

Geometric structure of the generic static traversable wormhole throat

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Traversable wormholes have traditionally been viewed as intrinsically topological entities in some multiply connected spacetime. Here, we show that topology is too limited a tool to accurately characterize a generic traversable wormhole: in general one needs geometric information to detect the presence of a wormhole, or more precisely to locate the wormhole throat. For an arbitrary static spacetime we shall define the wormhole throat in terms of a two-dimensional constant-time hypersurface of minimal area. (Zero trace for the extrinsic curvature plus a “flare-out” condition.) This enables us to severely constrain the geometry of spacetime at the wormhole throat and to derive generalized theorems regarding violations of the energy conditions, theorems that do not involve geodesic averaging but nevertheless apply to situations much more general than the spherically symmetric Morris-Thorne traversable wormhole. (For example, the null energy condition, when suitably weighted and integrated over the wormhole throat, must be violated.) The major technical limitation of the current approach is that we work in a static spacetime; this is already a quite rich and complicated system. [S0556-2821(97)02618-0]

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I. INTRODUCTION

Traversable wormholes [1–3] are often viewed as intrinsically topological objects, occurring only in multiply connected spacetimes. Indeed, the Morris-Thorne class of inter-universe traversable wormholes is even more restricted, requiring both exact spherical symmetry and the existence of two asymptotically flat regions in the spacetime. To deal with intrauniverse traversable wormholes, the Morris-Thorne analysis must be subjected to an approximation procedure wherein the two ends of the wormhole are distorted and forced to reside in the same asymptotically flat region. The existence of one or more asymptotically flat regions is an essential ingredient of the Morris-Thorne approach [1].

However, there are many other classes of geometries that one might still quite reasonably want to classify as wormholes, that either do not possess any asymptotically flat region [4], or have trivial topology [3], or exhibit both these phenomena.

A simple example of a wormhole lacking an asymptotically flat region is two closed Friedmann-Robertson-Walker spacetimes connected by a narrow neck (see Fig. 1); you might want to call this a “dumbbell wormhole.” A simple example of a wormhole with trivial topology is a single closed Friedmann-Robertson-Walker spacetime connected by a narrow neck to ordinary Minkowski space (see Fig. 2). A general taxonomy of wormhole exemplars may be found in [3], pp. 89–93, and discussions of wormholes with trivial topology may also be found in [3], pp. 53–74.

While the restricted viewpoint based on the Morris-Thorne analysis is acceptable for an initial discussion, the Morris-Thorne approach fails to capture the essence of large classes of wormholes that do not satisfy their simplifying assumptions.

In this paper we shall investigate the generic static tra-

versable wormhole. We make no assumptions about spherical symmetry (or axial symmetry, or even “exchange” symmetry), and we make no assumptions about the existence of any asymptotically flat region. We first have to define exactly what we mean by a wormhole, we find that there is a nice *geometrical* (not topological) characterization of the existence of, and location of, a wormhole “throat.” This characterization is developed in terms of a hypersurface of minimal area, subject to a “flare-out” condition that generalizes that of the Morris-Thorne analysis.

With this definition in place, we can develop a number of theorems about the existence of “exotic matter” at the

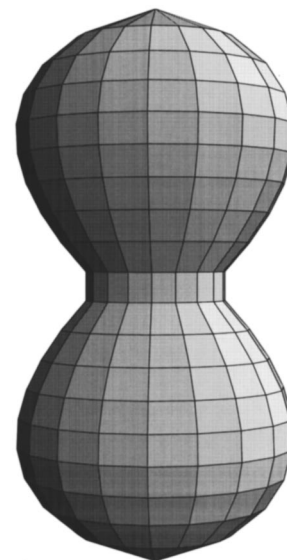


FIG. 1. A “dumbbell wormhole”: Formed (for example) by two closed Friedmann-Robertson-Walker spacetimes connected by a narrow neck.

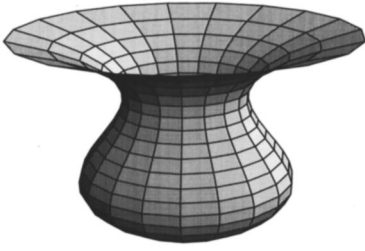


FIG. 2. A wormhole with trivial topology: Formed (for example) by connecting a single closed Friedmann-Robertson-Walker spacetime to Minkowski space by a narrow neck.

wormhole throat. These theorems generalize the original Morris-Thorne result by showing that the null energy condition (NEC) is generically violated at some points on or near the two-dimensional surface comprising the wormhole throat. These results should be viewed as complementary to the topological censorship theorem [5]. The topological censorship theorem tells us that in a spacetime containing a traversable wormhole the averaged null energy condition must be violated along at least some (not all) null geodesics, but the theorem provides very limited information on where these violations occur. The analysis of this paper shows that some of these violations of the energy conditions are concentrated in the expected place: on or near the throat of the wormhole. The present analysis, because it is purely local, also does not need the many technical assumptions about asymptotic flatness, future and past null infinities, and global hyperbolicity that are needed as ingredients for the topological censorship theorem [5].

The key simplifying assumption in the present analysis is that of taking a static wormhole. While we believe that a generalization to dynamic wormholes is possible, the situation becomes technically much more complex and one is rapidly lost in definitional subtleties and formalism.

II. STATIC SPACETIMES

In any static spacetime one can decompose the spacetime metric into block diagonal form [6–8]:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1}$$

$$= -\exp(2\phi) dt^2 + g_{ij} dx^i dx^j. \tag{2}$$

Greek indices run from 0 to 3 and refer to spacetime; latin indices from the middle of the alphabet (i, j, k, \dots) run from 1 to 3 and refer to space; latin indices from the beginning of the alphabet (a, b, c, \dots) will run from 1 to 2 and will be used to refer to the wormhole throat and directions parallel to the wormhole throat.

Being static tightly constrains the spacetime geometry in terms of the three-geometry of space on a constant time slice, and the manner in which this three-geometry is embedded into the spacetime. For example, from [6], p. 518 we have

$${}^{(3+1)}R_{ijkl} = {}^{(3)}R_{ijkl}, \tag{3}$$

$${}^{(3+1)}R_{\hat{t}abc} = 0, \tag{4}$$

$${}^{(3+1)}R_{\hat{t}ij} = \phi_{|ij} + \phi_{|i}\phi_{|j}. \tag{5}$$

The hat on the t index indicates that we are looking at components in the normalized t direction

$$X_{\hat{t}} = X_t \sqrt{-g^{tt}} = X_t \exp(-\phi). \tag{6}$$

This means we are using an orthonormal basis attached to the fiducial observers. We use $X_{;\alpha}$ to denote a spacetime covariant derivative, $X_{|i}$ to denote a three-space covariant derivative, and will shortly use $X_{;a}$ to denote two-space covariant derivatives taken on the wormhole throat itself.

Now taking suitable contractions,

$${}^{(3+1)}R_{ij} = {}^{(3)}R_{ij} - \phi_{|ij} - \phi_{|i}\phi_{|j}, \tag{7}$$

$${}^{(3+1)}R_{\hat{t}i} = 0, \tag{8}$$

$${}^{(3+1)}R_{\hat{t}\hat{t}} = g^{ij}[\phi_{|ij} + \phi_{|i}\phi_{|j}]. \tag{9}$$

So,

$${}^{(3+1)}R = {}^{(3)}R - 2g^{ij}[\phi_{|ij} + \phi_{|i}\phi_{|j}]. \tag{10}$$

To effect these contractions, we make use of the decomposition of the spacetime metric in terms of the spatial three-metric, the set of vectors e_i^μ tangent to the time slice, and the vector $V^\mu = \exp[\phi](\partial/\partial t)^\mu$ normal to the time slice:

$${}^{(3+1)}g^{\mu\nu} = e_i^\mu e_j^\nu g^{ij} - V^\mu V^\nu. \tag{11}$$

Finally, for the spacetime Einstein tensor (see [6], p. 552),

$${}^{(3+1)}G_{ij} = {}^{(3)}G_{ij} - \phi_{|ij} - \phi_{|i}\phi_{|j} + g_{ij}g^{kl}[\phi_{|kl} + \phi_{|k}\phi_{|l}], \tag{12}$$

$${}^{(3+1)}G_{\hat{t}i} = 0, \tag{13}$$

$${}^{(3+1)}G_{\hat{t}\hat{t}} = +\frac{1}{2}{}^{(3)}R. \tag{14}$$

This decomposition is generic to *any* static spacetime. (You can check this decomposition against various standard textbooks to make sure the coefficients are correct. For instance, see Sygne [9], p. 339, Fock [10], or Adler Bazin, and Schiffer [11].)

Suppose the strong energy condition (SEC) holds, then [3]

$$\text{SEC} \Rightarrow (\rho + g_{ij}T^{ij}) \geq 0 \tag{15}$$

$$\Rightarrow g^{ij}[\phi_{|ij} + \phi_{|i}\phi_{|j}] \geq 0 \tag{16}$$

$$\Rightarrow \phi \text{ has no isolated maxima.} \tag{17}$$

III. DEFINITION OF A GENERIC STATIC THROAT

We define a traversable wormhole throat Σ to be a two-dimensional hypersurface of *minimal* area taken in one of the constant-time spatial slices. Compute the area by taking

$$A(\Sigma) = \int \sqrt{{}^{(2)}g} d^2x. \tag{18}$$

Now use Gaussian normal coordinates, $x^i = (x^a; n)$, wherein the hypersurface Σ is taken to lie at $n=0$, so that

$${}^{(3)}g_{ij} dx^i dx^j = {}^{(2)}g_{ab} dx^a dx^b + dn^2. \quad (19)$$

The variation in surface area, obtained by pushing the surface $n=0$ out to $n = \delta n(x)$, is given by the standard computation

$$\delta A(\Sigma) = \int \frac{\partial \sqrt{{}^{(2)}g}}{\partial n} \delta n(x) d^2x, \quad (20)$$

which implies

$$\delta A(\Sigma) = \int \sqrt{{}^{(2)}g} \frac{1}{2} g^{ab} \frac{\partial g_{ab}}{\partial n} \delta n(x) d^2x. \quad (21)$$

In Gaussian normal coordinates the extrinsic curvature can be simply defined by

$$K_{ab} = -\frac{1}{2} \frac{\partial g_{ab}}{\partial n}. \quad (22)$$

[See [6], p. 552. We use Misner-Thorne-Wheeler (MTW) sign conventions. The convention in [3], p. 156 is opposite.] Thus,

$$\delta A(\Sigma) = - \int \sqrt{{}^{(2)}g} \operatorname{tr}(K) \delta n(x) d^2x. \quad (23)$$

[We use the notation $\operatorname{tr}(X)$ to denote $g^{ab} X_{ab}$.] Since this is to vanish for arbitrary $\delta n(x)$, the condition that the area be *extremal* is simply $\operatorname{tr}(K) = 0$. To force the area to be *minimal* requires (at the very least) the additional constraint $\delta^2 A(\Sigma) \geq 0$. (We shall also consider higher-order constraints below.) But by explicit calculation,

$$\delta^2 A(\Sigma) = - \int \sqrt{{}^{(2)}g} \left(\frac{\partial \operatorname{tr}(K)}{\partial n} - \operatorname{tr}(K)^2 \right) \delta n(x) \delta n(x) d^2x. \quad (24)$$

Extremality [$\operatorname{tr}(K) = 0$] reduces this minimality constraint to

$$\delta^2 A(\Sigma) = - \int \sqrt{{}^{(2)}g} \left(\frac{\partial \operatorname{tr}(K)}{\partial n} \right) \delta n(x) \delta n(x) d^2x \geq 0. \quad (25)$$

Since this is to hold for arbitrary $\delta n(x)$ this implies that at the throat we certainly require

$$\frac{\partial \operatorname{tr}(K)}{\partial n} \leq 0. \quad (26)$$

This is the generalization of the Morris-Thorne ‘‘flare-out’’ condition to arbitrary static wormhole throats.

In the following definitions, the two-surface referred to is understood to be embedded in a three-dimensional space, so that the concept of its extrinsic curvature (relative to that embedding space) makes sense.

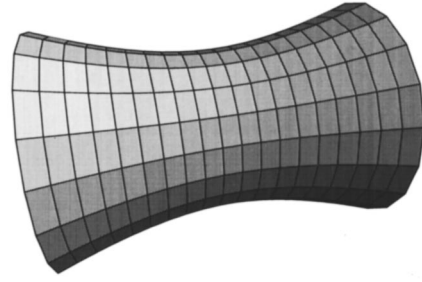


FIG. 3. Generically, we define the throat to be located at a true minimum of the area. The geometry should flare out on either side of the throat, but we make no commitment to the existence of any asymptotically flat region.

Definition: Simple flare-out condition

A two-surface satisfies the ‘‘simple flare-out’’ condition if and only if it is extremal, $\operatorname{tr}(K) = 0$, and also satisfies $\partial \operatorname{tr}(K) / \partial n \leq 0$.

This flare-out condition can be rephrased as follows: We have as an identity that

$$\frac{\partial \operatorname{tr}(K)}{\partial n} = \operatorname{tr} \left(\frac{\partial K}{\partial n} \right) + 2 \operatorname{tr}(K^2). \quad (27)$$

So, minimality implies

$$\operatorname{tr} \left(\frac{\partial K}{\partial n} \right) + 2 \operatorname{tr}(K^2) \leq 0. \quad (28)$$

Generically we would expect the inequality to be strict, in the sense that $\partial \operatorname{tr}(K) / \partial n < 0$, for at least some points on the throat. (See Fig. 3). This suggests the modified definition below.

Definition: Strong flare-out condition

A two-surface satisfies the ‘‘strong flare-out’’ condition at the point x if and only if it is extremal, $\operatorname{tr}(K) = 0$, everywhere satisfies $\partial \operatorname{tr}(K) / \partial n \leq 0$, and if at the point x on the surface the inequality is strict:

$$\frac{\partial \operatorname{tr}(K)}{\partial n} < 0. \quad (29)$$

It is sometimes sufficient to demand a weak integrated form of the flare-out condition.

Definition: Weak flare-out condition

A two-surface satisfies the ‘‘weak flare-out’’ condition if and only if it is extremal, $\operatorname{tr}(K) = 0$, and

$$\int \sqrt{{}^{(2)}g} \frac{\partial \operatorname{tr}(K)}{\partial n} d^2x < 0. \quad (30)$$

Note that the strong flare-out condition implies both the simple flare-out condition and the weak flare-out condition, but that the simple flare-out condition does not necessarily imply the weak flare-out condition. (The integral could be

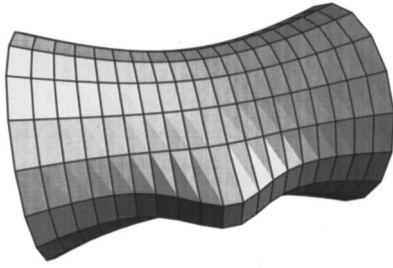


FIG. 4. Strong versus weak throats: Placing a small bump on a strong throat typically causes it to trifurcate into two strong throats plus a weak throat.

zero.) Whenever we do not specifically specify the type of flare-out condition being used we deem it to be the simple flare-out condition.

The conditions under which the definition of a weak flare out are appropriate arise, for instance, when one takes a Morris-Thorne traversable wormhole (which is symmetric under interchange of the two universes it connects) and distorts the geometry by placing a small bump on the original throat. (See Fig. 4).

The presence of the bump causes the old throat to trifurcate into three extremal surfaces: Two minimal surfaces are formed, one on each side of the old throat (these are minimal in the strong sense previously discussed), while the surface of symmetry between the two universes, though by construction still extremal, is no longer minimal in the strict sense. However, the surface of symmetry is often (but not always) minimal in the weak (integrated) sense indicated above.

A second situation in which the distinction between strong and weak throats is important is in the cut-and-paste construction for traversable wormholes [3,12,13]. In this construction one takes two (static) spacetimes $(\mathcal{M}_1, \mathcal{M}_2)$ and excises two geometrically identical regions of the form $\Omega_i \times \mathcal{R}$, Ω_i being compact spacelike surfaces with boundary and \mathcal{R} indicating the time direction. One then identifies the two boundaries $\partial\Omega_i \times \mathcal{R}$ thereby obtaining a single manifold $(\mathcal{M}_1 \# \mathcal{M}_2)$ that contains a wormhole joining the two regions $\mathcal{M}_i - \Omega_i \times \mathcal{R}$. We would like to interpret the junction $\partial\Omega_{1=2} \times \mathcal{R}$ as the throat of the wormhole.

If the sets Ω_i are convex, then there is absolutely no problem: the junction $\partial\Omega_{1=2} \times \mathcal{R}$ is by construction a wormhole throat in the strong sense enunciated above.

On the other hand, if the Ω_i are concave, then it is straightforward to convince oneself that the junction $\partial\Omega_{1=2} \times \mathcal{R}$ is not a wormhole throat in the strong sense. If one denotes the *convex hull* of Ω_i by $\text{conv}(\Omega_i)$ then the two regions $\partial[\text{conv}(\Omega_i)] \times \mathcal{R}$ are wormhole throats in the strong sense. The junction $\partial\Omega_{1=2} \times \mathcal{R}$ is at best a wormhole throat in the weak sense.

For these reasons it is useful to have this notion of a weak throat available as an alternative definition. Whenever we do not qualify the notion of wormhole throat it will refer to a throat in the simple sense. Whenever we refer to a throat in the weak sense or strong sense we will explicitly say so. Finally, it is also useful to make the following definition.

Definition: Weak f -weighted flare-out condition

A two-surface satisfies the “weak f -weighted flare-out” condition if and only if it is extremal, $\text{tr}(K) = 0$, and

$$\int \sqrt{{}^{(2)}g} f(x) \frac{\partial \text{tr}(K)}{\partial n} d^2x < 0. \tag{31}$$

[We will only be interested in this condition for $f(x)$ some positive function defined over the wormhole throat.]

The constraints on the extrinsic curvature embodied in these various definitions lead to constraints on the spacetime geometry, and consequently, constraints on the stress energy.

Technical point I: Degenerate throats

A class of wormholes for which we have to extend these definitions arises when the wormhole throat possesses an accidental degeneracy in the extrinsic curvature at the throat. The previous discussion has tacitly been assuming that near the throat we can write

$${}^{(2)}g_{ab}(x, n) = {}^{(2)}g_{ab}(x, 0) + n \frac{\partial [{}^{(2)}g_{ab}(x, n)]}{\partial n} \Big|_{n=0} + \frac{n^2}{2} \frac{\partial^2 [{}^{(2)}g_{ab}(x, n)]}{(\partial n)^2} \Big|_{n=0} + O(n^3), \tag{32}$$

with the linear term having trace zero (to satisfy extremality) and the quadratic term being constrained by the flare-out conditions.

Now if we have an accidental degeneracy with the quadratic (and possibly even higher-order terms) vanishing identically, we would have to develop an expansion such as

$${}^{(2)}g_{ab}(x, n) = {}^{(2)}g_{ab}(x, 0) + n \frac{\partial [{}^{(2)}g_{ab}(x, n)]}{\partial n} \Big|_{n=0} + \frac{n^{2N}}{(2N)!} \frac{\partial^{2N} [{}^{(2)}g_{ab}(x, n)]}{(\partial n)^{2N}} \Big|_{n=0} + O(n^{2N+1}). \tag{33}$$

Applied to the metric determinant this implies an expansion such as

$$\sqrt{{}^{(2)}g}(x, n) = \sqrt{{}^{(2)}g}(x, 0) \left(1 + \frac{n^{2N} k_N(x)}{(2N)!} + O(n^{2N+1}) \right), \tag{34}$$

where $k_N(x)$ denotes the first nonzero subdominant term in the above expansion, and we know by explicit construction that

$$k_N(x) = + \frac{1}{2} \text{tr} \left(\frac{\partial^{2N} [{}^{(2)}g_{ab}(x, n)]}{(\partial n)^{2N}} \Big|_{n=0} \right) \tag{35}$$

$$= - \text{tr} \left(\frac{\partial^{2N-1} K_{ab}(x, n)}{(\partial n)^{2N-1}} \Big|_{n=0} \right) \tag{36}$$

$$= - \left(\frac{\partial^{2N-1} K(x, n)}{(\partial n)^{2N-1}} \Big|_{n=0} \right), \tag{37}$$

since the trace is taken with ${}^{(2)}g^{ab}(x, 0)$ and this commutes with the normal derivative. We know that the first nonzero subdominant term in expansion (34) must be of even order in

n (i.e., n^{2N}), and cannot correspond to an odd power of n , since otherwise the throat would be a point of inflection of the area, not a minimum of the area. Furthermore, since $k_N(x)$ is by definition nonzero the flare-out condition must be phrased as the constraint $k_N(x) > 0$, with this now being a strict inequality. More formally, this leads to the definition below.

Definition: N -fold degenerate flare-out condition

A two-surface satisfies the “ N -fold degenerate flare-out” condition at a point x if and only if it is extremal, $\text{tr}(K) = 0$, if in addition the first $2N - 2$ normal derivatives of the trace of the extrinsic curvature vanish at x , and if finally at the point x one has

$$\frac{\partial^{2N-1} \text{tr}(K)}{(\partial n)^{2N-1}} < 0, \quad (38)$$

where the inequality is strict. [In the previous notation this is equivalent to the statement that $k_N(x) > 0$.]

Physically, at an N -fold degenerate point, the wormhole throat is seen to be extremal up to order $2N - 1$ with respect to normal derivatives of the metric, i.e., the flare out property is delayed spatially with respect to throats in which the flare out occurs at second order in n . The way we have set things up, the onefold degenerate flare-out condition is completely equivalent to the strong flare-out condition.

If we now consider the extrinsic curvature directly we see, by differentiating Eq. (34), first that

$$K(x, n) = -\frac{n^{2N-1} k_N(x)}{(2N-1)!} + O(n^{2N}), \quad (39)$$

and second that

$$\frac{\partial K(x, n)}{\partial n} = -\frac{n^{2N-2} k_N(x)}{(2N-2)!} + O(n^{2N-1}). \quad (40)$$

From the dominant $n \rightarrow 0$ behavior we see that if (at some point x) $2N$ happens to equal 2, then the flare-out condition implies that $\partial K(x, n)/\partial n$ must be negative *at and near the throat*. This can also be deduced directly from the equivalent strong flare-out condition: if $\partial K(x, n)/\partial n$ is negative and nonzero at the throat, then it must remain negative in some region surrounding the throat. On the other hand, if $2N$ is greater than 2 the flare-out condition only tells us that $\partial K(x, n)/\partial n$ must be negative *in some region surrounding the throat*, and does not necessarily imply that it is negative at the throat itself. (It could merely be zero at the throat.)

Thus for degenerate throats, the flare-out conditions should be rephrased in terms of the first nonzero normal derivative beyond the linear term. Analogous issues arise even for Morris-Thorne wormholes [1], p. 405, Eq. (56), see also the discussion presented in [3], p. 104–105 and 109. Even if the throat is nondegenerate (onefold degenerate) there are technical advantages to phrasing the flare-out conditions this way: It allows us to put constraints on the extrinsic curvature near but not on the throat.

Technical point II: Hyperspatial tubes

A second class of wormholes requiring even more technical fiddles arises when there is a central section which is completely uniform and independent of n . [So that $K_{ab} = 0$ over the whole throat for some finite range $n \in (-n_0, +n_0)$.] This central section might be called a “hyperspatial tube.” The flare-out condition should then be rephrased as stating that whenever extrinsic curvature first deviates from zero [at some point $(x, \pm n_0)$], one must formulate constraints such as

$$\left. \frac{\partial \text{tr}(K)}{\partial n} \right|_{\pm n_0^\pm} \leq 0. \quad (41)$$

In this case $\text{tr}(K)$ is by definition not an analytic function of n at n_0 , so the flare-out constraints have to be interpreted in terms of one-sided derivatives in the region outside the hyperspatial tube. [That is, we are concerned with the possibility that $\sqrt{g(x, n)}$ could be constant for $n < n_0$ but behave as $(n - n_0)^{2N}$ for $n > n_0$. In this case derivatives, at $n = n_0$, do not exist beyond order $2N$.]

IV. GEOMETRY OF A GENERIC STATIC THROAT

Using Gaussian normal coordinates in the region surrounding the throat,

$${}^{(3)}R_{abcd} = {}^{(2)}R_{abcd} - (K_{ac}K_{bd} - K_{ad}K_{bc}). \quad (42)$$

See [6], p. 514, Eq. (21.75). Because two dimensions are special this reduces to

$${}^{(3)}R_{abcd} = \frac{{}^{(2)}R}{2} (g_{ac}g_{bd} - g_{ad}g_{bc}) - (K_{ac}K_{bd} - K_{ad}K_{bc}). \quad (43)$$

Of course, we still have the standard dimension-independent results that

$${}^{(3)}R_{nabc} = -(K_{ab:c} - K_{ac:b}), \quad (44)$$

$${}^{(3)}R_{nanb} = \frac{\partial K_{ab}}{\partial n} + (K^2)_{ab}. \quad (45)$$

See [6], p. 514, Eq. (21.76) and [6], p. 516 Eq. (21.82). Here the index n refers to the spatial direction normal to the two-dimensional throat.

Thus far, these results hold both on the throat and in the region surrounding the throat: these results hold as long as the Gaussian normal coordinate system does not break down. (Such a breakdown being driven by the fact that the normal geodesics typically intersect after a certain distance.) In the interests of notational tractability we now particularize attention to the throat itself, but shall subsequently indicate that certain of our results can be extended off the throat itself into the entire region over which the Gaussian normal coordinate system holds sway.

Taking suitable contractions, *and using the extremality condition* $\text{tr}(K) = 0$,

$${}^{(3)}R_{ab} = \frac{{}^{(2)}R}{2} g_{ab} + \frac{\partial K_{ab}}{\partial n} + 2(K^2)_{ab}, \quad (46)$$

$${}^{(3)}R_{na} = -K_{ab}{}^{:b}, \quad (47)$$

$${}^{(3)}R_{nn} = \text{tr}\left(\frac{\partial K}{\partial n}\right) + \text{tr}(K^2) \quad (48)$$

$$= \frac{\partial \text{tr}(K)}{\partial n} - \text{tr}(K^2), \quad (49)$$

so that

$${}^{(3)}R = {}^{(2)}R + 2 \text{tr}\left(\frac{\partial K}{\partial n}\right) + 3 \text{tr}(K^2) \quad (50)$$

$$= {}^{(2)}R + 2 \frac{\partial \text{tr}(K)}{\partial n} - \text{tr}(K^2). \quad (51)$$

To effect these contractions, we make use of the decomposition of the three-space metric in terms of the throat two-metric and the set of two-vectors e_a^i tangent to the throat and the three-vector n^i normal to the two-surface:

$${}^{(2+1)}g^{ij} = e_a^i e_b^j g^{ab} + n^i n^j. \quad (52)$$

For the three-space Einstein tensor (cf. [6], p. 552), we see

$${}^{(3)}G_{ab} = \frac{\partial K_{ab}}{\partial n} + 2(K^2)_{ab} - g_{ab} \frac{\partial \text{tr}(K)}{\partial n} + \frac{1}{2} g_{ab} \text{tr}(K^2), \quad (53)$$

$${}^{(3)}G_{na} = -K_{ab}{}^{:b}, \quad (54)$$

$${}^{(3)}G_{nn} = -\frac{1}{2} {}^{(2)}R - \frac{1}{2} \text{tr}(K^2). \quad (55)$$

Note in particular that by the flare-out condition ${}^{(3)}R_{nn} \leq 0$. This implies that the three-space Ricci tensor ${}^{(3)}R_{ij}$ has at least one negative semidefinite eigenvalue everywhere on the throat. If we adopt the strong flare-out condition then the three-space Ricci tensor has at least one negative definite eigenvalue somewhere on the throat.

This decomposition now allows us to write down the various components of the spacetime Einstein tensor. For example,

$$\begin{aligned} {}^{(3+1)}G_{ab} &= -\phi_{|ab} - \phi_{|a}\phi_{|b} + g_{ab}g^{kl}[\phi_{|kl} + \phi_{|k}\phi_{|l}] + \frac{\partial K_{ab}}{\partial n} \\ &\quad + 2(K^2)_{ab} - g_{ab} \frac{\partial \text{tr}(K)}{\partial n} + \frac{1}{2} g_{ab} \text{tr}(K^2) \\ &= 8\pi G T_{ab}. \end{aligned} \quad (56)$$

But by the definition of the extrinsic curvature, and using the Gauss-Weingarten equations,

$$\phi_{|ab} = \phi_{:ab} + K_{ab}\phi_{|n}, \quad (57)$$

$$\phi_{|na} = K_a{}^b \phi_{:b}. \quad (58)$$

[See, for example, Eqs. (21.57) and (21.63) of [6].] Thus,

$$g^{kl}\phi_{|kl} = g^{ab}\phi_{:ab} + (g^{ab}K_{ab})\phi_{|n} + \phi_{|nn}. \quad (59)$$

But remember that $\text{tr}(K) = 0$ at the throat, so

$$g^{kl}\phi_{|kl} = g^{ab}\phi_{:ab} + \phi_{|nn}. \quad (60)$$

This finally allows us to write

$$\begin{aligned} {}^{(3+1)}G_{ab} &= -\phi_{:ab} - \phi_{:a}\phi_{:b} - K_{ab}\phi_{|n} + g_{ab}[g^{cd}(\phi_{:cd} \\ &\quad + \phi_{:c}\phi_{:d}) + \phi_{|nn} + \phi_{|n}\phi_{|n}] + \frac{\partial K_{ab}}{\partial n} + 2(K^2)_{ab} \\ &\quad - g_{ab} \frac{\partial \text{tr}(K)}{\partial n} + \frac{1}{2} g_{ab} \text{tr}(K^2) \\ &= 8\pi G T_{ab}. \end{aligned} \quad (61)$$

$${}^{(3+1)}G_{na} = -K_a{}^b \phi_{:b} - \phi_{|n}\phi_{:a} - K_{ab}{}^{:b} = 8\pi G T_{na}. \quad (62)$$

$$\begin{aligned} {}^{(3+1)}G_{nn} &= g^{cd}[\phi_{:cd} + \phi_{:c}\phi_{:d}] - \frac{1}{2} {}^{(2)}R - \frac{1}{2} \text{tr}(K^2) \\ &= -8\pi G \tau. \end{aligned} \quad (63)$$

$${}^{(3+1)}G_{ta} = 0. \quad (64)$$

$${}^{(3+1)}G_{tn} = 0. \quad (65)$$

$${}^{(3+1)}G_{tt} = \frac{{}^{(2)}R}{2} + \frac{\partial \text{tr}(K)}{\partial n} - \frac{1}{2} \text{tr}(K^2) = +8\pi G \rho. \quad (66)$$

Here τ denotes the *tension* perpendicular to the wormhole throat, it is the natural generalization of the quantity considered by Morris and Thorne, while ρ is simply the energy density at the wormhole throat.

V. CONSTRAINTS ON THE STRESS-ENERGY TENSOR

We can now derive several constraints on the stress energy. *First,*

$$\tau = \frac{1}{16\pi G} [{}^{(2)}R + \text{tr}(K^2) - 2g^{cd}(\phi_{:cd} + \phi_{:c}\phi_{:d})]. \quad (67)$$

(Unfortunately, the signs as given are correct. Otherwise we would have a lower bound on τ . We will need to be a little tricky when dealing with the ϕ terms.) The above is the generalization of the Morris-Thorne result that

$$\tau = \frac{1}{8\pi G r_0^2} \quad (68)$$

at the throat of the special class of model wormholes they considered. (With MTW conventions ${}^{(2)}R = 2/r_0^2$ for a two-sphere.) If you now integrate over the surface of the wormhole,

$$\int \sqrt{{}^{(2)}g} \tau d^2x = \frac{1}{16\pi G} \left[4\pi\chi + \int \sqrt{{}^{(2)}g} \{ \text{tr}(K^2) - 2g^{cd}\phi_{:c}\phi_{:d} \} d^2x \right]. \quad (69)$$

Here χ is the Euler characteristic of the throat, while the $g^{cd}\phi_{:c}\phi_{:d}$ term vanishes by partial integration, since the throat is a manifold without boundary.

Second,

$$\rho = \frac{1}{16\pi G} \left[{}^{(2)}R + 2 \frac{\partial \text{tr}(K)}{\partial n} - \text{tr}(K^2) \right]. \quad (70)$$

The second term is negative semidefinite by the flare-out condition, while the third term is manifestly negative semidefinite. Thus,

$$\rho \leq \frac{1}{16\pi G} {}^{(2)}R. \quad (71)$$

This is the generalization of the Morris-Thorne result that

$$\rho = \frac{b'(r_0)}{8\pi G r_0^2} \leq \frac{1}{8\pi G r_0^2} \quad (72)$$

at the throat of the special class of model wormholes they considered. (See [3], p. 107.)

Note in particular that if the wormhole throat does not have the topology of a sphere or torus then there *must* be places on the throat such that ${}^{(2)}R < 0$ and thus such that $\rho < 0$. Thus wormhole throats of a high genus will always have regions that violate the weak and dominant energy conditions. (The simple flare-out condition is sufficient for this result. For a general discussion of the energy conditions see [3] or [7].)

If the wormhole throat has the topology of a torus then it will generically violate the weak and dominant energy conditions; only for the very special case ${}^{(2)}R = 0$, $K_{ab} = 0$, $\partial \text{tr}(K)/\partial n = 0$ will it possibly satisfy (but still be on the verge of violating) the weak and dominant energy conditions. This is a particular example of a degenerate throat in the sense discussed previously.

Wormhole throats with the topology of a sphere will, provided they are convex, at least have a positive energy density, but we shall soon see that other energy conditions are typically violated.

If we now integrate over the surface of the wormhole,

$$\int \sqrt{{}^{(2)}g} \rho d^2x = \frac{1}{16\pi G} \left[4\pi\chi + \int \sqrt{{}^{(2)}g} \left\{ 2 \frac{\partial \text{tr}(K)}{\partial n} - \text{tr}(K^2) \right\} d^2x \right]. \quad (73)$$

So for a throat with the topology of a torus ($\chi = 0$) the simple flare-out condition yields

$$\int \sqrt{{}^{(2)}g} \rho d^2x \leq 0, \quad (74)$$

while the strong or weak flare-out condition yields

guaranteeing violation of the weak and dominant energy conditions. For a throat with higher genus topology ($\chi = 2 - 2g$), the simple flare-out condition is sufficient to yield

$$\int \sqrt{{}^{(2)}g} \rho d^2x \leq \frac{\chi}{4G} < 0. \quad (76)$$

Third,

$$\rho - \tau = \frac{1}{16\pi G} \left[+ 2 \frac{\partial \text{tr}(K)}{\partial n} - 2 \text{tr}(K^2) + 2g^{cd}(\phi_{:c}\phi_{:d}) \right]. \quad (77)$$

Note that the two-curvature ${}^{(2)}R$ has conveniently dropped out of this equation. As given, this result is valid only on the throat itself, but we shall soon see that a generalization can be constructed that will also hold in the region surrounding the throat. The first term is negative semidefinite by the simple flare-out condition (at the very worst when integrated over the throat it is negative by the weak flare-out condition). The second term is negative semidefinite by inspection. The third term integrates to zero though it may have either sign locally on the throat. The fourth term is unfortunately positive semidefinite on the throat which prevents us from deriving a truly general energy condition violation theorem without additional information.

Now because the throat is by definition a compact two-surface, we know that $\phi(x^a)$ must have a maximum somewhere on the throat. At the global maximum (or even at any local maximum) we have $\phi_{:a} = 0$ and $g^{ab}\phi_{:ab} \leq 0$, so at the maxima of ϕ one has

$$\rho - \tau \leq 0. \quad (78)$$

Generically, this inequality will be strict, and generically there will be points on the throat at which the null energy condition is violated.

Integrating over the throat we have

$$\int \sqrt{{}^{(2)}g} [\rho - \tau] d^2x = \frac{1}{16\pi G} \int \sqrt{{}^{(2)}g} \left[+ 2 \frac{\partial \text{tr}(K)}{\partial n} - 2 \text{tr}(K^2) + 2g^{cd}(\phi_{:c}\phi_{:d}) \right] d^2x. \quad (79)$$

Because of the last term we must be satisfied with the result

$$\int \sqrt{{}^{(2)}g} [\rho - \tau] d^2x \leq \int \sqrt{{}^{(2)}g} [2g^{cd}(\phi_{:c}\phi_{:d})] d^2x. \quad (80)$$

Fourth, we can rewrite the difference $\rho - \tau$ as

$$\rho - \tau = \frac{1}{16\pi G} \left[+2 \frac{\partial \text{tr}(K)}{\partial n} - 2 \text{tr}(K^2) + 2 \exp(-\phi) {}^{(2)}\Delta \exp(+\phi) \right]. \quad (81)$$

So if we multiply by $\exp(+\phi)$ before integrating, the two-dimensional Laplacian ${}^{(2)}\Delta$ vanishes by partial integration and we have

$$\int \sqrt{{}^{(2)}g} \exp(+\phi) [\rho - \tau] d^2x = \frac{1}{8\pi G} \int \sqrt{{}^{(2)}g} \exp(+\phi) \left[+ \frac{\partial \text{tr}(K)}{\partial n} - \text{tr}(K^2) \right] d^2x. \quad (82)$$

Thus the strong flare-out condition (or less restrictively, the weak e^ϕ -weighted flare-out condition) implies the violation of this “transverse-averaged null energy condition” (the NEC averaged over the throat):

$$\int \sqrt{{}^{(2)}g} \exp(+\phi) [\rho - \tau] d^2x < 0. \quad (83)$$

Fifth, we can define an average transverse pressure on the throat by

$$\begin{aligned} \bar{p} &\equiv \frac{1}{16\pi G} g^{ab(3+1)} G_{ab} \\ &= \frac{1}{16\pi G} \left[g^{cd} (\phi_{:cd} + \phi_{:c} \phi_{:d}) + 2 \phi_{|nn} + 2 \phi_{|n} \phi_{|n} - \frac{\partial \text{tr}(K)}{\partial n} + \text{tr}(K^2) \right]. \end{aligned} \quad (84)$$

The last term is manifestly positive semidefinite, and the penultimate term is positive semidefinite by the flare-out condition. The first and third terms are of indefinite sign while the second and fourth are also positive semidefinite. Integrating over the surface of the throat,

$$\int \sqrt{{}^{(2)}g} \bar{p} d^2x \geq \frac{1}{8\pi G} \int \sqrt{{}^{(2)}g} \phi_{|nn} d^2x. \quad (86)$$

A slightly different constraint, also derivable from the above, is

$$\int \sqrt{{}^{(2)}g} e^\phi \bar{p} d^2x \geq \frac{1}{8\pi G} \int \sqrt{{}^{(2)}g} (e^\phi)_{|nn} d^2x. \quad (87)$$

These inequalities relate transverse pressures to normal derivatives of the gravitational potential. In particular, if the throat lies at a minimum of the gravitational redshift the second normal derivative will be positive, so the transverse pressure (averaged over the wormhole throat) must be positive.

Sixth, now look at the quantities $\rho - \tau + 2\bar{p}$ and $\rho - \tau - 2\bar{p}$. We have

$$\rho - \tau + 2\bar{p} = \frac{1}{4\pi G} \{ g^{cd} (\phi_{:cd} + \phi_{:c} \phi_{:d}) + \phi_{|nn} + \phi_{|n} \phi_{|n} \} \quad (88)$$

$$= \frac{1}{4\pi G} \{ g^{ij} (\phi_{|ij} + \phi_{|i} \phi_{|j}) \}. \quad (89)$$

This serves as a nice consistency check. The combination of the stress-energy components appearing above is equal to $\rho + g^{ij} T_{ij}$ and is exactly that relevant to the strong energy condition. See Eqs. (15)–(17). See also Eqs. (12)–(14). Multiplying by e^ϕ and integrating,

$$\int \sqrt{{}^{(2)}g} e^\phi [\rho - \tau + 2\bar{p}] d^2x = \frac{1}{4\pi G} \int \sqrt{{}^{(2)}g} (e^\phi)_{|nn} d^2x. \quad (90)$$

This relates this transverse-integrated version of the strong energy condition to the normal derivatives of the gravitational potential.

On the other hand,

$$\rho - \tau - 2\bar{p} = \frac{1}{4\pi G} \left\{ -\phi_{|nn} - \phi_{|n} \phi_{|n} + \frac{\partial \text{tr}(K)}{\partial n} - \text{tr}(K^2) \right\}. \quad (91)$$

The second and fourth terms are negative semidefinite, while the third term is negative semidefinite by the flare-out condition.

Summarizing, there are a number of powerful constraints that can be placed on the stress-energy tensor at the wormhole throat simply by invoking the minimality properties of the wormhole throat. Depending on the precise form of the assumed flare-out condition, these constraints give the various energy condition violation theorems we are seeking. Even under the weakest assumptions (appropriate to a degenerate throat) they constrain the stress energy to at best be on the verge of violating the various energy conditions.

VI. SPECIAL CASE: THE ISOPOTENTIAL THROAT

Suppose we take $\phi_{:a} = 0$. This additional constraint corresponds to asserting that the throat is an *isopotential* of the gravitational redshift. In other words, $\phi(n, x^a)$ is simply a constant on the throat. For instance, all the Morris-Thorne model wormholes [1] possess this symmetry. Under this assumption there are numerous simplifications.

We will not present anew all the results for the Riemann curvature tensor but instead content ourselves with the Einstein tensor

$$\begin{aligned} {}^{(3+1)}G_{ab} &= +g_{ab} (\phi_{|nn} + \phi_{|n} \phi_{|n}) - K_{ab} \phi_{|n} + \frac{\partial K_{ab}}{\partial n} \\ &\quad + 2(K^2)_{ab} - g_{ab} \frac{\partial \text{tr}(K)}{\partial n} + \frac{1}{2} g_{ab} \text{tr}(K^2) \\ &= 8\pi G T_{ab}, \end{aligned} \quad (92)$$

$${}^{(3+1)}G_{na} = -K_{ab}{}^{:b} = 8\pi G T_{na}, \quad (93)$$

$${}^{(3+1)}G_{nn} = -\frac{1}{2} {}^{(2)}R - \frac{1}{2} \text{tr}(K^2) = -8\pi G \tau, \quad (94)$$

$${}^{(3+1)}G_{ta}^{\wedge} = 0, \quad (95)$$

$${}^{(3+1)}G_{tn}^{\wedge} = 0, \quad (96)$$

$${}^{(3+1)}G_{tt}^{\wedge} = \frac{{}^{(2)}R}{2} + \frac{\partial \operatorname{tr}(K)}{\partial n} - \frac{1}{2} \operatorname{tr}(K^2) = +8\pi G\rho. \quad (97)$$

Thus for an isopotential throat,

$$\tau = \frac{1}{16\pi G} [{}^{(2)}R + \operatorname{tr}(K^2)] \geq \frac{1}{16\pi G} {}^{(2)}R. \quad (98)$$

$$\rho = \frac{1}{16\pi G} \left[{}^{(2)}R + 2 \frac{\partial \operatorname{tr}(K)}{\partial n} - \operatorname{tr}(K^2) \right] \leq \frac{1}{16\pi G} {}^{(2)}R, \quad (99)$$

$$\rho - \tau = \frac{1}{16\pi G} \left[+2 \frac{\partial \operatorname{tr}(K)}{\partial n} - 2 \operatorname{tr}(K^2) \right] \leq 0. \quad (100)$$

This gives us a very powerful result: using only the simple flare-out condition, the NEC is on the verge of being violated everywhere on an isopotential throat.

By invoking the strong flare-out condition the NEC is definitely violated somewhere on an isopotential throat.

Invoking the weak flare-out condition we can still say that the surface-integrated NEC is definitely violated on an isopotential throat.

VII. SPECIAL CASE: THE EXTRINSICALLY FLAT THROAT

Suppose now that we take $K_{ab} = 0$. This is a much stronger constraint than simple minimality of the area of the wormhole throat and corresponds to asserting that the three-geometry of the throat is (at least locally) symmetric under interchange of the two regions it connects. For instance, all the Morris-Thorne model wormholes [1] possess this symmetry and have throats that are extrinsically flat. Under this assumption there are also massive simplifications. (Note that we are not making the isopotential assumption at this stage.)

Again, we will not present all the results but content ourselves with the Einstein tensor

$$\begin{aligned} {}^{(3+1)}G_{ab} &= -\phi_{:ab} - \phi_{:a}\phi_{:b} + g_{ab}[g^{cd}(\phi_{:cd} + \phi_{:c}\phi_{:d}) + \phi_{|nn} \\ &\quad + \phi_{|n}\phi_{|n}] + \frac{\partial K_{ab}}{\partial n} - g_{ab} \frac{\partial \operatorname{tr}(K)}{\partial n} \\ &= 8\pi GT_{ab}, \end{aligned} \quad (101)$$

$${}^{(3+1)}G_{na} = -\phi_{|n}\phi_{|a} = 8\pi GT_{na}, \quad (102)$$

$${}^{(3+1)}G_{nn} = g^{cd}[\phi_{:cd} + \phi_{:c}\phi_{:d}] - \frac{1}{2}{}^{(2)}R = -8\pi G\tau, \quad (103)$$

$${}^{(3+1)}G_{ta}^{\wedge} = 0, \quad (104)$$

$${}^{(3+1)}G_{tn}^{\wedge} = 0, \quad (105)$$

$${}^{(3+1)}G_{tt}^{\wedge} = \frac{{}^{(2)}R}{2} + \frac{\partial \operatorname{tr}(K)}{\partial n} = +8\pi G\rho. \quad (106)$$

Though the stress-energy tensor is now somewhat simpler than the general case, the presence of the $\phi_{:a}$ terms precludes the derivation of any truly new general theorems.

VIII. SPECIAL CASE: THE EXTRINSICALLY FLAT ISOPOTENTIAL THROAT

Finally, suppose we take both $K_{ab} = 0$ and $\phi_{:a} = 0$. A wormhole throat that is both extrinsically flat and isopotential is particularly simple to deal with, even though it is still much more general than the Morris-Thorne wormhole. Once again, we will not present all the results but content ourselves with the Einstein tensor

$$\begin{aligned} {}^{(3+1)}G_{ab} &= +g_{ab}(\phi_{|nn} + \phi_{|n}\phi_{|n}) + \frac{\partial K_{ab}}{\partial n} - g_{ab} \frac{\partial \operatorname{tr}(K)}{\partial n} \\ &= 8\pi GT_{ab}, \end{aligned} \quad (107)$$

$${}^{(3+1)}G_{na} = 0, \quad (108)$$

$${}^{(3+1)}G_{nn} = -\frac{1}{2}{}^{(2)}R = -8\pi G\tau, \quad (109)$$

$${}^{(3+1)}G_{ta}^{\wedge} = 0, \quad (110)$$

$${}^{(3+1)}G_{tn}^{\wedge} = 0, \quad (111)$$

$${}^{(3+1)}G_{tt}^{\wedge} = \frac{{}^{(2)}R}{2} + \frac{\partial \operatorname{tr}(K)}{\partial n} = +8\pi G\rho. \quad (112)$$

In this case $\rho - \tau$ is particularly simple:

$$\rho - \tau = \frac{1}{8\pi G} \frac{\partial \operatorname{tr}(K)}{\partial n}. \quad (113)$$

This quantity is manifestly negative semidefinite by the simple flare-out condition.

For the strong flare-out condition we deduce that the NEC must be violated somewhere on the wormhole throat.

Even for the weak flare-out condition we have

$$\int \sqrt{{}^{(2)}g} [\rho - \tau] d^2x < 0. \quad (114)$$

We again see that generic violations of the null energy condition are the rule.

IX. THE REGION SURROUNDING THE THROAT

Because the spacetime is static, one can unambiguously define the energy density everywhere in the spacetime by setting

$$\rho = \frac{{}^{(3+1)}G_{tt}^{\wedge}}{8\pi G}. \quad (115)$$

The normal tension, which we have so far defined only on the wormhole throat itself, can meaningfully be extended to the entire region where the Gaussian normal coordinate system is well defined by setting

$$\tau = - \frac{{}^{(3+1)}G_{nn}}{8\pi G}. \tag{116}$$

Thus, in particular,

$$\rho - \tau = \frac{{}^{(3+1)}G_{\hat{t}\hat{t}} + {}^{(3+1)}G_{nn}}{8\pi G} = \frac{{}^{(3+1)}R_{\hat{t}\hat{t}} + {}^{(3+1)}R_{nn}}{8\pi G}, \tag{117}$$

with this quantity being well defined throughout the Gaussian normal coordinate patch. (The last equality uses the fact that $g_{\hat{t}\hat{t}} = -1$ while $g_{nn} = +1$.) But we have already seen how to evaluate these components of the Ricci tensor. Indeed,

$${}^{(3+1)}R_{\hat{t}\hat{t}} = g^{ij}[\phi_{|ij} + \phi_{|i}\phi_{|j}], \tag{118}$$

$${}^{(3+1)}R_{nn} = {}^{(3)}R_{nn} - [\phi_{|nn} + \phi_{|n}\phi_{|n}] \tag{119}$$

$$= \frac{\partial \operatorname{tr}(K)}{\partial n} - \operatorname{tr}(K^2) - [\phi_{|nn} + \phi_{|n}\phi_{|n}], \tag{120}$$

where we have been careful to *not* use the extremality condition $\operatorname{tr}(K) = 0$. Therefore,

$$\rho - \tau = \frac{1}{8\pi G} \left[\frac{\partial \operatorname{tr}(K)}{\partial n} - \operatorname{tr}(K^2) + g^{ab}(\phi_{|ab} + \phi_{|a}\phi_{|b}) \right] \tag{121}$$

$$= \frac{1}{8\pi G} \left[\frac{\partial \operatorname{tr}(K)}{\partial n} - \operatorname{tr}(K^2) + \operatorname{tr}(K)\phi_{|n} + g^{ab}(\phi_{:ab} + \phi_{:a}\phi_{:b}) \right], \tag{122}$$

where in Eq. (122) we have used the Gauss-Weingarten equations.

If the throat is *isopotential*, where isopotential now means that near the throat the surfaces of constant gravitational potential coincide with the surfaces of fixed n , this simplifies to

$$\rho - \tau = \frac{1}{8\pi G} \left[\frac{\partial \operatorname{tr}(K)}{\partial n} - \operatorname{tr}(K^2) + \operatorname{tr}(K)\phi_{|n} \right]. \tag{123}$$

If the throat is nondegenerate and satisfies the simple flare-out condition, then at the throat the first and second terms are negative semidefinite, and the third is zero. Then the null energy condition is either violated or on the verge of being violated at the throat.

If the throat is nondegenerate and satisfies the strong flare-out condition at the point x , then the first term is negative definite, the second is negative semidefinite, and the third is zero. Then the null energy condition is violated at the point x on the throat.

If the throat satisfies the N -fold degenerate flare-out condition at the point x , then by the generalization of the flare-out conditions applied to degenerate throats the first term will be $O(n^{2N-2})$ and negative definite in some region surrounding the throat. The second term is again negative semidefinite. The third term can have either sign but will be $O(n^{2N-1})$. Thus there will be some region $n \in (0, n_*)$ in which the first term dominates. Therefore, the null energy condition is violated along the line $\{x\}(0, n_*)$. If at every point x on the throat the N -fold degenerate flare-out condition is satisfied for some *finite* N , then there will be an open region surrounding the throat on which the null energy condition is everywhere violated.

This is the closest one can get in generalizing to arbitrary wormhole shapes the discussion on p. 405 [Eq. (56)] of Morris-Thorne [1]. Note carefully their use of the phrase ‘‘at or near the throat.’’ In our parlance, they are considering a spherically symmetric extrinsically flat isopotential throat that satisfies the N -fold degenerate flare-out condition for some finite but unspecified N . See also p. 104, Eq. (11.12) and p. 109, Eq. (11.54) of [3], and contrast this with Eq. (11.56) of [3].

If the throat is not isopotential we multiply by $\exp(\phi)$ and integrate over surfaces of constant n . Then,

$$\begin{aligned} & \int \sqrt{{}^{(2)}g} \exp(\phi) [\rho - \tau] d^2x \\ &= \frac{1}{8\pi G} \int \sqrt{{}^{(2)}g} \exp(\phi) \left[\frac{\partial \operatorname{tr}(K)}{\partial n} - \operatorname{tr}(K^2) + \operatorname{tr}(K)\phi_{|n} \right] d^2x. \end{aligned} \tag{124}$$

This generalizes the previous version (82) of the transverse-averaged null energy condition to constant n hypersurfaces near the throat. For each point x on the throat, assuming the N -fold degenerate flare-out condition, we can by the previous argument find a range of values $[n \in (0, n_*(x))]$ that will make the integrand negative. Thus there will be a set of values of n for which the integral is negative. Again we deduce violations of the null energy condition.

X. DISCUSSION

We have presented a definition of a wormhole throat that is much more general than that of the Morris-Thorne wormhole [1]. The present definition works well in any static spacetime and nicely captures the essence of the idea of what we would want to call a wormhole throat.

We do not need to make any assumptions about the existence of any asymptotically flat region, nor do we need to assume that the manifold is topologically nontrivial. It is important to realize that the essence of the definition lies in the geometrical structure of the wormhole throat.

Starting from our definition we have used the theory of embedded hypersurfaces to place restrictions on the Riemann tensor and stress-energy tensor at the throat of the wormhole. We find, as expected, that the wormhole throat generically violates the null energy condition and we have provided several theorems regarding this matter. These theorems generalize the Morris-Thorne results on exotic matter [1], and are complementary to the topological censorship theorem [5].

Generalization to the time-dependent situation is clearly of interest. Unfortunately, we have encountered many subtleties of definition, notation, and formalism in this endeavor.

We defer the issue of time-dependent wormhole throats to a future publication.

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