How to produce exotic matter using classical fields

Dan N. Vollick
Department of Physics, Okanagan University College, 3333 College Way, Kelowna, British Columbia, Canada V1V 1V7
(Received 19 November 1996)

The field equations and energy-momentum tensor are derived for a charged ideal fluid coupled to a scalar field. It is shown that there exists an interaction energy-momentum tensor which can have negative energy densities even though the fluid and fields have positive energy densities. The energy density and mass of a fluid sphere are calculated in the weak field limit and it is shown that they can be negative if certain conditions are met. When the mass of the sphere is negative the entire interior consists of exotic matter (i.e., matter which violates the weak energy condition). [S0556-2821(97)07620-0]

PACS number(s): 04.40.Nr

INTRODUCTION

Recent work on wormholes [1–3] and the ''warp drive'' [4] has generated interest in matter that violates the weak energy condition, which states that $T_{\mu\nu}U^{\mu}U^{\nu}\geqslant 0$ for all timelike vectors U^{μ}. For such ''exotic matter'' there will be observers who see a negative energy density. The throat of a wormhole must be threaded by exotic matter to stay open [5]. For the warp drive exotic matter is needed to produce the contraction of space in front of the ship and the expansion of space behind it.

Most discussions of exotic matter involve quantum field theory effects, such as the Casimir effect [6]. Here I show that it is possible to generate exotic matter classically. This involves coupling matter to a scalar field in a very simple fashion. In addition to the usual matter and field parts, the energy-momentum tensor also contains an interaction part. This interaction part can have negative energy densities even though the matter and field parts have positive energy densities. To show that exotic matter can be created I consider a sphere composed of charged dust. The charge on the particles is required in order to produce a repulsive force which balances the attractive force produced by the scalar field. I show that it is possible to create a sphere which consists entirely of exotic matter and has a negative gravitational (and inertial) mass.

EQUATIONS OF MOTION AND THE ENERGY-MOMENTUM TENSOR

Consider a collection of charged timelike particles interacting with a scalar field ϕ. The action will be taken to be [7]

$$S=-\sum_n m_n\int\sqrt{-g_{\mu\nu}U_n^{\mu}U_n^{\nu}}d\tau_n+\frac{1}{2}\sum_n\int\lambda_n(\tau_n)[g_{\mu\nu}U_n^{\mu}U_n^{\nu}+1]d\tau_n-\sum_n\alpha_n\int\phi(x_n(\tau_n))d\tau_n \tag{1}$$

$$+\sum_n e_n\int A_{\mu}(x_n(\tau_n))U_n^{\mu}d\tau_n-\frac{1}{2}\int\nabla^{\mu}\phi\nabla_{\mu}\phi\sqrt{g}\,d^4x-\frac{1}{16\pi}\int F^{\mu\nu}F_{\mu\nu}\sqrt{g}\,d^4x, \tag{2}$$

where $x_n^{\mu}(\tau_n)$ and U_n^{μ} are the position and four-velocity of the nth particle, τ_n is the proper time along its world line, m_n and e_n are its rest mass and charge, $\lambda_n(\tau_n)$ are Lagrange multipliers, the α_n are coupling constants, A_{μ} is the vector potential, and $F_{\mu\nu}=\nabla_{\mu}A_{\nu}-\nabla_{\nu}A_{\mu}$.

The scalar field equations are found by varying the action with respect to $\phi(x)$ and are given by

$$\Box^2\phi=\frac{1}{\sqrt{g}}\sum_n\alpha_n\int\delta^4(x^{\mu}-x_n^{\mu}(\tau_n))d\tau_n, \tag{3}$$

where $\Box^2=\nabla^{\mu}\nabla_{\mu}$. The field equations for the electromagnetic field are found by varying the action with respect to A_{μ} and are given by

$$\nabla_{\mu}F^{\mu\nu}=-\frac{4\pi}{\sqrt{g}}\sum_n e_n\int\delta^4(x^{\mu}-x^{\mu}(\tau_n))U^{\nu}(\tau_n)d\tau_n. \tag{4}$$

The equations of motion for the particles are found by varying the action with respect to $x_n^{\mu}(\tau_n)$ and are given by

$$(m_n+\lambda_n)\left(\frac{dU_n^{\mu}}{d\tau_n}+\Gamma^{\mu}_{\alpha\beta}U_n^{\alpha}U_n^{\beta}\right)+\frac{d\lambda_n}{d\tau_n}U_n^{\mu}=-\alpha_n\nabla^{\mu}\phi+e_nF^{\mu}{}_{\nu}U_n^{\nu}. \tag{5}$$

Contracting with U_n^{μ} gives

$$\frac{d\lambda_n}{d\tau_n} = \alpha_n \frac{d\phi}{d\tau_n}. \tag{6}$$

Thus

$$\lambda_n = \alpha_n \phi. \tag{7}$$

The equations of motion for the particles are then

$$\frac{d}{d\tau_n}[(m_n+\alpha_n\phi)U_n^\mu]+(m_n+\alpha_n\phi)\Gamma^\mu_{\alpha\beta}U_n^\alpha U_n^\beta = -\alpha_n\nabla^\mu\phi+e_nF^\mu{}_\nu U_n^\nu. \tag{8}$$

The energy-momentum tensor of the field and particles is given by

$$T^{\mu\nu}=\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}}. \tag{9}$$

From Eq. (1) we have

$$T^{\mu\nu}=\sum_n \frac{1}{\sqrt{g}}\int (m_n+\alpha_n\phi)U_n^\mu U_n^\nu \delta^4(x-x_n(\tau_n))d\tau_n$$

$$+\nabla^\mu\phi\nabla^\nu\phi-\frac{1}{2}g^{\mu\nu}\nabla_\alpha\phi\nabla^\alpha\phi \tag{10}$$

$$+\frac{1}{4\pi}\left(F^{\alpha\mu}F_\alpha{}^\nu-\frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right). \tag{11}$$

There is therefore an interaction energy-momentum tensor given by

$$T^{\mu\nu}_{(I)}=\sum_n \frac{\alpha_n}{\sqrt{g}}\int \phi(x_n)U_n^\mu U_n^\nu \delta^4(x-x_n(\tau_n))d\tau_n. \tag{12}$$

This interaction energy-momentum tensor is necessary if

$$\nabla_\mu T^{\mu\nu}=0 \tag{13}$$

is to give the correct equations of motion for the particles.

Now consider a collection of identical particles. From Eqs. (3) and (11) it can be seen that it is the trace of the particle energy-momentum tensor which acts as the source of the scalar field. In the continuum limit Eqs. (3) and (11) become

$$\Box^2\phi=-\alpha^* T_f \tag{14}$$

and

$$T^{\mu\nu}=(1+\alpha^*\phi)[(\rho_m+P)U^\mu U^\nu+Pg^{\mu\nu}]+\nabla^\mu\phi\nabla^\nu\phi -\frac{1}{2}g^{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi+\frac{1}{4\pi}\left(F^{\alpha\mu}F_\alpha{}^\nu-\frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right), \tag{15}$$

where

$$T_f=3P-\rho_m \tag{16}$$

is the trace of the fluid energy-momentum tensor, ρ_m is the rest mass density, $\alpha^*=\alpha/m$, and P is the pressure. In the continuum limit Eq. (4) becomes

$$\nabla_\mu F^{\mu\nu}=-4\pi\rho_c U^\nu, \tag{17}$$

where ρ_c is the charge density.

WEAK FIELD SOLUTIONS FOR A SPHERE OF CHARGED DUST

In this section the energy density and the mass of a static sphere composed of charged dust will be found, in the weak field limit. Both the mass density and the charge density will be taken to be uniform throughout the sphere. The metric will be taken to be of the form

$$ds^2=-e^\nu dt^2+e^\lambda dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2), \tag{18}$$

where ν and λ are functions of r only.

First consider the field equations for the sources of the gravitational field. In the weak field limit they must satisfy their flat space-time equations. Thus the scalar field equation (14) becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)=\alpha^*\rho_m. \tag{19}$$

The solution in the exterior region, subject to the boundary condition $\phi\to 0$ as $r\to\infty$, is

$$\phi_e=-\frac{\alpha^* R^3}{3r}\rho_m, \tag{20}$$

where R is the radius of the sphere. The interior solution that matches the exterior solution at $r=R$ is

$$\phi_i=-\frac{1}{2}\alpha^*\rho_m\left(R^2-\frac{1}{3}r^2\right). \tag{21}$$

In the weak field limit the electromagnetic field equations reduce to

$$\vec\nabla\cdot\vec E=4\pi\rho_c. \tag{22}$$

The exterior solution is

$$\vec E_e=\frac{4\pi R^3\rho_c}{3r^2}\hat r \tag{23}$$

and the interior solution is

$$\vec E_i=\frac{4\pi\rho_c r}{3}\hat r. \tag{24}$$

The scalar field produces attractive forces between the volume elements of the sphere which must be compensated by repulsive electrostatic forces. Thus there must exist a relation between ρ_c and ρ_m. This relationship can be found from $\partial_\mu T^{\mu\nu}=0$. Considering the spatial components of this equation with $U^\mu=(1,\vec 0)$ gives

$$\rho_c\vec E-\alpha^*\rho_m\vec\nabla\phi=0. \tag{25}$$

Taking the divergence of the above gives

$$\rho_c = \pm \frac{\alpha^*}{\sqrt{4\pi}} \rho_m . \tag{26}$$

This is the relationship between ρ_c and ρ_m which is necessary to keep the sphere in equilibrium.

The only nonzero component of the energy-momentum tensor in the interior is T^{00}, which is given by

$$T^{00} = (1+\alpha^*\phi)\rho_m + \frac{1}{8\pi}E^2 + \frac{1}{2}\phi'^2. \tag{27}$$

Substituting Eqs. (21) and (24) into the above gives

$$T^{00} = \rho_m\left[1 - \frac{1}{2}\alpha^{*2}\rho_m\left(R^2 - \frac{5}{9}r^2\right)\right] \tag{28}$$

in the interior of the sphere. If the sphere is to contain exotic matter it is therefore necessary that

$$\alpha^{*2}\rho_m R^2 > 2. \tag{29}$$

For the sphere to be composed entirely of exotic matter it is necessary that

$$\alpha^{*2}\rho_m R^2 > \frac{9}{2}. \tag{30}$$

Outside the sphere the energy density is positive. To see if the total gravitational mass of the sphere is negative it is sufficient to examine the asymptotic structure of the space-time. In the weak field limit the Einstein field equations

$$G_{\mu\nu} = -8\pi G T_{\mu\nu} \tag{31}$$

give

$$\frac{d}{dr}(r\lambda) = 8\pi G r^2\left[(1+\alpha^*\phi)\rho_m + \frac{1}{2}\phi'^2 + \frac{1}{8\pi}E^2\right], \tag{32}$$

$$r\nu' - \lambda = 4\pi G r^2\left[\phi'^2 - \frac{1}{4\pi}E^2\right], \tag{33}$$

and

$$\nu'' + \frac{1}{r}(\nu' - \lambda') = 8\pi G\left[\frac{1}{4\pi}E^2 - \phi'^2\right]. \tag{34}$$

In the exterior region the solution for λ which satisfies the boundary condition $\lambda_e \to 2GM/r$ as $r\to\infty$ is

$$\lambda_e = \frac{2GM}{r} - \frac{8\pi G\alpha^{*2}R^6\rho_m^2}{9r^2}. \tag{35}$$

In the interior region

$$\lambda_i = \frac{8\pi G}{r}\int_0^r r^2\left[(1+\alpha^*\phi)\rho_m + \frac{1}{8\pi}E^2 + \frac{1}{2}\phi'^2\right]dr. \tag{36}$$

The continuity of λ across $r=R$ gives

$$M = 4\pi\int_0^R r^2\left[(1+\alpha^*\phi)\rho_m + \frac{1}{8\pi}E^2 + \frac{1}{2}\phi'^2\right]dr + \frac{4\pi\alpha^{*2}R^5\rho_m^2}{9}. \tag{37}$$

Substituting Eqs. (21) and (24) into the above and integrating gives

$$M = \frac{4\pi}{3}\rho_m R^3\left[1 - \frac{1}{6}\alpha^{*2}\rho_m R^2\right]. \tag{38}$$

Thus the gravitational mass of the sphere will be negative if

$$\alpha^{*2}\rho_m R^2 > 6. \tag{39}$$

The inertial mass is given by $4\pi\int_0^\infty T^{00}r^2dr$, and is equal to the gravitational mass, as expected. Thus for $\alpha^{*2}\rho_m R^2 > 6$ both the gravitational and inertial masses are negative. Note that when the mass of the sphere is negative the entire interior consists of exotic matter.

For the weak field approximation to be valid $|\lambda| \ll 1$ everywhere. The maximum value of $|\lambda|$ is of order

$$|\lambda(R)| = \frac{\pi G}{\alpha^{*2}}\eta^2 \tag{40}$$

for $\alpha^{*2}\rho_m R^2 = \eta > 2$. Thus if exotic matter is present α^* must satisfy

$$\alpha^{*2} \gg G \tag{41}$$

for the weak field limit to be valid. A similar result also holds for ν, which appears in the metric (18). Thus, to produce exotic matter and to remain in the weak field limit it is necessary to have the scalar field coupled more strongly to matter than gravity. This result follows intuitively from the requirement that $\alpha^*\phi < -1$ for exotic matter to exist and from the requirement that $|h_{\mu\nu}| \ll 1$ for the weak field limit to be valid, where $h_{\mu\nu}$ is the perturbation to the Minkowski metric. From this argument one also expects that $\alpha^* \gtrsim G$ will be necessary to produce exotic matter even in moderately strong gravitational fields. Since there are no known long range scalar fields stronger than gravity the scalar field cannot couple to the particle's mass. The scalar field must couple to some ''scalar charge'' whose net value is near zero for common macroscopic objects. This could be accomplished if the main constituents of matter (i.e., protons, neutrons, and electrons) do not couple to the scalar field or if they have opposite scalar charges which cancel for macroscopic objects.

CONCLUSION

The field equations and the energy-momentum tensor for a charged ideal fluid coupled to a scalar field were derived. The energy-momentum tensor for the system contains an interaction part which can have negative energy densities even though the fluid and fields have positive energy densities.

The mass and energy density for a sphere composed of charged dust was calculated. It was shown that the energy density would be negative throughout the interior of the sphere if

$$\alpha^{*2}\rho_m R^2 > \frac{9}{2} \tag{42}$$

and that the mass would be negative if

$$\alpha^{*2}\rho_m R^2 > 6. \tag{43}$$

It was also shown that it is necessary that $\alpha^{*2} \geqslant G$ to produce exotic matter and remain within the weak field limit. Even for moderately strong gravitational fields it is necessary that $\alpha^{*2} \gtrsim G$. Since no such scalar field is known it must couple to some ‘‘scalar charge’’ whose net value is near zero for common macroscopic objects. This could be accomplished if the main constituents of matter (i.e., protons, neutrons, and electrons) do not couple to the field or if they have opposite scalar charges which cancel for macroscopic objects.

[1] M. S. Morris and K. S. Thorne, Am. J. Phys. **56**, 395 (1988).
[2] M. S. Morris, K. S. Thorne, and U. Yurtsever, Phys. Rev. Lett. **61**, 1446 (1988).
[3] V. P. Frolov and I. D. Novikov, Phys. Rev. D **42**, 1057 (1990).
[4] M. Alcubierre, Class. Quantum Grav. **11**, L73 (1994).
[5] Technically, the energy-momentum tensor maintaining the wormhole must violate the average weak energy condition. Thus there must be null geodesics passing through the wormhole, with tangent vectors $k^\alpha = dx^\alpha/d\sigma$, which have $\int_0^\infty T_{\alpha\beta}k^\alpha k^\beta d\sigma < 0$.
[6] C. Itzykson and J. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980), see Sec. 3-2-4.
[7] D. N. Vollick, Phys. Rev. D **52**, 3576 (1995).