Hadamard renormalization, conformal anomaly, and cosmological event horizons

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The Hadamard renormalization prescription is used to derive a two-dimensional analog of the renormalized stress tensor for a minimally coupled scalar field in Schwarzschild–de Sitter space-time. In the two-dimensional analog the minimal coupling reduces to the conformal coupling and the stress tensor is found to be determined by the (nonlocal) contribution of the anomalous trace and some additional parameters in close relation to the work by Christensen and Fulling. To properly relate the stress tensor to the state of outwards signals coming from the direction of the black hole at late times we propose a cutoff hypothesis that excludes the contribution of the anomalous trace close to the black hole horizon. The corresponding cutoff scale is found to be related to the equilibrium temperature of the cosmological horizon in a leading order estimate. Finally, we establish a relation between the radiation temperature of the black hole horizon at large distance from the hole and the anomalous trace and determine the correction term to the Hawking temperature due to the presence of the cosmological horizon. [S0556-2821(97)05918-3]

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I. INTRODUCTION

A central problem in quantum field theory in curved space-time is the computation of the renormalized expectation value of the stress tensor operator [2]. Usually one is inclined to expect that the stress tensor at some point in a curved space-time can be measured by a well-defined local operator. However, the usual expression for the stress tensor operator involves singular products of the field operator at the same space-time point, and it seems clear that such singular products do not allow the definition of a well-defined local operator. Renormalization theory of the stress tensor was originally designed to solve this problem. But, it must be remarked that the usual scheme of renormalization involves complicated, often ambiguous, steps and it is by no means apparent that the resulting final expressions actually correspond to the expectation value of a well-defined local operator acting on the Hilbert space of states. In principle, one should recognize that there is no conceptual support for a local measure of energy-momentum of some given state without reference to any global construct. In fact, even in Minkowski space energy-momentum is measured relative to a global construct, namely the Minkowski vacuum. We emphasize that the conceptual basis of the renormalization theory, as it is currently understood, is still ill defined.

Despite these difficulties, the usual renormalization prescriptions have some power, in that they satisfy some general requirements, such as the covariant conservation law, and in the case of the conformal invariant coupling the general requirement on the anomalous trace. It is just for this reason that the study of the usual renormalization prescriptions can still be justified. In this article we first clarify this aspect in connection with the Hadamard renormalization prescription developed in [3–5] (see also [6]). We then apply the results to the two-dimensional analog of Schwarzschild–de Sitter space-time and derive the leading order approximation of equilibrium temperature and radiation temperature associated with the cosmological event horizon and the black hole horizon, respectively. In our presentation the equilibrium temperature of the cosmological event horizon basically emerges from a physical cutoff which close to the black hole horizon excludes the contribution of the anomalous trace to radiation part of the stress tensor. In essence, such an approach has a certain similarity to the recent publications [7–9] in which dissatisfactions were expressed with the use of the singular modes escaping from a region close to the black hole horizon in the derivation of the Hawking effect. We believe that our presentation, although it will not use the notion of mode to generate the Hawking effect, is instructive because it suggests that in a future theory the physical cutoff for a black hole may have an intimate connection to the presence of an associated cosmological horizon.

II. HADAMARD RENORMALIZATION

We consider a linear scalar quantum field ϕ propagating on a curved space-time with the action of the standard form [2]

$$S[\phi]=-\frac{1}{2}\int d^4x g^{1/2}(g_{\alpha\beta}\phi^{;\alpha}\phi^{;\beta}+\xi R\phi^2+m^2\phi^2), \quad (1)$$

where m and ξ are parameters, and R is the scalar curvature. (In the following the semicolon and ∇ indicates covariant differentiation.) The corresponding field equation is

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$$(\Box - m^2 - \xi R)\phi(x) = 0. \tag{2}$$

The choice of the parameter m and ξ depends on the particular type of coupling we wish to consider. For example, the minimal coupling corresponds to $(m=0,\xi=0)$ and the conformal coupling in four dimensions corresponds to $(m=0,\xi=\frac{1}{6})$.

The energy-momentum of ϕ is defined by the singular expression

$$T^{\mu\nu}(x) = (1-2\xi)\nabla^{\mu}\phi\nabla^{\nu}\phi + \left(2\xi - \frac{1}{2}\right)g^{\mu\nu}\nabla_{\beta}\phi\nabla^{\beta}\phi + \xi\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\phi^2 + 2\xi\phi(g^{\mu\nu}\Box\phi - \nabla^{\mu}\nabla^{\nu}\phi) - \frac{1}{2}m^2 g^{\mu\nu}\phi^2. \tag{3}$$

We shall deal with a particularly useful version of Eq. (3) in terms of anticommutator, namely

$$T^{\mu\nu}(x) = \frac{1}{2}(1-2\xi)\{\nabla^{\mu}\phi,\nabla^{\nu}\phi\} + \left(\xi - \frac{1}{4}\right)g^{\mu\nu}\{\nabla^{\beta}\phi,\nabla_{\beta}\phi\} + \frac{1}{2}\xi\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\{\phi,\phi\} + \xi g^{\mu\nu}\{\phi,\Box\phi\} - \xi\{\phi,\nabla^{\mu}\nabla^{\nu}\phi\} - \frac{1}{4}m^2 g^{\mu\nu}\{\phi,\phi\}. \tag{4}$$

A state of ϕ is characterized by a hierarchy of Wightman functions

$$\langle \phi(x_1), \ldots, \phi(x_n)\rangle. \tag{5}$$

The ''operator'' $T^{\mu\nu}$ takes a singular expectation value $\langle T^{\mu\nu}\rangle$ in a given state. Using the point-splitting method [10], this singular expectation value can most conveniently be represented by

$$\langle T^{\mu\nu}\rangle = \lim_{x'\to x} D^{\mu\nu}(x,x')\{G^{+}(x,x')\}. \tag{6}$$

Here $G^{+}(x,x')$ is the symmetric two-point function, $D^{\mu\nu}(x,x')$ is the bilocal differential operator

$$D^{\mu\nu}(x,x') = \left(\frac{1}{2} - \xi\right)\{g^{\mu}_{\mu'}\nabla^{\mu'}\nabla^{\nu} + g^{\nu}_{\nu'}\nabla^{\mu}\nabla^{\nu'}\} + \left(2\xi - \frac{1}{2}\right)g^{\mu\nu}g^{\beta}_{\beta'}\nabla_{\beta}\nabla^{\beta'} + \xi\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) + \xi g^{\mu\nu}\{\Box + \Box'\} - \xi\{\nabla^{\mu}\nabla^{\nu} + g^{\mu}_{\mu'}g^{\nu}_{\nu'}\nabla^{\mu'}\nabla^{\nu'}\} - \frac{1}{2}m^2 g^{\mu\nu} \tag{7}$$

and $g^{\beta}_{\beta'}$ is the bivector of parallel transport. This expression makes explicit that the singular character of the operator $T^{\mu\nu}$ emerges as a consequence of the short-distance singularity of the symmetric two-point function $G^{+}(x,x')$. This function satisfies Eq. (2) in each argument.

We remark that for a linear theory the antisymmetric part of the two-point function is common to all states in the same representation. It is just the universal commutator function. Thus, in our case all the relevant informations about the state-dependent part of the two-point function are encoded in $G^{+}(x,x')$. Equivalence principle suggests that the leading singularity of $G^{+}(x,x')$ should have a close correspondence to singularity structure of the two-point function of a free massless field in Minkowski space [11]. In general the entire singularity of $G^{+}(x,x')$ may have a more complicated structure. Usually one assumes that $G^{+}(x,x')$ has a singular structure represented by the Hadamard expansions. This means that in a normal neighborhood of a point x the function $G^{+}(x,x')$ can be written

$$G^{+}(x,x') = \frac{1}{8\pi^2}\left\{\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')} + V(x,x')\ln\sigma(x,x') + W(x,x')\right\}, \tag{8}$$

where $2\sigma(x,x')$ is the square of the distance along the geodesic joining x and x' and $\Delta(x,x')$ is the van Vleck determinant

$$\Delta(x,x') = -g^{-1/2}(x)\mathrm{Det}\{-\sigma_{;\mu\nu'}\}g^{-1/2}(x'),\qquad g(x) = \mathrm{Det}g_{\alpha\beta}. \tag{9}$$

The functions $V(x,x')$ and $W(x,x')$ have the following representations as power series:

$$V(x,x') = \sum_{n=0}^{+\infty} V_n(x,x')\sigma^n, \tag{10}$$

$$W(x,x') = \sum_{n=0}^{+\infty} W_n(x,x')\sigma^n, \tag{11}$$

in which the coefficients are determined by applying Eq. (2) to $G^{+}(x,x')$, yielding the recursion relations

$$(n+1)(n+2)V_{n+1} + (n+1)V_{n+1;\alpha}\sigma^{;\alpha} - (n+1)V_{n+1}\Delta^{-1/2}\Delta^{1/2}_{;\alpha}\sigma^{;\alpha} + \frac{1}{2}(\Box - m^2 - \xi R)V_n = 0, \tag{12}$$

$$(n+1)(n+2)W_{n+1} + (n+1)W_{n+1;\alpha}\sigma^{;\alpha} - (n+1)W_{n+1}\Delta^{-1/2}\Delta^{1/2}_{;\alpha}\sigma^{;\alpha} + \frac{1}{2}(\Box - m^2 - \xi R)W_n + (2n+3)V_{n+1} + V_{n+1;\alpha}\sigma^{;\alpha} - V_{n+1}\Delta^{-1/2}\Delta^{1/2}_{;\alpha}\sigma^{;\alpha} = 0 \tag{13}$$

together with the boundary condition

$$V_0 + V_{0;\alpha}\sigma^{;\alpha} - V_0\Delta^{-1/2}\Delta^{1/2}_{;\alpha}\sigma^{;\alpha} + \frac{1}{2}(\Box - m^2 - \xi R)\Delta^{1/2} = 0. \tag{14}$$

From these relations one can determine the function $V(x,x')$ uniquely in terms of local geometry. It takes therefore the same universal form for all states. But the biscalar $W_0(x,x')$ remains arbitrary. Its specification depends significantly on the choice of a state.

Let us now explain the standard Hadamard renormalization prescription. The basic strategy is, in the first place, to extract the finite part of $G^+(x,x')$ by subtracting from $G^+(x,x')$ a local symmetric two-point function $G_L^+(x,x')$ with the same short-distance singularity of the Hadamard expansion and, in the second place, to define the renormalized expectation value of the stress tensor as

$$\langle T^{\mu\nu}\rangle_{\rm ren}=\lim_{x'\to x} D^{\mu\nu}(x,x')\{G^+(x,x')-G_L^+(x,x')\}. \tag{15}$$

The result is apparently finite. But there is a fundamental ambiguity concerning the choice of $G_L^+(x,x')$. As a general criterion one reasonably assumes that $G_L^+(x,x')$ is a function of local geometry.

This criterion does not eliminate the ambiguity concerning the choice of the function $G_L^+(x,x')$, but the renormalization theory replaces this ambiguity by another one, namely the freedom to add to $\langle T^{\mu\nu}\rangle_{\rm ren}$ a state-independent conserved tensor. We explain the underlying reasoning. Using the definitions (7), (8), and (15), one can write the decomposition

$$\langle T^{\mu\nu}\rangle_{\rm ren}=\lim_{x'\to x} D^{\mu\nu}(x,x')\{(8\pi^2)^{-1}W(x,x')\}+\Sigma^{\mu\nu} \tag{16}$$

in which the first term on the right-hand side represents the finite state-dependent contribution of the function $W(x,x')$ in the Hadamard expansion of $G^+(x,x')$, and the second term is imagined to incorporate the finite state-independent contribution of $G^+(x,x')$ together with the finite state-independent contribution of $G_L^+(x,x')$. Now the point is that the conservation law determines the tensor $\Sigma^{\mu\nu}$ up to a divergenceless state-independent tensor. Thus the ambiguity concerning the choice of $G_L^+(x,x')$ yields the freedom to add to the tensor $\Sigma^{\mu\nu}$ a conserved state-independent tensor.

The decomposition (16) is, however, incomplete without specifying the nature of the tensor $\Sigma^{\mu\nu}$. In the renormalization theory one uses a decomposition in which the tensor $\Sigma^{\mu\nu}$ comes out to be divergenceless. To find the corresponding decomposition we apply the conservation law to $\langle T^{\mu\nu}\rangle_{\rm ren}$ and find for the divergence of $\Sigma^{\mu\nu}$ the expression

$$\nabla_\mu\Sigma^{\mu\nu}=-\nabla_\mu\Gamma^{\mu\nu}[W], \tag{17}$$

where

$$\Gamma^{\mu\nu}[W]=\lim_{x'\to x} D^{\mu\nu}(x,x')\{(8\pi^2)^{-1}W(x,x')\}. \tag{18}$$

Now expanding $W(x,x')$ into a covariant power series [6,10,12], namely

$$W(x,x')=W(x)-\frac{1}{2}W_{;\alpha}(x)\sigma^\alpha+\frac{1}{2}W_{\alpha\beta}(x)\sigma^\alpha\sigma^\beta +\frac{1}{4}\left\{\frac{1}{6}W_{;\alpha\beta\gamma}-W_{\alpha\beta;\gamma}\right\}\sigma^\alpha\sigma^\beta\sigma^\gamma+O(\sigma^2), \tag{19}$$

the tensor $\Gamma^{\mu\nu}[W]$ can be calculated to yield

$$\Gamma^{\mu\nu}[W]=(8\pi^2)^{-1}\left\{\frac{1}{2}(1-2\xi)W^{;\mu\nu}(x) +\frac{1}{2}\left(2\xi-\frac{1}{2}\right)\Box W(x)g^{\mu\nu}+\xi\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R\right) \times W(x)-\frac{1}{2}m^2g^{\mu\nu}W(x)-\left(W^{\mu\nu}-\frac{1}{2}g^{\mu\nu}W^\gamma_\gamma\right)\right\}. \tag{20}$$

In this expression the scalar $W(x)$ is arbitrary, but once it has been chosen the choice of the tensor $W_{\mu\nu}(x)$ must be subjected to the constraint

$$\nabla_\mu\left[W^{\mu\nu}-\frac{1}{2}g^{\mu\nu}\left(W^\gamma_\gamma-m^2W+\frac{1}{2}\Box W\right)\right] =2v_1^{;\nu}+\frac{1}{2}R^{\mu\nu}W_{;\mu}-\frac{1}{2}\xi R W^{;\nu} \tag{21}$$

which, using the differential identity,

$$\Box(\nabla^\nu)W=\nabla^\nu\Box W+R^{\mu\nu}\nabla_\mu W, \tag{22}$$

follows from the symmetry property of the biscalar $W(x,x')$ together with (see [6,12])

$$(\Box-m^2-\xi R)W(x,x')=-6v_1(x)+2v_{1;\alpha}\sigma^{;\alpha}+O(\sigma). \tag{23}$$

From the constraint (21) one finds

$$\nabla_\mu\Sigma^{\mu\nu}=2(8\pi^2)^{-1}\nabla_\mu g^{\mu\nu}v_1(x), \qquad \lim_{x'\to x} V_1(x,x')=v_1(x). \tag{24}$$

This has the implication that the incorporation of a compensating term proportional to $g^{\mu\nu}v_1(x)$ into the tensor $\Gamma^{\mu\nu}[W]$ will make $\Sigma^{\mu\nu}$ divergenceless. The corresponding decomposition used by the renormalization theory is

$$\langle T^{\mu\nu}\rangle_{\rm ren}=\overline{\Gamma^{\mu\nu}}[W]+\Sigma^{\mu\nu}, \tag{25}$$

where

$$\overline{\Gamma^{\mu\nu}}[W]=\Gamma^{\mu\nu}[W]+2(8\pi^2)^{-1}g^{\mu\nu}v_1(x). \tag{26}$$

The merits of such a decomposition is that each term becomes now divergenceless.

For the calculations, the tensor $\overline{\Gamma^{\mu\nu}}[W]$ is very important because, first, it provides a conserved tensor which contains all the relevant informations about the state-dependent part of $\langle T^{\mu\nu}\rangle_{\rm ren}$ and, second, in the case of conformal coupling it produces the usual restriction imposed on the anomalous trace of $\langle T^{\mu\nu}\rangle_{\rm ren}$ [1,13]. In the following we shall exclusively deal with the tensor $\overline{\Gamma^{\mu\nu}}[W]$. From Eqs. (20) and (26) one gets its explicit expression as

$$\overline{\Gamma^{\mu\nu}}[W]=(8\pi^2)^{-1}\left\{\frac{1}{2}(1-2\xi)W^{;\mu\nu}(x)+\frac{1}{2}\left(2\xi-\frac{1}{2}\right)\Box W(x)g^{\mu\nu}+\xi\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R\right)W(x)-\frac{1}{2}m^2g^{\mu\nu}W(x)-\left(W^{\mu\nu}-\frac{1}{2}g^{\mu\nu}W^{\gamma}_{\gamma}\right)+2g^{\mu\nu}v_1(x)\right\}. \tag{27}$$

III. THE APPROXIMATE STRESS TENSOR IN THE PRESENCE OF THE COSMOLOGICAL CONSTANT

We study now the case of minimal coupling $\xi=m=0$, and proceed to find the approximate form of the tensor $\overline{\Gamma^{\mu\nu}}[W]$ in a space-time with a metric given by a vacuum solution of Einstein's equations in the presence of a small cosmological constant Λ. The metric arises as a solution of the equations

$$G^{\mu\nu}+\Lambda g^{\mu\nu}=0, \tag{28}$$

from which one finds

$$R^{\mu\nu}=O(\Lambda),\quad R^{\alpha\beta\gamma\delta}=O(\Lambda),\quad R=O(\Lambda), \tag{29}$$

where $O(\Lambda)$ indicates the order of the tensors involved with respect to the cosmological constant (see Appendix A). We now consider the construction of the tensor $W^{\mu\nu}$ for the particularly simple case in which the scalar $W(x)$ is taken as slowly varying space-time function. In this case the function $W(x)$ can approximately be replaced by an almost constant mean value $\overline{W}$, so we can neglect its derivatives. Correspondingly, the divergence relation (21) results in the following constraint:

$$\nabla_\mu\left(W^{\mu\nu}-\frac{1}{2}g^{\mu\nu}W^{\alpha}_{\alpha}\right)=O(\Lambda^2). \tag{30}$$

Using the relations (23) one can obtain a further constraint on the trace of the tensor $W^{\mu\nu}$. One finds

$$W^{\alpha}_{\alpha}=O(\Lambda^2), \tag{31}$$

which together with Eq. (30) implies

$$\nabla_\mu W^{\mu\nu}=O(\Lambda^2). \tag{32}$$

Thus, up to terms of order Λ^2, the construction of the tensor $\overline{\Gamma^{\mu\nu}}[W]$ amounts to finding the traceless conserved tensor $W^{\mu\nu}$. For our purpose it is convenient to use for $W^{\mu\nu}$ a decomposition of the form

$$W^{\mu\nu}=-\alpha\overline{W}G^{\mu\nu}-S^{\mu\nu}, \tag{33}$$

where α is a constant parameter. As a consequence of Bianchi identity $\nabla_\mu G^{\mu\nu}=0$ one gets then from Eqs. (30)–(32) the corresponding constraints on the tensor $S^{\mu\nu}$, namely

$$\nabla_\mu S^{\mu\nu}=O(\Lambda^2),\qquad S^{\alpha}_{\alpha}=\alpha\overline{W}R+O(\Lambda^2). \tag{34}$$

Our approximation now consists in neglecting terms of order Λ^2. One finds from Eqs. (27), (33), and (34) the approximate expression of the tensor $\overline{\Gamma^{\mu\nu}}[W]$ in terms of $S^{\mu\nu}$, namely

$$\overline{\Gamma^{\mu\nu}}[W]\approx S^{\mu\nu}+\alpha\overline{W}G^{\mu\nu}, \tag{35}$$

where the tensor $S^{\mu\nu}$ is a conserved tensor with the trace $S^{\alpha}_{\alpha}=\alpha\overline{W}R$.

IV. DIMENSIONAL REDUCTION

Our goal now is to arrive at a (suitably defined) two-dimensional analog of the approximate stress tensor (35). In two dimensions the field ϕ is a dimensionless quantity. Correspondingly, the stress tensor takes the dimension of a length to the power -2. Thus, to arrive at a two-dimensional analog of the stress tensor we first replace the field ϕ by the dimensionless quantity $\overline{W}^{-1/2}\phi$. Correspondingly, we replace the tensor $\overline{\Gamma^{\mu\nu}}[W]$ by $\overline{W}^{-1}\overline{\Gamma^{\mu\nu}}[W]$. Denoting this latter quantity by $\overline{\Gamma^{\mu\nu}_2}[W]$ and taking into account that in two dimensions the tensor $G^{\mu\nu}$ is identically vanishing, we define the two-dimensional analog of Eq. (35) as

$$\overline{\Gamma^{\mu\nu}_2}[W]\approx S^{\mu\nu}_2,\qquad S^{\mu\nu}_2=\overline{W}^{-1}S^{\mu\nu}, \tag{36}$$

in which the conserved tensor $S^{\mu\nu}_2$ takes now the trace

$$S^{\alpha}_{2\,\alpha}=\alpha R. \tag{37}$$

The still unknown parameter α in Eq. (37) can be determined by a general requirement. We remark that the minimal coupling in two dimensions reduces to the conformal coupling. Thus, α can be determined by the requirement that the trace of $S^{\mu\nu}_2$ shall reproduce the general restriction on the anomalous trace in two dimensions [1,13], yielding $\alpha=1/24\pi$. We conclude that, in our two-dimensional analog of the problem, the determination of the tensor $\overline{\Gamma^{\mu\nu}}[W]$ amounts to finding a tensor $S^{\mu\nu}$ satisfying the constraints (we suppress the subscript 2),

$$\nabla_\mu S^{\mu\nu}=0,\qquad S^{\alpha}_{\alpha}=\frac{1}{24\pi}R. \tag{38}$$

These constraints correspond exactly to the well-known constraints imposed on the two-dimensional stress tensor of a conformally invariant field. Here we have shown that, restricting ourselves to solutions of Eq. (28), these constraints can also be found from a (suitably defined) dimensional reduction of the state-dependent part of the renormalized stress tensor of a minimally coupled field in four dimensions plus some approximation.

V. THERMAL RADIATION AND THE COSMOLOGICAL EVENT HORIZONS

As an illustration we shall apply the results of the previous sections to a particular solution of the equations (28), namely the Schwarzschild–de Sitter space-time on which the metric in the static and spherical symmetric form is given by

$$ds^2=-\left(1-\frac{2M}{r}-\frac{\Lambda r^2}{3}\right)dt^2+\left(1-\frac{2M}{r}-\frac{\Lambda r^2}{3}\right)^{-1}dr^2 +r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{39}$$

This metric describes a Schwarzschild-like black hole in the presence of the cosmological constant Λ [14]. In the following we shall restricts ourselves to a typical situation in which $\Lambda>0$ and $\Lambda M^2\ll 1$. There are two solutions of $g_{tt}=0$ corresponding to a black hole horizon and a cosmological horizon. The black hole horizon can be obtained if we approximate g_{tt} for $r-2M\ll 2M$ by

$$g_{tt}\approx-\left(1-\frac{2M+\Lambda(2M)^3/3}{r}\right) \tag{40}$$

from which one infers that g_{tt} becomes zero at a value $r_b\approx 2M(1+\frac{4}{3}\Lambda M^2)$. This is the position of black hole horizon. It increases with respect to the Schwarzschild radius $r=2M$ by a term of the relative order $(\Lambda/3)(2M)^2$. The cosmological horizon can be obtained if we approximate g_{tt} for $\sqrt{3/\Lambda}-r\ll\sqrt{3/\Lambda}$ by

$$g_{tt}\approx-\left[1-\left(\frac{2M}{(\sqrt{3/\Lambda})^3}+\frac{\Lambda}{3}\right)r^2\right] \tag{41}$$

from which one infers that g_{tt} becomes zero at a value $r_c\approx\sqrt{3/\Lambda}(1-M\sqrt{\Lambda/3})$. This is the position of cosmological horizon. It decreases with respect to the de Sitter radius $r=\sqrt{3/\Lambda}$ by a term of the relative order $\sqrt{\Lambda M^2/3}$.

In the following we shall deal with the two-dimensional analog of the metric (39), namely

$$ds^2=-\left(1-\frac{2M}{r}-\frac{\Lambda r^2}{3}\right)dt^2+\left(1-\frac{2M}{r}-\frac{\Lambda r^2}{3}\right)^{-1}dr^2 \tag{42}$$

for which the positions of event horizons are the same as those for the four-dimensional case. The metric (42) can be written in the conformally-flat form

$$ds^2=\Omega(r)(-dt^2+dr^{*2}), \tag{43}$$

with

$$\Omega(r)=1-\frac{2M}{r}-\frac{\Lambda r^2}{3},\quad \frac{dr}{dr^*}=\Omega(r). \tag{44}$$

In the following our main objective is the determination of the tensor S_μ^ν defined by Eq. (38) for the metric of Eq. (42). For the nonzero christoffel symbols of the metric (42) we have in (t,r^*) coordinates

$$\Gamma^{r^*}_{tt}=\Gamma^t_{tr^*}=\Gamma^t_{r^*t}=\Gamma^{r^*}_{r^*r^*}=\frac{1}{2}\frac{d}{dr}\Omega(r). \tag{45}$$

Under the assumptions that S_μ^ν is time independent and spherically symmetric, the conservation equation takes the form

$$\partial_{r^*}S_t^{r^*}+\Gamma^t_{tr^*}S_t^{r^*}-\Gamma^{r^*}_{tt}S^t_{r^*}=0, \tag{46}$$

$$\partial_{r^*}S_{r^*}^{r^*}+\Gamma^t_{tr^*}S_{r^*}^{r^*}-\Gamma^t_{tr^*}S^t_t=0, \tag{47}$$

with

$$S^t_{r^*}=-S_t^{r^*},$$

$$S^t_t=S^\alpha_\alpha-S^{r^*}_{r^*}, \tag{48}$$

where S^α_α is trace anomaly in two dimensions. Using Eqs. (45)–(48) one can show that

$$\frac{d}{dr}\{\Omega(r)S_t^{r^*}\}=0 \tag{49}$$

and

$$\frac{d}{dr}\{\Omega(r)S^{r^*}_{r^*}\}=\frac{1}{2}\left\{\frac{d}{dr}\Omega(r)\right\}S^\alpha_\alpha. \tag{50}$$

Equation (49) leads

$$S_t^{r^*}=\alpha\Omega^{-1}(r), \tag{51}$$

where α is a constant of integration. The solution of Eq. (50) may be written in the following form:

$$S^{r^*}_{r^*}(r)=[H(r)+\beta]\Omega^{-1}(r),\quad \beta=\Omega(L)S^{r^*}_{r^*}(L), \tag{52}$$

where

$$H(r)=\frac{1}{2}\int_L^r S^\alpha_\alpha(r')\frac{d}{dr'}\Omega(r')dr', \tag{53}$$

with L being an arbitrary scale of length, and

$$S^\alpha_\alpha(r)=\frac{1}{24\pi}R=\frac{M}{6\pi r^3}+\frac{1}{36\pi}\Lambda. \tag{54}$$

Given a length scale L, the function $H(r)$ incorporates the corresponding (nonlocal) contribution of the trace $S^\alpha_\alpha(r)$ to the tensor S_μ^ν. The choice of L needs careful considerations. It does not appear possible to include the contribution of a region very close to the black hole horizon to the off-diagonal components of S_μ^ν, if the latter is taken as properly describing the late time (steady state) behavior of outwards signals coming from the direction of the black hole[1] In fact, an ‘‘off-diagonal’’ contribution of a region very close to the black hole horizon cannot be sharply defined with respect to the state of outwards signals at late times because the infinite gravitational redshift at the black hole horizon connects the latter state at asymptotic times with the physical situations in the vicinity of the horizon where the quantum fluctuation of the horizon (and the corresponding change of the gravitational field) can no longer be neglected. To accurately describe the outwards signals at late time by S_μ^ν our criterion is to exclude in the definition of $H(r)$ the contribution of the trace very close to the black hole horizon using a characteristic cutoff length l_c. Since the scales of the problem are set by the mass of the black hole and the cosmological constant, it should be possible to define the cutoff in terms of M and Λ. The least arbitrary way to do this is to relate the cutoff to the actual shift of the black hole horizon with respect to the Schwarzschild-radius $2M$ which has been previously determined to be of the relative order $(\Lambda/3)(2M)^2$. Thus, we shall subject the choice of L in Eq. (53) to a condition of the type

$$L=r_b+l_c, \quad l_c\approx\frac{\Lambda}{3}(2M)^3. \tag{55}$$

Using Eqs. (51) and (52) one can show that S_μ^ν takes the form (in t,r^* coordinates)

$$S_\mu^\nu(r)=\begin{pmatrix} S_\alpha^\alpha(r)-\Omega^{-1}(r)H(r) & 0\\ 0 & \Omega^{-1}(r)H(r)\end{pmatrix}+\Omega^{-1}(r)\begin{pmatrix}-\beta & -\alpha\\ \alpha & \beta\end{pmatrix}. \tag{56}$$

Now, defining $Q=\alpha+\beta$ and $K=\alpha$, the tensor S_μ^ν takes the form

$$S_\mu^\nu=S_\mu^{(r)\nu}+S_\mu^{(eq)\nu}, \tag{57}$$

with

$$S_\mu^{(r)\nu}=\begin{pmatrix} S_\alpha^\alpha(r)-\Omega^{-1}(r)H(r) & 0\\ 0 & \Omega^{-1}(r)H(r)\end{pmatrix}+K\Omega^{-1}(r)\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix} \tag{58}$$

and

$$S_\mu^{(eq)\nu}=Q\Omega^{-1}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}. \tag{59}$$

[1]Of course, from Eq. (52) it follows that the function $H(r)$ has no explicit contribution to the off-diagonal components of the stress tensor. But we shall see later, Eq. (66), that $H(r)$ has an implicit ‘‘off-diagonal’’ contribution to the radiation temperature of the black hole through the parameter α appearing in Eq. (51).

Both tensors in Eq. (57) satisfy the conservation law. Note that only $S_\mu^{(r)\nu}$ has off-diagonal (flux) components.

Now we should determine the constants Q and K. For the determination of Q we require the regularity of S_μ^ν at the black hole horizon in a coordinate system which is regular there. This results in a relation (Appendix B)

$$Q+H(r)\to 0, \quad \text{as } r\to r_b, \tag{60}$$

which together with Eq. (53) implies

$$Q=\frac{1}{2}\int_{r_B}^{L}S_\alpha^\alpha(r')\frac{d}{dr'}\Omega(r')dr'. \tag{61}$$

Using Eq. (54), the approximate value (we neglect terms of higher orders in Λ) of this integral can be found to be

$$Q\approx\frac{\Lambda}{72\pi} \tag{62}$$

from which one infers that in quasiflat regions of space-time $r\approx r_{\rm qf}$ where

$$r_b\ll r_{\rm qf}\ll r_c, \quad \Omega(r_{\rm qf})\approx 1, \tag{63}$$

the tensor $S_\mu^{(eq)\nu}$ in Eq. (57) describes an equilibrium gas with a temperature $T_c=1/2\pi\sqrt{\Lambda/3}$. This follows if one compares the tensor $S_\mu^{(eq)\nu}$ with the stress tensor of an equilibrium gas, namely

$$\frac{\pi}{12}(kT)^2\begin{pmatrix}-2 & 0\\ 0 & 2\end{pmatrix}. \tag{64}$$

The equilibrium temperature T_c corresponds to the leading-order estimate of the temperature of the cosmological event horizon [14].

We proceed now to describe the radiation temperature of the black hole. In the present case an outwards flux of thermal radiation in quasiflat regions can be described by the stress tensor

$$\frac{\pi}{12}(kT)^2\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}, \tag{65}$$

where T is the temperature. For such a stress tensor the energy density and flux are numerically equal. This latter condition if applied in Eq. (57) to the tensor $S_\mu^{(r)\nu}$, leads in quasiflat regions to the relation

$$K=\frac{1}{2}\{H(r_{\rm qf})-S_\alpha^{(r)\alpha}(r_{\rm qf})\}, \tag{66}$$

in which $H(r_{\rm qf})=(\pi/6)(8\pi M)^{-2}+O(\Lambda)$, as may be verified from Eq. (53) by a simple calculation. Therefore $S_\mu^{(r)\nu}$ takes in quasiflat regions the form

$$S_\mu^{(r)\nu}(r\to r_{\rm qf})=\frac{\pi}{12}(8\pi M)^{-2}\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}+O(\Lambda), \tag{67}$$

from which one infers that $S_\mu^{(r)\nu}$ describes an outward radiation with the temperature

$$T_b = (8\pi M)^{-1} + O(\Lambda M). \tag{68}$$

The term $O(\Lambda M)$ is a correction term to the Hawking temperature $T_H = (8\pi M)^{-1}$ [15] which is the temperature of the hole in the absence of the cosmological event horizon. In terms of the cutoff length l_c the correction term takes the form $O(l_c/M^2)$. Thus the correction to the Hawking temperature is a term of the relative order l_c/M. In our case this makes no significant difference for thermal predictions because our assumption $\Lambda M^2 \ll 1$ means that the cutoff l_c is much smaller than the Schwarzschild-radius $2M$.

VI. CONCLUDING REMARKS

We have seen that the existence of a cutoff excluding the contribution of the anomalous trace to the stress tensor in a neighborhood of the black hole horizon can be connected to the equilibrium temperature of a background heat bath of the cosmological event horizon. For the corresponding temperature we have found an estimate in terms of the contribution of the anomalous trace close to the black hole horizon; see Eqs. (61) and (62). It is important to note that, while the latter contribution seems to be unphysical with respect to the radiation temperature coming from the black hole at late times, it does determine the leading order estimate of the equilibrium temperature. Is there any justification for regarding the contribution of the anomalous trace close to the black hole horizon as physical with respect to the equilibrium temperature? We emphasize the distinct character of the equilibrium temperature as compared to the radiation temperature. The former is not expected to be sensitive to the outward signals at late times coming from the direction of the black hole, so dissatisfaction with the role of the infinite gravitational redshift at the black hole horizon may not be expressed in this case.

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APPENDIX A

Let n be an arbitrary real number. A tensor $H^{\alpha\beta\cdots}_{\gamma\delta\ldots}$ is said to be of the covariant order Γ^n with respect to some parameter Γ, if $\Gamma^{-n}H^{\alpha\beta\cdots}_{\gamma\delta\ldots}$ can be factorized in the metric tensor $g_{\mu\nu}$. We shall denote such a situation by $H^{\alpha\beta\cdots}_{\gamma\delta\ldots} = O(\Gamma^n)$. (For scalars the usual meaning is understood.) For simplicity the attribute ''covariant'' has been suppressed throughout the paper. From Eq. (28) one gets

$$R = 4\Lambda = O(\Lambda) \tag{A1}$$

and

$$R^{\mu\nu} = R^{\mu\lambda\nu}_{\lambda} = g_{\lambda\gamma}R^{\mu\lambda\nu\gamma} = \Lambda g^{\mu\nu} = O(\Lambda). \tag{A2}$$

From the last equation it follows

$$R^{\mu\lambda\nu\gamma} = \Lambda g^{\mu\lambda}g^{\gamma\nu} = O(\Lambda). \tag{A3}$$

We also find

$$v_1(x) = \lim_{x'\to x} V_1(x,x') = \frac{1}{720}\{\Box R - R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\lambda}R^{\alpha\beta\gamma\lambda}\}$$
$$= O(\Lambda^2). \tag{A4}$$

APPENDIX B

An analysis similar to that presented in [1] for the Schwarzschild metric shows that S^{ν}_{μ}, as measured in a local Kruskal coordinate system at black hole horizon, will be finite if S_{vv}, and $S^t_t + S^{r*}_{r*}$ are finite as $r \to r_b$ and

$$\lim_{r\to r_b} (r-r_b)^{-2}|S_{uu}| < \infty, \tag{B1}$$

where u and v are null coordinates. We find easily

$$S_{uu} = \frac{1}{4}(S_{tt} + S_{r*r*} - 2S_{tr*}). \tag{B2}$$

Using Eqs. (57)–(59), this gives

$$S_{uu} = \frac{1}{2}\left\{H(r) + Q - \frac{1}{2}\Omega(r)S^{\alpha}_{\alpha}(r)\right\}. \tag{B3}$$

Therefore, the condition (B1) is equivalent to

$$H(r) + Q \to 0 \quad \text{as} \quad r \to r_b. \tag{B4}$$

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