^*}.\tag{2.3}$$

The constant appearing in front of $e^{-i\omega r^*}$ determines the overall normalization of the Regge-Wheeler function. Because our final results will be independent of this normalization, we shall leave this constant arbitrary. At infinity, $X_l^H(\omega;r)$ describes a superposition of incoming and outgoing waves. Consequently,

$$X_l^H(\omega;r\to\infty)\sim\mathcal{A}_l^{\text{in}}(\omega)e^{-i\omega r^*}+\mathcal{A}_l^{\text{out}}(\omega)e^{i\omega r^*}.\tag{2.4}$$

The amplitudes $\mathcal{A}_l^{\text{in}}(\omega)$ and $\mathcal{A}_l^{\text{out}}(\omega)$ are determined by solving the differential equation.

We wish to integrate Eq. (2.1) in the low-frequency limit, for $M|\omega|\ll 1$. Without loss of generality, we henceforth take ω to be positive; the negative-frequency case can easily be recovered from the relation $X_l^H(-\omega)=\overline{X_l^H}(\omega)$, where an overbar denotes complex conjugation. To facilitate the calculations, we define the small (positive) quantity

$$\varepsilon\equiv 2M\omega,\tag{2.5}$$

and introduce a new dependent variable

$$z=\omega r.\tag{2.6}$$

After substitution and expansion in powers of ε, the generalized Regge-Wheeler equation becomes

$$\left\{\frac{d^2}{dz^2}+1-\frac{l(l+1)}{z^2}+\frac{\varepsilon}{z}\left[\frac{1}{z}\frac{d}{dz}+2-\frac{l(l+1)+1-s^2}{z^2}\right]+O(\varepsilon^2)\right\}X_l(z)=0. \tag{2.7}$$

It should be noted that when expanding in powers of ε, it was implicitly assumed that $z \gg \varepsilon$. Our low-frequency solution will therefore be restricted to the domain $r \gg 2M$. Our task is now to integrate Eq. (2.7) to first order in ε. We proceed by iteration, by writing $X_l = X_l^{(0)} + \varepsilon X_l^{(1)} + O(\varepsilon^2)$, substituting into Eq. (2.7), and solving order by order [32].

Because we are restricted to the domain $z \gg \varepsilon$, which excludes the event horizon at $z=\varepsilon$, Eq. (2.3) cannot be imposed directly, and the issue of boundary conditions must be reexamined. This question was addressed by Poisson and Sasaki [33], who integrated the $s=2$ Regge-Wheeler equation in the domain $z \ll 1$ (which includes the horizon), imposed the correct boundary condition at $z=\varepsilon$, and then matched the resulting function to the general solution of Eq. (2.7) in the common domain $\varepsilon \ll z \ll 1$. Such an analysis will not be repeated here. It suffices to state the conclusion: To be compatible with the incoming-wave boundary condition at the horizon, the solution to Eq. (2.7) must be regular in the (unphysical, and unrealized) limit $z \to 0$.

With this in mind, the desired zeroth-order solution to Eq. (2.7) is

$$X_l^{H(0)}(z) = z j_l(z), \tag{2.8}$$

where $j_l(z)$ are the spherical Bessel functions of the first kind. It should be noted that Eq. (2.8) provides a particular choice for the overall normalization of $X_l^H(z)$. The first-order solution is then determined by solving

$$\left[\frac{d^2}{dz^2}+1-\frac{l(l+1)}{z^2}\right]X_l^{(1)}(z) = -W_l(z), \tag{2.9}$$

where

$$W_l(z)=\frac{1}{z}\left[\frac{1}{z}\frac{d}{dz}+2-\frac{l(l+1)+1-s^2}{z^2}\right]z j_l(z). \tag{2.10}$$

The general solution to Eq. (2.9) is

$$X_l^{(1)}(z)=z j_l(z)\left[a+\int^z z' n_l(z') W_l(z')dz'\right]+z n_l(z)\left[b-\int^z z' j_l(z') W_l(z')dz'\right], \tag{2.11}$$

where $n_l(z)$ are the spherical Bessel functions of the second kind, and a and b are constants which must be chosen so that $X_l^{(1)}(z)$ satisfies the regularity condition at $z=0$.

The integrations of Eq. (2.11) can be carried out explicitly. First, we use the recurrence relations among spherical Bessel functions (Ref. [34], p. 439) to write $W_l(z)$ in the form

$$zW_l(z)=2zj_l(z)-\frac{(l-s)(l+s)}{2l+1}j_{l-1}(z)-\frac{(l-s+1)(l+s+1)}{2l+1}j_{l+1}(z). \tag{2.12}$$

Second, we evaluate the integrals using the results found in the Appendix of Ref. [35]. After straightforward manipulations, we arrive at

$$X_l^{(1)}(z)=[a-A_l(z)]zj_l(z)+[b+\gamma+B_l(z)]zn_l(z)-\frac{(l-s)(l+s)}{2l(2l+1)}zj_{l-1}(z)+\frac{(l-s+1)(l+s+1)}{2(l+1)(2l+1)}zj_{l+1}(z), \tag{2.13}$$

where

$$A_l(z)=\mathrm{Si}(2z)+z^2 n_0(z) j_0(z)+\sum_{p=1}^{l-1}\left(\frac{1}{p}+\frac{1}{p+1}\right)z^2 n_p(z) j_p(z) \tag{2.14}$$

and

$$B_l(z)=\mathrm{Ci}(2z)-\gamma-\ln(2z)+z^2 {j_0}^2(z)+\sum_{p=1}^{l-1}\left(\frac{1}{p}+\frac{1}{p+1}\right)z^2 {j_p}^2(z). \tag{2.15}$$

Here, Si and Ci are the sine and cosine integral functions, respectively, and $\gamma \simeq 0.577\,21$ is Euler's number.

In Appendix B, the functions $A_l(z)$ and $B_l(z)$ are evaluated in the limit $z \to 0$. We find

$$A_l(z)=\frac{z}{l}+O(z^3),$$

$$B_l(z)=-\frac{z^{2l+2}}{l(l+1)(2l-1)!!(2l+1)!!}+O(z^{2l+4}). \tag{2.16}$$

It follows that $X^{(1)}(z)$ goes to zero in the limit $z \to 0$ provided that $b=-\gamma$. Otherwise, the function diverges. This choice for b therefore selects $X_l^{H(1)}(z)$, the desired solution. The constant a remains arbitrary, because it affects only the overall normalization of the solution; a can be set to zero without loss of generality.

Integration of the generalized Regge-Wheeler function, to first order in ε, is now completed. We have pointed out that our answer incorporates a specific choice of overall normalization which is provided by Eq. (2.8). We now wish to form a normalization-independent quantity $X_l^H(z)/\mathcal{A}_l^{\mathrm{in}}$, which will be required in the following sections of this paper. We must therefore calculate $\mathcal{A}_l^{\mathrm{in}}$. This involves the evaluation of

$X_l^H(z)$, as given by Eqs. (2.8) and (2.13)–(2.15), in the limit $z\to\infty$, and a comparison with the low-frequency limit of Eq. (2.4). Such a calculation is carried out in Appendix C. The final result is quoted here:

$$\frac{X_l^H(z)}{\mathcal{A}_l^{\rm in}}=2(-i)^{l+1}e^{i\varepsilon(\ln 2\varepsilon-\beta_l)}\left(1+\frac{\pi}{2}\varepsilon\right)z\bigg\{[1-\varepsilon A_l(z)]j_l(z)+\varepsilon B_l(z)n_l(z)-\varepsilon\frac{(l-s)(l+s)}{2l(2l+1)}j_{l-1}(z)+\varepsilon\frac{(l-s+1)(l+s+1)}{2(l+1)(2l+1)}j_{l+1}(z)+O(\varepsilon^2)\bigg\}, \quad (2.17)$$

where

$$\beta_l=\psi(l+1)+\frac{1}{2}-\frac{s^2}{2l(l+1)}, \quad (2.18)$$

with $\psi(l+1)=-\gamma+\Sigma_{k=1}^{l}k^{-1}$ denoting the digamma function.

Finally, we evaluate Eq. (2.17) in the limit $z\ll 1$, to a degree of accuracy sufficient for our purposes in the following sections of this paper. For this, we use Eq. (2.16) and the series expansions for the spherical Bessel functions. We arrive at

$$\frac{X_l^H(\omega;r)}{\mathcal{A}_l^{\rm in}(\omega)}=\frac{2}{(2l+1)!!}e^{2iM\omega(\ln 4M|\omega|-\beta_l)}(1+\pi M|\omega|)\times(-i\omega r)^{l+1}\bigg\{1-\frac{(\omega r)^2}{2(2l+3)}-\frac{(l-s)(l+s)}{l}\frac{M}{r}+O[(\omega r)^4,M\omega^2 r,(M/r)^2]\bigg\}. \quad (2.19)$$

This equation holds both for positive and negative frequencies.

III. SCALAR RADIATION

A. Wave equation

We begin our study of radiative multipole moments in curved spacetime with the simplest case, that of a real scalar field $\Phi(\boldsymbol{x})$ obeying the wave equation

$$\Box\Phi(\boldsymbol{x})=-4\pi\rho(\boldsymbol{x}). \quad (3.1)$$

Here, $\Box=g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ is the curved spacetime wave operator, $\rho(\boldsymbol{x})$ is an unspecified source function, and $\boldsymbol{x}$ collectively designates all spacetime coordinates. The spacetime is assumed to be Schwarzschild (with mass M), and the usual coordinates $\{t,r,\theta,\phi\}$ are adopted.

Because the spacetime is static and spherically symmetric, the scalar field can be decomposed according to

$$\Phi(\boldsymbol{x})=\frac{1}{r}\int d\omega\sum_{lm}R_{lm}(\omega;r)Y_{lm}(\theta,\phi)e^{-i\omega t}, \quad (3.2)$$

where the sums over l and m are restricted by $l\geqslant 0$, $|m|\leqslant l$. Substituting this into Eq. (3.1), we obtain the following ordinary differential equation for the radial function $R_{lm}(\omega;r)$:

$$\left\{\frac{d^2}{dr^{*2}}+\omega^2-f\left[\frac{l(l+1)}{r^2}+\frac{2M}{r^3}\right]\right\}R_{lm}(\omega;r)=fT_{lm}(\omega;r), \quad (3.3)$$

where $d/dr^*=f\,d/dr$ and $f=1-2M/r$. The source term is given by

$$T_{lm}(\omega;r)=-4\pi r\int\tilde{\rho}(\omega,\vec{x})\overline{Y_{lm}}(\theta,\phi)d\Omega, \quad (3.4)$$

where $d\Omega=d\cos\theta\,d\phi$, an overbar denotes complex conjugation, and

$$\tilde{\rho}(\omega,\vec{x})=\frac{1}{2\pi}\int\rho(t,\vec{x})e^{i\omega t}dt \quad (3.5)$$

is the Fourier transform of $\rho(\boldsymbol{x})$. The symbol $\vec{x}$ collectively designates all spatial coordinates.

B. Solution

Equation (3.3) has a Sturm-Liouville form, and it can therefore be solved in terms of a Green's function constructed from two linearly independent solutions to the homogeneous problem. Which solutions are selected depends on the boundary conditions we wish to impose on $R_{lm}(\omega;r)$. The appropriate choice here is dictated by the physical requirement that the scalar field must represent waves which are purely ingoing at the black hole horizon ($r=2M$) and purely outgoing at $r=\infty$. This amounts to integrating Eq. (3.1) with a no-incoming-radiation initial condition.

We therefore seek functions $R_l^H(\omega;r)$ and $R_l^\infty(\omega;r)$, solutions to

$$\left\{\frac{d^2}{dr^{*2}}+\omega^2-f\left[\frac{l(l+1)}{r^2}+\frac{2M}{r^3}\right]\right\}R_l(\omega;r)=0, \quad (3.6)$$

and such that

$$R_l^H(\omega;r\to 2M)\sim e^{-i\omega r^*},$$
$$R_l^H(\omega;r\to\infty)\sim\mathcal{Q}_l^{\rm in}(\omega)e^{-i\omega r^*}+\mathcal{Q}_l^{\rm out}(\omega)e^{i\omega r^*}, \quad (3.7)$$
$$R_l^\infty(\omega;r\to\infty)\sim e^{i\omega r^*},$$

where $r^*=r+2M\ln(r/2M-1)$. Equation (3.7) indicates that $R_l^H(\omega;r)$ describes waves which are purely ingoing at the black-hole horizon, while $R_l^\infty(\omega;r)$ describes waves which are purely outgoing at infinity. The behavior of $R_l^\infty(\omega;r)$ near $r=2M$ will not be needed.

In terms of these functions, the solution to Eq. (3.3) takes the form

$$R_{lm}(\omega;r)=\frac{1}{2i\omega\mathcal{Q}_l^{\rm in}(\omega)}\times\left[R_l^{\infty}(\omega;r)\int_{2M}^{r}T_{lm}(\omega;r')R_l^{H}(\omega;r')dr'+R_l^{H}(\omega;r)\int_{r}^{\infty}T_{lm}(\omega;r')R_l^{\infty}(\omega;r')dr'\right],\tag{3.8}$$

where the factor $2i\omega\mathcal{Q}_l^{\rm in}(\omega)$ is the conserved Wronskian of the functions $R_l^H(\omega;r)$ and $R_l^\infty(\omega;r)$.

We are interested in the radiative part of the field, which dominates at large distances from the source. More precisely, we define the radiative field by $r\Phi_{\rm rad}(\boldsymbol{x})=\lim_{r\to\infty}r\Phi(\boldsymbol{x})$, which expresses the fact that at large distances, $\Phi(\boldsymbol{x})=\Phi_{\rm rad}(\boldsymbol{x})+O(1/r^2)$. Evaluating Eq. (3.8) in the limit $r\to\infty$ [assuming that $T_{lm}(\omega;r)$ has compact support], we obtain

$$R_{lm}(\omega;r\to\infty)\sim\tilde{\mathcal{Z}}_{lm}(\omega)e^{i\omega r^*},\tag{3.9}$$

where

$$\tilde{\mathcal{Z}}_{lm}(\omega)\equiv\frac{1}{2i\omega\mathcal{Q}_l^{\rm in}(\omega)}\int_{2M}^{\infty}T_{lm}(\omega;r)R_l^H(\omega;r)dr.\tag{3.10}$$

Finally, substituting this into Eq. (3.2) yields

$$\Phi_{\rm rad}(t,\vec{x})=\frac{1}{r}\sum_{lm}\mathcal{Z}_{lm}(u)Y_{lm}(\theta,\phi),\tag{3.11}$$

where $u=t-r^*$ is retarded time, and

$$\mathcal{Z}_{lm}(u)=\int\tilde{\mathcal{Z}}_{lm}(\omega)e^{-i\omega u}d\omega.\tag{3.12}$$

The quantities $\mathcal{Z}_{lm}(u)$, or their Fourier transform $\tilde{\mathcal{Z}}_{lm}(\omega)$, will be referred to as the *radiative multipole moments* of the scalar field $\Phi(\boldsymbol{x})$.

C. Slow-motion approximation

In Eq. (3.10), the radiative multipole moments are written in exact form in terms of the source $T_{lm}(\omega;r)$ and the function $R_l^H(\omega;r)/\mathcal{Q}_l^{\rm in}(\omega)$. While the source function will be left unspecified, we now wish to find an expression for $R_l^H(\omega;r)/\mathcal{Q}_l^{\rm in}(\omega)$. To do this we must resort to approximations, because Eq. (3.6) cannot be integrated in closed form.

We will derive approximate expressions for the radiative multipole moments, and these will be valid in weak-field, slow-motion situations. To formulate this approximation precisely, we introduce a characteristic radius r_c, to be thought of as the radial coordinate of a typical portion of the source. [In other words, $T_{lm}(\omega;r)$ is assumed to be appreciably different from zero only for values of r comparable to r_c.] We introduce also a characteristic time $1/\omega_c$, to be thought of as the typical time scale over which the source moves. [In other words, $T_{lm}(\omega;r)$ is assumed to be appreciably different from zero only for values of ω comparable to ω_c.] Finally, we introduce a characteristic velocity $v\ll 1$, which will be the smallness parameter of our approximation. In terms of these quantities, the requirement that the source motions must be slow translates to

$$\omega_c r_c=O(v).\tag{3.13}$$

The virial theorem for gravitationally bound systems then implies that the gravitational field must be weak inside the source:

$$M/r_c=O(v^2).\tag{3.14}$$

Finally, the slow-motion approximation implies that the scalar waves produced by the source's motion must have low frequencies:

$$M\omega_c=O(v^3).\tag{3.15}$$

Our calculation of the radiative multipole moments will be carried out to order v^3 beyond the leading-order expressions.

We now proceed. It is evident that Eq. (3.6) is nothing but Eq. (2.1), the generalized Regge-Wheeler equation, with $s=0$. So we have, immediately,

$$\frac{R_l^H(\omega;r)}{\mathcal{Q}_l^{\rm in}(\omega)}=\frac{X_l^H(\omega;r)}{\mathcal{A}_l^{\rm in}(\omega)}.\tag{3.16}$$

Equation (2.19) may therefore be substituted into Eq. (3.10). After using Eq. (3.4), we obtain

$$\tilde{\mathcal{Z}}_{lm}(\omega)=\frac{4\pi}{(2l+1)!!}e^{2iM\omega(\ln 4M|\omega|-\beta_l)}(1+\pi M|\omega|)(-i\omega)^l\times\int\left[1-\frac{(\omega r)^2}{2(2l+3)}-l\frac{M}{r}+O(v^4)\right]\times\tilde{\rho}(\omega,\vec{x})r^l\overline{Y_{lm}}(\theta,\phi)d\vec{x},\tag{3.17}$$

where

$$\beta_l=\psi(l+1)+\frac{1}{2}\tag{3.18}$$

and $d\vec{x}=r^2drd\Omega$.

The physical interpretation of this result comes more easily if we first invert the Fourier transform. The ω-dependent prefactors complicate this procedure slightly, but an explicit expression for $\mathcal{Z}_{lm}(u)$ can nevertheless be found (see Appendix D). We obtain

$$\mathcal{Z}_{lm}(u)=Z_{lm}(u)+2M\int_{-\infty}^{u}\left[\ln\left(\frac{u-u'}{4M}\right)+\beta_l+\gamma\right]\times\ddot{Z}_{lm}(u')du',\tag{3.19}$$

where overdots indicate differentiation with respect to u' and

$$Z_{lm}(u)=\frac{4\pi}{(2l+1)!!}\left(\frac{d}{du}\right)^l\int\left[1+\frac{(r\partial_u)^2}{2(2l+3)}-l\frac{M}{r}+O(v^4)\right]\rho(u,\vec{x})r^l\overline{Y_{lm}}(\theta,\phi)d\vec{x}.\tag{3.20}$$

Equation (3.20) indicates that to leading order, $\mathcal{Z}_{lm}(u)$ is given by the lth retarded-time derivative of $\int \rho r^l \overline{Y_{lm}} d\vec{x}$. This justifies our referring to these quantities as multipole moments. Equation (3.20) also shows that all corrections of order v^2 appear *inside* the spatial integral, and that they depend on the detailed behavior of the source function. These corrections are *near-zone* corrections, and they are purely local in time: As with the leading-order term, they involve the value of the source at the retarded time u only. This is not so for the corrections of order v^3, as is shown by Eq. (3.19): These involve the value of the source at *all times* prior to the retarded time u. Furthermore, the $O(v^3)$ corrections appear *outside* the spatial integral, and they are independent of the detailed behavior of the source. These are *wave-propagation* corrections. Equations (3.19) and (3.20), with β_l given by Eq. (3.18), are equivalent to Eqs. (1.4)–(1.6).

We recognize the important distinction between near-zone and wave-propagation corrections. Near-zone corrections depend on the detailed behavior of the source and are local in time. Wave-propagation corrections, on the other hand, do not depend on the detailed behavior of the source, and are nonlocal in time. This nonlocality is heuristically understood as arising from the scattering of the radiation by the spacetime curvature surrounding the source. This scattering causes part of the information about the state of the source to be delayed further than what is strictly required by causality. The integral term in Eq. (3.19) is often called the *tail* term, and wave-propagation corrections are often called *tail* corrections.

IV. ELECTROMAGNETIC RADIATION

A. Teukolsky equation

In this section, we derive expressions for the radiative multipole moments of an electromagnetic field in Schwarzschild spacetime. The Maxwell equations for this spacetime have been cast, by Teukolsky [36], in a form convenient for our purposes. In the Teukolsky formalism, the radiative part of the electromagnetic field is represented by the complex quantity $\Phi_2 = F^{\alpha\beta}\overline{m}_\alpha n_\beta$, where $F^{\alpha\beta}$ is the field tensor, $n_\alpha = -1/2(f,1,0,0)$ a null vector pointing radially inward, and $\overline{m}_\alpha = (0,0,r,-ir\sin\theta)/\sqrt{2}$ a spatial vector with zero norm. As before, an overbar denotes complex conjugation.

The field $\Phi_2(\boldsymbol{x})$ has spin weight $s=-1$ (see Appendix A), and it can be decomposed according to

$$\Phi_2(\boldsymbol{x}) = \frac{1}{r^2}\int d\omega \sum_{lm} R_{lm}(\omega;r)\, {}_{-1}Y_{lm}(\theta,\phi) e^{-i\omega t}, \tag{4.1}$$

where ${}_{-1}Y_{lm}(\theta,\phi)$ are spherical harmonics of spin weight -1 (see Appendix A). The sums over l and m are restricted by $l \geq 1$, $|m| \leq l$. The radial function then satisfies the inhomogeneous Teukolsky equation [36]

$$\left\{ r^2 f \frac{d^2}{dr^2} + \frac{1}{f}[(\omega r)^2 - 2i\omega r(1-3M/r)] - l(l+1) \right\} \times R_{lm}(\omega;r) = T_{lm}(\omega;r), \tag{4.2}$$

where $f = 1-2M/r$.

The source term $T_{lm}(\omega;r)$ is constructed as follows from $J^\alpha(\boldsymbol{x})$, the (unspecified) current density. One first forms the contractions

$${}_0J = -J^\alpha n_\alpha, \quad {}_{-1}J = -J^\alpha \overline{m}_\alpha, \tag{4.3}$$

then evaluates the Fourier transforms

$${}_s\tilde{J}(\omega,\vec{x}) = \frac{1}{2\pi}\int {}_sJ(t,\vec{x}) e^{i\omega t} dt, \tag{4.4}$$

and takes the projections

$${}_s\tilde{J}_{lm}(\omega;r) = \int {}_s\tilde{J}(\omega,\vec{x})\, {}_s\overline{Y_{lm}}(\theta,\phi) d\Omega. \tag{4.5}$$

The source term is finally given by [36]

$$T_{lm}(\omega;r) = 2\pi \sum_s {}_sp_{ls} \mathcal{D}_s \tilde{J}_{lm}(\omega;r), \tag{4.6}$$

where the sum runs from $s=0$ to $s=-1$,

$${}_sp_l = \begin{cases} \sqrt{2l(l+1)}, & s=0 \\ 1, & s=-1, \end{cases} \tag{4.7}$$

and

$${}_s\mathcal{D} = \begin{cases} r^3, & s=0, \\ r\mathcal{L}r^3, & s=-1, \end{cases} \tag{4.8}$$

with $\mathcal{L} = f d/dr + i\omega$.

B. Solution

Equation (4.2) is integrated by means of a Green's function, in a manner similar to what was done in Sec. III. We introduce two functions $R_l^H(\omega;r)$ and $R_l^\infty(\omega;r)$, solutions to the homogeneous Teukolsky equation [Eq. (4.2) with $T_{lm}(\omega;r)=0$], with asymptotic behavior [36]

$$\begin{gathered} R_l^H(\omega;r\to 2M) \sim f e^{-i\omega r^*}, \\ R_l^H(\omega;r\to\infty) \sim \mathcal{Q}_l^{\text{in}}(\omega)(i\omega r)^{-1} e^{-i\omega r^*} + \mathcal{Q}_l^{\text{out}}(\omega)(i\omega r) e^{i\omega r^*}, \\ R_l^\infty(\omega;r\to\infty) \sim (i\omega r) e^{i\omega r^*}, \end{gathered} \tag{4.9}$$

where $r^* = r + 2M\ln(r/2M-1)$. In terms of these, the radial function is given by

$$R_{lm}(\omega;r) = \frac{1}{2i\omega \mathcal{Q}_l^{\text{in}}(\omega)} \times \left[R_l^\infty(\omega;r)\int_{2M}^r \frac{T_{lm}(\omega;r')R_l^H(\omega;r')}{r'^2 f'}\, dr' + R_l^H(\omega;r)\int_r^\infty \frac{T_{lm}(\omega;r')R_l^\infty(\omega;r')}{r'^2 f'}\, dr' \right], \tag{4.10}$$

where $f' = 1 - 2M/r'$.

As before, we are mostly interested in the behavior of the radial function near $r=\infty$. Equation (4.10) gives

$$R_{lm}(\omega;r\to\infty) \sim \tilde{\tilde{\mathcal{Z}}}(\omega)(i\omega r)e^{i\omega r^*}, \tag{4.11}$$

where

$$\tilde{\tilde{\mathcal{Z}}}_{lm}(\omega) \equiv \frac{1}{2i\omega \mathcal{Q}_l^{\rm in}(\omega)} \int_{2M}^{\infty} \frac{T_{lm}(\omega;r) R_l^H(\omega;r)}{r^2 f} dr. \tag{4.12}$$

These quantities are the (Fourier transform of the) radiative multipole moments of the electromagnetic field.

C. Adjoint operators and Chandrasekhar transformation

Equation (4.12) will not be our final expression for the radiative multipole moments. In Eq. (4.12), $T_{lm}(\omega;r)$ is obtained by applying the differential operators ${}_s\mathcal{D}$ on ${}_s\tilde{J}_{lm}(\omega;r)$, the projections of the current density. To have to take derivatives of these functions is an inconvenience, and we would like to express the moments directly in terms of ${}_s\tilde{J}_{lm}(\omega;r)$. It is easy to show, by straightforward integration by parts, that if we define the adjoint operators

$${}_s\mathcal{D}^\dagger = \begin{cases} r^3, & s=0, \\ -r^5 \bar{\mathcal{L}} r^{-1}, & s=-1, \end{cases} \tag{4.13}$$

where $\bar{\mathcal{L}} = f d/dr - i\omega$, then Eq. (4.12) is equivalent to

$$\tilde{\tilde{\mathcal{Z}}}_{lm}(\omega) = \frac{\pi}{i\omega \mathcal{Q}_l^{\rm in}(\omega)} \sum_s {}_s p_l \int_{2M}^{\infty} \frac{{}_s\tilde{J}_{lm}(\omega;r)\, {}_s\mathcal{D}^\dagger R_l^H(\omega;r)}{r^2 f} dr. \tag{4.14}$$

Although $\tilde{\tilde{\mathcal{Z}}}_{lm}(\omega)$ is now expressed directly in terms of ${}_s\tilde{J}_{lm}(\omega;r)$, Eq. (4.14) still will not be our final expression for the radiative multipole moments. We now want to write $R_l^H(\omega;r)$ in terms of $X_l^H(\omega;r)$, the solution to the generalized Regge-Wheeler equation (with $s=1$) considered in Sec. II. The relationship between these functions was trivial in the case of scalar radiation [cf. Eq. (3.16)]. That such a relationship exists in the case of gravitational radiation was shown by Chandrasekhar [37], who also provided it explicitly. We show in Appendix E that for the case of electromagnetic radiation, the Chandrasekhar transformation is given by

$$\frac{R_l^H(\omega;r)}{\mathcal{Q}_l^{\rm in}(\omega)} = \frac{-2}{l(l+1)} r\mathcal{L} \frac{X_l^H(\omega;r)}{\mathcal{A}_l^{\rm in}(\omega)}. \tag{4.15}$$

Substituting this into Eq. (4.14) and taking into account the fact that $\bar{\mathcal{L}}\mathcal{L} = l(l+1)f/r^2$ when acting on $X_l^H(\omega;r)$, we arrive at

$$\tilde{\tilde{\mathcal{Z}}}_{lm}(\omega) = \frac{-2\pi}{l(l+1) i\omega \mathcal{A}_l^{\rm in}(\omega)} \sum_s {}_s p_l \times \int_{2M}^{\infty} {}_s\tilde{J}_{lm}(\omega;r)\, {}_s\Gamma X_l^H(\omega;r) dr, \tag{4.16}$$

where we have introduced the operators

$${}_s\Gamma = \begin{cases} r^2 f^{-1} \mathcal{L}, & s=0, \\ -l(l+1)r, & s=-1. \end{cases} \tag{4.17}$$

We emphasize that $X_l^H(\omega;r)$ denotes the solution to the $s=1$ generalized Regge-Wheeler equation with boundary conditions (2.3) and (2.4). Equation (4.16) *will* be our final expression for the radiative multipole moments.

D. Vector potential

The physical meaning of the quantities $\tilde{Z}_{lm}(\omega)$ becomes more transparent if we use Eqs. (4.1) and (4.11) to construct $A_\alpha^{\rm rad}(\boldsymbol{x})$, the vector potential describing the radiative part of the field. [This is defined similarly to $\Phi_{\rm rad}(\boldsymbol{x})$, encountered in Sec. III.]

We may choose a gauge in which $A_\alpha^{\rm rad}(\boldsymbol{x})$ is purely transverse to the direction of propagation, which, at large distances from the source, is radially outward. This implies that the vector potential may be expressed as

$$A_\alpha^{\rm rad} = A m_\alpha + \overline{A m_\alpha}, \tag{4.18}$$

where $m_\alpha = (0,0,r,ir\sin\theta)/\sqrt{2}$ is complex conjugate to $\bar{m}_\alpha$. It should be noted that the quantity $A(\boldsymbol{x})$ is complex, but that $A_\alpha^{\rm rad}(\boldsymbol{x})$ is real.

The radiative part of the electromagnetic field tensor, $F_{\rm rad}^{\alpha\beta}(\boldsymbol{x})$, can easily be computed from Eq. (4.18), and the Teukolsky field $\Phi_2(\boldsymbol{x}) = F_{\rm rad}^{\alpha\beta} \bar{m}_\alpha n_\beta$ follows immediately. Keeping in mind that we are working near $r=\infty$, we find

$$\Phi_2 = -A_{,\alpha} n^\alpha = -\frac{\partial A}{\partial u}, \tag{4.19}$$

where $u = t - r^*$. Combining this with Eqs. (4.1) and (4.11) yields

$$A(t,\vec{x}) = \frac{1}{r} \sum_{lm} \mathcal{Z}_{lm}(u)\, {}_{-1}Y_{lm}(\theta,\phi), \tag{4.20}$$

where

$$\mathcal{Z}_{lm}(u) = \int \tilde{Z}_{lm}(\omega) e^{i\omega u} d\omega. \tag{4.21}$$

This shows that $\mathcal{Z}_{lm}(u)$ are indeed the radiative multipole moments of the electromagnetic field.

The vector potential is obtained by substituting Eq. (4.20) into Eq. (4.18). The spin-weighted spherical harmonics then combine with the vectors m_α and $\bar{m}_\alpha$ to form the vectorial spherical harmonics described in Appendix A. [See Eq. (A16); Eq. (A15) must also be used.] We find

$$A_\alpha^{\rm rad}(t,\vec{x})=\frac{1}{r}\sum_{lm}\,[\mathcal{I}_{lm}(u)Y_\alpha^{E,lm}(\theta,\phi)+\mathcal{S}_{lm}(u)Y_\alpha^{B,lm}(\theta,\phi)],\tag{4.22}$$

where we have introduced the *charge* multipole moments

$$\mathcal{I}_{lm}(u)=\frac{1}{\sqrt{2}}[\mathcal{Z}_{lm}(u)+(-1)^m\overline{\mathcal{Z}_{l,-m}}(u)]\tag{4.23}$$

and the *current* multipole moments

$$\mathcal{S}_{lm}(u)=\frac{i}{\sqrt{2}}[\mathcal{Z}_{lm}(u)-(-1)^m\overline{\mathcal{Z}_{l,-m}}(u)].\tag{4.24}$$

The reason for using this terminology will become clear below. For the time being we may mention that the first group of terms in Eq. (4.22), that involving the charge moments, has electric-type parity, while the second group, involving the current moments, has magnetic-type parity (see Appendix A). The fact that two sets of multipole moments are needed to form $A_\alpha^{\rm rad}(\boldsymbol{x})$ is related to the fact that the electromagnetic field possesses two radiative degrees of freedom.

E. Slow-motion approximation

We now compute the radiative multipole moments (4.16) in the slow-motion approximation. The calculation is similar to the one presented in Sec. III C.

We begin by substituting Eq. (2.19), with $s=1$, into Eq. (4.16). After using Eqs. (4.3), (4.5), (4.7), and (4.17), we obtain the lengthy expression

$$\begin{aligned}\tilde{\mathcal{Z}}_{lm}(\omega)=\frac{4\pi}{(2l+1)!!}\mathcal{T}_l(\omega)\,(-i\omega)^l\Bigg(\sqrt{\frac{l+1}{2l}}\int\Bigg\{1-\frac{l+3}{2(l+1)(2l+3)}(\omega r)^2-(l-1)\frac{M}{r}\\+\frac{i\omega r}{l+1}\left[1-\frac{(\omega r)^2}{2(2l+3)}-\frac{l^2-2l-1}{l}\frac{M}{r}\right]+O(v^4)\Bigg\}(f\tilde{\rho}+\tilde{J}^r)r^l{}_0\overline{Y_{lm}}d\vec{x}\\-\int\left[1-\frac{(\omega r)^2}{2(2l+3)}-\frac{l^2-1}{l}\frac{M}{r}+O(v^4)\right]_{-1}\tilde{J}r^l{}_{-1}\overline{Y_{lm}}d\vec{x}\Bigg).\end{aligned}\tag{4.25}$$

Here, $\tilde{\rho}(\omega,\vec{x})\equiv\tilde{J}^t(\omega,\vec{x})$ is the Fourier transform of the charge density, and

$$\mathcal{T}_l(\omega)=e^{2iM\omega(\ln 4M|\omega|-\beta_l)}(1+\pi M|\omega|),\tag{4.26}$$

where β_l is given by Eq. (2.18) with $s=1$:

$$\beta_l=\psi(l+1)+\frac{1}{2}-\frac{1}{2l(l+1)}.\tag{4.27}$$

Inspecting Eq. (4.25), we notice that it does not have the same mathematical structure as Eq. (3.17), which gives the radiative multipole moments of a scalar field. In particular, we see that $\tilde{\mathcal{Z}}_{lm}(\omega)$ possesses correction terms that are linear in v [the terms $i\omega r\tilde{\rho}/(l+1)$ and $\tilde{J}^r$, the latter being one power of v smaller than $\tilde{\rho}$], as well as many third-order terms that depend explicitly on r, and which cannot be taken outside the integral. This apparently contradicts our expectation that $\tilde{\mathcal{Z}}_{lm}(\omega)$ should come with only near-zone corrections of order v^2 and wave-propagation corrections of order v^3.

However, expression (4.25) is not unique, and we may use the continuity equation $J^\alpha{}_{;\alpha}=0$ to remove the unwanted terms. When written out explicitly, this reads

$$\rho_{,t}=-\frac{1}{r^2}(r^2J^r)_{,r}-\frac{1}{\sqrt{2}r}(\hat{\eth}_{-1}J+\check{\eth}_1J),\tag{4.28}$$

where $\hat{\eth}$ and $\check{\eth}$ are the ''edth'' differential operators described in Appendix A, and

$${}_1J=-J^\alpha m_\alpha.\tag{4.29}$$

The continuity equation gives rise to an integral identity if we multiply both sides by $r^n{}_0\overline{Y_{lm}}(\theta,\phi)$ and integrate over $d\vec{x}$. After a Fourier transform and several partial integrations [using Eqs. (A13) and (A14)], we obtain

$$-i\omega\int\tilde{\rho}r^n{}_0\overline{Y_{lm}}d\vec{x}=n\int\tilde{J}^rr^{n-1}{}_0\overline{Y_{lm}}d\vec{x}-\sqrt{\frac{l(l+1)}{2}}\int({}_{-1}\tilde{J}\,{}_{-1}\overline{Y_{lm}}-{}_1\tilde{J}\,{}_1\overline{Y_{lm}})r^{n-1}d\vec{x}.\tag{4.30}$$

We now use this identity to remove all terms proportional to $i\omega\tilde{\rho}(\omega,\vec{x})$ in Eq. (4.25). After some remarkable cancellations, wherein all unwanted terms disappear, we arrive at our final expression for the radiative multipole moments:

$$\tilde{\tilde{\mathcal{Z}}}_{lm}(\omega)=\frac{4\pi}{(2l+1)!!}\mathcal{T}_l(\omega)\ (-i\omega)^l\left\{\sqrt{\frac{l+1}{2l}}\int\left[1-\frac{l+3}{2(l+1)(2l+3)}(\omega r)^2-(l-1)\frac{M}{r}+O(v^4)\right]\left(\tilde{\rho}+\frac{i\omega r}{l+1}\tilde{J}^r\right)r^l{}_0\overline{Y}_{lm}d\vec{x}\right.$$
$$\left.-\frac{1}{2}\int\left[1-\frac{(\omega r)^2}{2(2l+3)}-\frac{l^2-1}{l}\frac{M}{r}+O(v^4)\right]({}_{-1}\tilde{J}_{-1}\overline{Y}_{lm}+{}_1\tilde{J}_1\overline{Y}_{lm})r^l d\vec{x}\right\}. \tag{4.31}$$

We see that this expression has the expected form, with all $O(v^2)$ corrections occurring inside the spatial integrals, and all $O(v^3)$ corrections occurring outside.

We now separate $\tilde{\tilde{\mathcal{Z}}}_{lm}(\omega)$ into charge and current moments, according to the Fourier transform of Eqs. (4.23) and (4.24). This gives

$$\tilde{\mathcal{I}}_{lm}(\omega)=\frac{4\pi}{(2l+1)!!}\sqrt{\frac{l+1}{l}}\mathcal{T}_l(\omega)(-i\omega)^l\int\left[1-\frac{l+3}{2(l+1)(2l+3)}(\omega r)^2-(l-1)\frac{M}{r}+O(v^4)\right]$$
$$\times\left[\tilde{\rho}(\omega,\vec{x})+\frac{i\omega r}{l+1}\tilde{J}^r(\omega,\vec{x})\right]r^l{}_0\overline{Y}_{lm}(\theta,\phi)d\vec{x} \tag{4.32}$$

and

$$\tilde{\mathcal{S}}_{lm}(\omega)=-\frac{2\sqrt{2}i\pi}{(2l+1)!!}\mathcal{T}_l(\omega)(-i\omega)^l\int\left[1-\frac{(\omega r)^2}{2(2l+3)}-\frac{l^2-1}{l}\frac{M}{r}+O(v^4)\right][{}_{-1}\tilde{J}(\omega,\vec{x})_{-1}\overline{Y}_{lm}(\theta,\phi)$$
$$+{}_1\tilde{J}(\omega,\vec{x})_1\overline{Y}_{lm}(\theta,\phi)]r^l d\vec{x}. \tag{4.33}$$

The corresponding expressions in the time domain (see Appendix D) are

$$\mathcal{I}_{lm}(u)=I_{lm}(u)+2M\int_{-\infty}^{u}\left[\ln\left(\frac{u-u'}{4M}\right)+\beta_l+\gamma\right]\ddot{I}_{lm}(u')du' \tag{4.34}$$

and

$$\mathcal{S}_{lm}(u)=S_{lm}(u)+2M\int_{-\infty}^{u}\left[\ln\left(\frac{u-u'}{4M}\right)+\beta_l+\gamma\right]\ddot{S}_{lm}(u')du', \tag{4.35}$$

where overdots indicate differentiation with respect to u'. We have defined

$$I_{lm}(u)=\frac{4\pi}{(2l+1)!!}\sqrt{\frac{l+1}{l}}\left(\frac{d}{du}\right)^l\int\left[1+\frac{l+3}{2(l+1)(2l+3)}(r\partial_u)^2-(l-1)\frac{M}{r}+O(v^4)\right]\left[\rho(u,\vec{x})-\frac{r\partial_u}{l+1}J^r(u,\vec{x})\right]r^l{}_0\overline{Y}_{lm}(\theta,\phi)d\vec{x} \tag{4.36}$$

and

$$S_{lm}(u)=-\frac{2\sqrt{2}i\pi}{(2l+1)!!}\left(\frac{d}{du}\right)^l\int\left[1+\frac{(r\partial_u)^2}{2(2l+3)}-\frac{l^2-1}{l}\frac{M}{r}+O(v^4)\right][{}_{-1}J(u,\vec{x})_{-1}\overline{Y}_{lm}(\theta,\phi)+{}_1J(u,\vec{x})_1\overline{Y}_{lm}(\theta,\phi)]r^l d\vec{x}. \tag{4.37}$$

Equations (4.34)–(4.37) have the same mathematical structure as Eqs. (3.19) and (3.20), which give the radiative multipole moments of a scalar field. The physical meaning of these equations is therefore exactly the same as in Sec. III, and the discussion appearing at the end of Sec. III E need not be repeated. Equations (4.34)–(4.37), with β_l given by Eq. (4.27), are equivalent to Eqs. (1.8)–(1.12), once the spin-weighted spherical harmonics have been converted into the vectorial harmonics of Eq. (A16).

V. GRAVITATIONAL RADIATION

A. Teukolsky equation

In this section, we derive expressions for the radiative multipole moments of a gravitational-wave field in Schwarzschild spacetime. Specifically, we consider a tensor field $h_{\alpha\beta}(\boldsymbol{x})$, defined as the difference between the metric of the perturbed spacetime and the Schwarzschild metric. Field equations for $h_{\alpha\beta}(\boldsymbol{x})$ are obtained by linearizing the Einstein equations for the full metric. It is therefore assumed that the perturbation is small. Teukolsky [36] has cast the field equations for $h_{\alpha\beta}(\boldsymbol{x})$ in a form convenient for our purposes. We briefly summarize this formulation here.

In the Teukolsky formalism, the radiative part of $h_{\alpha\beta}(\boldsymbol{x})$ is represented by the complex-valued function $\Psi_4 = -C_{\alpha\beta\gamma\delta} n^\alpha \overline{m^\beta} n^\gamma \overline{m^\delta}$, where $C_{\alpha\beta\gamma\delta}$ is the perturbed Weyl tensor, and n^α and $\overline{m^\alpha}$ are defined as in Sec. IV. The field $\Psi_4(\boldsymbol{x})$ has spin weight $s=-2$ (see Appendix A), and it can be decomposed according to

$$\Psi_4 = \frac{1}{r^4} \int d\omega \sum_{lm} R_{lm}(\omega;r)\, {}_{-2}Y_{lm}(\theta,\phi) e^{-i\omega t}, \quad (5.1)$$

where ${}_{-2}Y_{lm}(\theta,\phi)$ are spin-weighted spherical harmonics (see Appendix A). The sums over l and m are restricted by $l \geqslant 2$ and $|m| \leqslant l$. The radial function then satisfies the inhomogeneous Teukolsky equation [36]

$$\left[r^2 f \frac{d^2}{dr^2} - 2(r-M)\frac{d}{dr} + U(\omega;r) \right] R_{lm}(\omega;r) = T_{lm}(\omega;r), \quad (5.2)$$

where $f = 1-2M/r$ and $U(\omega;r) = f^{-1}[(\omega r)^2 - 4i\omega(r-3M)] - (l-1)(l+2)$.

The source term on the right-hand side of Eq. (5.2) is constructed as follows from $T^{\alpha\beta}(\boldsymbol{x})$, the (unspecified) stress-energy tensor responsible for the perturbation. The first step is to form the contractions

$${}_0T = T_{\alpha\beta} n^\alpha n^\beta, \quad {}_{-1}T = T_{\alpha\beta} n^\alpha \overline{m^\beta}, \quad {}_{-2}T = T_{\alpha\beta} \overline{m^\alpha}\, \overline{m^\beta}. \quad (5.3)$$

One then evaluates the Fourier transforms

$${}_s\tilde{T}(\omega,\vec{x}) = \frac{1}{2\pi} \int {}_sT(t,\vec{x})\, e^{i\omega t}\, dt \quad (5.4)$$

and takes the projections

$${}_s\tilde{T}_{lm}(\omega;r) = \int {}_s\tilde{T}(\omega,\vec{x})\, {}_s\overline{Y_{lm}}(\theta,\phi) d\Omega. \quad (5.5)$$

Finally, $T_{lm}(\omega;r)$ is given by [36]

$$T_{lm}(\omega;r) = 2\pi \sum_s {}_sp_l\, {}_s\mathcal{D}\, {}_s\tilde{T}_{lm}(\omega;r), \quad (5.6)$$

where

$${}_sp_l = \begin{cases} 2\sqrt{(l-1)l(l+1)(l+2)}, & s=0, \\ 2\sqrt{2(l-1)(l+2)}, & s=-1, \\ 1, & s=-2, \end{cases} \quad (5.7)$$

and

$${}_s\mathcal{D} = \begin{cases} r^4, & s=0, \\ r^2 f \mathcal{L} r^3 f^{-1}, & s=-1, \\ r f \mathcal{L} r^4 f^{-1} \mathcal{L} r, & s=-2. \end{cases} \quad (5.8)$$

Here, $\mathcal{L} = f d/dr + i\omega$.

B. Solution

The inhomogeneous Teukolsky equation (5.2) can be integrated by means of a Green's function, in a manner similar to what was done in Sec. IV. Here also, the form of the radial function can be simplified by introducing adjoint operators ${}_s\mathcal{D}^\dagger$, and by expressing it in terms of $X_l^H(\omega;r)/\mathcal{A}_l^{\rm in}(\omega)$, where $X_l^H(\omega;r)$ is the solution to the Regge-Wheeler equation — Eq. (2.1) with $s=2$ — with boundary conditions (2.3) and (2.4). These manipulations are described in detail in Ref. [33], and they will not be displayed here. The conclusion is that at large distances, the radial function is given by

$$R_{lm}(\omega;r\to\infty) \sim \frac{1}{2}\omega^2 \tilde{\mathcal{Z}}_{lm}(\omega) r^3 e^{i\omega r^*}, \quad (5.9)$$

where

$$\tilde{\mathcal{Z}}_{lm}(\omega) = \frac{-2\pi}{i\omega\kappa_l(\omega)\mathcal{A}_l^{\rm in}(\omega)} \sum_s {}_sp_l \times \int_{2M}^\infty r f^{-2}\, {}_s\tilde{T}_{lm}(\omega;r)\, {}_s\Gamma X_l^H(\omega;r). \quad (5.10)$$

Here,

$$\kappa_l(\omega) = \frac{1}{4}[(l-1)l(l+1)(l+2) - 12iM\omega] \quad (5.11)$$

and

$$\begin{aligned} {}_0\Gamma &= 2(1-3M/r+i\omega r) r f \frac{d}{dr} + f[l(l+1) - 6M/r] \\ &\quad + 2i\omega r(1-3M/r+i\omega r), \\ {}_{-1}\Gamma &= -f\Big\{ [l(l+1)+2i\omega r] r f \frac{d}{dr} \\ &\quad + l(l+1)(f+i\omega r) - 2(\omega r)^2 \Big\}, \\ {}_{-2}\Gamma &= f^2 \Big\{ 2[(l-1)(l+2)+6M/r] r f \frac{d}{dr} \\ &\quad + (l-1)(l+2)[l(l+1)+2i\omega r] + 12 f M/r \Big\}. \end{aligned} \quad (5.12)$$

The quantities $\tilde{\mathcal{Z}}_{lm}(\omega)$ are the multipole moments of the radiative part of $h_{\alpha\beta}(\boldsymbol{x})$.

C. Metric perturbation

The gravitational-wave field $h_{\alpha\beta}^{\rm rad}(x)$ can be obtained from the behavior of $\Psi_4(\boldsymbol{x})$ at large distances [33]. Choosing the θ and ϕ directions as polarization axes, the two fundamental polarizations of the gravitational waves are given by

$$h_+(t,\vec{x})-ih_\times(t,\vec{x})=\frac{1}{r}\sum_{lm}\mathcal{Z}_{lm}(u)\,{}_{-2}Y_{lm}(\theta,\phi), \tag{5.13}$$

where $u=t-r^*$ and

$$\mathcal{Z}_{lm}(u)=\int_{-\infty}^{\infty}\tilde{\mathcal{Z}}_{lm}(\omega)e^{-i\omega u}d\omega. \tag{5.14}$$

This shows that $\mathcal{Z}_{lm}(u)$ are indeed the multipole moments of the radiative field. It should be noted that while these quantities are complex, $h_+(\boldsymbol{x})$ and $h_\times(\boldsymbol{x})$ are real.

In the traceless-transverse gauge, the gravitational-wave tensor is given by

$$h^{\rm rad}_{\alpha\beta}=(h_+-ih_\times)m_\alpha m_\beta+(h_++ih_\times)\overline{m_\alpha}\,\overline{m_\beta} \tag{5.15}$$

or, after substituting Eq. (5.13),

$$h^{\rm rad}_{\alpha\beta}(t,\vec{x})=\frac{1}{r}\sum_{lm}\left[\mathcal{I}_{lm}(u)\;T^{E2,lm}_{\alpha\beta}(\theta,\phi)+\mathcal{S}_{lm}(u)\;T^{B2,lm}_{\alpha\beta}(\theta,\phi)\right]. \tag{5.16}$$

Here, $h^{\rm rad}_{\alpha\beta}(\boldsymbol{x})$ is expressed in terms of the tensorial spherical harmonics described in Appendix A. The *mass* multipole moments $\mathcal{I}_{lm}(u)$ and the *current* multipole moments $\mathcal{S}_{lm}(u)$ are related to $\mathcal{Z}_{lm}(u)$ by the same equations as Eqs. (4.23) and (4.24).

D. Slow-motion approximation

We now calculate the radiative multipole moments in the slow-motion approximation. We proceed as in Sec. IV E. Substituting Eq. (2.19), with $s=2$, into Eq. (5.10), and using Eqs. (5.5), (5.7), (5.11), and (5.12) yields

$$\tilde{\mathcal{Z}}_{lm}(\omega)=\frac{16\pi}{(2l+1)!!}\mathcal{T}_l(\omega)\;(-i\omega)^l\sum_s\;{}_s\tilde{P}_{lm}(\omega), \tag{5.17}$$

where

$$\mathcal{T}_l(\omega)=e^{2iM\omega(\ln4M|\omega|-\mu_l)}(1+\pi M|\omega|), \tag{5.18}$$

with

$$\mu_l=\beta_l-\frac{6}{(l-1)l(l+1)(l+2)}, \tag{5.19}$$

while β_l is given by Eq. (2.18) with $s=2$:

$$\beta_l=\psi(l+1)+\frac{1}{2}-\frac{2}{l(l+1)}. \tag{5.20}$$

We have also introduced

$$\begin{aligned}{}_0\tilde{P}_{lm}(\omega)=2\sqrt{\frac{(l+1)(l+2)}{(l-1)l}}\int\Bigg\{&1-\frac{l+9}{2(l+1)(2l+3)}(\omega r)^2-(l+2)\frac{M}{r}+\frac{2i\omega r}{l+1}\Bigg[1-\frac{l+4}{2(l+2)(2l+3)}(\omega r)^2\\&-\frac{l^3+3l^2+l-4}{l(l+2)}\frac{M}{r}\Bigg]+O(v^4)\Bigg\}\frac{{}_0\tilde{T}}{f^2}r^l\,{}_0\overline{Y_{lm}}d\vec{x},\end{aligned} \tag{5.21}$$

$$\begin{aligned}{}_{-1}\tilde{P}_{lm}(\omega)=-2\sqrt{\frac{2(l+2)}{l-1}}\int\Bigg\{&1-\frac{l^2+3l+6}{2l(l+1)(2l+3)}(\omega r)^2-\frac{(l-1)(l+2)}{l}\frac{M}{r}+\frac{i\omega r}{l}\Bigg[1-\frac{l^2+3l+6}{2(l+1)(l+2)(2l+3)}(\omega r)^2\\&-\frac{l^3+3l^2-8}{(l+1)(l+2)}\frac{M}{r}\Bigg]+O(v^4)\Bigg\}\frac{{}_{-1}\tilde{T}}{f}r^l\,{}_{-1}\overline{Y_{lm}}d\vec{x},\end{aligned} \tag{5.22}$$

$${}_{-2}\tilde{P}_{lm}(\omega)=\frac{l+2}{l}\int\left[1+\frac{2i\omega r}{(l+1)(l+2)}+O(v^2)\right]_{-2}\tilde{T}r^l\,{}_{-2}\overline{Y_{lm}}d\vec{x}. \tag{5.23}$$

To better keep track of the relative importance of each term in Eq. (5.17), we decompose the stress-energy tensor according to

$$\rho=T^{tt}, \tag{5.24}$$

$${}_0j=T^{tr},\quad {}_{-1}j=-T^{t\alpha}\overline{m_\alpha},\quad {}_1j=-T^{t\alpha}m_\alpha, \tag{5.25}$$

$${}_0p=T^{rr},\quad {}_{-1}p=-T^{r\alpha}\overline{m_\alpha},\quad {}_1p=-T^{r\alpha}m_\alpha, \tag{5.26}$$

$${}_0t=T^{\alpha\beta}m_\alpha\overline{m_\beta},\quad {}_{-2}t=T^{\alpha\beta}\overline{m_\alpha}\,\overline{m_\beta},\quad {}_2t=T^{\alpha\beta}m_\alpha m_\beta. \tag{5.27}$$

Thus, if ρ is considered to be a quantity of order unity, then ${}_sj=O(v)$, ${}_sp=O(v^2)$, and ${}_st=O(v^2)$. Inspection of Eqs.

(5.21)–(5.23) — in which we substitute ${}_0T/f^2=1/4(\rho+2{}_0j/f+{}_0p/f^2)$, ${}_{-1}T/f=\frac{1}{2}({}_{-1}j+{}_{-1}p/f)$, and ${}_{-2}T={}_{-2}t$ — then reveals that as it stands, $\tilde{\mathcal{Z}}_{lm}(\omega)$ contains many unwanted terms of the sort encountered in Sec. IV E: terms which are first order in v, and $O(v^3)$ terms that cannot be pulled outside of the spatial integrals.

In Sec. IV E, the unwanted terms were removed by invoking the continuity equation $J^{\alpha}{}_{;\alpha}=0$. Here, they are removed with the help of the energy-momentum conservation equations $T^{\alpha\beta}{}_{;\beta}=0$. When written out explicitly, these become

$$\rho_{,t}+\frac{1}{r^2 f}(r^2 f {}_0j)_{,r}+\frac{1}{\sqrt{2}r}(\hat{\partial}{}_{-1}j+\check{\partial}{}_1j)=0, \quad (5.28)$$

$${}_0j_{,t}+\frac{\sqrt{f}}{r^2}\left(\frac{r^2}{\sqrt{f}}{}_0p\right)_{,r}+\frac{1}{\sqrt{2}r}(\hat{\partial}{}_{-1}p+\check{\partial}{}_1p)+\frac{1}{2}ff'\rho-\frac{2f}{r}{}_0t=0, \quad (5.29)$$

$${}_{-1}j_{,t}+\frac{1}{r^3}(r^3{}_{-1}p)_{,r}+\frac{1}{\sqrt{2}r}(\hat{\partial}{}_{-2}t+\check{\partial}{}_0t)=0, \quad (5.30)$$

$${}_1j_{,t}+\frac{1}{r^3}(r^3{}_1p)_{,r}+\frac{1}{\sqrt{2}r}(\hat{\partial}{}_0t+\check{\partial}{}_{-2}t)=0, \quad (5.31)$$

where $\hat{\partial}$ and $\check{\partial}$ are the ''edth'' differential operators described in Appendix A and $f'=df/dr=2M/r^2$.

Equations (5.28)–(5.31) give rise to a number of integral identities, which we write in the frequency domain, and which are easily established by partial integration, using Eqs. (A13) and (A14). We shall need the following two identities, which involve no approximation:

$$i\omega\int \tilde{\rho}r^n{}_0\overline{Y_{lm}}\,d\vec{x}=-\int(n-rf'/f){}_0\tilde{J}r^{n-1}{}_0\overline{Y_{lm}}d\vec{x}+\sqrt{\frac{l(l+1)}{2}}\int({}_{-1}\tilde{J}{}_{-1}\overline{Y_{lm}}-{}_1\tilde{J}{}_1\overline{Y_{lm}})r^{n-1}d\vec{x} \quad (5.32)$$

and

$$i\omega\int {}_{-1}\tilde{J}\,r^n{}_{-1}\overline{Y_{lm}}d\vec{x}=-(n-1)\int {}_{-1}\tilde{p}r^{n-1}{}_{-1}\overline{Y_{lm}}d\vec{x}+\sqrt{\frac{(l-1)(l+2)}{2}}\int {}_{-2}\tilde{t}\,r^{n-1}{}_{-2}\overline{Y_{lm}}\,d\vec{x}-\sqrt{\frac{l(l+1)}{2}}\int {}_0\tilde{t}r^{n-1}{}_0\overline{Y_{lm}}d\vec{x}. \quad (5.33)$$

We shall also need the following identities, which are valid in the slow-motion approximation:

$$\int[(\omega r)^2-lM/r]{}_0\tilde{J}r^l{}_0\overline{Y_{lm}}d\vec{x}=-\sqrt{\frac{l(l+1)}{2}}\int\frac{M}{r}({}_{-1}\tilde{J}{}_{-1}\overline{Y_{lm}}-{}_1\tilde{J}{}_1\overline{Y_{lm}})r^l d\vec{x}+\int i\omega r[(l+2){}_0\tilde{p}+2{}_0\tilde{t}]r^l{}_0\overline{Y_{lm}}d\vec{x}-\sqrt{\frac{l(l+1)}{2}}\int i\omega r({}_{-1}\tilde{p}{}_{-1}\overline{Y_{lm}}-{}_1\tilde{p}{}_1\overline{Y_{lm}})r^l d\vec{x}, \quad (5.34)$$

where terms of order v^5 and higher (with $\tilde{\rho}$ taken to be of order unity) have been discarded, and

$$\int[iM\omega+O(v^5)]{}_{-1}\tilde{J}r^l{}_{-1}\overline{Y_{lm}}d\vec{x}=-(l-1)\int[M/r+O(v^4)]{}_{-1}\tilde{p}r^l{}_{-1}\overline{Y_{lm}}d\vec{x}, \quad (5.35)$$

which follows from Eq. (5.33) after multiplying both sides by M.

These identities are used to remove all unwanted terms from $\tilde{\mathcal{Z}}_{lm}(\omega)$, as given by Eqs. (5.17)–(5.23). After a rather long calculation (which spans several pages), we arrive at the following expression for the radiative multipole moments:

$$\tilde{\mathcal{Z}}_{lm}(\omega)=\frac{8\pi}{(2l+1)!!}\mathcal{T}_l(\omega)(-i\omega)^l\Bigg\{\sqrt{\frac{(l+1)(l+2)}{(l-1)l}}\int\left[1-\frac{l+9}{2(l+1)(2l+3)}(\omega r)^2-(l+2)\frac{M}{r}+O(v^4)\right]$$
$$\times\left(\tilde{\rho}+{}_0\tilde{p}+2{}_0\tilde{t}+\frac{4i\omega r}{l+1}{}_0\tilde{J}\right)r^l{}_0\overline{Y_{lm}}d\vec{x}-\sqrt{\frac{2(l+2)}{l-1}}\int\left[1-\frac{l+4}{2(l+2)(2l+3)}(\omega r)^2-\frac{(l-1)(l+2)}{l}\frac{M}{r}\right.$$
$$\left.-\frac{24iM\omega}{(l-1)l(l+1)(l+2)}+O(v^4)\right]\left({}_{-1}\tilde{J}+\frac{i\omega r}{l+2}{}_{-1}\tilde{p}\right)r^l{}_{-1}\overline{Y_{lm}}d\vec{x}-\sqrt{\frac{2(l+2)}{l-1}}\int\left[1-\frac{l+4}{2(l+2)(2l+3)}(\omega r)^2\right.$$
$$\left.-\frac{(l-1)(l+2)}{l}\frac{M}{r}+O(v^4)\right]\left({}_1\tilde{J}+\frac{i\omega r}{l+2}{}_1\tilde{p}\right)r^l{}_1\overline{Y_{lm}}d\vec{x}\Bigg\}. \tag{5.36}$$

Notice that the second and third integrals differ by a term proportional to $iM\omega$. We see that this expression for the radiative multipole moments has the expected form, with all $O(v^2)$ correction terms occurring inside the spatial integrals and all $O(v^3)$ corrections occurring outside.

We now separate $\tilde{\mathcal{Z}}_{lm}(\omega)$ into mass and current moments, according to the Fourier transform of Eqs. (4.23) and (4.24). (As was pointed out in Sec. IV C, those equations are valid also in the case of gravitational radiation.) We find that the mass moments are given by

$$\tilde{\mathcal{I}}_{lm}(\omega)=\frac{16\pi}{(2l+1)!!}\sqrt{\frac{(l+1)(l+2)}{2(l-1)l}}\mathcal{T}_l(\omega)(-i\omega)^l\int\left[1-\frac{l+9}{2(l+1)(2l+3)}(\omega r)^2-(l+2)\frac{M}{r}+O(v^4)\right]$$
$$\times\left[\tilde{\rho}(\omega,\vec{x})+{}_0\tilde{p}(\omega,\vec{x})+2{}_0\tilde{t}(\omega,\vec{x})+\frac{4i\omega r}{l+1}{}_0\tilde{J}(\omega,\vec{x})\right]r^l{}_0\overline{Y_{lm}}(\theta,\phi)d\vec{x}, \tag{5.37}$$

and that the current moments are given by

$$\tilde{\mathcal{S}}_{lm}(\omega)=-\frac{16\pi i}{(2l+1)!!}\sqrt{\frac{l+2}{l-1}}\mathcal{T}_l^\sharp(\omega)(-i\omega)^l\int\left[1-\frac{l+4}{2(l+2)(2l+3)}(\omega r)^2-\frac{(l-1)(l+2)}{l}\frac{M}{r}+O(v^4)\right]$$
$$\times\left\{\left[{}_{-1}\tilde{J}(\omega,\vec{x})+\frac{i\omega r}{l+2}{}_{-1}\tilde{p}(\omega,\vec{x})\right]{}_{-1}\overline{Y_{lm}}(\theta,\phi)+\left[{}_1\tilde{J}(\omega,\vec{x})+\frac{i\omega r}{l+2}{}_1\tilde{p}(\omega,\vec{x})\right]{}_1\overline{Y_{lm}}(\theta,\phi)\right\}r^l d\vec{x}, \tag{5.38}$$

where

$$\mathcal{T}_l^\sharp(\omega)=e^{2iM\omega(\ln 4M|\omega|-\beta_l)}(1+\pi M|\omega|). \tag{5.39}$$

Notice that different constants (μ_l for the mass moments, β_l for the current moments) appear in $\mathcal{T}_l(\omega)$ and $\mathcal{T}_l^\sharp(\omega)$; these are defined by Eqs. (5.19) and (5.20).

The corresponding expressions in the time domain (see Appendix D) are

$$\mathcal{I}_{lm}(u)=I_{lm}(u)+2M\int_{-\infty}^{u}\left[\ln\left(\frac{u-u'}{4M}\right)+\mu_l+\gamma\right]\ddot{I}_{lm}(u')du' \tag{5.40}$$

and

$$\mathcal{S}_{lm}(u)=S_{lm}(u)+2M\int_{-\infty}^{u}\left[\ln\left(\frac{u-u'}{4M}\right)+\beta_l+\gamma\right]\ddot{S}_{lm}(u')du', \tag{5.41}$$

where overdots indicate differentiation with respect to u'. We have defined

$$I_{lm}(u)=\frac{16\pi}{(2l+1)!!}\sqrt{\frac{(l+1)(l+2)}{2(l-1)l}}\left(\frac{d}{du}\right)^l\int\left[1+\frac{l+9}{2(l+1)(2l+3)}(r\partial_u)^2-(l+2)\frac{M}{r}+O(v^4)\right]$$
$$\times\left[\rho(u,\vec{x})+{}_0p(u,\vec{x})+2{}_0t(u,\vec{x})-\frac{4r\partial_u}{l+1}{}_0j(u,\vec{x})\right]r^l{}_0\overline{Y_{lm}}(\theta,\phi)d\vec{x} \tag{5.42}$$

and

$$S_{lm}(u)=-\frac{16\pi i}{(2l+1)!!}\sqrt{\frac{l+2}{l-1}}\left(\frac{d}{du}\right)^l\int\left[1+\frac{l+4}{2(l+2)(2l+3)}(r\partial_u)^2-\frac{(l-1)(l+2)}{l}\frac{M}{r}+O(v^4)\right]$$

$$\times\left\{\left[{}_{-1}j(u,\vec{x})-\frac{r\partial_u}{l+2}{}_{-1}p(u,\vec{x})\right]{}_{-1}\overline{Y_{lm}}(\theta,\phi)+\left[{}_{1}j(u,\vec{x})-\frac{r\partial_u}{l+2}{}_{1}p(u,\vec{x})\right]{}_{1}\overline{Y_{lm}}(\theta,\phi)\right\}r^l d\vec{x}. \tag{5.43}$$

The interpretation of these results is exactly the same as for the cases (scalar and electromagnetic radiation) considered previously. Equations (5.40)–(5.43), with $\beta_l \equiv \beta_l^{\rm grav,current}$ and $\mu_l \equiv \beta_l^{\rm grav,mass}$ given by Eqs. (5.20) and (5.19), respectively, are equivalent to Eqs. (1.14)–(1.19), once the spin-weighted spherical harmonics have been converted into the vectorial harmonics of Eq. (A16).

VI. PHYSICAL ORIGIN OF THE TAIL TERM

A survey of Secs. III, IV, and V reveals that for scalar, electromagnetic, and gravitational radiation, the tail correction to the radiative multipole moments takes the universal form [cf. Eqs. (3.19), (4.34), (4.35), (5.40), and (5.41)]

$$\mathcal{M}_{lm}(u)=M_{lm}(u)+2M\int_{-\infty}^{u}\left[\ln\left(\frac{u-u'}{4M}\right)+c_l+\gamma\right]\times\ddot{M}_{lm}(u')du'. \tag{6.1}$$

Here, $\mathcal{M}_{lm}(u)$ stands for $\mathcal{Z}_{lm}(u)$ in the case of scalar radiation, and for $\mathcal{I}_{lm}(u)$ and $\mathcal{S}_{lm}(u)$ in the case of electromagnetic and gravitational radiation. Similarly, $M_{lm}(u)$ stands for either $Z_{lm}(u)$, $I_{lm}(u)$, or $S_{lm}(u)$. The constant c_l stands for β_l [cf. Eq. (2.18)], except for the mass multipole moments of the gravitational-wave field, for which c_l stands for μ_l [cf. Eq. (5.19)].

The physical interpretation of Eq. (6.1) is clear, and was first given at the end of Sec. III C. Equation (6.1) shows that while the correction terms of order v^2 that appear in $M_{lm}(u)$ are near-zone corrections that depend on the detailed behavior of the source, the correction terms of order v^3 — the terms under the integral sign, or tail terms — are due to wave-propagation effects, and are independent of the detailed behavior of the source. And while the $O(v^2)$ corrections are local in time, the $O(v^3)$ corrections introduce a nonlocality in the radiative multipole moments. This nonlocality is understood as arising from the scattering of the radiation field off the spacetime curvature generated by the mass M, and as Eq. (6.1) shows, the tail term is indeed proportional to M.

The mass parameter appears in two places in the Schwarzschild metric,

$$ds^2=-f dt^2+f^{-1}dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2), \tag{6.2}$$

where $f=1-2M/r$. It enters in g_{tt} and in g_{rr}, which are both involved in the calculation of the tail correction. Because the mass parameter is the same in both components of the metric, it is impossible to tell, on the basis of our previous calculations, whether it is g_{tt} that is ‘‘mostly responsible’’ for the tail effect or whether it is g_{rr}, or whether both components are ‘‘equally responsible.’’ In other words, our previous calculations cannot tell us how the temporal and spatial curvatures separately contribute to the tail effect. This is the question we now wish to examine. We shall answer it by artificially introducing an additional mass parameter in the description of our spacetime. The resulting metric, of course, will no longer be a solution to the Einstein equations. This, however, does not prevent us from examining the scalar wave equation in this spacetime. Extension of the following considerations to the case of electromagnetic radiation would be straightforward. However, in the absence of field equations, an eventual extension to the case of gravitational waves would be ambiguous.

We consider a static, spherically symmetric spacetime with a line element of the most general form

$$ds^2=-f dt^2+g^{-1}dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2), \tag{6.3}$$

where f and g are two arbitrary functions of r obeying the following restrictions. First, the spacetime must be asymptotically flat, so that the metric functions must behave as

$$f\sim 1-2M/r, \quad g\sim 1-2\hat{\gamma}M/r, \tag{6.4}$$

at large distances ($r\gg M$). Here, M is the gravitational mass of the system responsible for the gravitational field, and $\hat{\gamma}$ is a parameter that measures the failure of the metric to match the Schwarzschild form at large distances; $\hat{\gamma}M$ can be thought of as the system's inertial mass, and the Schwarzschild behavior is recovered by putting $\hat{\gamma}=1$. (This parameter has the same meaning as γ in the parametrized post-Newtonian formalism [38]. We nevertheless use the notation $\hat{\gamma}$ to distinguish this quantity from the Euler number γ.) Second, we assume for concreteness that the metric describes a black-hole spacetime, so that both $f(r)$ and $g(r)$ vanish at a common radius r_0. Regularity of the spacetime at the event horizon further demands that the ratio f/g be finite and nonvanishing at $r=r_0$. (Our conclusions are insensitive to this second set of assumptions.) Apart from these requirements, $f(r)$ and $g(r)$ will be left unspecified.

We consider the scalar wave equation $\Box\Phi(\boldsymbol{x})=-4\pi\rho(\boldsymbol{x})$ in a spacetime with line element (6.3). After separation of the variables, according to Eqs. (3.2), (3.4), and (3.5), the radial function is found to satisfy

$$\left\{\frac{d^2}{dr^{*2}}+\omega^2-f\left[\frac{l(l+1)}{r^2}+\sqrt{\frac{g}{f}}\frac{(\sqrt{fg})'}{r}\right]\right\}R_{lm}(\omega;r)=fT_{lm}(\omega;r), \tag{6.5}$$

where $d/dr^*=\sqrt{fg}\, d/dr$ and a prime indicates differentiation with respect to r. This equation is integrated by means of a Green's function, constructed from two linearly independent solutions to the homogeneous equation. These are denoted $R_l^H(\omega;r)$ and $R_l^\infty(\omega;r)$, and are defined as in Eq. (3.7), with $r^*=\int dr/\sqrt{fg}$, and with $r=r_0$ replacing $r=2M$. The solution at large distances is then given by Eq. (3.9), with

$$\tilde{\tilde{\mathcal{Z}}}_{lm}(\omega)=\frac{1}{2i\omega\mathcal{Q}_l^{\rm in}(\omega)}\int_{r_0}^{\infty}\sqrt{f/g}\,T_{lm}(\omega;r)R_l^H(\omega;r)dr. \tag{6.6}$$

Substituting this into Eq. (3.2) yields

$$\Phi_{\rm rad}(t,\vec{x})=\frac{1}{r}\sum_{lm}\mathcal{Z}_{lm}(u)Y_{lm}(\theta,\phi), \tag{6.7}$$

where $\mathcal{Z}_{lm}(u)$ is the inverse Fourier transform of $\tilde{\tilde{\mathcal{Z}}}_{lm}(\omega)$ and $u=t-r^*$ is retarded time. According to Eqs. (6.4), r^* is now given by

$$r^*\sim r+2\sigma M\ln(r/2M) \tag{6.8}$$

at large distances, where

$$\sigma=\frac{1}{2}(1+\hat{\gamma}). \tag{6.9}$$

Equation (6.8) agrees with the Schwarzschild definition when $\hat{\gamma}=\sigma=1$.

We now wish to calculate $\tilde{\tilde{\mathcal{Z}}}_{lm}(\omega)$, the radiative multipole moments, in the slow-motion approximation. The first step is to integrate the homogeneous version of Eq. (6.5) in the low-frequency limit. The calculation proceeds as in Sec. II and Appendix C, and uses the approximations (6.4) for $f(r)$ and $g(r)$; these steps will not be duplicated here. Defining $\varepsilon=2M\omega$ and $z=\omega r$, we eventually find

$$\frac{R_l^H(z)}{\mathcal{Q}_l^{\rm in}}=2(-i)^{l+1}e^{i\sigma\varepsilon(\ln 2\varepsilon-\beta_l)}\left(1+\frac{\pi}{2}\sigma\varepsilon\right)\times z\Bigg\{[1-\sigma\varepsilon A_l(z)]j_l(z)+\sigma\varepsilon B_l(z)n_l(z)-\varepsilon\frac{(l+1)\hat{\gamma}-\sigma}{2(2l+1)}j_{l-1}(z)+\varepsilon\frac{l\hat{\gamma}+\sigma}{2(2l+1)}j_{l+1}(z)+O(\varepsilon^2)\Bigg\}, \tag{6.10}$$

where $A_l(z)$ and $B_l(z)$ are defined by Eqs. (2.14) and (2.15), respectively, and

$$\beta_l=\psi(l+1)+\frac{\hat{\gamma}}{2\sigma}. \tag{6.11}$$

Evaluating Eq. (6.10) for $z\ll 1$ yields

$$\frac{R_l^H(\omega;r)}{\mathcal{Q}_l^{\rm in}(\omega)}=\frac{2}{(2l+1)!!}e^{2i\sigma M\omega(\ln 4M|\omega|-\beta_l)}(1+\pi\sigma M|\omega|)\times(-i\omega r)^{l+1}\Bigg\{1-\frac{(\omega r)^2}{2(2l+3)}-[(l+1)\hat{\gamma}-\sigma]\frac{M}{r}+O(v^4)\Bigg\}. \tag{6.12}$$

The second step is to substitute Eq. (6.12) into Eq. (6.6), using Eq. (6.4) once more. We obtain

$$\tilde{\tilde{\mathcal{Z}}}_{lm}(\omega)=\frac{4\pi}{(2l+1)!!}e^{2i\sigma M\omega(\ln 4M|\omega|-\beta_l)}(1+\pi\sigma M|\omega|)\times(-i\omega)^l\int\left[1-\frac{(\omega r)^2}{2(2l+3)}-\frac{(2l-1)\hat{\gamma}+1}{2}\frac{M}{r}+O(v^4)\right]\tilde{\rho}(\omega,\vec{x})r^l\overline{Y_{lm}}(\theta,\phi)d\vec{x}. \tag{6.13}$$

The corresponding expression in the time domain is

$$\mathcal{Z}_{lm}(u)=Z_{lm}(u)+2\sigma M\int_{-\infty}^{u}\left[\ln\left(\frac{u-u'}{4M}\right)+\beta_l+\gamma\right]\times\ddot{Z}_{lm}(u')du', \tag{6.14}$$

where

$$Z_{lm}(u)=\frac{4\pi}{(2l+1)!!}\left(\frac{d}{du}\right)^l\int\left[1+\frac{(r\partial_u)^2}{2(2l+3)}-\frac{(2l-1)\hat{\gamma}+1}{2}\frac{M}{r}+O(v^4)\right]\times\rho(u,\vec{x})r^l\overline{Y_{lm}}(\theta,\phi)d\vec{x}. \tag{6.15}$$

These are the radiative multipole moments of a scalar field in a spacetime with line element (6.3). Equations (6.14) and (6.15), with β_l given by Eq. (6.11), are equivalent to Eqs. (1.20) and (1.21).

We see from Eqs. (6.14) and (6.15) that $\hat{\gamma}$ and σ both appear in the near-zone and wave-propagation correction terms. In particular, the tail integral is now proportional to $\sigma\equiv 1/2(1+\hat{\gamma})$. This allows us to conclude that in general relativity, temporal and spatial curvatures contribute *equally* to the tail correction. This result is striking, because the same conclusion is known to hold in two other situations: the deflection and time delay of light by the gravitational field of a massive body. Indeed, in a parametrized post-Newtonian calculation [38], the deflection angle and the time delay are both found to be proportional to $1/2(1+\gamma)$. (Here, γ is the parameter that measures how much spatial curvature is produced by a unit rest mass; it is equal to unity in general relativity.) The statement that temporal and spatial curvatures contribute equally therefore applies to two very different physical situations. While the deflection and time delay of light are both high-frequency, geometric-optics phenomena, the tail effect is very much a low-frequency, wavelike phenomenon, and the discovery of such a similarity in such different situations could not have been expected on physical grounds. However, this similarity is not entirely surprising on mathematical grounds: The factor of σ that appears in front of the tail integral is essentially the same σ that appears in the new definition of r^*, Eq. (6.8); since radial light rays propagate along curves of constant $t-r^*$ or $t+r^*$, it is perhaps not surprising that σ should also appear in expressions for the deflection angle and the time delay.

VII. SPACETIME APPROACH

A. DeWitt-Brehme Green's function

The mathematical methods employed in the previous sections of this paper to derive expressions for the radiative multipole moments of integer-spin fields were based upon a separation of variables approach made possible by the symmetries of the Schwarzschild solution. Although the physical interpretation of our results, in terms of near-zone and wave-propagation corrections, is quite clear, we cannot claim that the physical picture is particularly well represented by the mathematics involved in bringing the problem to a solution. Indeed, the physical meaning of our expressions became clear only *after* performing the inverse Fourier transform that gave the multipole moments in the time domain; by themselves, the frequency-domain expressions did not have a very compelling interpretation.

In this section, we offer an alternative derivation of the radiative multipole moments in which the mathematics reflects the physics every step of the way. For simplicity we shall again restrict attention to the case of scalar radiation.

Following the seminal work by Hadamard [1], DeWitt and Brehme [3] considered the scalar wave equation, $\Box\Phi(\boldsymbol{x})=-4\pi\rho(\boldsymbol{x})$, and its Green's function satisfying $\Box G(\boldsymbol{x},\boldsymbol{x}')=-\delta(\boldsymbol{x},\boldsymbol{x}')$, in an arbitrary spacetime with metric $g_{\alpha\beta}$. These equations imply that the scalar field can be expressed as

$$\Phi(\boldsymbol{x})=4\pi\int G(\boldsymbol{x},\boldsymbol{x}')\rho(\boldsymbol{x}')d\boldsymbol{x}', \tag{7.1}$$

where $d\boldsymbol{x}'=|g(x')|^{1/2}d^4x'$, with $g=\det(g_{\alpha\beta})$. Assuming that the field point $\boldsymbol{x}$ belongs to the normal convex neighborhood of the source points $\boldsymbol{x}'$, DeWitt and Brehme found that the retarded Green's function takes the form

$$G(\boldsymbol{x},\boldsymbol{x}')=\frac{1}{4\pi}\theta(\boldsymbol{x},\boldsymbol{x}')[u(\boldsymbol{x},\boldsymbol{x}')\delta(\sigma)-v(\boldsymbol{x},\boldsymbol{x}')\theta(-\sigma)]. \tag{7.2}$$

Here, $\sigma(\boldsymbol{x},\boldsymbol{x}')$ is the world function first introduced by Synge [39], and equal to one-half the squared geodesic distance between $\boldsymbol{x}$ and $\boldsymbol{x}'$; σ is positive if the points are spacelike related, negative if the relation is timelike, and zero if $\boldsymbol{x}$ and $\boldsymbol{x}'$ are joined by a null geodesic. The functions $u(\boldsymbol{x},\boldsymbol{x}')$ and $v(\boldsymbol{x},\boldsymbol{x}')$ are nonsingular in the limit $\sigma\to 0$, and are obtained by substituting Eq. (7.2) into the differential equation for the Green's function. Finally, $\theta(\boldsymbol{x},\boldsymbol{x}')$ is a time-ordering function, equal to unity if $\boldsymbol{x}$ is in the causal future of $\boldsymbol{x}'$, and zero otherwise.

As can be seen from Eq. (7.2), the retarded Green's function splits naturally into a direct part (the first term), which has support on, and only on, the past light cone of $\boldsymbol{x}$ (all points $\boldsymbol{x}'$ such that $\sigma=0$), and a tail part (the second term), which has support inside the past light cone (all points $\boldsymbol{x}'$ such that $\sigma<0$). This, in turn, implies that $\Phi(\boldsymbol{x})$ will also be split into direct and tail parts, as was observed in Sec. III. We therefore see that contrary to our previous mathematical formulation, Eqs. (7.1) and (7.2) reflect the physical picture quite closely.

In the remainder of this section, we calculate $G(\boldsymbol{x},\boldsymbol{x}')$ in a weak-field approximation (relying on previous work by DeWitt and DeWitt [40]), and derive an expression for the radiative multipole moments of the scalar field. Not surprisingly, our answer will agree with what was obtained in Sec. III, Eqs. (3.19) and (3.20). Although this calculation tells us nothing new in terms of the final answer, it is still instructive, because of the fact that the mathematical origin of the tail correction is clear from the outset — it follows directly from the tail term in the Green's function.

B. Direct term

We begin with the calculation of the direct part of the field,

$$\Phi_{\text{direct}}(\boldsymbol{x})=\int \theta(\boldsymbol{x},\boldsymbol{x}')u(\boldsymbol{x},\boldsymbol{x}')\delta(\sigma)\rho(\boldsymbol{x}')d\boldsymbol{x}'. \tag{7.3}$$

This involves the evaluation of $\sigma(\boldsymbol{x},\boldsymbol{x}')$ and $u(\boldsymbol{x},\boldsymbol{x}')$. We shall work in the weak-field approximation (that is, linearized gravity in harmonic coordinates), and express the metric as $g_{\alpha\beta}(\boldsymbol{x})=\eta_{\alpha\beta}+h_{\alpha\beta}(\boldsymbol{x})$, where $\eta_{\alpha\beta}$ is the metric of flat spacetime in Cartesian coordinates, and

$$h_{\alpha\beta}(\boldsymbol{x})=\frac{2M}{|\vec{x}|}\delta_{\alpha\beta}. \tag{7.4}$$

Here, $\delta_{\alpha\beta}$ is the Kronecker delta, and for any three-vector $\vec{s}$, $|\vec{s}|^2=\vec{s}\cdot\vec{s}=\delta_{ab}s^a s^b$. It is assumed that the source of the gravitational field is a point mass located at the origin of the coordinates.

The world function is given by [41]

$$\sigma(\boldsymbol{x},\boldsymbol{x}')=\frac{1}{2}\int_{\mathcal{C}}g_{\alpha\beta}\frac{d\xi^\alpha}{d\lambda}\frac{d\xi^\beta}{d\lambda}d\lambda, \tag{7.5}$$

where $\mathcal{C}$ is the geodesic relating the points $\boldsymbol{x}'$ and $\boldsymbol{x}$, $\xi^\alpha(\lambda)$ the equation of this geodesic, and λ an affine parameter on the geodesic, normalized so that $\xi^\alpha(0)=x'^\alpha$ and $\xi^\alpha(1)=x^\alpha$. Equation (7.5) follows immediately from the geometric meaning of the world function. Because Eq. (7.5) is an action principle for the geodesic equation, an error of order ϵ in the specification of $\mathcal{C}$ is translated into an error of order ϵ^2 in $\sigma(\boldsymbol{x},\boldsymbol{x}')$. Since we wish to evaluate $\sigma(\boldsymbol{x},\boldsymbol{x}')$ accurately to *first order* in the formally small parameter M, it is sufficient to approximate $\mathcal{C}$ by the straight path [41]

$$\xi^\alpha(\lambda)=x'^\alpha+\lambda(x^\alpha-x'^\alpha). \tag{7.6}$$

Substituting this into Eq. (7.5) and discarding all $O(M^2)$ terms, we obtain

$$\sigma(\boldsymbol{x},\boldsymbol{x}')=-\frac{1}{2}\left(1-2M\int_0^1\frac{d\lambda}{\xi}\right)(t'-t_-)(t'-t_+), \tag{7.7}$$

where

$$t_\pm=t\pm|\vec{x}-\vec{x}'|\pm 2M|\vec{x}-\vec{x}'|\int_0^1\frac{d\lambda}{\xi} \tag{7.8}$$

and $\xi \equiv |\vec{\xi}|$. Equation (7.7) implies

$$\delta(\sigma) = \frac{1}{|\vec{x}-\vec{x}'|}[\delta(t'-t_-)+\delta(t'-t_+)], \tag{7.9}$$

and the second term vanishes when $\delta(\sigma)$ is multiplied by $\theta(\boldsymbol{x},\boldsymbol{x}')$. To evaluate the integral in Eq. (7.8), we use Eq. (7.6) to write

$$\xi = \sqrt{r'^2 + 2\lambda \vec{x}'\cdot(\vec{x}-\vec{x}') + \lambda^2 |\vec{x}-\vec{x}'|^2}, \tag{7.10}$$

where $r' = |\vec{x}'|$; we also define $r = |\vec{x}|$ and $\vec{n} = \vec{x}/r$. The integration is elementary, and Eq. (7.8) becomes

$$t_\pm = t \pm |\vec{x}-\vec{x}'| \pm 2M \ln(2r/s'), \tag{7.11}$$

where $s' \equiv r' + \vec{n}\cdot\vec{x}'$ and where terms of order unity have been discarded in the logarithm.

In the weak-field approximation, $u(\boldsymbol{x},\boldsymbol{x}')$ is given by [41]

$$u(\boldsymbol{x},\boldsymbol{x}') = 1 + \frac{1}{2}(x^\alpha - x'^\alpha)(x^\beta - x'^\beta)\int_{\mathcal{C}} R_{\alpha\beta}\lambda(1-\lambda)d\lambda, \tag{7.12}$$

where $R_{\alpha\beta}$ is the Ricci tensor. Because $R_{\alpha\beta}$ is already proportional to M, the geodesic $\mathcal{C}$ can once again be approximated by the straight path (7.6). Now, $R_{\alpha\beta} \propto \delta(\vec{x})$ in the point-mass approximation, and there is only one path $\mathcal{C}'$ which gives rise to a nonvanishing integral in Eq. (7.12) — the path for which $\vec{x}'$ and $\vec{x}$ are diametrically opposite. Because $\mathcal{C}'$ forms a set of measure zero in the space of all paths connecting source points $\boldsymbol{x}'$ to a given field point $\boldsymbol{x}$, the fact that $u(\boldsymbol{x},\boldsymbol{x}') \neq 1$ for this path has no effect on $\Phi_{\text{direct}}(\boldsymbol{x})$. Therefore, we can safely set

$$u(\boldsymbol{x},\boldsymbol{x}') = 1 \tag{7.13}$$

in the following [42].

Substituting Eqs. (7.9), (7.11), and (7.13) into Eq. (7.3) yields

$$\Phi_{\text{direct}}(t,\vec{x}) = \int \frac{\rho(t_-,\vec{x}')}{|\vec{x}-\vec{x}'|}\, d\vec{x}', \tag{7.14}$$

where $d\vec{x}' \equiv |g(\boldsymbol{x}')|^{1/2}\, d^3x = (1+2M/r')d^3x$. At large distances, this becomes

$$\Phi^{\text{rad}}_{\text{direct}}(t,\vec{x}) = \frac{1}{r}\int \rho[u + \vec{n}\cdot\vec{x}' + 2M\ln(s'/2c),\vec{x}']d\vec{x}', \tag{7.15}$$

where

$$u = t - r - 2M\ln(r/c) \tag{7.16}$$

is retarded time, with c an arbitrary constant. This definition of retarded time is similar to the Schwarzschild expression, and c will eventually be chosen so that the two definitions agree.

We now invoke the slow-motion approximation and expand ρ in a Taylor series about u. (The approximation ensures that the series converges.) This gives

$$\Phi^{\text{rad}}_{\text{direct}}(t,\vec{x}) = \frac{1}{r}\sum_{n=0}^{\infty}\frac{1}{n!}\int \rho^{(n)}(u,\vec{x}')[\vec{n}\cdot\vec{x}' + 2M\ln(s'/2c)]^n d\vec{x}', \tag{7.17}$$

where $\rho^{(n)} \equiv \partial^n\rho/\partial u^n$. After discarding all terms of second and higher order in M, we arrive at

$$\Phi^{\text{rad}}_{\text{direct}}(t,\vec{x}) = \frac{1}{r}\sum_{n=0}^{\infty}\frac{1}{n!}\int \rho^{(n)}(u,\vec{x}')(\vec{n}\cdot\vec{x}')^n d\vec{x}' + \frac{2M}{r}\sum_{n=0}^{\infty}\frac{1}{n!}\int \rho^{(n+1)}(u,\vec{x}')\ln(s'/2c) \times(\vec{n}\cdot\vec{x}')^n d\vec{x}'. \tag{7.18}$$

This is our final expression for the direct part of the radiative field.

C. Tail term

The tail part of the scalar field is

$$\Phi_{\text{tail}}(\boldsymbol{x}) = -\int \theta(\boldsymbol{x},\boldsymbol{x}')v(\boldsymbol{x},\boldsymbol{x}')\theta(-\sigma)\rho(\boldsymbol{x}')d\boldsymbol{x}', \tag{7.19}$$

and it is now our task to evaluate this.

An expression for $v(\boldsymbol{x},\boldsymbol{x}')$, accurate to first order in M in a weak-field approximation, was derived by DeWitt and DeWitt [40], who find

$$v(\boldsymbol{x},\boldsymbol{x}') = -\frac{2M}{|\vec{x}-\vec{x}'|}\frac{\partial^2}{\partial t'\partial t}\Bigg[\theta(r+r'+t'-t) \times\ln\frac{r+r'+|\vec{x}-\vec{x}'|}{r+r'-|\vec{x}-\vec{x}'|} + \theta(t-t'-r'-r) \times\ln\frac{t-t'+|\vec{x}-\vec{x}'|}{t-t'-|\vec{x}-\vec{x}'|}\Bigg]. \tag{7.20}$$

For large r, this reduces to

$$v(\boldsymbol{x},\boldsymbol{x}') = -\frac{2M}{r}\left[\frac{\delta(u-u'-s')}{s'} - \frac{\theta(u-u'-s')}{(u-u')^2}\right], \tag{7.21}$$

where $s' = r' + \vec{n}\cdot\vec{x}'$,

$$u' = t' - r + |\vec{x}-\vec{x}'| \simeq t' - \vec{n}\cdot\vec{x}', \tag{7.22}$$

and $u = t - r$ is retarded time. [The true retarded time is given by Eq. (7.16) and differs from $t-r$ by a term $2M\ln(r/c)$. Nevertheless, $u = t - r$ is the appropriate expression to use in the calculation of the tail term when working to first order in M, because $v(\boldsymbol{x},\boldsymbol{x}')$ is already proportional to M.]

We now substitute Eq. (7.21) into Eq. (7.19), taking into account that $\theta(\boldsymbol{x},\boldsymbol{x}')\theta(-\sigma)=\theta(t-|\vec{x}-\vec{x}'|-t') \quad =\theta(u-u')$, according to Eqs. (7.7) and (7.8). We obtain

$$\Phi^{\rm rad}_{\rm tail}(t,\vec{x})=\frac{2M}{r}\int\int_{-\infty}^{u}\left[\frac{\delta(u-u'-s')}{s'}-\frac{\theta(u-u'-s')}{(u-u')^2}\right]\rho(u'+\vec{n}\cdot\vec{x}',\vec{x}')du'd\vec{x}', \tag{7.23}$$

where $d\vec{x}'=|g(\boldsymbol{x}')|^{1/2}\,d^3x'=[1+O(M)]\,d^3x'$. After two partial integrations and a few lines of algebra, this becomes

$$\Phi^{\rm rad}_{\rm tail}(t,\vec{x})=\frac{2M}{r}\int\Bigg[-\dot{\rho}(u-r',\vec{x}')\ln s'-\int_{-\vec{n}\cdot\vec{x}'}^{r'}\ddot{\rho}(u-\zeta,\vec{x}')\ln(\zeta+\vec{n}\cdot\vec{x}')d\zeta+\int_{-\infty}^{u}\ddot{\rho}(u'+\vec{n}\cdot\vec{x}',\vec{x}')\times\ln(u-u')du'\Bigg]d\vec{x}, \tag{7.24}$$

where overdots indicate differentiation with respect to either u or u'.

The ζ integral on the right-hand side of Eq. (7.24) can be evaluated explicitly if $\ddot{\rho}(u-\zeta,\vec{x}')$ is expanded in a Taylor series about $\zeta=0$. (Again, the slow-motion approximation ensures that this series converges.) This results in an infinite sum of terms involving the integrals $\int\zeta^n\ln(\zeta+\vec{n}\cdot\vec{x}')\,d\zeta$, which can be expressed in closed form (Ref. [43], p. 205). After a rather long but straightforward calculation, we find that this integral is equal to

$$\dot{\rho}(u-r',\vec{x}')\ln s'+\sum_{n=0}^{\infty}\frac{1}{n!}\Bigg\{[-\ln s'+\psi(n+1)+\gamma]\rho^{(n+1)}(u,\vec{x}')-\sum_{p=n+1}^{\infty}\frac{(-1)^{p-n}n!}{p!(p-n)}\rho^{(p+1)}(u,\vec{x}')r'^{p-n}\Bigg\}(\vec{n}\cdot\vec{x}')^n. \tag{7.25}$$

Notice that the first term cancels out the first term on the right-hand side of Eq. (7.24). The rest of Eq. (7.25) is simplified by invoking the slow-motion approximation. Because it involves an additional (retarded) time derivative, the first term in the sum over p is smaller than $\rho^{(n+1)}$ by a factor of order v, and the remaining terms are smaller still. Now, $M\rho^{(n+1)}$ is already a factor of order v^3 smaller than $\rho^{(n)}$, which appears in the direct part of the radiative field. This means that in Eq. (7.25), the sum over p is $O(v^4)$, and therefore, it will be neglected.

After substituting Eq. (7.25) into Eq. (7.24), and expanding the third term on the right-hand side of this equation in a Taylor series about $\vec{n}\cdot\vec{x}'=0$, we arrive at

$$\Phi^{\rm rad}_{\rm tail}(t,\vec{x})=\frac{2M}{r}\sum_{n=0}^{\infty}\frac{1}{n!}\int\Bigg\{[-\ln s'+\psi(n+1)+\gamma]\rho^{(n+1)}(u,\vec{x}')+\int_{-\infty}^{u}\rho^{(n+2)}(u',\vec{x}')\ln(u-u')du'\Bigg\}\times(\vec{n}\cdot\vec{x}')^n d\vec{x}'. \tag{7.26}$$

This is our final expression for the tail part of the radiative field.

D. Total radiative field

The total radiative field is obtained by adding the direct and tail terms. Combining Eqs. (7.18) and (7.26), we obtain

$$\Phi_{\rm rad}(t,\vec{x})=\frac{1}{r}\sum_{n=0}^{\infty}\frac{1}{n!}\int\Bigg\{\rho^{(n)}(u,\vec{x}')+2M\int_{-\infty}^{u}\Bigg[\ln\left(\frac{u-u'}{2c}\right)+\psi(n+1)+\gamma\Bigg]\rho^{(n+2)}(u',\vec{x}')du'\Bigg\}\times(\vec{n}\cdot\vec{x}')^n\,d\vec{x}'. \tag{7.27}$$

We recall that $d\vec{x}'=(1+2M/r')d^3x'$, $u=t-r-2M\ln(r/c)$, and that Eq. (7.27) has been derived on the basis of a weak-field, slow-motion approximation; it is valid to first order in M, and neglects terms of order v^4.

The constant c appearing in Eq. (7.27) is the same one which enters in the definition of the retarded time u, Eq. (7.16). The radiative field does not actually depend on the numerical value of this constant. To see this, let $c\to\lambda c$, where λ is a scaling constant. Then Eq. (7.16) implies $u\to u+2M\ln\lambda$, and we have $\rho^{(n)}(u,\vec{x}')\to\rho^{(n)}(u,\vec{x}')+2M\ln\lambda\rho^{(n+1)}(u,\vec{x}')+O(M^2)$. Substituting these relations into Eq. (7.27) and discarding all terms of order M^2 shows that, indeed, $\Phi_{\rm rad}(\boldsymbol{x})$ is invariant under this transformation.

Our current expression for the radiative field has a mathematical structure similar to that of Eqs. (3.11), (3.19), and (3.20), but there appear to be some differences. We now show that these are only apparent, and that in fact, Eq. (7.27) is entirely equivalent to the results of Sec. III.

We first reintroduce the spherical harmonics, with the relation [44]

$$(\vec{n}\cdot\vec{x}')^n=4\pi n!r'^n\sum_{l=0}^{l}{}'\sum_{m=-l}^{l}\times\frac{2^l[(n+l)/2]!}{(n+l+1)![(n-l)/2]!}\times\overline{Y_{lm}}(\theta',\phi')Y_{lm}(\theta,\phi), \tag{7.28}$$

where the sum over l includes even values only if n is even, and odd values only if n is odd. The angles θ' and ϕ' are the polar angles of the source point $\vec{x}'$, and θ and ϕ belong to the field point $\vec{x}$. Substituting this into Eq. (7.27) and reordering the sums, we obtain

$$\Phi(t,\vec{x}) = \frac{4\pi}{r}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\overline{Y_{lm}}(\theta',\phi')\, Y_{lm}(\theta,\phi) \times \sum_{n=l}^{\infty}{}' \frac{2^l[(n+l)/2]!}{(n+l+1)![(n-l)/2]!}\int\Biggl\{\rho^{(n)}(u,\vec{x}') + 2M\int_{-\infty}^{u}\left[\left(\ln\frac{u-u'}{2c}\right)+\psi(n+1)+\gamma\right]\rho^{(n+2)} \times(u',\vec{x}')\, du'\Biggr\} r'^n d\vec{x}'. \tag{7.29}$$

The slow-motion approximation now demands that we keep only the first two terms ($n=l$ and $n=l+2$) in the sum over n. After some algebra, we arrive at

$$\Phi_{\rm rad}(t,\vec{x}) = \frac{1}{r}\sum_{lm}\mathcal{Z}_{lm}(u)Y_{lm}(\theta,\phi), \tag{7.30}$$

which is the same as Eq. (3.11). Here,

$$\mathcal{Z}_{lm}(u) = Z_{lm}(u) + 2M\int_{-\infty}^{u}\left[\ln\left(\frac{u-u'}{2c}\right)+\psi(l+1) + \gamma\right]\ddot{Z}_{lm}(u')du' \tag{7.31}$$

and

$$Z_{lm}(u) = \frac{4\pi}{(2l+1)!!}\left(\frac{d}{du}\right)^l\int\left[1+\frac{(r\partial_u)^2}{2(2l+3)} + O(v^4)\right]\rho(u,\vec{x})r^l\overline{Y_{lm}}(\theta,\phi)d\vec{x}. \tag{7.32}$$

This is almost, but not quite, the same as Eqs. (3.19) and (3.20).

To properly compare our results with those of Sec. III, we must account for the different choices of coordinate systems. The coordinates used in this section, and those for which Eq. (7.4) holds, are the harmonic coordinates $\{t,x,y,z\}$. From these we have constructed the spherical coordinates $\{t,r,\theta,\phi\}$ in the usual way, and in this coordinate system, $d\vec{x}=(1+2M/r)r^2dr\, d\Omega$, where $d\Omega = d\cos\theta\, d\phi$. These coordinates are distinct from the Schwarzschild coordinates used in Sec. III, which we now denote $\{\bar{t},\bar{r},\bar{\theta},\bar{\phi}\}$. The transformation between the two coordinate systems is [45]

$$\bar{t}=t, \quad \bar{r}=r+M, \quad \bar{\theta}=\theta, \quad \bar{\phi}=\phi. \tag{7.33}$$

We therefore have $d\vec{x}=\bar{r}^2 d\bar{r}\, d\bar{\Omega}\equiv d\bar{\vec{x}}$, which is the volume element of Sec. III. We also have $r^l=\bar{r}^l(1-lM/\bar{r})$, and substituting this into Eq. (7.31) yields

$$Z_{lm}(u) = \frac{4\pi}{(2l+1)!!}\left(\frac{d}{du}\right)^l\int\left[1+\frac{(\bar{r}\partial_u)^2}{2(2l+3)} - l\frac{M}{\bar{r}} + O(v^4)\right]\rho(u,\bar{\vec{x}})\,\bar{r}^l\,\overline{Y_{lm}}(\bar{\theta},\bar{\phi})d\bar{\vec{x}}. \tag{7.34}$$

This is the same as Eq. (3.20).

Finally, a specific choice for c can be made by demanding that $u=t-r-2M\ln(r/c)$ be equal to $\bar{u}=\bar{t}-\bar{r}-2M\ln(\bar{r}/2M)$, which is the retarded time encountered in Sec. III. (We have approximated $\bar{r}/2M-1$ by $\bar{r}/2M$ in the logarithm.) A short calculation gives

$$c = 2Me^{-1/2}, \tag{7.35}$$

and with this choice, Eq. (7.31) becomes

$$\mathcal{Z}_{lm}(u) = Z_{lm}(u) + 2M\int_{-\infty}^{u}\left[\ln\left(\frac{u-u'}{4M}\right)+\psi(l+1)+\frac{1}{2}+\gamma\right] \times\ddot{Z}_{lm}(u')du'. \tag{7.36}$$

This is the same as Eq. (3.19). We therefore have precise agreement between the results of this section and those of Sec. III.

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APPENDIX A: TENSOR FIELDS ON A TWO-SPHERE

We gather, for the benefit of the reader, several results pertaining to the ''edth'' differential operators and the associated spin-weighted spherical harmonics. The discussion follows closely Ref. [46], but it is essentially self-contained.

We consider S^2, a spherical two-dimensional space with the metric

$$ds^2 = g_{ab}d\theta^a d\theta^b = r^2(d\theta^2+\sin^2\theta\, d\phi^2), \tag{A1}$$

on which fields of various tensorial types are defined. For simplicity, all geometric objects considered in this appendix will be confined to this space. However, extension of the following considerations to four-dimensional, spherically symmetric spacetimes is immediate.

It is useful to introduce a set of basis vectors m^a and $\overline{m^a}$ (an overbar denotes complex conjugation), which satisfy the relations

$$m_a m^a = \overline{m_a}\,\overline{m^a} = 0, \quad m_a\overline{m^a} = 1. \tag{A2}$$

A particular choice is

$$m_a = \frac{r}{\sqrt{2}}(1, i\sin\theta), \quad \overline{m_a} = \frac{r}{\sqrt{2}}(1,-i\sin\theta). \tag{A3}$$

This choice is not unique, because the relations (A2) are preserved under the transformation

$$m^a \to e^{i\psi}m^a, \quad \overline{m^a}\to e^{-i\psi}\overline{m^a}, \tag{A4}$$

where ψ is any constant.

We may use the basis to decompose tensor fields. For example, a vector field V^a may be expressed as

$$V_a = {}_{-1}V m_a + {}_1 V \overline{m_a}, \tag{A5}$$

where

$${}_1V = V^a m_a, \quad {}_{-1}V = V^a \overline{m_a}. \tag{A6}$$

Similarly, a symmetric tensor field T^{ab} is decomposed as

$$T_{ab} = {}_{-2}T m_a m_b + 2\,{}_0T m_{(a}\overline{m_{b)}} + {}_2T \overline{m_a}\,\overline{m_b}, \tag{A7}$$

where

$${}_2T = T^{ab} m_a m_b, \quad {}_0T = T^{ab} m_a \overline{m_b}, \quad {}_{-2}T = T^{ab}\overline{m_a}\,\overline{m_b}. \tag{A8}$$

The *spin weight* of a field is determined by how the field transforms under Eqs. (A4). By definition, a field has spin weight s and is denoted ${}_s\eta$, if

$${}_s\eta \rightarrow e^{is\psi}{}_s\eta \tag{A9}$$

under the transformation. For example, ${}_{-1}V$ has spin weight $s=-1$, while ${}_2T$ has spin weight $s=2$.

The covariant derivatives (with respect to g_{ab}) of the base vectors are given by

$$m_{a;b} = -\frac{1}{\sqrt{2}r}\cot\theta\, m_a(m_b - \overline{m_b}) \tag{A10}$$

and its complex conjugate. It follows that the covariant derivatives of arbitrary tensor fields can be conveniently expressed in terms of the ''edth'' differential operators $\hat{\partial}$ and $\check{\partial}$, which are defined by

$$\hat{\partial} = -\left(\frac{\partial}{\partial\theta} + i\csc\theta\frac{\partial}{\partial\phi} - s\cot\theta\right),$$

$$\check{\partial} = -\left(\frac{\partial}{\partial\theta} - i\csc\theta\frac{\partial}{\partial\phi} + s\cot\theta\right). \tag{A11}$$

It should be noted that these operators depend on s, the spin weight of the object on which they act. (The original notation [46] for these operators was $\eth$ and $\overline{\eth}$, respectively.) For example,

$$V_{a;b} = -\frac{1}{\sqrt{2}r}[(\check{\partial}_{-1}V)m_a m_b + (\hat{\partial}_{-1}V)m_a\overline{m_b} + (\check{\partial}_1 V)\overline{m_a}m_b + (\hat{\partial}_1 V)\overline{m_a}\,\overline{m_b}]. \tag{A12}$$

From this relation it is clear that $\hat{\partial}$ raises the spin weight by one unit, while $\check{\partial}$ lowers it by one unit. For example, $\hat{\partial}_1 V = -\sqrt{2}rV_{a;b}m^a m^b$ has spin weight $s=2$.

The ''edth'' operators can be manipulated efficiently when working under an integral sign. Given two smooth, complex functions ${}_{s-1}f(\theta,\phi)$ and ${}_sg(\theta,\phi)$, the following identities are easily established by straightforward partial integration:

$$\int (\hat{\partial}_{s-1}f)\,\overline{{}_sg}\,d\Omega = -\int {}_{s-1}f\,(\overline{\check{\partial}_s g})\,d\Omega,$$

$$\int (\check{\partial}_s g)\,\overline{{}_{s-1}f}\,d\Omega = -\int {}_sg\,(\overline{\hat{\partial}_{s-1}f})\,d\Omega, \tag{A13}$$

where $d\Omega = d\cos\theta\, d\phi$.

The ''edth'' operators can be used to generate sets of spin-weighted spherical-harmonic functions, denoted ${}_sY_{lm}(\theta,\phi)$. Each set (corresponding to a fixed value of s) is complete, and members of a given set obey the usual orthonormality relations. The defining relations are ${}_0Y_{lm} \equiv Y_{lm}$ (the usual spherical harmonics), and

$$\hat{\partial}_s Y_{lm} = \sqrt{(l-s)(l+s+1)}\,{}_{s+1}Y_{lm},$$

$$\check{\partial}_s Y_{lm} = -\sqrt{(l+s)(l-s+1)}\,{}_{s-1}Y_{lm}. \tag{A14}$$

The spin-weighted spherical harmonics also satisfy the relations

$${}_{-s}\overline{Y_{l,-m}} = (-1)^{s+m}{}_sY_{lm}. \tag{A15}$$

The spin-weighted spherical harmonics can be combined with basis vectors to form tensorial spherical harmonics [29]. For example,

$$Y_a^{E,lm} = \frac{1}{\sqrt{2}}({}_{-1}Y_{lm}m_a - {}_1Y_{lm}\overline{m_a}),$$

$$Y_a^{B,lm} = -\frac{i}{\sqrt{2}}({}_{-1}Y_{lm}m_a + {}_1Y_{lm}\overline{m_a}), \tag{A16}$$

are vectorial spherical harmonics. The superscript E indicates that under a parity transformation, $Y_a^{E,lm}$ has electric-type parity, $Y_a^{E,lm} \rightarrow (-1)^l Y_a^{E,lm}$; the superscript B indicates a magnetic-type parity, $Y_a^{B,lm} \rightarrow (-1)^{l+1}Y_a^{B,lm}$. Similarly,

$$T_{ab}^{E2,lm} = \frac{1}{\sqrt{2}}({}_{-2}Y_{lm}m_a m_b + {}_2Y_{lm}\overline{m_a}\,\overline{m_b}),$$

$$T_{ab}^{B2,lm} = -\frac{i}{\sqrt{2}}({}_{-2}Y_{lm}\, m_a m_b - {}_2Y_{lm}\,\overline{m_a}\,\overline{m_b}), \tag{A17}$$

are tensorial spherical harmonics.

APPENDIX B: EVALUATION OF TWO FUNCTIONS

We evaluate, in the limit $z \rightarrow 0$, the functions $A_l(z)$ and $B_l(z)$ defined by Eqs. (2.14) and (2.15).

To evaluate $A_l(z)$ is easy. By using the expansions $\mathrm{Si}(2z) = 2z + O(z^3)$ and $z^2 n_p j_p = -z/(2p+1) + O(z^3)$, we quickly arrive at

$$A_l(z) = z - z\sum_{p=1}^{l-1}\left(\frac{1}{p} - \frac{1}{p+1}\right) + O(z^3). \tag{B1}$$

The sum evaluates to $1-1/l$, and $A_l(z)$ reduces to the result quoted in the text — Eq. (2.16).

To evaluate $B_l(z)$ requires more work. We begin by recalling the series expansions for the cosine integral (Ref. [34], p. 232),

$$\mathrm{Ci}(2z)=\gamma+\ln(2z)+\sum_{n=1}^{\infty}\frac{(-1)^n}{2n(2n)!}(2z)^{2n}, \qquad \text{(B2)}$$

the squared sine,

$$\sin^2 z=-\sum_{n=1}^{\infty}\frac{(-1)^n}{2(2n)!}(2z)^{2n}, \qquad \text{(B3)}$$

and the squared spherical Bessel functions of the first kind (Ref. [43], p. 960),

$$z^2 j_p{}^2(z)=\pi\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(2p+2+2k)}{k!\,\Gamma(2p+2+k)\Gamma^2(p+\frac{3}{2}+k)}\times\left(\frac{z}{2}\right)^{2p+2+2k}. \qquad \text{(B4)}$$

Substituting these into Eq. (2.15) gives the series

$$B_l(z)=\sum_{n=1}^{\infty}b_n z^{2n}, \qquad \text{(B5)}$$

where, after some rearranging,

$$b_n=\frac{(-1)^{n+1}2^{2n}(n-1)}{2n(2n)!}+\frac{\pi\Gamma(2n)}{2^{2n}\Gamma^2(n+\frac{1}{2})}\times\sum_{p=1}^{P}\left(\frac{1}{p}+\frac{1}{p+1}\right)\frac{(-1)^{n-p-1}}{(n-p-1)!\,\Gamma(n+p+1)}, \qquad \text{(B6)}$$

with $P=\min(n-1,l-1)$. Additional manipulations bring b_n to the form

$$b_n=\frac{(-1)^{n+1}2^{2n}(n-1)}{2n(2n)!}\left[1+\frac{n!^2}{n-1}\sum_{p=1}^{P}\left(\frac{1}{p}+\frac{1}{p+1}\right)\times\frac{(-1)^p}{(n-p-1)!(n+p)!}\right]. \qquad \text{(B7)}$$

The sum evaluates to

$$-\frac{n-1}{n!^2}+\frac{(-1)^P(n-P-1)}{n(P+1)(n+P)!(n-P-1)!}, \qquad \text{(B8)}$$

which implies that $b_n=0$ for $n\leqslant l$ (because $P=n-1$), while

$$b_n=\frac{(-1)^{n+l}2^{2n-1}(n-1)!^2}{l(2n)!(n-l-1)!(n+l-1)!} \qquad \text{(B9)}$$

for $n\geqslant l+1$. It is then easy to show that Eq. (B5) reduces to the result quoted in the text — Eq. (2.16).

APPENDIX C: REGGE-WHEELER FUNCTION IN THE ASYMPTOTIC LIMIT

We wish to evaluate the Regge-Wheeler function $X_l^H(z)$, as given by Eqs. (2.8) and (2.13) (with $a=0$ and $b=-\gamma$), in the asymptotic limit $z\to\infty$. By comparing with the low-frequency limit of Eq. (2.4), we will then compute $\mathcal{A}_l^{\mathrm{in}}$ in the normalization provided by Eq. (2.8).

To express Eqs. (2.8) and (2.13) in the limit $z\to\infty$, we use such asymptotic relations as $\mathrm{Si}(2z)\sim\pi/2$, $\mathrm{Ci}(2z)\sim 0$, $z^3(n_l j_p-j_l n_p)j_p\sim\frac{1}{2}[1-(-1)^{l-p}]zn_l$, $j_{l-1}\sim -n_l$, and $j_{l+1}\sim n_l$. After some algebra, we obtain

$$X_l^H\sim\left(1-\varepsilon\frac{\pi}{2}\right)zj_l-\varepsilon[\ln(2z)-\beta_l]zn_l+O(\varepsilon^2), \qquad \text{(C1)}$$

where β_l is given by Eq. (2.18).

We must now compare this result with the low-frequency limit of Eq. (2.4), which we rewrite as

$$X_l^H\sim\mathcal{A}_l^{\mathrm{in}}e^{-iz*}+\mathcal{A}_l^{\mathrm{out}}e^{iz*}, \qquad \text{(C2)}$$

where $z*=z+\varepsilon\ln(z/\varepsilon-1)$. Expanding the phase factors in powers of ε, and using the asymptotic relations $e^{\pm iz}\sim(\pm i)^{l+1}(zj_l\pm izn_l)$, yields

$$X_l^H\sim(1+\varepsilon\mathcal{A}_l^+)zj_l+\varepsilon(\mathcal{A}_l^- -\ln z)zn_l+O(\varepsilon^2), \qquad \text{(C3)}$$

where

$$1+\varepsilon\mathcal{A}_l^+=(-i)^{l+1}\mathcal{A}_l^{\mathrm{in}}e^{i\varepsilon\ln\varepsilon}+(i)^{l+1}\mathcal{A}_l^{\mathrm{out}}e^{-i\varepsilon\ln\varepsilon}, \qquad \text{(C4)}$$

$$i\varepsilon\mathcal{A}_l^-=(-i)^{l+1}\mathcal{A}_l^{\mathrm{in}}e^{i\varepsilon\ln\varepsilon}-(i)^{l+1}\mathcal{A}_l^{\mathrm{out}}e^{-i\varepsilon\ln\varepsilon}.$$

Finally, comparing Eqs. (C1) and (C3), and using the relations (C4), we arrive at

$$\mathcal{A}_l^{\mathrm{in}}=\frac{1}{2}(i)^{l+1}e^{-i\varepsilon(\ln 2\varepsilon-\beta_l)}\left[1-\frac{\pi}{2}\varepsilon+O(\varepsilon^2)\right]. \qquad \text{(C5)}$$

From this and Eqs. (2.8) and (2.13), we obtain Eq. (2.17).

APPENDIX D: INVERSE FOURIER TRANSFORM OF TAIL CORRECTIONS

We wish to take the inverse Fourier transform of the function

$$\tilde{\mathcal{F}}(\omega)=e^{2iM\omega(\ln 4M|\omega|-c)}(1+\pi M|\omega|)\tilde{F}(\omega), \qquad \text{(D1)}$$

where c is a constant and $\tilde{F}(\omega)$ an arbitrary, square-integrable function. In other words, we wish to compute the function $\mathcal{F}(u)$ given by

$$\mathcal{F}(u)=\int\tilde{\mathcal{F}}(\omega)e^{-i\omega u}d\omega. \qquad \text{(D2)}$$

We shall do so in the spirit of the slow-motion approximation, by formally treating M as a small parameter. We follow closely the derivation found in Appendix A of Ref. [19].

We first expand the exponential factor in Eq. (D1) to linear order in M, and combine the result with the $(1+\pi M|\omega|)$ factor. We then substitute the identity $i\omega\ln|\omega|+\pi|\omega|/2=i\omega\ln(-i\omega)$. After a few lines of algebra, we obtain

$$\mathcal{F}(u) = F(u) - 2M(\ln 4M - c)\dot{F}(u) + 2M \times \int \widetilde{F}(\omega)\; i\omega \ln(-i\omega) e^{-i\omega u} d\omega, \tag{D3}$$

where $F(u)$ is the inverse Fourier transform of $\widetilde{F}(\omega)$, and a dot indicates differentiation with respect to u.

To evaluate the integral, we write $\ln(-i\omega)$ in a different form by using the identity (Ref. [43], p. 573)

$$\ln\mu = -\gamma - \mu \int_0^\infty e^{-\mu x} \ln x\, dx, \tag{D4}$$

with $\mu = -i\omega$; $\gamma \simeq 0.577\,21$ is Euler's number. Strictly speaking, this identity is valid only if the real part of μ is positive. This problem can be circumvented by introducing a regulator $\epsilon > 0$ and setting $\mu = -i\omega + \epsilon$. The limit $\epsilon \to 0$ can be taken after integrating over ω, which yields

$$\mathcal{F}(u) = F(u) - 2M(\ln 4M - c - \gamma)\dot{F}(u) + 2M \times \int_0^\infty \ln x \ddot{F}(u-x)\, dx. \tag{D5}$$

We write this in its final form as

$$\mathcal{F}(u) = F(u) + 2M \int_{-\infty}^{u} \left[\ln\left(\frac{u-u'}{4M} \right) + c + \gamma \right] \ddot{F}(u')\, du'. \tag{D6}$$

This is the desired result.

APPENDIX E: CHANDRASEKHAR TRANSFORMATION FOR THE ELECTROMAGNETIC FIELD

We derive the relation between $R_l^H(\omega;r)$, the solution to the homogeneous version of Eq. (4.2) with boundary conditions (4.9), and $X_l^H(\omega;r)$, the solution to the $s=1$ version of Eq. (2.1) with boundary conditions (2.3) and (2.4). For convenience, in this Appendix we set to unity the arbitrary constant appearing in Eq. (2.3).

Direct substitution shows that if $X_l(\omega;r)$ satisfies the generalized Regge-Wheeler equation (with $s=1$), then $R_l(\omega;r) = r\mathcal{L}X_l(\omega;r)$ satisfies the homogeneous Teukolsky equation. Here, $\mathcal{L} = f d/dr + i\omega$. The desired relation must therefore have the form

$$R_l^H(\omega;r) = \chi\; r\mathcal{L}X_l^H(\omega;r), \tag{E1}$$

where the constant χ must be chosen so that the normalization of $R_l^H(\omega;r)$ agrees with Eq. (4.9).

To find χ and to relate $\mathcal{Q}_l^{\text{in}}(\omega)$ to $\mathcal{A}_l^{\text{in}}(\omega)$, we need expressions for $X_l^H(\omega;r)$ that are more accurate than Eqs. (2.3) and (2.4). By solving the generalized Regge-Wheeler equation, we find that

$$X_l^H(\omega;r) = \left[1 + \frac{l(l+1)}{1-4iM\omega} f + O(f^2) \right] e^{-i\omega r^*} \tag{E2}$$

near $r=2M$, while

$$X_l^H(\omega;r) = \mathcal{A}_l^{\text{in}}(\omega) \left\{ 1 + \frac{l(l+1)}{2i\omega r} + O[(\omega r)^{-2}] \right\} e^{-i\omega r^*} + \cdots \tag{E3}$$

near $r=\infty$, where the ellipsis designates terms proportional to $e^{i\omega r^*}$. We are now in a position to verify that near $r=2M$, $r\mathcal{L}e^{-i\omega r^*} \sim l(l+1) f e^{-i\omega r^*}/(1-4iM\omega)$, and that near $r=\infty$, $r\mathcal{L}e^{-i\omega r^*} = -\frac{1}{2} l(l+1)(i\omega r)^{-1} e^{-i\omega r^*}$.

Combining these results with Eqs. (2.3), (2.4), (4.9), and (E1), we find

$$\chi = \frac{1-4iM\omega}{l(l+1)} \tag{E4}$$

and

$$\mathcal{Q}_l^{\text{in}}(\omega) = -\frac{1}{2}(1-4iM\omega)\mathcal{A}_l^{\text{in}}(\omega). \tag{E5}$$

Equation (4.15) follows immediately.

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