

Post-Newtonian Lagrangian planetary equations

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We present a method to study the time variation of the orbital parameters of a post-Keplerian binary system undergoing a generic external perturbation. The method is the relativistic extension of the planetary Lagrangian equations. The theory only assumes the smallness of the external perturbation while relativistic effects are already included in the unperturbed problem. This is the major advantage of this novel approach over classical Lagrangian methods. [S0556-2821(97)05120-5]

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I. INTRODUCTION

Since the early stages of classical celestial mechanics, a very large amount of effort has been made in order to find exact or approximate solutions to the problem of N pointlike interacting bodies. It is well known that a general solution exists only for $N=2$ (Keplerian solution). In general, when N exceeds two searching for even approximate solutions becomes a very difficult task. Fortunately many N -body systems of interest in celestial mechanics may be considered a two-body problem, and the interaction of the other $N-2$ bodies can be regarded as a perturbation. In these cases the distance of the $N-2$ bodies, together with the relative magnitude of masses, allows one to resort to standard perturbative methods (see for instance [1], Chap. 6). Moreover when a two-body system suffers perturbations such as drag or radiation damping forces, oblateness of one of the two bodies and so on, the usual Keplerian solution must be viewed only as a zeroth-order approximation.

In the framework of Newtonian gravity, all these kinds of perturbations may be handled using the Lagrangian planetary equations through which the time dependence of orbital elements (otherwise constant) is achieved. This way the motion of the system is formally Keplerian but the orbital elements are allowed to vary with time. This procedure has been widely used from physicists and astronomers to study Newtonian binary systems; but what happens if relativistic effects are to be taken into account together with the external perturbation? A first way to approach the problem is to assume that relativistic effects and the external perturbation are roughly of the same order of magnitude; in this *semiclassical* standpoint they are both considered as a perturbation to Keplerian motion.

We propose a novel procedure to account for relativistic effects in the unperturbed problem, in which the only perturbation is the external one. For this scheme we divide our work in two steps; first the relativistic two-body problem needs to be solved (this has already been done up to the first post-Newtonian (1PN) and 2PN orders beyond the classical limit by various authors (see, for instance, [2,3]); this is the

so-called post-Keplerian solution); then our task, that is the main result of this paper, is to find out the relativistic version of Lagrangian planetary equations, giving the time dependence of relativistic orbital elements.

II. POST-KEPLERIAN SOLUTION

Let us consider a system of two bodies, moving under mutual gravitational attraction. Since we want to focus our attention on systems whose relativistic effects are not negligible, we have to introduce a post-Newtonian parameter quantifying the relevance of these effects; this parameter is defined as

$$\epsilon_{\text{PN}} \stackrel{\text{def}}{=} \frac{v}{c}. \quad (1)$$

In the above expression v is the typical speed of the bodies of the system and c is the speed of light; using relativistic Einstein equations it can be shown that the equations of motion up to 1PN order, i.e., retaining only the terms in $(\epsilon_{\text{PN}})^2$, may be derived from the following Lagrangian, in a reference frame with the 1PN center of mass at rest (see, for instance, [2]):

$$\begin{aligned} \mathcal{L}_0 = & \frac{1}{2} \frac{\mathbf{v}^2}{c^2} + \frac{Gm}{c^2 r} + \frac{1}{8} (1-3\nu) \frac{\mathbf{v}^4}{c^4} \\ & + \frac{Gm}{2c^2 r} \left((3+\nu) \frac{\mathbf{v}^2}{c^2} + \nu \frac{(\mathbf{v} \cdot \mathbf{x})^2}{c^2 r^2} \right) - \frac{G^2 m^2}{2c^4 r^2}, \end{aligned} \quad (2)$$

where

$$\begin{aligned} \mathbf{x} &= \mathbf{r}_2 - \mathbf{r}_1, \\ \mathbf{r}_B &= \frac{\mathcal{E}_1}{\mathcal{E}_1 + \mathcal{E}_2} \mathbf{r}_1 + \frac{\mathcal{E}_2}{\mathcal{E}_1 + \mathcal{E}_2} \mathbf{r}_2, \\ \mathcal{E}_1 &= m_1 c^2 + \frac{m_1}{2} \mathbf{v}_1^2 - \frac{Gm_1 m_2}{2r}, \\ \mathcal{E}_2 &= m_2 c^2 + \frac{m_2}{2} \mathbf{v}_2^2 - \frac{Gm_1 m_2}{2r}, \end{aligned}$$

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$$m = m_1 + m_2, \quad \nu = \frac{m_1 m_2}{m^2},$$

$$r = |\mathbf{x}|, \quad \mathbf{v} = \frac{d\mathbf{x}}{dt}.$$

In the above formulas, \mathbf{r}_1 and \mathbf{r}_2 are the radius vectors of the two bodies, m_1 and m_2 denote their masses, and \mathbf{r}_B is the position of the post-Newtonian center of mass [see [4], Eqs. (5.4.5)–(5.4.9)]. In the following the usefulness of the Hamiltonian approach with respect to the Lagrangian one will become clear; therefore we calculate the canonical momenta deriving from Eq. (2):

$$\mathbf{p} = \frac{\partial \mathcal{L}_0}{\partial \mathbf{v}} = \frac{\mathbf{v}}{c^2} + \frac{1}{2}(1-3\nu) \frac{\mathbf{v}^2 \mathbf{v}}{c^4} + \frac{Gm}{c^2 r} (3+\nu) \mathbf{v} + \frac{G\nu m}{c^2} \frac{(\mathbf{v} \cdot \mathbf{x})}{r^3} \mathbf{x}. \quad (3)$$

The Hamiltonian function of the system is defined as usual:

$$\mathcal{H}_0 = \mathbf{p} \cdot \mathbf{v} - \mathcal{L}_0. \quad (4)$$

Now, from Eqs. (2) and (3) we are in a position to express explicitly such a Hamiltonian in terms of the canonical variables \mathbf{x} and \mathbf{p} , obtaining

$$\mathcal{H}_0 = \frac{c^2}{2} \mathbf{p}^2 - \frac{Gm}{c^2 r} - \frac{c^4}{8} (1-3\nu) \mathbf{p}^4 - \frac{Gm}{2r} (3+\nu) \mathbf{p}^2 - \frac{Gm\nu}{2r^3} (\mathbf{x} \cdot \mathbf{p})^2 + \frac{G^2 m^2}{2c^4 r^2}. \quad (5)$$

Hence, the Hamilton equations are

$$\frac{d\mathbf{p}}{dt} = -\frac{Gm}{c^2 r^3} \mathbf{x} - \frac{Gm}{2r^3} (3+\nu) \mathbf{p}^2 \mathbf{x} - \frac{3Gm\nu}{2r^5} (\mathbf{x} \cdot \mathbf{p})^2 \mathbf{x} + \frac{Gm\nu}{r^3} (\mathbf{x} \cdot \mathbf{p}) \mathbf{p} + \frac{G^2 m^2}{c^4 r^4} \mathbf{x}, \quad (6)$$

$$\frac{d\mathbf{x}}{dt} = c^2 \mathbf{p} - \frac{c^4}{2} (1-3\nu) \mathbf{p}^2 \mathbf{p} - \frac{Gm}{r} (3+\nu) \mathbf{p} - \frac{Gm\nu}{r^3} (\mathbf{x} \cdot \mathbf{p}) \mathbf{x}. \quad (7)$$

The exact solution (called the post-Keplerian), up to 1PN order, has been found by Damour and Deruelle (see [2], and also [3] for its extension to the 2PN level). The post-Keplerian solution may be written in a form very similar with respect to the classical one as follows:

$$\begin{aligned} x^1 &= (\cos\omega\cos\Omega - \sin\omega\sin\Omega\cos i) \tilde{x}^1 \\ &\quad - (\sin\omega\cos\Omega + \cos\omega\sin\Omega\cos i) \tilde{x}^2, \\ x^2 &= (\cos\omega\sin\Omega + \sin\omega\cos\Omega\cos i) \tilde{x}^1 \\ &\quad - (\sin\omega\sin\Omega - \cos\omega\cos\Omega\cos i) \tilde{x}^2, \\ x^3 &= \sin\omega\sin i \tilde{x}^1 + \cos\omega\sin i \tilde{x}^2, \end{aligned} \quad (8)$$

where

$$\begin{aligned} \tilde{x}^1 &= r \cos \left\{ 2(1+\kappa) \arctan \left[\sqrt{\frac{1+e_3}{1-e_3}} \tan\left(\frac{\eta}{2}\right) \right] \right\} = r \cos \tilde{\eta}, \\ \tilde{x}^2 &= r \sin \left\{ 2(1+\kappa) \arctan \left[\sqrt{\frac{1+e_3}{1-e_3}} \tan\left(\frac{\eta}{2}\right) \right] \right\} = r \sin \tilde{\eta}, \end{aligned} \quad (9)$$

and

$$n(t-T) = \eta - e_1 \sin \eta, \quad (10)$$

$$r = a(1 - e_2 \cos \eta), \quad (11)$$

$$n = \sqrt{\frac{Gm}{a^3}} \left(1 - \frac{(9-\nu) Gm}{2c^2 a} \right), \quad (12)$$

$$\kappa = \frac{3Gm}{c^2 a(1-e^2)}, \quad (13)$$

$$e_1 = \left(1 - \frac{(8-3\nu) Gm}{2c^2 a} \right) e, \quad (14)$$

$$e_2 = e, \quad (15)$$

$$e_3 = \left(1 + \frac{\nu Gm}{2c^2 a} \right) e. \quad (16)$$

In these equations ω , Ω , and i are Euler angles, defining the rotation that connects the observation reference frame with the intrinsic frame of the motion. In celestial mechanics they are usually referred to as *argument of periastron* (the angle in orbital plane from the line of nodes (see [5], Sec. 4.4) to the perihelion point), *longitude of the ascending node* (the angle measured from the positive x axis of the observer to the line of nodes) and *inclination of the orbit* (the angle between the orbital plane and the x - y plane of the observer), respectively. The other elements of the orbit are the *semimajor axis* of the ellipse a , the *eccentricity* e , and the *time of periastron passage* T .

III. 1PN LAGRANGIAN BRACKETS

In last section we have reviewed the post-Newtonian solution of a binary system of bodies. Such a solution is quite correct provided that either the system is completely isolated or the external perturbation induces effects whose order of magnitude is less than or equal to the 2PN ones; if we quantify the weakness of external perturbation to the parameter ϵ_{ext} , our last assumption is

$$\epsilon_{\text{ext}} \leq (\epsilon_{\text{PN}})^2. \quad (17)$$

In the framework of the validity of the above expression the simple post-Keplerian solution is correct. When the external perturbation is so strong that the above expression does not hold true anymore, we have to take into account its effect on the 1PN relative motion of the two bodies. To this purpose we have developed a perturbation method enabling us to calculate the time dependence of orbital parameters due to external perturbation. Such a method may be viewed as the relativistic extension of the planetary Lagrangian equations,

which are widely being used in classical celestial mechanics to evaluate the time evolution of the orbital elements for Keplerian motion. Indeed it is our aim to calculate the time evolution of orbital parameters for post-Keplerian motion. To summarize we assume that the solution of the perturbed problem has the same functional dependence on orbital parameters and on time explicitly as the unperturbed problem.

In the presence of an external perturbation, the Hamiltonian describing the system can be written as

$$\mathcal{H} = \mathcal{H}_0(\mathbf{x}, \mathbf{p}) + \mathcal{H}_1(\mathbf{x}, \mathbf{p}), \quad (18)$$

where \mathcal{H}_0 is the usual two-body Hamiltonian, which is exact up to 1PN order, while \mathcal{H}_1 describes the most general perturbation. This latter is a sort of *disturbing function* with the only difference that the usual classical disturbing function (see [1], Chaps. 5, 6 and [6] if the external perturbation is due to the gravitational attraction of other bodies) is a perturbation to the *Lagrangian*, while the one concerned through this paper is a perturbation to the *Hamiltonian*. With the above assumptions, the solution satisfies the following equations:

$$\frac{\partial \mathbf{x}}{\partial t} = \frac{\partial \mathcal{H}_0}{\partial \mathbf{p}}, \quad \frac{\partial \mathbf{p}}{\partial t} = - \frac{\partial \mathcal{H}_0}{\partial \mathbf{x}}, \quad (19)$$

where $\mathbf{x} = \mathbf{x}(a, e, T, \omega, \Omega, i, t)$ and $\mathbf{p} = \mathbf{p}(a, e, T, \omega, \Omega, i, t)$.

As in the classical case, we need to find out the equations governing the time dependence of orbital parameters $(a, e, T, \omega, \Omega, i)$ due to an external perturbation. First of all we have to calculate the relativistic Lagrangian brackets defined as ([5], Sec. 9-4)

$$[C_j, C_k] \stackrel{\text{def}}{=} \frac{\partial \mathbf{x}}{\partial C_j} \cdot \frac{\partial \mathbf{p}}{\partial C_k} - \frac{\partial \mathbf{x}}{\partial C_k} \cdot \frac{\partial \mathbf{p}}{\partial C_j}, \quad (20)$$

where C_j with $1 \leq j \leq 6$ are the orbital elements $a, e, T, \omega, \Omega, i$. On the strength of Eqs. (19) it is easy to show that the Lagrangian brackets do not depend explicitly upon time (for the classical version of this property see [1], Sec. 6-2).

From definition (20) and the Hamilton equations of motion pertaining to Eq. (18), we find, after straightforward calculation (see, for instance, [5], Sec. 11-2),

$$\sum_{k=1}^{k=6} [C_j, C_k] \frac{dC_k}{dt} = - \frac{\partial \mathcal{H}_1}{\partial C_j}. \quad (21)$$

We remark that these latter equations are rather similar to the classical ones [see [1], p. 133, Eq. (6-19)], where \mathcal{H}_1 plays the role of disturbing function (the sign is reversed since our 1PN disturbing function is ascribed to a Hamiltonian).

Since 1PN Lagrangian brackets do not depend explicitly upon time, it suffices to evaluate them at the periastron (i.e., at $t = T$). In doing so we follow the same procedure as in the classical case (see, for instance, [1], Sec. 6-3), i.e., we expand in powers of $t - T$, up to second-order terms, the relative radius vector \mathbf{x} and the canonical momentum \mathbf{p} , then we compute Lagrangian brackets (20); finally we set $t = T$. After a very long but straightforward calculation we achieve the only nonvanishing brackets:

$$[a, T] = \frac{Gm}{2c^2 a^2} \left[1 + \frac{Gm}{2c^2 a} (\nu - 7) \right], \quad (22)$$

$$[a, \omega] = - \frac{a \sqrt{1-e^2}}{2c^2} \sqrt{\frac{Gm}{a^3}} \left[1 - \frac{2Gm}{c^2 a (1-e^2)} \left(1 + \frac{e^2}{2} - \frac{\nu e^2}{4} \right) \right], \quad (23)$$

$$[a, \Omega] = - \frac{a \sqrt{1-e^2}}{2c^2} \sqrt{\frac{Gm}{a^3}} \left[1 - \frac{2Gm}{c^2 a (1-e^2)} \left(1 + \frac{e^2}{2} - \frac{\nu e^2}{4} \right) \right] \cos i, \quad (24)$$

$$[e, \omega] = \frac{a^2 e}{c^2 \sqrt{1-e^2}} \sqrt{\frac{Gm}{a^3}} \left[1 - \frac{4Gm}{c^2 a (1-e^2)} \left(1 - \frac{e^2}{4} - \frac{\nu}{4} + \frac{\nu e^2}{8} \right) \right], \quad (25)$$

$$[e, \Omega] = \frac{a^2 e}{c^2 \sqrt{1-e^2}} \sqrt{\frac{Gm}{a^3}} \left[1 - \frac{4Gm}{c^2 a (1-e^2)} \left(1 - \frac{e^2}{4} - \frac{\nu}{4} + \frac{\nu e^2}{8} \right) \right] \cos i, \quad (26)$$

$$[\Omega, i] = - \frac{a^2}{c^2} \sqrt{1-e^2} \sqrt{\frac{Gm}{a^3}} \left[1 + \frac{2Gm}{c^2 a (1-e^2)} \left(1 + \frac{e^2}{2} - \frac{\nu e^2}{4} \right) \right] \sin i. \quad (27)$$

IV. 1PN PLANETARY LAGRANGIAN EQUATIONS

Using the above expressions for the brackets and inverting the system of equations (21), we find the post-Newtonian Lagrangian planetary equations:

$$\frac{da}{dt} = \frac{2c^2a^2}{Gm} \left[1 + \frac{Gm}{c^2a} \left(\frac{7}{2} - \nu \right) \right] \frac{\partial \mathcal{H}_1}{\partial T}, \quad (28)$$

$$\frac{de}{dt} = \frac{c^2a(1-e^2)}{Gme} \left[1 + \frac{Gm}{c^2a} \left(\frac{11}{2} - \frac{3\nu}{2} \right) \right] \frac{\partial \mathcal{H}_1}{\partial T} + \frac{c^2\sqrt{1-e^2}}{e\sqrt{Gma}} \left[1 + \frac{Gm}{c^2a(1-e^2)} \left(4 - \nu - e^2 + \frac{\nu e^2}{2} \right) \right] \frac{\partial \mathcal{H}_1}{\partial \omega}, \quad (29)$$

$$\frac{dT}{dt} = -\frac{2c^2a^2}{Gm} \left[1 + \frac{Gm}{c^2a} \left(\frac{7}{2} - \nu \right) \right] \frac{\partial \mathcal{H}_1}{\partial a} - \frac{c^2a(1-e^2)}{Gme} \left[1 + \frac{Gm}{c^2a} \left(\frac{11}{2} - \frac{3\nu}{2} \right) \right] \frac{\partial \mathcal{H}_1}{\partial e}, \quad (30)$$

$$\frac{d\omega}{dt} = -\frac{c^2\sqrt{1-e^2}}{e\sqrt{Gma}} \left[1 + \frac{Gm}{c^2a(1-e^2)} \left(4 - \nu - e^2 + \frac{\nu e^2}{2} \right) \right] \frac{\partial \mathcal{H}_1}{\partial e} + \frac{c^2 \cot i}{\sqrt{Gma}\sqrt{1-e^2}} \left[1 - \frac{Gm}{c^2a(1-e^2)} \left(2 + e^2 - \frac{\nu e^2}{2} \right) \right] \frac{\partial \mathcal{H}_1}{\partial i}, \quad (31)$$

$$\frac{d\Omega}{dt} = -\frac{c^2}{\sqrt{Gma}\sqrt{1-e^2} \sin i} \left[1 - \frac{Gm}{c^2a(1-e^2)} \left(2 + e^2 - \frac{\nu e^2}{2} \right) \right] \frac{\partial \mathcal{H}_1}{\partial i}, \quad (32)$$

$$\frac{di}{dt} = -\frac{c^2 \cot i}{\sqrt{Gma}\sqrt{1-e^2}} \left[1 - \frac{Gm}{c^2a(1-e^2)} \left(2 + e^2 - \frac{\nu e^2}{2} \right) \right] \frac{\partial \mathcal{H}_1}{\partial \omega} + \frac{c^2}{\sqrt{Gma}\sqrt{1-e^2} \sin i} \left[1 - \frac{Gm}{c^2a(1-e^2)} \left(2 + e^2 - \frac{\nu e^2}{2} \right) \right] \frac{\partial \mathcal{H}_1}{\partial \Omega}. \quad (33)$$

Once the external perturbation has been assigned, it is possible to evaluate its effects on the orbital parameters by solving the system (28)–(33); as in classical mechanics, the first-order time variation of orbital elements is simply obtained after substitution of the unperturbed values into the right-hand sides of Eqs. (28)–(33). This could be done without assuming that the magnitude of ϵ_{PN} is a first-order term because the 1PN effects have already been taken into account using the post-Keplerian parametrization instead of the Keplerian one.

In celestial mechanics, the parameter T , is often replaced by (e.g., [1,6,7]):

$$\sigma = -nT, \quad (34)$$

which is just the constant related to the *mean anomaly* $M = nt + \sigma$, i.e., the angle which the radius vector would have described if it had been moving uniformly with the average rate n .

This newly defined orbital element allows one to see that all Lagrangian brackets are not changed except for this one:

$$[\sigma, a] = \frac{a}{2c^2} \sqrt{\frac{Gm}{a^3}} \left(1 + \frac{Gm}{c^2a} \right), \quad (35)$$

which takes the place of Eq. (22). Using this new set of elements the 1PN planetary equations become

$$\frac{da}{dt} = -\frac{2c^2a^2}{Gm} \sqrt{\frac{Gm}{a^3}} \left(1 - \frac{Gm}{c^2a} \right) \frac{\partial \mathcal{H}_1}{\partial \sigma}, \quad (36)$$

$$\frac{de}{dt} = -\frac{c^2a(1-e^2)}{Gme} \sqrt{\frac{Gm}{a^3}} \left(1 + \frac{Gm}{c^2a} (1-\nu) \right) \frac{\partial \mathcal{H}_1}{\partial \sigma} + \frac{c^2\sqrt{1-e^2}}{e\sqrt{Gma}} \left[1 + \frac{Gm}{c^2a(1-e^2)} \left(4 - \nu - e^2 + \frac{\nu e^2}{2} \right) \right] \frac{\partial \mathcal{H}_1}{\partial \omega}, \quad (37)$$

$$\frac{d\sigma}{dt} = \frac{2c^2a^2}{Gm} \sqrt{\frac{Gm}{a^3}} \left(1 - \frac{Gm}{c^2a} \right) \frac{\partial \mathcal{H}_1}{\partial a} + \frac{c^2a(1-e^2)}{Gme} \sqrt{\frac{Gm}{a^3}} \left(1 + \frac{Gm}{c^2a} (1-\nu) \right) \frac{\partial \mathcal{H}_1}{\partial e}, \quad (38)$$

$$\frac{d\omega}{dt} = -\frac{c^2\sqrt{1-e^2}}{e\sqrt{Gma}} \left[1 + \frac{Gm}{c^2a(1-e^2)} \left(4 - \nu - e^2 + \frac{\nu e^2}{2} \right) \right] \frac{\partial \mathcal{H}_1}{\partial e} + \frac{c^2a \cot i}{Gm\sqrt{1-e^2}} \sqrt{\frac{Gm}{a^3}} \left[1 - \frac{Gm}{c^2a(1-e^2)} \left(2 + e^2 - \frac{\nu e^2}{2} \right) \right] \frac{\partial \mathcal{H}_1}{\partial i}, \quad (39)$$

$$\frac{d\Omega}{dt} = -\frac{c^2a}{Gm\sqrt{1-e^2} \sin i} \sqrt{\frac{Gm}{a^3}} \left[1 - \frac{Gm}{c^2a(1-e^2)} \left(2 + e^2 - \frac{\nu e^2}{2} \right) \right] \frac{\partial \mathcal{H}_1}{\partial i}, \quad (40)$$

$$\frac{di}{dt} = -\frac{c^2a \cot i}{Gm\sqrt{1-e^2}} \sqrt{\frac{Gm}{a^3}} \left[1 - \frac{Gm}{c^2a(1-e^2)} \left(2 + e^2 - \frac{\nu e^2}{2} \right) \right] \frac{\partial \mathcal{H}_1}{\partial \omega}$$

$$+ \frac{c^2 a}{Gm \sqrt{1-e^2} \sin i} \sqrt{\frac{Gm}{a^3}} \left[1 - \frac{Gm}{c^2 a (1-e^2)} \left(2 + e^2 - \frac{\nu e^2}{2} \right) \right] \frac{\partial \mathcal{H}_1}{\partial \Omega}. \quad (41)$$

Usually the disturbing function \mathcal{H}_1 depends upon σ only through the phase $M = nt + \sigma$, therefore it is suitable to express \mathcal{H}_1 in terms of M . Furthermore \mathcal{H}_1 depends upon a both explicitly and implicitly through M [in fact $n = n(a)$], so that the derivative of the disturbing function taken with respect to a , can be written as

$$\frac{\partial \mathcal{H}_1}{\partial a} = \left(\frac{\partial \mathcal{H}_1}{\partial a} \right)_M + \frac{\partial \mathcal{H}_1}{\partial M} t \frac{dn}{da}, \quad (42)$$

the first term on the right-hand side is the derivative of \mathcal{H}_1 with respect to a while keeping M (and then n) as a constant, and the second term takes into account the dependence of \mathcal{H}_1 upon a through n . From the above expression and using Eq. (38) we get

$$\begin{aligned} \frac{dM}{dt} = n + \frac{da}{dt} t \frac{dn}{da} + \frac{2c^2 a^2}{Gm} \sqrt{\frac{Gm}{a^3}} \left(1 - \frac{Gm}{c^2 a} \right) \left(\frac{\partial \mathcal{H}_1}{\partial a} \right)_M + \frac{2c^2 a^2}{Gm} \sqrt{\frac{Gm}{a^3}} \left(1 - \frac{Gm}{c^2 a} \right) \frac{\partial \mathcal{H}_1}{\partial M} t \frac{dn}{da} \\ + \frac{c^2 a (1-e^2)}{Gme} \sqrt{\frac{Gm}{a^3}} \left(1 + \frac{Gm}{c^2 a} (1-\nu) \right) \frac{\partial \mathcal{H}_1}{\partial e}. \end{aligned} \quad (43)$$

Now, since $\partial/\partial M = \partial/\partial \sigma$, Eq. (36) becomes

$$\frac{da}{dt} = - \frac{2c^2 a^2}{Gm} \sqrt{\frac{Gm}{a^3}} \left(1 - \frac{Gm}{c^2 a} \right) \frac{\partial \mathcal{H}_1}{\partial M}, \quad (44)$$

while the time derivative of M can be written

$$\frac{dM}{dt} = n + \frac{2c^2 a^2}{Gm} \sqrt{\frac{Gm}{a^3}} \left(1 - \frac{Gm}{c^2 a} \right) \left(\frac{\partial \mathcal{H}_1}{\partial a} \right)_M + \frac{c^2 a (1-e^2)}{Gme} \sqrt{\frac{Gm}{a^3}} \left(1 + \frac{Gm}{c^2 a} (1-\nu) \right) \frac{\partial \mathcal{H}_1}{\partial e}. \quad (45)$$

Equations (44) and (45) take the place of Eqs. (36) and (38) when M is used as orbital parameter (instead of σ or T).

V. APPLICATION

As an example of application we consider the interaction of a close binary system with a third body. For the sake of simplicity we assume the third body at a distance $R_B \gg a$ from the binary center of mass with mass $m_3 \gg m$. In this way the resulting motion could be safely approximated by the perturbed IPN motion of the binary system around its center of mass \mathbf{r}_B which, in turn, has a Keplerian orbit around m_3 , considered motionless. The time evolution of \mathbf{r}_B is therefore known, this latter not being a variable of the problem. The discussion made at the beginning of Sec. III showed that, in order to achieve a consistent picture, the effect of the perturbation must be greater than that of the 2PN one. The most interesting case is when the effect of the perturbation is of the same order of magnitude as the IPN effect; as it will be seen later this occurs when

$$\frac{m_3}{R_B^3} \sim \frac{Gm^2}{c^2 a^4} \sim 5 \left(\frac{M_\odot}{\text{a.u.}^3} \right), \quad (46)$$

where the second relation holds true if $m \sim M_\odot$ and $a \sim 2 R_\odot$.

Therefore, in the framework of previous assumptions, the perturbation to the Hamiltonian (5) is given by

$$\mathcal{H}_1 = - \frac{m_3 G}{2c^2 R_B^3} \left[r^2 - 3 \frac{(\mathbf{x} \cdot \mathbf{R}_B)^2}{R_B^2} \right], \quad (47)$$

where \mathbf{R}_B is the radius vector from m_3 to the IPN center of mass of the binary system. We assume that the binary center of mass performs a circular orbit around m_3 , which is coplanar with the orbital plane of the binary system. Hence we set $i = 0$, $\Omega = 0$, and $\mathbf{R}_B = A_B (\cos \omega_3 t, \sin \omega_3 t, 0)$, where $\omega_3 = \sqrt{Gm_3/A_B^3}$. The Hamiltonian perturbation thus becomes

$$\mathcal{H}_1(a, e, M, \omega, t) = - \frac{m_3 G}{4 c^2 A_B^3} a^2 (1 - e \cos \eta)^2 [1 + 3 \cos 2(\omega_3 t - \omega - \tilde{\eta})], \quad (48)$$

where $\eta = \eta(a, e, M)$ and $\tilde{\eta} = \tilde{\eta}(a, \eta(a, e, M), e)$ [see Eqs. (9) and (10), together with the definition of M after Eq. (34)] and

$$\frac{\partial \eta}{\partial a} = \frac{(e - e_1) \sin \eta}{a(1 - e_1 \cos \eta)}, \quad \frac{\partial \eta}{\partial e} = \frac{e_1 \sin \eta}{e(1 - e_1 \cos \eta)}, \quad \frac{\partial \eta}{\partial M} = \frac{1}{1 - e_1 \cos \eta}. \quad (49)$$

From Eqs. (13) and (16) one also has $\partial \kappa / \partial a = -\kappa/a$, $\partial \kappa / \partial e = 2e\kappa/(1 - e^2)$, $\partial e_3 / \partial a = (e - e_3)/a$, $\partial e_3 / \partial e = e_3/e$. In order to obtain the equations providing the temporal derivatives of the orbital elements, it suffices to calculate the partial derivatives of \mathcal{H}_1 ; in this way the problem of motion is solved. In order to better understand the behavior of the solution let us suppose $e = 0$. In this case the Hamiltonian perturbation greatly simplifies:

$$\mathcal{H}_1(a, M, \omega, t) = -\frac{m_3 G}{4c^2 A_B^3} a^2 \{1 + 3 \cos 2[\omega_3 t - \omega - (1 + \kappa)M]\}. \quad (50)$$

Finally the equations for the orbital elements a and M are

$$\frac{da}{dt} = \frac{3a^2 m_3}{A_B^3} \sqrt{\frac{Ga}{m}} (1 + \kappa) \left(1 - \frac{Gm}{c^2 a}\right) \sin 2[\omega_3 t - \omega - (1 + \kappa)M], \quad (51)$$

$$\frac{dM}{dt} = n - \frac{m_3 a}{A_B^3} \sqrt{\frac{Ga}{m}} \left(1 - \frac{Gm}{c^2 a}\right) \{1 + 3 \cos 2[\omega_3 t - \omega - (1 + \kappa)M]\}, \quad (52)$$

while

$$\frac{d(e^2)}{dt} = 0, \quad \frac{d\omega}{dt} = 0, \quad \frac{d\Omega}{dt} = 0, \quad \frac{di}{dt} = 0, \quad (53)$$

that is e , ω , Ω , and i are constants of the motion. By substituting the unperturbed values of the orbital elements on the right-hand side of Eqs. (51) and (52), the first-order time variation of the parameters is achieved. As far as relation (46) is fulfilled, being $m_3 a^3 / (m A_B^3) \sim Gm / (c^2 a)$, the solution can be written as

$$a(t) = a_0 \left(1 + \frac{3m_3 a_0^3}{2m A_B^3} \{\cos 2[\omega_3 t - \omega - (1 + \kappa)M_0] - \cos 2\omega\}\right), \quad (54)$$

$$M(t) = \left(1 - \frac{m_3 a_0^3}{m A_B^3}\right) M_0 + \frac{3m_3 a_0^3}{2m A_B^3} \{\sin 2[\omega_3 t - \omega - (1 + \kappa)M_0] - \sin 2\omega\}, \quad (55)$$

where a_0 and M_0 are the unperturbed values.

Accurate timing of the binary system could then provide the orbital profile through which [via Eqs. (54) and (55)] m_3 and A_B are evaluated. This example has shown how the method proposed in this paper allowed one to achieve an already existing result in a relatively simple way.

VI. CONCLUSIONS

We have developed a post-Newtonian extension of the planetary Lagrangian equations in order to describe the motion of a relativistic binary system under the influence of a generic external perturbation. An important class of phenomena that could be treated is the perturbation caused by external bodies to a binary system. We also have provided an

example featuring the capability of the method.

Our approach is a method to solve in a perturbative way the equations governing the variation of orbital parameters. The theory only assumes the smallness of the external perturbation while relativistic effects are already included in the unperturbed problem. This is the major advantage of our approach over classical Lagrangian methods. In the present paper the problem is solved at the 1PN level, but this approach should also be suitable to 2PN or higher-accuracy orders in the relativistic expansion.

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