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Closet non-Gaussianity of anisotropic Gaussian fluctuations

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In this paper we explore the connection between anisotropic Gaussian fluctuations and isotropic non-Gaussian fluctuations. We first set up a large angle framework for characterizing non-Gaussian fluctuations: large angle non-Gaussian spectra. We then consider anisotropic Gaussian fluctuations in two different situations. First we look at anisotropic space-times and propose a prescription for superimposed Gaussian fluctuations; we argue against accidental symmetry in the fluctuations and that therefore the fluctuations should be anisotropic. We show how these fluctuations display previously known non-Gaussian effects both in the angular power spectrum and in non-Gaussian spectra. Secondly we consider the anisotropic Grischuk-Zel'dovich effect. We construct a flat space time with anisotropic, nontrivial topology and show how Gaussian fluctuations in such a space-time look non-Gaussian. In particular we show how non-Gaussian spectra may probe superhorizon anisotropy. [S0556-2821(97)07220-2]

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I. INTRODUCTION

Anisotropic models of the Universe have been often considered in the past (e.g., [1,2]). In recent times globally anisotropic space-times have attracted attention for their thought provoking value, as primordial anisotropy would appear to contradict inflation [3]. It is therefore important to find experimental evidence for, or constraints on, primordial anisotropy. The cosmic microwave background (CMB) is the cleanest and most accurate experimental probe in current cosmology. Thus it makes sense to explore the impact of anisotropic expansion on the CMB. For homogeneous space times this was largely done in [4–6]. In the more sophisticated analysis in [6] the effects of the unperturbed anisotropic expansion were combined with a spectrum of superposed Gaussian fluctuations. An admitted shortcoming of this analysis is the assumption that while the unperturbed model leaves an anisotropic pattern in the sky, the Gaussian fluctuations around it are isotropic. Should the Gaussian fluctuations in such models be anisotropic one may expect a more stringent statistical bound on anisotropy, if the Universe is indeed isotropic. One can consider another class of models where the background space-time is homogeneous and isotropic but anisotropic topological identifications lead to anisotropic Gaussian fluctuations. Some of these universes have been considered before [7] and an example of the patterns in an open universe has been presented in [8].

The apparently unrelated issue of large-angle CMB non-Gaussianity has also been considered recently, both as an experimental matter [9], and as a possible prediction in topological defect theories [10–16]. In [10], in particular, an outline is given of a comprehensive formalism for encoding large angle non-Gaussianity based on the spherical harmonic coefficients a^l_m in the expansion

$$\frac{\Delta T(n)}{T}=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}a^l_m Y^l_m(n). \tag{1}$$

In [10] it is also stated that ''any non-Gaussian theory is to some extent anisotropic, favoring particular directions in the sky and some m's over others.'' The converse statement follows: that Gaussian anisotropic fluctuations will appear as non-Gaussian fluctuations from the standpoint of an isotropic theory. This establishes an interesting link between the search for cosmological anisotropy and the search for non-Gaussian signatures.

Let us consider Gaussian theories which favor an axis, where Ω are angles defining this axis. Then the probability distribution conditional to this axis $P(a^l_m|\Omega)$ is Gaussian. Isotropy is violated, but the resulting theory is Gaussian within the reduced set of symmetries the theory now must satisfy. However from an isotropic point of view the full ensemble is made up of all the ensembles which favor an axis, but allowing the axis to be uniformly distributed. Such a superensemble would undoubtedly be isotropic, but it would also be non-Gaussian. Marginalizing with respect to the axis reveals a non-Gaussian theory, that is

$$P(a^l_m)=\int d\Omega P(\Omega)P(a^l_m|\Omega),$$

$$P(\Omega)=\frac{\sin\,\theta}{4\pi} \tag{2}$$

is non-Gaussian. This identifies the origin of the Gaussian/non-Gaussian switch. Conditionalizing to an axis renders the theory Gaussian (and anisotropic). Marginalizing with respect to the axis reveals a non-Gaussian theory (but an isotropic ensemble).

This phenomenon turns out to be a particular case of the general phenomenon discussed in connection with the texture analytical model in [11,12]. In that model it is found that the temperature anisotropies are very non-Gaussian. The theory has C_l cosmic variance error bars above their Gaussian value, and there are strong correlations among C_l. It turns out, however, that these large-angle non-Gaussian effects are largely due to the last texture (as in the texture closest to us, or the texture at lower redshift). The culprit identified, one then notices that conditionalizing the theory to the last texture redshift z_1 reveals a Gaussian ensemble, that is, the probability distribution $P(a^l_m|z_1)$ is Gaussian. Marginalizing with respect to z_1, however, produces a non-Gaussian ensemble, that is the probability

$$P(a^l_m)=\int dz_1 P(z_1)P(a^l_m|z_1) \tag{3}$$

is non-Gaussian.

The picture is then clear [17]. We come up with a construction where the full ensemble is made up of subensembles which are Gaussian. Each subensemble is however labeled by an index which from the point of view of the full ensemble is a random variable. Marginalizing with respect to this variable reveals a non-Gaussian ensemble. Conditionalizing with respect to this index renders the theory Gaussian. Such an index was called in [12] the random index, and it was conjectured[1] in that paper that non-Gaussianity could often be characterized by a set of such indices labeling Gaussian ensembles. Within such a construction the strategy for predicting experiment must be modified. One should now not provide a direct statistical description of the full ensemble (that is, marginal distributions), which would be plagued by all sorts of non-Gaussian effects. Rather it makes more sense to supply information on all the Gaussian subensembles, plus the distribution function of their random indices.

Hence we may use a subclass of the comprehensive formalism for encoding large-angle non-Gaussianity outlined in [10] to describe anisotropic Gaussian fluctuations. This is essentially a large-angle generalization of [19] and is described in Sec. II. The idea is to complement the angular power spectrum C_l with a set of multipole shape spectra B_m describing how the power is distributed among the m's for a given scale l. The B_m encode information on the shape of large angle structures. They are uniformly distributed in a Gaussian isotropic theory, meaning its fluctuations are shapeless. However, as we shall see in Sec. V, preferred shapes emerge in non-Gaussian isotropic theories, as well as in Gaussian anisotropic theories, where the B_m are not uniformly distributed. Non-Gaussian spectra then appear as a natural predictive tool for these theories.

In this paper we study the disguised non-Gaussianity of anisotropic Gaussian fluctuations along two lines. Firstly, in Sec. III, we propose a simple method for defining anisotropic Gaussian fluctuations. Breaking isotropy essentially amounts to choosing an alternative symmetry group under which the covariance matrix should be invariant, and which picks a favored direction in the sky. We can then write down the most general form for the covariance matrix of the theory simply by studying the representation theory of the symmetry group. We argue that the accidental symmetry allowing anisotropic fluctuations to be isotropic is a model dependent and unnatural assumption. Hence Gaussian fluctuations in anisotropic universes should be anisotropic too. Although we concentrate on anisotropic fluctuations with an SO(2) symmetry, the definition and considerations given in Sec. II are quite general, as explained in more detail in Appendix A.

We then show how anisotropic Gaussian theories induce well known non-Gaussian effects in the relation between the observed and the predicted angular power spectrum C_l. These effects include larger cosmic variance error bars, and also the phenomenon of cosmic covariance, that is correlations between the observed C_l. Cosmic covariance allows for more structure to exist in each realization than in the predicted average power spectrum and complicates comparison between theory and experiment. These effects are shown to be present for anisotropic Gaussian theories in Sec. IV.

Then, in Sec. V, we show how anisotropic Gaussian fluctuations render non-Gaussian spectra nonuniformly distributed, as announced above. We also find the most general class of isotropic non-Gaussian theories into which anisotropic Gaussian fluctuations may be mapped. As a concrete example in Sec. VI we proceed to characterize the non-Gaussian spectra for the relevant, globally anisotropic space times.

Along a totally different line in Sec. VIII we construct a simple example of a topologically nontrivial space time and show how the non-Gaussian spectra will indicate anisotropic topological identifications. We propose this as an *anisotropic* Grischuk-Zel'Dovich effect: from subhorizon, large angle observables we can characterize super-horizon anisotropies.

In Sec. IX we discuss the implications of our results and their practical implementation.

II. LARGE-ANGLE NON-GAUSSIANITY

We now set up a formalism for describing large-angle non-Gaussianity which is based on [19], but makes use of a^l_m coefficients rather than Fourier components, and so is suitable for mapping large-angle non-Gaussianity. Again the idea is to map the $\{a^l_m\}$ into a set of spectra which for a Gaussian isotropic theory are independent random variables. One of these spectra is the angular power spectrum C_l, and should be a χ^2_{2l+1} for a Gaussian isotropic theory. The other variables make up non-Gaussian spectra which should be uniformly distributed for a Gaussian isotropic theory.

The transformation proposed is defined as follows. First we split the complex modes into moduli and phases

$$a^l_0=s^l_0\rho^l_0,$$

$$a^l_m=\frac{\rho^l_m}{\sqrt{2}}e^{i\phi^l_m}, \tag{4}$$

where $s^l_0=\pm 1$ is simply the sign of a^l_0. The fact that the $m=0$ mode is real introduces a slight modification to the construction in [19]. There are now $l+1$ moduli, but there are only l phases (the index m starts at 1 for the phases). Working out the Jacobian of the transformation shows that for a Gaussian theory the distribution of the $\{\rho^l_m,\phi^l_m,s^l_0\}$ is

[1] This conjecture can in fact be promoted to a mathematical theorem; see [18].

$$F(\rho_m^l,\phi_m^l,s_0^l)=\frac{2\exp\left(-\frac{\Sigma_0^l\rho_m^2}{2C_l}\right)}{(2\pi)^{1/2}C_l^{l+1/2}}\left(\prod_1^l\rho_m\right)\frac{1}{(2\pi)^l}\frac{1}{2}. \tag{5}$$

The phases ϕ_m^l are uniformly distributed in $[0,2\pi]$. The sign s_0^l has a uniform discrete distribution. The moduli ρ_m^l are χ_2^2 distributed except for ρ_0^l which is χ_1^2 distributed. Since ρ_0 now does not appear in the Jacobian of the transformation, the only way one can proceed with the construction in [19] is by ordering the ρ's by decreasing order of m, and then introduce polars:

$$\begin{aligned}\rho_l^l&=r\cos\theta_1,\\ \rho_{l-1}^l&=r\sin\theta_1\cos\theta_2,\\ &\dots\\ \rho_1^l&=r\sin\theta_1\dots\cos\theta_l,\\ \rho_0^l&=r\sin\theta_1\dots\sin\theta_l.\end{aligned} \tag{6}$$

Again, working out the Jacobian of the transformation implies that for a Gaussian isotropic theory the distribution of these variables is

$$F(r,\theta_m,s_0^l,\phi_m)=\frac{\exp(-r^2/2C_l)r^{2l}}{(\pi/2)^{1/2}C_l^{l+1/2}}\times\prod_1^l\cos\theta_m(\sin\theta_m)^{2(l-i)}\frac{1}{2}\frac{1}{(2\pi)^l}. \tag{7}$$

One can then define shape spectra B_m^l as

$$B_m^l=(\sin\theta_m)^{2(l-m)+1} \tag{8}$$

so that for a Gaussian isotropic theory one has

$$F(r,B_m^l,s_0^l,\phi_m^l)=\frac{\exp(-r^2/2C_l)r^{2l}}{(\pi/2)^{1/2}C_l^{l+1/2}(2l-1)!!}\frac{1}{2}\frac{1}{(2\pi)^l}. \tag{9}$$

The angular power spectrum C^l seen as a random variable is then related to r by

$$C^l=\frac{r^2}{2l+1} \tag{10}$$

and is a χ_{2l+1}^2. The multipole shape spectra B_m^l may be obtained from the moduli ρ_m^l according to

$$B_m^l=\left(\frac{\rho_{m-1}^{l2}+\cdots+\rho_0^{l2}}{\rho_m^{l2}+\cdots+\rho_0^{l2}}\right)^{m-1/2} \tag{11}$$

and are uniformly distributed in [0,1]. Finally the phases ϕ_m^l are uniformly distributed in $[0,2\pi]$, and the sign s_0^l is a discrete uniform distribution over $\{-1,+1\}$.

As in [19] we define non-Gaussian structure in terms of departures from uniformity and independence in the $\{B_m^l,\phi_m^l\}$. Gaussian theories can only allow for modulation, that is, a nonconstant power spectrum. The most general power spectrum has as much information as Gaussian theories can carry. White noise is the only type of fluctuations which is more limited in terms of structure than Gaussian fluctuations.[2] In isotropic Gaussian theories there is no structure in the $\{B_m^l,\phi_m^l\}$ since these are independent and uniformly distributed. By allowing the B_m^l to be not uniformly distributed, or to be constrained by correlations amongst themselves and with the power spectrum, one adds shape to the multipoles. This is because the B_m^l tell us how the power in multipole l given by C^l (or r) is distributed among the various $|m|$ modes, which reflect the shape of the fluctuations. Indeed the $m=0$ mode (zonal mode) has no azimuthal dependence. It corresponds to fluctuations with strict cylindrical symmetry (rather than statistical symmetry). The $|m|>0$ modes correspond to the various azimuthal frequencies allowed for the scale l. Each of these modes represent a way in which strict cylindric symmetry may be broken. The relative intensities of all the m modes carry information on the shape of the random structures at least as seen by the scale l. In a Gaussian theory all the m modes must have the same intensity, something which can be rephrased by the statement that the B_m^l are independent and uniformly distributed. Hence Gaussian fluctuation display shapeless multipoles. Any departure from this distribution in the B_m^l may then be regarded as a evidence for more or less random shape in the fluctuations.

On the other hand the phases ϕ_m^l transform under azimuthal rotations. Therefore they carry information on the localization of the fluctuations. If the phases are independent and uniformly distributed then the perturbations are delocalized.

Finally there may be correlations between the various scales defined by l. In the language of [19] this is what is called connectivity of the fluctuations. These correlations measure how much coherent interference is allowed between different scales, a phenomenon required for the rather abstract shapes and localization on each scale to become something visually recognizable as shapeful or localized. As in [19] this may be cast into inter-l correlators. As we shall see these are in fact quite complicated for general anisotropic Gaussian theories. Therefore we have chosen not to dwell on this aspect of large-scale non-Gaussianity in this paper.

III. A POSSIBLE METHOD FOR INTRODUCING GAUSSIAN FLUCTUATIONS IN ANISOTROPIC UNIVERSES

We now present a possible way of introducing Gaussian fluctuations in anisotropic universes such as the Bianchi models. In Sec. VIII we will present another context in

[2]It is curious to note that white noise has less structure than generic Gaussian fluctuations, but it also has more symmetry. It is tempting to associate reduction of symmetry and addition of structure. Anisotropic fluctuations have less symmetry than isotropic fluctuations, but they also have more structure, reflected in their non-Gaussian structure.

which anisotropy appears: periodic universes. There we shall present more specific calculations of anisotropic Gaussian perturbations. Here we shall however use a method which relies simply on inspecting the reduced symmetry group anisotropic Gaussian perturbations must satisfy. This is a simple, if somewhat phenomenological, way of introducing the most general Gaussian perturbation which can live in an anisotropic background. Without actually performing a detailed perturbation analysis of these spacetimes, one can refine the analysis of [6] by using this prescription and possibly find more stringent constraints.

Let an all-sky temperature anisotropy map be decomposed into spherical harmonics as in Eq. (1). Then, for a general Gaussian theory, the a^l_m are Gaussian random variables specified by a covariance matrix which must satisfy the symmetries of the underlying theory. In Friedman models the symmetry group is SO(3), but the symmetry group may be smaller. Anisotropic Gaussian fluctuations may be defined as Gaussian fluctuations with a covariance matrix satisfying a symmetry group which picks a favoured direction in the sky. We concentrate on anisotropic fluctuations with an SO(2) symmetry, that is, with cylindrical symmetry.

The general form of the covariance matrix may be obtained just from the representation theory of the symmetry group. The symmetry group breaks the $\{a^l_m\}$ space into irreducible representations (irreps). The a^l_m may then be reexpressed in a basis adapted to these irreps. Using Schur's Lemmas [20] one knows (see Appendix A for more detail) that the covariance matrix of the theory must be a multiple of the identity within each irrep.[3] Furthermore correlations between different a^l_m can only occur for elements of different but equivalent irreps. Hence, for any Gaussian theory subject to a symmetry which does not lead to equivalent irreps, the spherical harmonic coefficients, expressed in a basis adapted to the partition into irreps, must be independent random variables, and their variance must be a function only of the irrep they belong to. As we shall see it may happen that the variance is the same for a set of irreps. This degeneracy then leads to an accidental enlarged symmetry. If some of the irreps are equivalent then in principle one may also have correlations between coefficients belonging to different but equivalent irreps.

As an example consider an isotropic theory. Then the $\{a^l_m\}$ for each l are an irrep of the symmetry group SO(3) represented by the D matrices

$$R(\psi,\theta,\phi)a^l_m = D^l_{mm'}(\psi,\theta,\phi)a^l_{m'}, \qquad (12)$$

where (ψ,θ,ϕ) are Euler angles. None of these irreps is equivalent, as indeed none of them have the same dimension. Hence for a Gaussian isotropic theory the a^l_m must have a covariance matrix of the form

$$\langle a^l_m a^{l'*}_{m'}\rangle = \delta_{ll'}\delta_{mm'}C_l . \qquad (13)$$

If the angular power spectrum C_l happens to be a constant (white-noise) over a certain section of the spectrum then this degeneracy increases the symmetry group of the theory: rotations among different l's are now an extra symmetry. This is an accidental symmetry resulting from the degeneracy displayed by the particular model considered (white noise) and not required by the underlying theory.

Now suppose that the symmetry group is SO(2), that is, the unperturbed model supporting the fluctuations is cylindrically symmetric. Then there is a favored axis in the universe and with respect to this axis the symmetry transformations are

$$R(\phi)a^l_m = e^{im\phi}a^l_m . \qquad (14)$$

The irreps are now indexed by l,m with $m \geq 0$. They are one-dimensional complex irreps for $m>0$, and one dimensional real (and trivial) irreps for $m=0$. For the same m irreps with different l are equivalent irreps. For each l we have a single irrep of SO(3) which splits into $l+1$ irreps of SO(2). The covariance matrix of the theory now has the general form

$$\langle a^l_m a^{l'*}_{m'}\rangle = \delta_{mm'}C^{ll'}_{|m|} \qquad (15)$$

and we may call the diagonal terms C_{lm} of $C^{ll'}_{|m|}$ the cylindrical power spectrum. It may now happen that $C^{ll'}_{|m|} = \delta^{ll'}C^l_{|m|}$, and furthermore that a given model displays the degeneracy $C_{l|m|}=C_l$, that is the cylindrical power spectrum is white noise in m. In this case the SO(3) symmetry is accidentally restored. However this is no different from the white-noise model $C_l=$const referred to above. It is merely an accidental enlarged symmetry displayed by a concrete model and not a fundamental symmetry imposed by the underlying model.

Accidental symmetries (e.g., family symmetry in particle physics) are always regarded with horror. If they happen to exist, sooner or later a fundamental principle is sought which will promote them from accidental to fundamental symmetries. If they do not happen to exist *a priori*, such as in the case of fluctuations in anisotropic models, then better not postulate them in the first place.

IV. NON-GAUSSIAN EFFECTS ON THE ANGULAR POWER SPECTRUM

Gaussian anisotropic theories display many of the novelties present in non-Gaussian theories, such as the texture models considered in [11,12]. They trade their added predictivity in terms of non-Gaussian spectra for larger cosmic variance error bars in the angular power spectrum. Also the observed C_l may be correlated, a phenomenon called cosmic covariance and present in the texture models in [11,12]. Cosmic covariance (or C^l aliasing) induces great mess when comparing predicted and observed power spectra. Correlations allow for each observed power spectrum to have more structure than the average power spectrum. This may result in the average power spectrum corresponding to nothing that

[3]Schurs' Lemma only applies to finite dimensional representations, such as the ones offered by the a^l_m. If one instead looks at the real space maps $\delta T/T$, then the representation space is S^2. This is infinite dimensional, and indeed the covariance matrix of Gaussian theories is not diagonal, and is specified by the two-point correlation function $C(\theta)$.

any observer ever sees. More subtle methods for predicting power spectra are then necessary. Two prescriptions are given in [12].

A. Cosmic variance surplus

For a Gaussian isotropic theory the angular power spectrum

$$C^l=\frac{1}{2l+1}\sum_{m=-l}^{l}|a^l_m|^2 \tag{16}$$

has the variance

$$\sigma^2(C^l)=\frac{2C_l^2}{2l+1}. \tag{17}$$

Here we use the notation C^l to denote the random variable and C_l to denote its ensemble average. For a Gaussian anisotropic theory this variance is

$$\sigma^2(C^l)=\frac{2}{(2l+1)^2}\sum_{m=-l}^{l}C^2_{lm}. \tag{18}$$

If we define the average cylindrical power spectrum by

$$C_l=\frac{1}{2l+1}\sum_{m=-l}^{l}C_{lm}, \tag{19}$$

then

$$\sigma^2(C^l)\geqslant\frac{2C_l^2}{2l+1}. \tag{20}$$

It is a simple analysis exercise to prove this inequality and show that it is saturated only when $C_{lm}=C_l$, that is when the fluctuations are isotropic.

Generally we may interpret this result as a reduction in the number of degrees of freedom in the χ^2 induced by anisotropy. Suppose, for instance, that a theory is strongly anisotropic so that only a few m modes among the available $2l+1$ contribute to the power spectrum C_l, for a given l. Then, effectively, the observed power spectrum C^l is the result of these few modes. Since these are still Gaussian variables the observed power spectrum is a χ^2, but with an effective number of degrees of freedom equal to the number of predominant modes. If for example all the power is concentrate on the $m=0$ mode, then the C^l is a χ^2_1. If all the power is in a $m>0$ mode, the C^l is a χ^2_2.

We may use the ratio between the actual cosmic variance of the theory and its Gaussian prediction to quantify how anisotropic the fluctuations are. Quantitatively let us call anisotropy in the multipole l to the quantity

$$A_l=\frac{\sigma^2_{\rm GA}(C^l)}{\sigma^2_{\rm GI}(C^l)}=\frac{1}{2l+1}\sum_{m=-l}^{l}\left(\frac{C_{lm}}{C_l}\right)^2, \tag{21}$$

which varies between $A^l=1$ for isotropic theories to $A^l=2l+1$ for cylindrically symmetric multipoles (for which all the power is in the $m=0$ mode).

B. Cosmic covariance

There are also correlations between different C^l. For $l\neq l'$ we have that

$$\mathrm{cov}(C^l,C^{l'})=\frac{1}{(2l+1)(2l'+1)}\sum_{m,m'}\mathrm{cov}(|a^l_m|^2,|a^{l'}_{m'}|^2). \tag{22}$$

For two (possibly correlated) complex Gaussian random variables z_1 and z_2 with uncorrelated real and imaginary parts, it can be shown that $\mathrm{cov}[|z_1|^2,|z_2|^2]=\langle z_1z_2^*\rangle^2+\langle z_1z_2\rangle^2$, and so

$$\mathrm{cov}(C^l,C^{l'})=\frac{1}{(2l+1)(2l'+1)}\sum_m C^{ll'2}_m, \tag{23}$$

where m in the summation runs from $-\min(l,l')$ to $\min(l,l')$. The off-diagonal elements (in l,l') in $C^{ll'}_m$ therefore induce correlations among the various observed C^l. A possible, but model dependent, way to do away with these correlations is to rotate the C^l among themselves so as to diagonalize the covariance matrix (23). These rotated C^l will then be independent, and so their average value is a good prediction for what each observer will see. Also, as shown in [12], in the rotated basis the cosmic variance error bars tend to be smaller and approach their Gaussian minimum. Therefore cosmic covariance, and larger cosmic variance error bars can be dealt with by means of this trick. However this trick does depend on each particular model, and is not a universal prescription applicable to every model.

V. THE NON-GAUSSIAN STRUCTURES EXHIBITED BY ANISOTROPIC GAUSSIAN THEORIES

Anisotropic Gaussian theories also display non-Gaussian structure in the senses given at the end of Sec. II, that is they produce nontrivial non-Gaussian spectra. Here we shall find the most general type of isotropic non-Gaussian structure which can be mapped from these theories.

We shall consider the anisotropic covariance matrix in more detail. Let the matrix $C^{ll'}_m$ be split into its diagonal and its off-diagonal $X^{ll'}_m$ parts

$$C^{ll'}_m=\delta^{ll'}C_{l|m|}+X^{ll'}_m. \tag{24}$$

Then $X^{ll'}_m\ll C_{l|m|}$, and so the bilinear form in the exponent of the Gaussian distribution

$$F(a^l_m)\propto\exp\left(-\sum_m\sum_{l,l'}a^l_mM^{ll'}_ma^{l'}_m\right) \tag{25}$$

is

$$M^{ll'}_m=C^{ll'-1}_m=\frac{\delta^{ll'}}{C_{l|m|}}-\frac{X^{ll'}_m}{C_lC_{l'}} \tag{26}$$

and so the distribution factorizes into a factor which reveals the structure inside each multipole, and a factor which reveals correlations between different multipoles. We shall analyze these two factors in turn.

Let us first assume that $X_m^{ll'}=0$. Repeating the transformation presented in Sec. II but using a covariance matrix of the form (15) one ends up with a rather complex distribution which has the form

$$F(C^l,B_m^l,s_0^l,\phi_m^l)=F(C^l,B_m^l)\,\frac{1}{2}\,\frac{1}{(2\pi)^l}. \tag{27}$$

Unless $C_{lm}=C_l$, the B_m^l are not uniformly distributed. Also the C^l will in general not be a χ^2_{2l+1}, and the function $F(C^l,B_m^l)$ will not factorize. This means that not only will correlations exist between the B_m^l but the B_m^l will also be correlated with the angular power spectrum. The phases ϕ_m^l on the other hand will still be uniformly distributed and independent. The phases tell us nothing about Gaussian anisotropic fluctuations.

Hence anisotropic Gaussian fluctuations, when seen from the point of view of an isotropic formalism, are an example of delocalized shapeful fluctuations (explored in some detail in [19]). In the next two sections we will explore in more detail the particular type of non-Gaussian effects which Gaussian anisotropic fluctuations may induce. The shapes exhibited by these theories are not the most general shapes, because there must be a scale transformation in the ρ_m^l which would render the B_m^l uniformly distributed again. Clearly not all shapes have this property.

On top of this if $X_m^{ll'}\neq 0$ the distribution $F(a_m^l)$ does not factorize into factors which only depend on one l. Correlations between the different l will then appear, which in the language of [19] amount to the emergence of connected structures: different scales are allowed to interfere constructively. In this paper we will not explore this side of the problem in depth. Nevertheless we have identified the non-Gaussian structures into which anisotropic Gaussian fluctuations are mapped. These are the delocalized shapeful (and possibly connected) structures defined in [19], or rather, a subclass thereof.

We should note that although the $\{C^l,B_m^l,\phi_m^l\}$ decomposition is not SO(3) invariant, the $\{C^l,B_m^l\}$ already are SO(2) invariant.[4] Since the phases contain no information whatsoever on Gaussian anisotropic fluctuations they do not count as a device for making predictions in these theories (as much as one does not compute B_m^l for Gaussian isotropic theories). Hence the set of variables $\{C^l,B_m^l\}$ is suitable for representing invariantly the most general form of non-Gaussian fluctuation which can be mapped from Gaussian anisotropic fluctuations.

VI. GLOBALLY ANISOTROPIC UNIVERSES

A useful set of models in which to explore these concepts are the homogeneous, anisotropic cosmologies, also know as the Bianchi models [1]. One can describe Bianchi cosmologies in terms of the metric

$$g_{\mu\nu}=-n_\mu n_\nu+a^2[\exp(2\beta)]_{AB}e^A_\mu e^B_\nu, \tag{28}$$

where n_α is the normal to spatial hypersurfaces of homogeneity, a is the conformal scale factor, β_{AB} is a three matrix only dependent on cosmic time, t, and e^A_μ are invariant covector fields on the surfaces of homogeneity, which obey the commutation relations

$$e^A_{\mu;\nu}-e^A_{\nu;\mu}=C^A_{BC}e^B_\mu e^C_\nu. \tag{29}$$

The structure constants C^A_{BC} can be used to classify the different models. We shall focus on open or flat models which are asymptotically Friedman. These can be obtained by taking different limits of the type VII_h model which has structure constants

$$C^2_{31}=C^3_{21}=1,\quad C^2_{21}=C^3_{31}=\sqrt{h}. \tag{30}$$

It is convenient to define the parameter $x=\sqrt{h/(1-\Omega_0)}$, which determines the scale on which the principal axes of shear and rotation change orientation. By taking combinations of limits of Ω and x one can obtain Bianchi type-I, V, and VII_0 cosmologies.

We are interested in large-scale anisotropies so it suffices to evaluate the peculiar redshift a photon will feel from the epoch of last scattering (ls) until now (0):

$$\Delta T_A(\hat r)=(\hat r^i u_i)_0-(\hat r^i u_i)_{\rm ls}-\int_{\rm ls}^0 \hat r^j\hat r^k\sigma_{jk}d\tau, \tag{31}$$

where $\hat{\mathbf r}=(\cos\theta\sin\phi,\sin\theta\sin\phi,\cos\phi)$ is the direction vector of the incoming null geodesic, $\mathbf u$ is the spatial part of the fluid four-velocity vector and to first order, the shear is $\sigma_{ij}=\partial_\tau\beta_{ij}$. To evaluate expression (31), one must first of all determine a parameterization of geodesics on this spacetime. This is given by

$$\tan\left(\frac{\phi(\tau)}{2}\right)=\tan\left(\frac{\phi_0}{2}\right)\exp[-(\tau-\tau_0)\sqrt{h}],$$

$$\theta(\tau)=\theta_0+(\tau-\tau_0),$$

$$-\frac{1}{\sqrt{h}}\ln\left\{\sin^2\left(\frac{\phi_0}{2}\right)+\cos^2\left(\frac{\phi_0}{2}\right)\exp[2(\tau-\tau_0)\sqrt{h}]\right\}. \tag{32}$$

Solving Einstein's equations (and assuming that matter is a pressureless fluid) one can determine $\mathbf u$ and σ_{ij}. A general expression for Eq. (31) was determined in [5]:

$$\Delta T_A(\hat{\mathbf r})=\left(\frac{\sigma}{H}\right)_0\frac{2\sqrt{1-\Omega_0}}{\Omega_0}\left\{[\sin\phi_0\cos\theta_0-\sin\phi_{\rm ls}\cos\theta_{\rm ls}(1+z_{\rm ls})]-\int_{\tau_{\rm ls}}^{\tau_0}\frac{3h(1-\Omega_0)}{\Omega_0}\sin2\phi[\cos(\theta)+\sin(\theta)]\frac{d\tau}{\sinh^4(\sqrt{h}\tau/2)}\right\}. \tag{33}$$

---

[4]We are assuming that not only the universe is anisotropic but that we know, *a priori*, what its symmetry axis is; e.g., by the detection of a Hubble-size coherent magnetic field. Alternatively we leave the Euler angles of this axis free, to be estimated by some Maximum Likelihood Estimator.

A useful discussion of the different cosmic microwave background (CMB) patterns imprinted by the unperturbed anisotropic expansion is presented in [5]. The patterns can be roughly said to be constructed out of two ingredients: a focusing of the quadrupole when $\Omega<1$ and a spiral pattern when x is finite. The Bianchi type-VII$_h$ is most general form of homogeneous, anisotropic universes in an $\Omega\leqslant 1$ which are asymptotically Friedmann-Robertson-Walker. The pattern is of the form

$$\frac{\Delta T}{T}=f_1(\theta)\cos[\phi-\tilde{\phi}(\theta)]. \tag{34}$$

In each $\theta=$const circle the pattern has a dependence in ϕ of the form $\cos(\phi-\tilde{\phi})$. The phase $\tilde{\phi}$ depends on θ, and hence the spiralling of the simple cold and hot bump induced by the $\cos\phi$ dependence. The functions $f_1(\theta)$ and $\tilde{\phi}(\theta)$ are rather complicated functions which have to be evaluated numerically, and depend on various details of the particular Bianchi model within the type we have chosen. It is curious to note, however, that only the power spectrum C_l and the phases ϕ are sensitive to these details. All the spirals imprinted by Bianchi type-VII$_h$ models have moduli of the form

$$\rho^l_m=\delta_{m1}f_2(l,x). \tag{35}$$

Therefore their shape spectra will always be

$$B^l_m=1,\quad \text{for } 2\leqslant m\leqslant l,$$

$$B^l_m=0,\quad \text{for } m=1. \tag{36}$$

The background patterns in Bianchi type-VII$_h$ models are all localized, shapeful, and connected structures. Depending on the model they will however have different positions, power spectra, and connectivity. Nevertheless, their shape spectra is always the same exact shape, of form (36), without any cosmic variance error bars. Confusion with a Gaussian is zero. Confusion with the shape of a perfect texture hot spot is zero as well. These have a non-Gaussian spectrum of the form

$$B^l_m=1,\quad \text{for } 1\leqslant m\leqslant l. \tag{37}$$

Although the shape spectrum is the same up to the last B^l_m, the confusion between the two theories is zero. Of course real textures are not perfect circular spots. Cosmic variance in their irregularities will further complicate the problem. Nevertheless some sort of peak around this B^l_m prediction should exist in real life.

VII. ANISOTROPIC GAUSSIAN FLUCTUATIONS IN BIANCHI MODELS

In the previous section we explored the fact that in Bianchi models the CMB is not isotropic even before fluctuations are introduced. A rather non-Gaussian spiral pattern of form (34) is imprinted by the universal rotation on the CMB. This pattern is obviously non-Gaussian and is a localized feature. It is therefore not hard to place tight constraints on anisotropy on the grounds of this prediction. A possibility remains however that anisotropy might manifest itself in a more subtle way. Anisotropic expansion would certainly affect the production of Gaussian fluctuations. From what we have said in Sec. V it is clear that anisotropic Gaussian fluctuations could escape detection much more easily. They are delocalized non-Gaussian features, and most non-Gaussian tests, including our eyes, target localization. As shown in [19] delocalized features escape traditional methods of non-Gaussian detection. These include skewness and kurtosis, the three-point correlation function, density of peaks above a given height, genus number of isotemperature lines, and the ubiquitous plotting of pixel histograms. We will reiterate this point here with one class of anisotropic Gaussian fluctuations.

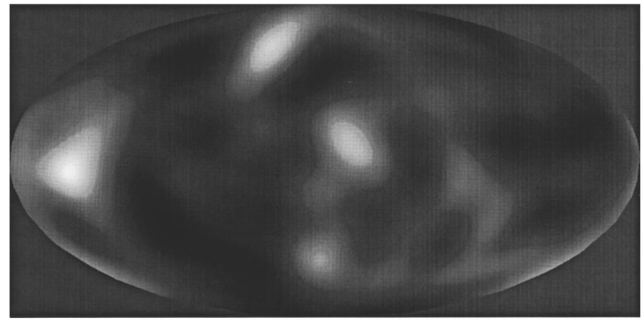

FIG. 1. An isotropic Gaussian map as seen with a resolution of about 20°. The power spectrum was assumed to be scale invariant.

Let us consider anisotropic Gaussian fluctuations with a covariance matrix of form

$$\langle a^l_m a^{l'*}_{m'}\rangle=\delta_{mm'}\delta_{ll'}C_{l|m|}. \tag{38}$$

These theories are characterized by an angular power spectrum which is now a function of two indices, l and m. Isotropic Gaussian theories may be regarded in this context as a particular type of spectrum in the m dimension: they are white noise in m that is, the power is constant as a function of m. Anisotropy manifests itself in the form of departures from white noise in the m dimension of the power spectrum. These departures may in principle take any functional form, but for simplicity let us consider a linear dependence, that is, we merely tilt the white noise spectrum. Then

$$C_{l|m|}=C_l(n|m|-\beta), \tag{39}$$

where β is defined so that C_l is indeed the angular power spectrum. We call n the m-tilt. An isotropic theory has zero m-tilt. A positive m-tilt will favor high azimuthal frequencies, a negative tilt will favor modes which disrupt strict cylindrical symmetry the least, that is low m modes. The larger the m-tilt the more anisotropic the fluctuations are. For every l a critical tilt n^* exists beyond which some m modes do not receive any power. We may consider exceeding this tilt unreasonable.

In Fig. 1 we produced an isotropic Gaussian map with a resolution of 20° and a scale-invariant power spectrum. In Fig. 2 we produced an anisotropic Gaussian map with the same power spectrum, and using the same random numbers. The tilt is the critical negative tilt n^*. It is curious to note how similar the two maps are. The point is that it is the phases that determine what the maps look like and they are the same random phases in both maps. Clearly it will be

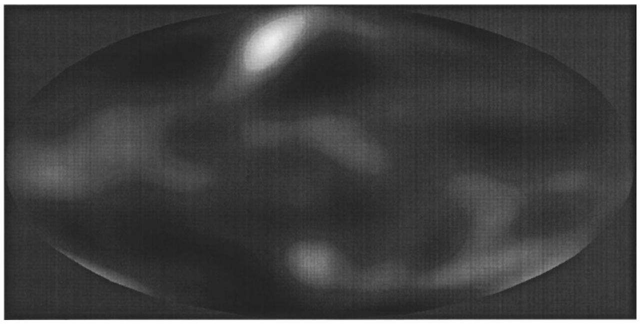

FIG. 2. An anisotropic Gaussian map obtained with the same random numbers as in Fig. 1. However this theory is maximally tilted in m in all scales. Since the azimuthal phases are random, and they are the same in both realizations the outcome looks the same, although this map is extremely anisotropic, or, from the point of view of an isotropic formalism, extremely non-Gaussian.

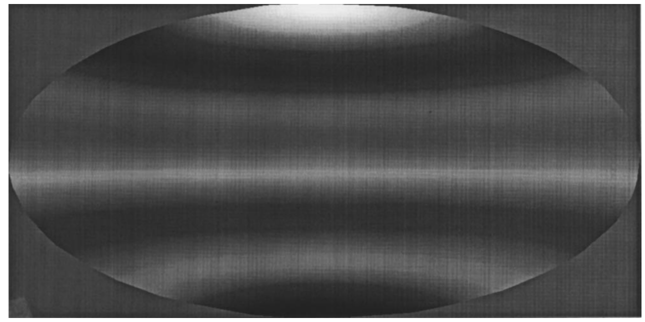

FIG. 3. A theory with exact cylindrical symmetry in all realizations is an extreme case of Gaussian anisotropic fluctuation. This map has still the same random numbers as before. Now it does become obvious that the theory is non-Gaussian isotropic, or anisotropic in the first place.

difficult to identify the anisotropy, or equivalently the non-Gaussianity, of m-tilted maps. They will look very Gaussian and isotropic even for the large tilt n^*.

One would have to do something extreme, such as consider $n=-\infty$, for the anisotropy of m-tilted theories to become obvious. The map in Fig. 3 has the same power spectrum and random number as the two previous maps but it has $n=-\infty$. This means that the mode $m=0$ receives all the power for any l. The surprise now is that the map obtained is Gaussian at all, in some sense. The features shown in this maps appear to be not only blatantly anisotropic, but also very non-Gaussian.

It should not come as a surprise that for any theory with a reasonable m-tilt classic Gaussian tests will fail to detect the non-Gaussianity. We exemplify this with skewness γ_3 and kurtosis γ_4. In all anisotropic Gaussian theories with $n<n^*$ the average γ_3 and γ_4 are always well within the cosmic variance error bar for a Gaussian. In no case could we decide between isotropic and anisotropic Gaussian theories on the grounds of their γ_3 and γ_4. The situation is even worse if n is not too much larger then $n^*/2$. Then the distributions of γ_3 and γ_4 are in practice the same for isotropic and anisotropic Gaussian fluctuations. We show this fact in Fig. 4, where we plotted histograms of skewness and kurtosis from an isotropic Gaussian theory, and an anisotropic Gaussian theory with the same power spectrum and m-tilt $n=-n^*/2/2$. This plot shows that not only that γ_3 and γ_4 cannot be used to distinguish between the theories, but also the the theories are statistically the same as far as γ_3 and γ_4 are concerned.

In spite of the failure of classical tests, the B^l_m should still detect the non-Gaussianity in m-tilted maps. They contain all degrees of freedom in the a^l_m apart from the phases ϕ^l_m which are Gaussian in these theories. Therefore if the initial a^l_m are non-Gaussian the B^l_m have to accuse their non-Gaussianity. In Fig. 5 we plot B^l_m spectra for an isotropic Gaussian theory and for a theory with $n=-n^*/2$. We chose

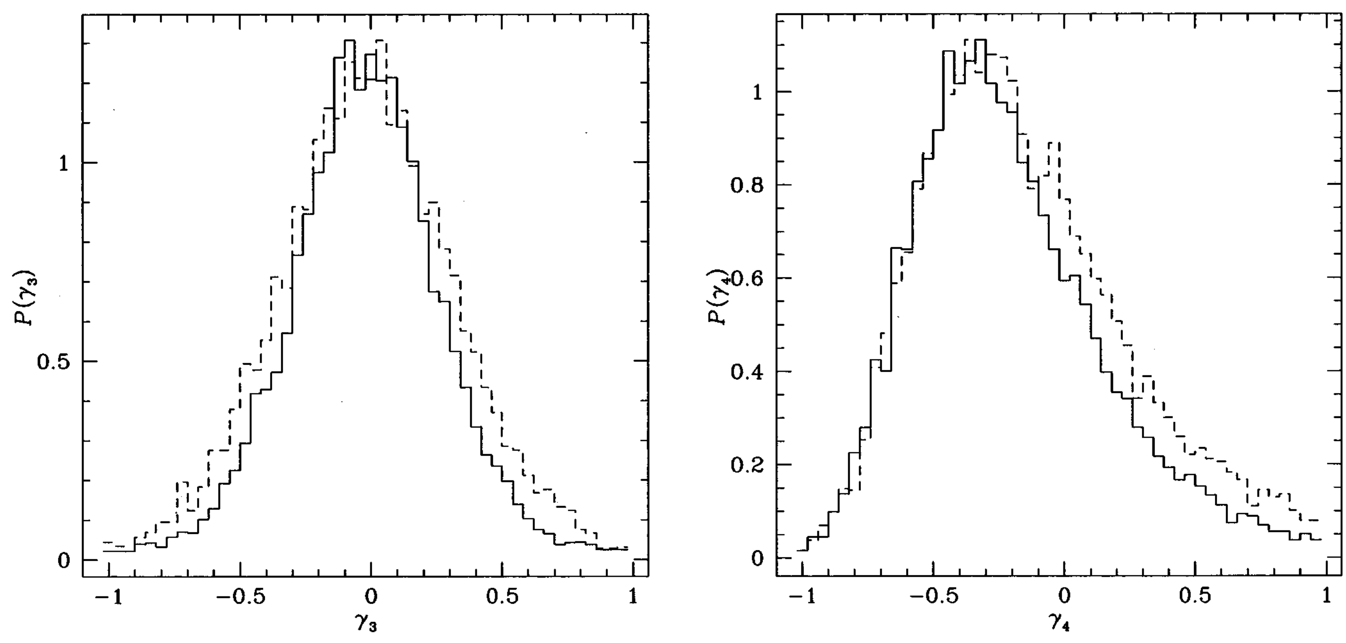

FIG. 4. Histograms of skewness (left) and kurtosis (right) obtained from 8000 realizations for an isotropic Gaussian theory (line) and an anisotropic Gaussian theory with half the maximal transverse tilt n^* on all scales.

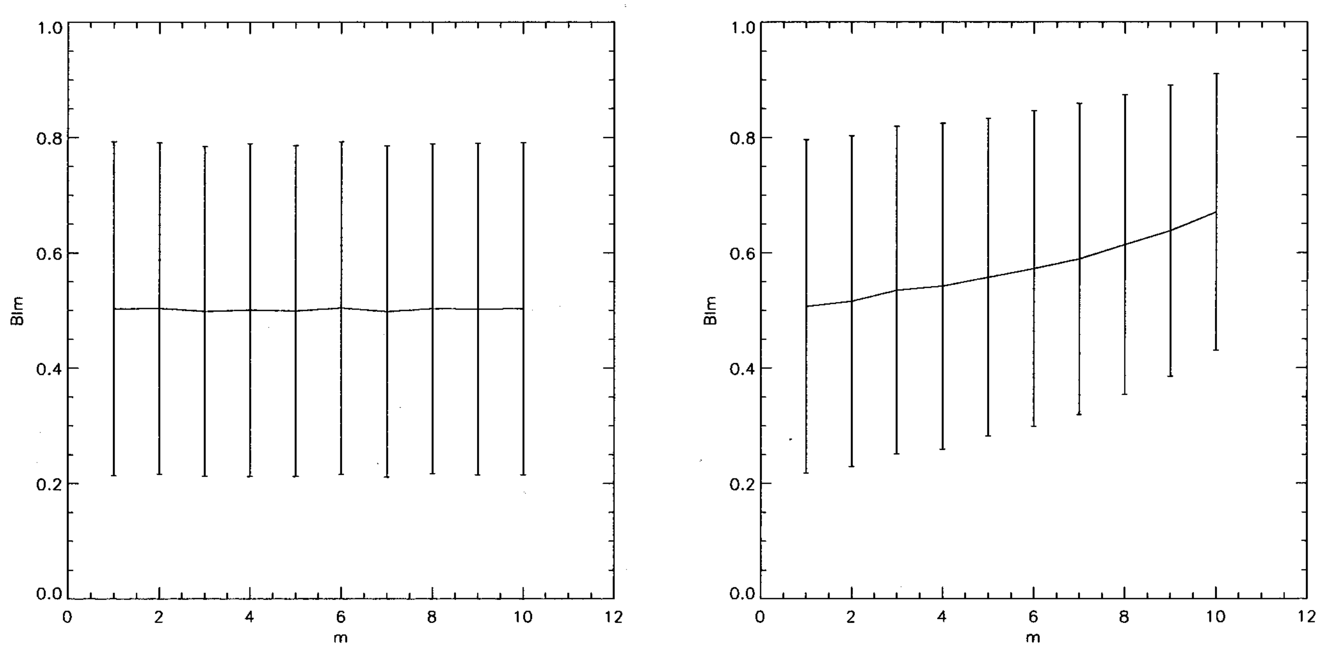

FIG. 5. The B_m^{10} spectra with errorbars for an isotropic Gaussian theory and for an anisotropic Gaussian theory with $n = -n^*/2$. The error bars are misleading as the distributions may not be peaked. In particular the B_m^l distribution in the Gaussian case is simply uniform. Nevertheless it should be obvious that a peaked distribution is emerging in some B_m^l in the bottom plot. These correspond to the emergence of preferred shapes in these theories.

multipole $l=10$ and in analogy with C_l spectra plotted our results in the form of an average and a 1σ error bar. This may be misleading. For instance in the isotropic Gaussian case the distributions are simply uniform distributions, for which the average 1/2 does not represent any favored value where the distribution peaks. Also the distributions for anisotropic Gaussian theories are very skewed and tend to be more peaked than their error bars suggest. Hence this method over rates confusion with a Gaussian. Even so it is clear that departures from uniformity are being observed, at least in some m's. Peaked distributions are emerging and concept of B_m^l spectrum is acquiring a meaning, in the same way C_l spectra are meaningful.

In Fig. 6 we plot histograms of various B_m^l for a tilted

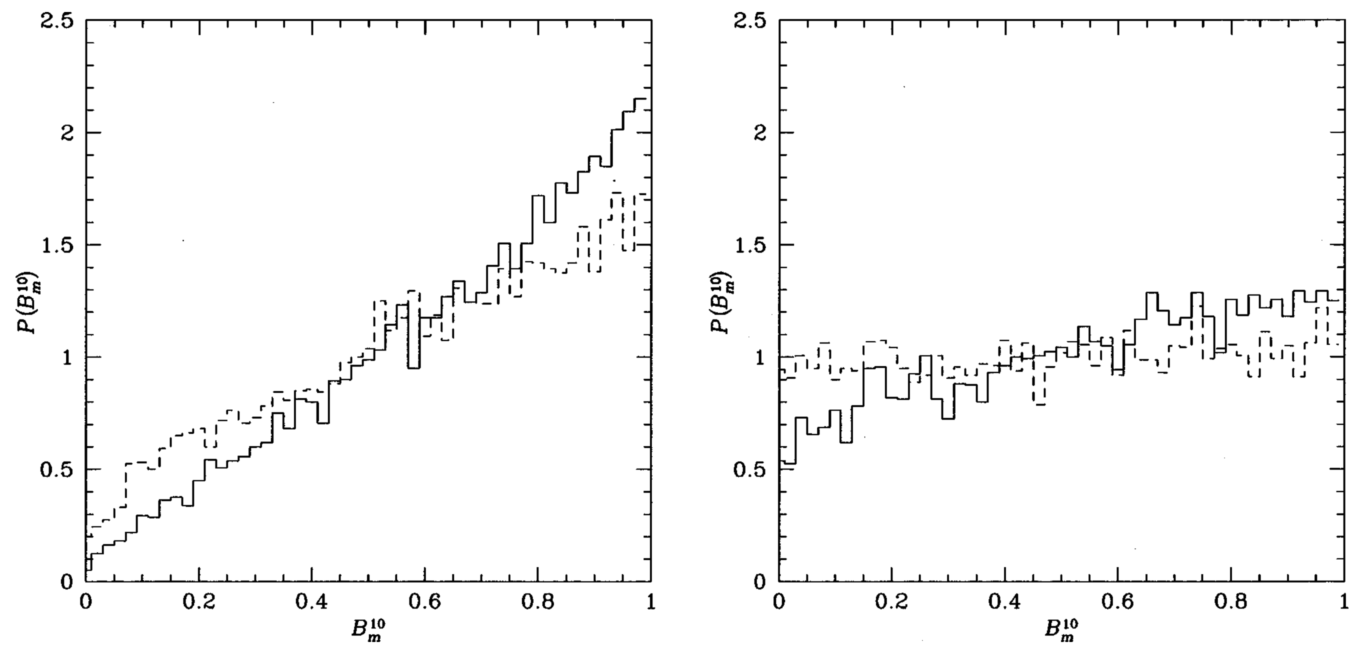

FIG. 6. The distributions of B_9^{10} (line, left), B_7^{10} (dash, left), B_5^{10} (line, right), and B_1^{10} (dash, right), for a tilted theory with $n = -n^*/2$. Although for small m the B_m^l are still very Gaussian, for m close to l the distributions peak around 1. Recall that for an isotropic Gaussian theory all B_m^l should be uniformly distributed.

theory with $n=-n^{*}/2$. These show much more clearly where the peak value is obtained and what the error bar should be for the B^l_m. For m close to l the B^l_m distributions peak at 1, falling off more steeply the closer to l the m. It is only for low m that the distributions become again very close to uniform.

Although we have chosen $n=-n^{*}/2$ in the above discussion the method described applies for lower values of transverse tilt. It is important to realize that the B^l_m contain all the non-Gaussian degrees of freedom in the a^l_m for anisotropic Gaussian theories, and therefore they *have to* detect non-Gaussianity should there be any. One must realize that the B^l_m are not a single statistic but a whole spectrum of statistics, and that even a faint signal in only some of the l and m would combine into a rather powerful signal when the whole spectrum is taken into account.

VIII. AN ANISOTROPIC GRISCHUK-ZEL'DOVICH EFFECT

We shall now consider an example of a flat homogeneous and isotropic universe with topological identification along one axis. This example is simpler than most of those considered in the literature but illustrates one of the key features of such models: the breaking of statistical isotropy in the fluctuations. Let us consider a universe with a topological identification along the z axis. All functions defined on such a space satisfy

$$\Phi(x,y,z)=\Phi(x,y,z+L). \tag{40}$$

By considering a flat universe we can restrict ourselves to calculating the Sachs-Wolfe effect. The temperature anisotropy from the surface of last scattering will be given by

$$\frac{\Delta T}{T}(\mathbf{n})=-\frac{1}{2}\frac{H_0^2}{c^2}\int d^2k\sum_{j=-\infty}^{+\infty}\delta_{\mathbf{k}j}\frac{e^{i\Delta\mathbf{n}\cdot\mathbf{q}}}{q^2}, \tag{41}$$

where $\Delta=\eta_0-\eta_{\rm ls}$ is radius of the surface of last scatter and $q=[k\cos\phi,k\sin\phi,2\pi(j/L)]$.

The a_{lm}'s are given by

$$a_{lm}=-\frac{i^{-l}}{2}\frac{H_0^2}{c^2}\int d^2k\sum_j\delta_{\mathbf{k}j}\frac{j_l(\Delta q)}{q^2}Y^*_{lm}(\hat{q}). \tag{42}$$

We now assume statistical homogeneity and isotropy of δ and a scale invariant power spectrum

$$\langle\delta^*_{\mathbf{k}j}\delta_{\mathbf{k}'j'}\rangle=\delta^2(\mathbf{k}-\mathbf{k}')\delta_{jj'}q^{-1}\qquad q=\sqrt{k^2+(2\pi j/L)^2} \tag{43}$$

This leads to the covariance matrix for the a_{lm}'s:

$$\langle a^*_{lm}a_{l'm'}\rangle=\frac{i^{l'-l}}{4}\frac{H_0^4}{c^4}\int d^2k\sum_j\frac{j_l(q\Delta)j_{l'}(q\Delta)}{q^3}\times Y^*_{lm}(\hat{q})Y_{l'm'}(\hat{q}). \tag{44}$$

Expressing $Y_{lm}(\mathbf{n})=\tilde{P}_{lm}(\cos\theta)e^{im\phi}$ and performing the azimuthal integral, one immediately finds that the covariance matrix is diagonal in m, so one has

$$\langle a^*_{lm}a_{l'm'}\rangle=\frac{i^{l'-l}}{4}\frac{H_0^4}{c^4}\delta_{mm'}\int kdk\sum_j\frac{j_l(q\Delta)j_{l'}(q\Delta)}{q^3}\times\tilde{P}_{lm}(\hat{q})\hat{p}_{l'm'}(\hat{q}). \tag{45}$$

It is convenient to define $\chi=\Delta k$ and $\mu_j=2\pi\alpha j$ where $\alpha=\Delta/L$, i.e., the ration of our horizon to the topological identification scale. If we now define $y=\mu_j/\sqrt{\mu_j^2+\chi^2}$ the expression simplifies to

$$\langle a^*_{lm}a_{l'm'}\rangle=\frac{i^{l'-l}}{4}\frac{H_0^4\Delta}{c^4}\delta_{mm'}\int_0^1 dy\sum_j\frac{j_l\left(\frac{\mu_j}{y}\right)j_{l'}\left(\frac{\mu_j}{y}\right)}{\mu_j}\times\tilde{P}_{lm}(y)\tilde{P}_{l'm'}(y). \tag{46}$$

In the limit where the identification scale goes to infinity we get the standard result

$$\lim_{\alpha\to0}\langle a^*_{lm}a_{l'm'}\rangle\propto\int_0^1 dy\,j_l^2\left(\frac{1}{y}\right)\delta_{ll'}\delta_{mm'}\propto\frac{1}{l^2}\delta_{ll'}\delta_{mm'}, \tag{47}$$

i.e., a scale invariant, diagonal covariance matrix. In the case of finite α this is not the case. Consider the quadrupole. The ring spectra has two components, B_1 and B_2 with a probability distribution function

$$F(r,B_1,B_2)=\frac{r^4}{(\pi/2)^{1/2}\sigma_2^3 3!!}\exp\{-(r^2/2\sigma_2^2)[c_0B_1^2B_2^{3/2}+c_1B_2^{3/2}(1-B_1^2)+c_2(1-B^{3/2}]\}, \tag{48}$$

where we have defined

$$\sigma_2^2=(\langle|a_{20}|^2\rangle\langle|a_{21}|^2\rangle\langle|a_{22}|^2\rangle)^{1/3},$$

$$c_i=\frac{\sigma_2^2}{\langle|a_{2i}|^2\rangle}. \tag{49}$$

By exploring the dependence of c_0, c_1, and c_2 on α we can see how the probability distribution function of the B's change with topology scale; to a very good approximation we find

$$c_0=\frac{1+5(2\alpha)^2}{1+2(2\alpha)^2},\quad c_1=\frac{1+3(2\alpha)^8}{1+5(2\alpha)^8},$$

$$c_2=\frac{1+7(2\alpha)^2}{1+10(2\alpha)^2}. \tag{50}$$

We can see the signature for non-Gaussianity arising here. For a nonzero α there are correlations between the three statistical quantities. The probability distribution function for B_1 and B_2 (defined on $[0,1]\times[0,1]$) becomes peaked at 1. We have focused on the quadrupole where the effect is easy to see. The method is systematic however, and one can construct the probability distribution function of the high order ring spectra in the same way.

To exemplify the strength of the technique we shall contrast it with two standard measures of non-Gaussianity, the

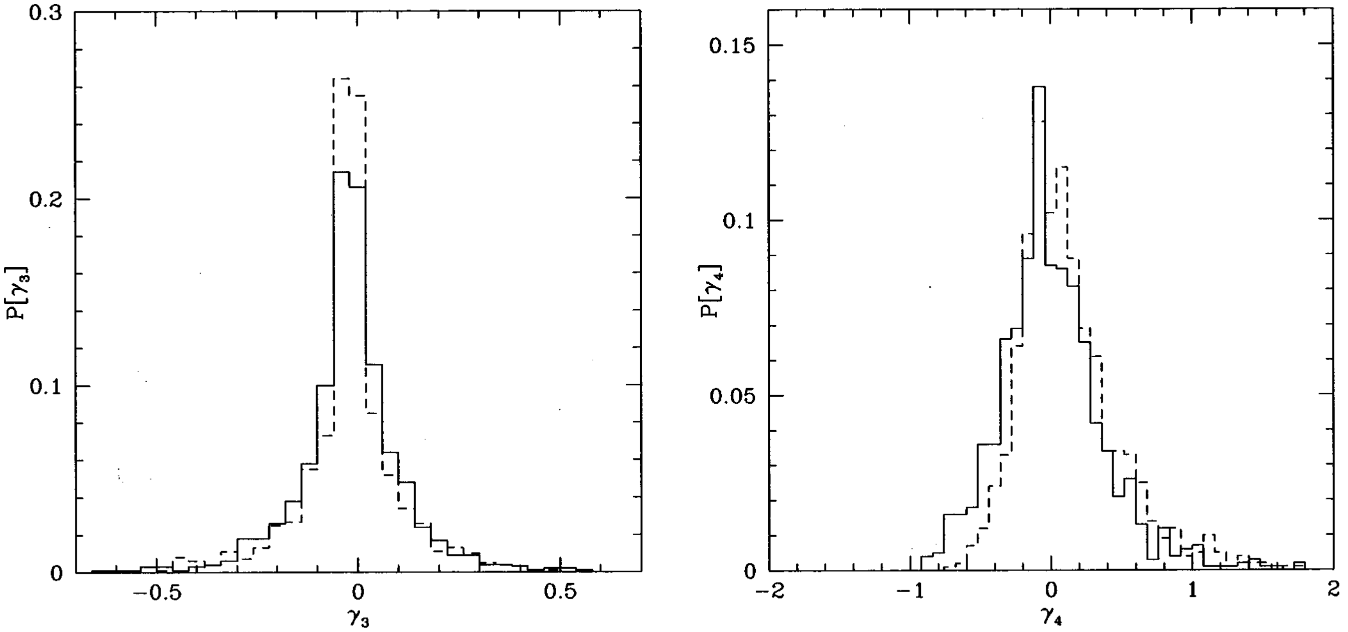

FIG. 7. A comparison of the probability distribution function of skewness (left panel) and kurtosis (right panel) for an anisotropic ensemble (solid line) with $\alpha=1$ and am isotropic, Guassian ensemble (dashed line) with the same power spectrum.

skewness, γ_3, and kurtosis, γ_4. Unlike with our method, where one can calculate the probability distribution function analytically, for these more conventional methods one has to resort to Monte Carlo simulations. The procedure for generating topologically anisotropic skies is a slight modification of the one proposed in [21] for generating isotropic skies. In the latter, one generates an uncorrelated set of a_{lm} (the covariance matrix is diagonal). Given a nondiagonal covariance matrix $\mathcal{C}_{ii'}=\langle a^*_{lm}a_{l'm'}\rangle$ where $i=l(l+1)+m+1$ [$i'=l'(l'+1)+m'+1$], one can Cholesky decompose it, $\mathcal{C}=LL^T$. One can then generate a set of uncorrelated random numbers, **r** from a Gaussian distribution with 0 mean and unit variance. The vector $\mathbf{s}=L\mathbf{r}$ will satisfy $\langle s_i s'_i\rangle=\mathcal{C}_{ii'}$ and therefore is a set of a_{lm} with the correct covariance matrix.

In Fig. 7 we show the results for two ensembles of 1000 realizations. In one ensemble (A) we generate skies with an $\alpha=2$, each realization having a random orientation. The rival, Gaussian ensemble (I) has as many Gaussian realizations of skies with the same power spectrum as ensemble A. Clearly we are well within the region where the probability distribution function for the shape spectra deviates from unity ($c_0\simeq 5/2$, $c_2\simeq 3/5$, and $c_1\simeq 7/10$). In Fig. 7, we plot the histograms for γ_3 and γ_4. Although γ_3 for ensemble A has a marginally larger variance than ensemble B it is clear that in practice it is difficult to disentangle the two. The probability distribution functions for γ_4 are, in practice, indistinguishable. This to be compared to Fig. 8 where we plot the ratio of the joint probability distribution function (for a fixed r) of the completely anisotropic case relative to the completely isotropic case. There is a clear departure from uniformity.

This a curious application of the idea put forward in [22], an anisotropic Grischuk-Zel'Dovich effect. By looking at the shape of the low l multipole moments we can constrain the degree of statistical anisotropy outside the current horizon. Note that, already for $\alpha<1$ there are deformations in the covariance matrix which may be statistically significant with current data. This will be pursued in a future publication [23].

IX. CONCLUSIONS

In this paper we have presented a new technique for quantifying non-Gaussianity on large scales. It is the extension of the non-Gaussian spectra developed in [19] to the surface of the two sphere. As we have shown in Sec. II the construction is slightly different to take into account the particularities of

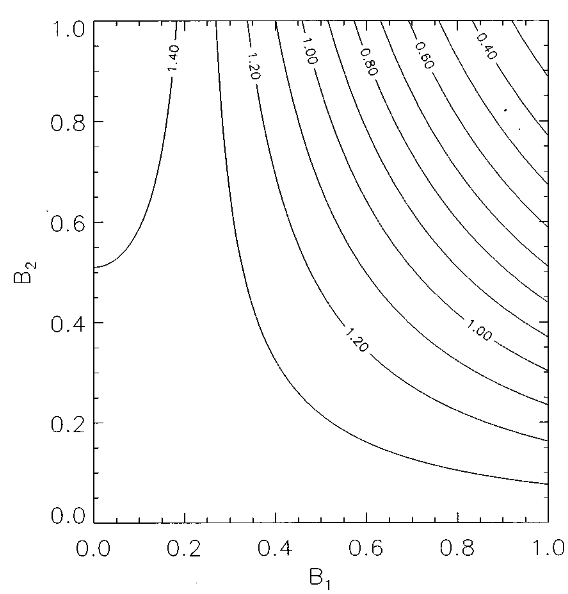

FIG. 8. A comparison of the joint probability distribution function of B_1 and B_2 for the isotropic and completely anisotropic case. For $r^2/(2\sigma_2^2)=1$ we plot ratio of the anisotropic likelihood function to the isotropic likelihood function. Recall that for an isotropic theory the B's are uniformly distributed.

the spherical harmonic basis. However the qualititative interpretation of the different levels of non-Gaussianity follows through, exactly as in [19]. One can identify the information contained in the ring, interring and phase spectra with shape, connectivity and localization.

An interesting and untapped application is to universes with statistically anisotropic fluctuations. Developing the idea put forward in [10] we explain how statistical anisotropy and non-Gaussianity are intimately related. From this one can infer some novel properties of the covariance matrix of fluctuations in statistically anisotropic spacetimes. In particular, features which appear in non-Gaussian theories of structure formation, like textures [11,12] will appear here: a surplus of cosmic variance and cosmic *co*variance of the power spectra.

There are a number of situations where these results are applicable. One is in the case of anisotropic universes, i.e., universes which are not Friedmann-Robertson-Walker universes. There are a number of known examples [1,2]. Without actually doing perturbation theory on them we argue for a natural prescription for adding fluctuations in the CMB to such models. It consists of finding the reduced symmetry group of the temperature patterns and constructing anisotropic Gaussian random fields with such properties (in which the covariance matrix satisfies those symmetries, and not more). In fact it can be shown that these symmetries can be deduced from the geodesic structure of the space time [24]. This can be a first step in extending the prescription used in [6] for constraining general anisotropic models with the Cosmic Background Explorer (COBE) four year data. A brief analysis is made of the relevant Bianchi models for which we present the non-Gaussian spectra.

Another, different application is the case of homogeneous isotropic models where an anisotropic topological identification has been imposed. As an example we identify one direction in space. One finds that statistical isotropy is broken. This can be easily from the following: if we look along the axis of identification, and the identification scale is smaller than our horizon, one will find strong correlations between patches of the microwave sky which are reflected about the uncompactified plane [7]. By looking at the structure of the covariance matrix one can see that this anisotropy will manifest itself by inducing not only non-Gaussian ring spectra but also inter-ring spectra. This non-Gaussian manifestation may persist if we consider the identification scale to be large than our horizon. We name this effect the *anisotropic* Grischuk-Zel'Dovich effect.

Although we now have a high quality measurement of anisotropies on large angular scales we are confronted with the hardships of the real world. Galactic contamination leads one to consider an anisotropic rendition of the sky and considerably complicates the analysis of the COBE four year data. It is well known that one of the consequences is that the quadrupole measurement should be viewed with scepticism. Unfortunately it is the quadrupole which could supply us with a probe of primordial CMB on the largest angular scales. There may be ways around these shortcomings. One can try and reformulate our non-Gaussian spectra on the largest angular scales using the techniques put forward in [25]. This would involve a proper likelihood analysis and to make the problem tractable one would need to find an adequate parametrization of the non-Gaussian spectra. The fact that we have devised a consistent method for characterizing non-Gaussianity on all scales may allow us to use the cumulative information of all $l>2$ to infer the behavior on large scales; all modes will be affected to some extent by large scale anisotropy. Finally it would be interesting to analyze in more detail the observability of the anisotropic Grischuk-Zel'Dovich effect, taking into consideration issues of cosmic variance.

ACKNOWLEDGMENTS

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APPENDIX A: GAUSSIAN FLUCTUATIONS WITH A SYMMETRY

In this appendix we detail the group theory argument sketched in Sec. III. This argument is not necessary in the SO(2) case targeted in this paper, where the covariance matrix may be easily derived directly. However, it opens doors to more general anisotropic symmetry groups. It also shows how the general form of the covariance matrix depends not on the symmetry group, but only on its irreducible representations (irreps).

Fluctuations (Gaussian or not) in any universe must be subject to the symmetries of the underlying cosmological model. However, due to the random nature of the fluctuations, they must satisfy these symmetries only statistically. By this we mean that the statistical ensemble of fluctuations, and not each realization, should be subject to the symmetries. If a symmetry transformation is applied to each member of the ensemble, then each member may change, but the ensemble should remain the same. For instance, for the SO(3) symmetry group only realizations containing only a monopole $l=0$ are left unchanged by rotations. Nevertheless much more general fluctuations respect statistical isotropy. Similarly, only the $m=0$ modes are cylindrically symmetric, but more general fluctuations are SO(2) statistically invariant.[5]

Gaussian fluctuations are fully specified by their covariance matrix

$$C^{l_1 l_2}_{m_1 m_2} = \langle a^{l_1}_{m_1} a^{l_2 *}_{m_2} \rangle, \tag{A1}$$

[5]Some texture models exhibit approximate SO(2) symmetry at low l in each realization. Then an axes system exists in which the $m=0$ mode has much more power than any other. This is of course a very non-Gaussian effect which cannot be simply reproduced by anisotropic Gaussian fluctuations.

which may be seen as a bilinear form on the $\{a^l_m\}$ space. Hence the statistical symmetries of a Gaussian theory are equivalent to the requirement that the covariance matrix is left unchanged by any symmetry transformation. Let G be the symmetry group of the underlying cosmological model as projected on the sky. Let us first suppose that G breaks the $\{a^l_m\}$ into a set on nonequivalent irreps. Then, let us find a new basis a^L_M adapted to G, where L now labels the irrep the basis element belongs to, and M the actual element. Then G is represented by a set of matrices $G^L_{MM'}$ acting on a^L_M as

$$\tilde{a}^L_M = G a^L_M = G^L_{MM'} a^L_{M'} . \tag{A2}$$

The covariance matrix for the a^L_M,

$$C^{L_1L_2}_{M_1M_2} = \langle a^{L_1}_{M_1} a^{L_2*}_{M_2} \rangle, \tag{A3}$$

must remain unchanged by the transformation (A2), so that

$$\tilde{C}^{\,L_1L_2}_{\,M_1M_2} = \langle \tilde{a}^{\,L_1}_{\,M_1} \tilde{a}^{\,L_2*}_{\,M_2} \rangle = G^{L_1}_{M_1M'_1} G^{L_2*}_{M_2M'_2} C^{L_1L_2}_{M'_1M'_2} = C^{L_1L_2}_{M_1M_2}, \tag{A4}$$

which for unitary representations amounts to the commutation relation

$$G^{L_1} C^{L_1L_2} = C^{L_1L_2} G^{L_2}. \tag{A5}$$

Let us now recall Schur's Lemmas [20].

Schur's Lemma 1. Let Γ and Γ' be two irreps of a group G with dimensions d and d', and let there be a $d\times d'$ matrix A such that

$$\Gamma(g) A = A \Gamma'(g) \tag{A6}$$

for all group elements g. Then either $A=0$ or $d=d'$ and $\det A \neq 0$.

It follows that if $A\neq 0$ then Γ and Γ' are equivalent.

Schur's Lemma 2. If Γ is a d-dimensional irrep of a group G and B is a $d\times d$ matrix such that

$$\Gamma(g) B = B \Gamma(g) \tag{A7}$$

for all group elements g, then $B=\lambda 1$.

Combining the two Lemmas we can then find that the covariance matrix must take the form

$$C^{L_1L_2}_{M_1M_2} = \langle \tilde{a}^{\,L_1}_{\,M_1} \tilde{a}^{\,L_2*}_{\,M_2} \rangle = \delta^{L_1L_2} \delta_{M_1M_2} C_{L_1}. \tag{A8}$$

An interesting result is that if G breaks the $\{a^l_m\}$ space into non-equivalent irreps spanned by $\{a^L_M\}$, then these must be independent random variables with a variance which can only depend on the irrep they belong to. This argument applies for instance if the symmetry group is SO(3), in which case $L=l$ and $M=m$.

The argument just present breaks down however, if some of the irreps defined by G are equivalent. Let the $\{a^l_m\}$ space now be spanned by an adapted basis $\{a^{LD}_M\}$, where L labels each class of equivalent representations, D labels the actual irrep within this class, and M labels the elements within each Irrep. Then, from the second Schur's Lemma we know that within the same irrep we still have

$$\langle a^{LD}_{M_1} a^{LD*}_{M_2} \rangle = \delta_{M_1M_2} C^{LD}, \tag{A9}$$

but for different but equivalent irreps ($D_1\neq D_2$) we now have

$$\langle a^{LD_1}_{M_1} a^{LD_2*}_{M_2} \rangle = C^{LD_1D_2}_{M_1M_2} \tag{A10}$$

with $\det C\neq 0$, whereas for nonequivalent irreps ($L_1\neq L_2$) we still have

$$\langle a^{L_1D_1}_{M_1} a^{L_2D_2*}_{M_2} \rangle = 0. \tag{A11}$$

Therefore, although the a^{LD}_M are independent within each irrep and among different non-equivalent irreps, correlations may exist between different but equivalent irreps. Of course one may always rotate the a^{LD}_M within each class of equivalent irreps so as to diagonalize the covariance matrix. However such a rotation is model dependent, and cannot be determined from the symmetries. This situation happens for instance in the case of SO(2), with $L=m$ and $D=l$ (no M index, since the irreps are one dimensional). Each m provides a class of equivalent representations, with the same m but different l. Then the covariance matrix takes the general form

$$\langle a^{l_1}_{m_1} a^{l_2*}_{m_2} \rangle = \delta_{m_1m_2} C^{l_1l_2}_{m_1} \tag{A12}$$

and as we see although correlations among different m are not allowed, now we may have correlations between different l, for the same m. We could rotate the a^l_m in l for each fixed m so as to diagonalize the covariance matrix, but such procedure naturally would depend on the covariance matrix one starts from, and would therefore be model dependent.

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