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Quantum cosmological multidimensional Einstein-Yang-Mills model in an $R\times S^3\times S^d$ topology

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The quantum cosmological version of the multidimensional Einstein-Yang-Mills model in an $\mathbf{R}\times S^3\times S^d$ topology is studied in the framework of the Hartle-Hawking proposal. In contrast with previous work in the literature, we consider Yang-Mills field configurations with nonvanishing time-dependent components in both S^3 and S^d spaces. We obtain stable compactifying solutions that do correspond to extrema of the Hartle-Hawking wave function of the Universe. Subsequently, we also show that the regions where the 4-dimensional metric behaves classically or quantum mechanically (i.e., regions where the metric is Lorentzian or Euclidean) will depend on the number d of compact space dimensions. [S0556-2821(97)08220-9]

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I. INTRODUCTION

The issue of compactification is central in multidimensional theories of unification, such as generalized Kaluza-Klein, supergravity, and superstring theories. Consistency with known phenomenology requires that the extra dimensions in these theories be of Planck size and stable. A necessary condition for the latter is the presence of matter with repulsive stresses to counterbalance the collapsing thrust of gravity. For this purpose, magnetic monopoles [1], Casimir forces [2], and Yang-Mills fields [3,4] have been considered. The situation with Yang-Mills fields is particularly interesting as it illustrates well the importance of considering nonvanishing external-space components of the gauge fields, a point that has been disregarded in previous work in the literature. In fact, it was shown in Ref. [4] that it is precisely this feature that renders compactifying solutions classically as well as semiclassicaly stable.

The main motivation for considering our study of compactification in the context of quantum cosmology lies in ascertaining how this process takes place. Indeed, this is crucial for extracting classical predictions from any multidimensional unifying theories. In fact, no cosmological description can be considered complete until specifying the set of initial conditions for integration of the classical equations of motion. Furthermore, since the quantum cosmological approach of Hartle and Hawking [5] allows for a well-defined program for establishing this set of initial conditions, it is quite natural to extend this approach to the study of the issue of compactification in higher-dimensional theories. This program has been already applied to many different quantum models of interest such as massive scalar fields [6], Yang-Mills fields [7], and massive vector fields [8] as well as in supersymmetric models (see Ref. [9] for a review and a complete set of references) and to the lowest-order gravity-dilaton theory arising from string theory [10]. The generalization of the Hartle-Hawking program to higher spacetime dimensions has been considered previously for the 6-dimensional Einstein-Maxwell theory [11], for gravity coupled with a $(D-4)$th rank antisymmetric tensor field [12], where the stability of compactification was achieved thanks to the presence of a magnetic-monopole-type configuration, and also to 11-dimensional supergravity [13].

In this work a rather general and realistic setting to study the compactification process is considered in the context of the Einstein-Yang-Mills multidimensional model of Ref. [4] with an SO(N) gauge field in $D=4+d$ dimensions and an homogeneous and (partially) isotropic spacetime with a $\mathbf{R}\times S^3\times S^d$ topology. We aim to study the quantum mechanics of the coset compactification of the D-dimensional spacetime $\mathcal{M}^D$:

$$\mathcal{M}^D=\mathbf{R}\times G^{\rm ext}/H^{\rm ext}\times G^{\rm int}/H^{\rm int}, \tag{1}$$

where $G^{\rm ext(int)}=\mathrm{SO}(4)[\mathrm{SO}(d+1)]$ and $H^{\rm ext(int)}=\mathrm{SO}(3)[\mathrm{SO}(d)]$ are, respectively, the homogeneity and isotropy groups in $3(d)$ dimensions. For this purpose we will seek compactifying solutions of the Wheeler-DeWitt equation for the Einstein-Yang-Mills cosmological model of Ref. [4] in the framework of the Hartle-Hawking proposal.

In contrast with previous work in the literature [11,12] we consider Yang-Mills field configurations with *nonvanishing* time-dependent components in *both* S^3 and S^d spaces. We then derive an effective model by restricting the fields to be homogeneous and isotropic. This construction will allow us to study in detail the issue of compactification, which as discussed in Ref. [4], depends crucially in the contribution of the *external* gauge field components. Our analysis of the resulting Wheeler-DeWitt equation indicates that the regions

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where the metric is Lorentzian or Euclidean do depend on the number d of internal dimensions *and* on the potentials for the external and internal components of the gauge field. Furthermore, we show that stable compactifying solutions do indeed correspond to the extrema of the wave function of the Universe, implying a correlation between compactification of the extra dimensions and expansion of the macroscopic spacetime. We should mention that an attractive feature of our model is that it can be regarded as the bosonic sector of some general unifying theories, implying that possibly most of the conclusions of our quantum-mechanical analysis of the compactification process and of its stability will remain valid in those theories as well.

This paper is then organized as follows. In the next section we present our *Ansätze* for the metric and for the gauge field (see Refs. [4,14] for a general discussion) as well as the resulting effective action which is the starting point of our analysis. We also obtain in that section the Wheeler-DeWitt equation of our effective model. In Sec. III we present and discuss compactifying solutions of the Wheeler-DeWitt equation and in Sec. IV we discuss their interpretation. In Sec. V we present our conclusions. We also include an Appendix where the mathematical aspects of extending the Hartle-Hawking proposal to higher-dimensional spacetimes is described, with emphasis in our model where hypersurfaces are of $\Sigma_{D-1}\sim S^3\times S^d$ type.

II. EFFECTIVE MODEL AND WHEELER-DE WITT EQUATION

We shall describe in this section our multidimensional Einstein-Yang-Mills quantum cosmological model. Special emphasis will be given to the differences between our model and others present in the literature [11–13]. Namely, the gauge field in our reduced model will have time-dependent spatial components on the 3-dimensional physical space. This contrasts with previous work on the subject where either static magnetic-monopole-type configurations, whose only nonvanishing components were the internal d-dimensional ones [11,12], or scalar fields [15] were considered. Our approach provides therefore a somewhat more realistic model to study the influence of higher dimensions on the evolution of the 4-dimensional physical spacetime. In addition, we shall also see how *different* values for d, the number of internal space dimensions, may induce fairly different physical situations.

Our model is derived from the generalized Kaluza-Klein action:

$$S[\hat{g}_{\hat{\mu}\hat{\nu}},\hat{A}_{\hat{\mu}},\hat{\chi}]=S_{\rm gr}[\hat{g}_{\hat{\mu}\hat{\nu}}]+S_{\rm gf}[\hat{g}_{\hat{\mu}\hat{\nu}},\hat{A}_{\hat{\mu}}]+S_{\rm inf}[\hat{g}_{\hat{\mu}\hat{\nu}},\hat{\chi}], \tag{2}$$

with

$$S_{\rm gr}[\hat{g}_{\hat{\mu}\hat{\nu}}]=\frac{1}{16\pi\hat{k}}\int_{\mathcal{M}^D}d\hat{x}\sqrt{-\hat{g}}(\hat{R}-2\hat{\Lambda}), \tag{3}$$

$$S_{\rm gf}[\hat{g}_{\hat{\mu}\hat{\nu}},\hat{A}_{\hat{\mu}}]=\frac{1}{8\hat{e}^2}\int_{\mathcal{M}^D}d\hat{x}\sqrt{-\hat{g}}{\rm Tr}\hat{F}_{\hat{\mu}\hat{\nu}}\hat{F}^{\hat{\mu}\hat{\nu}}, \tag{4}$$

$$S_{\rm inf}[\hat{g}_{\hat{\mu}\hat{\nu}},\hat{\chi}]=-\int_{\mathcal{M}^D}d\hat{x}\sqrt{-\hat{g}}\left[\frac{1}{2}(\partial_{\hat{\mu}}\hat{\chi})^2+\hat{U}(\hat{\chi})\right], \tag{5}$$

where $\hat{g}$ is $\det(\hat{g}_{\hat{\mu}\hat{\nu}})$, $\hat{g}_{\hat{\mu}\hat{\nu}}$ is the $D=4+d$ dimensional metric, $\hat{R}$, $\hat{e}$, $\hat{k}$, and $\hat{\Lambda}$ are, respectively, the scalar curvature, gauge coupling, gravitational, and cosmological constants in D dimensions. In addition, the following field variables are defined in $\mathcal{M}^D$: $\hat{F}_{\hat{\mu}\hat{\nu}}=\partial_{\hat{\mu}}\hat{A}_{\hat{\nu}}-\partial_{\hat{\nu}}\hat{A}_{\hat{\mu}}+[\hat{A}_{\hat{\mu}},\hat{A}_{\hat{\nu}}]$ is the field strength, $\hat{A}_{\mu}$ denotes the components of the gauge field, and $\hat{\chi}$ is the inflaton responsible for the inflationary expansion of the external space with $\hat{U}(\hat{\chi})$ being the potential for $\hat{\chi}$. We assume that the potential $\hat{U}(\hat{\chi})$ is bounded from below, that it has a global minimum, and without loss of generality that $\hat{U}_{\rm min}=0$. As first suggested in Ref. [16], the splitting of the internal and external dimensions of space in the generalized Kaluza-Klein theory (2) can have its origin in the spontaneous symmetry-breaking process, which is due to vacuum solutions corresponding to a factorization of spacetime in a product of spaces. Assuming that is indeed the case, then

$$\mathcal{M}^D=M^4\times I^d, \tag{6}$$

M^4 being the 4-dimensional Minkowski spacetime and I^d a Planck-size d-dimensional compact space. For the cosmological setting we are interested in consider instead

$$\mathcal{M}^{4+d}={\bf R}\times G^{\rm ext}/H^{\rm ext}\times G^{\rm int}/H^{\rm int}, \tag{7}$$

admiting local coordinates $\hat{x}^{\hat{\mu}}=(t,x^i,\xi^m)$, where $\hat{\mu}=0,1,\ldots,3+d$, $i=1,2,3$; $m=4,\ldots,d+3$, where $\bf R$ denotes a timelike direction and $G^{\rm ext}/H^{\rm ext}(G^{\rm int}/H^{\rm int})$ the space of external (internal) spatial dimensions realized as a coset space of the external (internal) isometry group $G^{\rm ext}\,(G^{\rm int})$.

We restrict ourselves to spatially homogeneous and (partially) isotropic field configurations, which means that these are symmetric under the action of the group $G^{\rm ext}\times G^{\rm int}$. Let the gauge group $\hat{K}$ of the D-dimensional theory be a simple Lie group. For definiteness, let us consider the case with the gauge group $\hat{K}={\rm SO}(N)$, $N\geqslant 3+d$ and

$$\mathcal{M}^{4+d}={\bf R}\times S^3\times S^d, \tag{8}$$

where $S^3(S^d)$ is the 3- (d-) dimensional sphere. The group of spatial homogeneity and isotropy is, in this case,

$$G^{\rm HI}={\rm SO}(4)\times{\rm SO}(d+1), \tag{9}$$

while the group of spatial isotropy is

$$H^{\rm I}={\rm SO}(3)\times{\rm SO}(d), \tag{10}$$

which allows for the alternative realization of $\mathcal{M}^{4+d}$:

$$\begin{aligned}\mathcal{M}^{4+d}&={\bf R}\times{\rm SO}(4)/SO(3)\times{\rm SO}(d+1)/{\rm SO}(d)\\&={\bf R}\times[{\rm SO}(4)\times{\rm SO}(d+1)]/[{\rm SO}(3)\times{\rm SO}(d)].\end{aligned} \tag{11}$$

The field configurations associated with the above geometry were described in Ref. [4], using the theory of symmetric fields (see also Refs. [7,17,18]). The most general form of a SO(4)×SO($d+1$)-invariant metric in E^{4+d} as Eq. (8) reads

$$\hat{g}=-\tilde{N}^2(t)dt^2+\tilde{a}^2(t)\Sigma_{i=1}^{3}\omega^i\omega^i+b^2(t)\Sigma_{m=4}^{d+3}\omega^m\omega^m, \tag{12}$$

where the scale factors $\tilde{a}(t)$, $b(t)$ and the lapse function $\tilde{N}(t)$ are arbitrary nonvanishing functions of time. Moreover, ω^α denote local moving coframes in $S^3\times S^d$, and $\Sigma_{i=1}^3\omega^i\omega^i$ and $\Sigma_{m=4}^{d+3}\omega^m\omega^m$ coincide with the standard metrics $d\Omega_3^2$ and $d\Omega_d^2$ of 3- and d-dimensional spheres with local coordinates (x^i,ξ^m), respectively.

The SO(4)×SO($d+1$)-invariant *Ansatz* for the inflaton field $\hat{\chi}$ reads

$$\hat{\chi}(t,x^i,\xi^m)=\hat{\chi}(t). \tag{13}$$

As for the SO(4)×SO($d+1$)-*symmetric* gauge field, the following *Ansatz* is considered:

$$\hat{A}=\frac{1}{2}\Sigma_{p,q=1}^{N-3-d}B^{pq}(t)\mathcal{T}^{(N)}_{3+d+p\ 3+d+q}dt+\frac{1}{2}\Sigma_{1\le i<j\le 3}\mathcal{T}^{(N)}_{ij}\omega^{ij}$$
$$+\frac{1}{2}\Sigma_{4\le m<n\le 3}\mathcal{T}^{(N)}_{mn}\tilde{\omega}^{m-3\ n-3}$$
$$+\Sigma_{i=1}^{3}\left[\frac{1}{4}f_0(t)\Sigma_{j,k=1}^{3}\epsilon_{jik}\mathcal{T}^{(N)}_{jk}\right.$$
$$\left.+\frac{1}{2}\Sigma_{p=1}^{N-3-d}f_p(t)\mathcal{T}^{(N)}_{i\ d+3+p}\right]\omega^i$$
$$+\Sigma_{m=4}^{d+3}\left[\frac{1}{2}\Sigma_{q=1}^{N-3-d}g_q(t)\mathcal{T}^{(N)}_{m\ d+3+q}\right]\omega^m, \tag{14}$$

where $f_0(t)$, $f_p(t),p=1,\dots,N-3-d;g_q(t),q=1,\dots,N-3-d;B^{pq}(t),1\le p<q\le N-3-d$ are arbitrary functions; and $\mathcal{T}^{(N)}_{pq},1\le p<q\le N$ are the generators of the gauge group SO(N). We have used the decomposition

$$\omega=\Sigma_{\alpha=1}^{d+3}\omega^\alpha T_\alpha+\Sigma_{1\le i<j\le 3}\omega^{ij}\frac{T_{ij}^{(4)}}{2}+\Sigma_{1\le m<n\le d}\tilde{\omega}^{mn}\frac{\tilde{T}_{mn}^{(d+1)}}{2} \tag{15}$$

for the Cartan's one-form in $S^3\times S^d$. Here $T_{ij}^{(4)}$ and $\tilde{T}_{mn}^{(d+1)}$ form a basis of the Lie algebra of G^{HI}, $T_\alpha=T_{\alpha 4}^{(4)}/2$, $\alpha=1,2,3$ and $T_\alpha=T^{(d+1)}_{\alpha-3\ d=1}/2$, $\alpha=4,\dots,d+3$.

Substituting the *Ansätze* (12), (13), and (14) into action (2) and performing the conformal changes

$$\tilde{N}^2(t)=\left[\frac{b_0}{b(t)}\right]^d N^2(t), \tag{16}$$

$$\tilde{a}^2(t)=\left[\frac{b_0}{b(t)}\right]^d a^2(t), \tag{17}$$

where b_0 denotes the equilibrium value of b, we obtain a 1-dimensional effective reduced action for the functions of time that parametrize the *symmetric* field configurations [4]:

$$S_{\mathrm{eff}}[a,\psi,f_0,\mathbf{f},\mathbf{g},\chi,N,\hat{B}]$$
$$=16\pi^2\int dtNa^3\left\{-\frac{3}{8\pi k}\frac{1}{a^2}\left[\frac{\dot{a}}{N}\right]^2+\frac{3}{32\pi k}\frac{1}{a^2}+\frac{1}{2}\left[\frac{\dot{\psi}}{N}\right]^2\right.$$
$$+\frac{1}{2}\left[\frac{\dot{\chi}}{N}\right]^2+e^{d\beta\psi}\frac{3}{4e^2}\frac{1}{a^2}\left(\frac{1}{2}\left[\frac{\dot{f}_0}{N}\right]^2+\frac{1}{2}\left[\frac{\mathcal{D}_t\mathbf{f}}{N}\right]^2\right)$$
$$\left.+e^{-2\beta\psi}\frac{d}{4e^2}\frac{1}{b_0^2}\frac{1}{2}\left[\frac{\mathcal{D}_t\mathbf{g}}{N}\right]^2-W(a,\psi,f_0,\mathbf{f},\mathbf{g},\chi)\right\}, \tag{18}$$

with $k=\hat{k}/v_d b_0^d, e^2=\hat{e}^2/v_d b_0^d, \beta=\sqrt{16\pi k/d(d+2)}, v_d$ is the the volume of S^d for $b=1$, and where we have used $\psi=\beta^{-1}\ln(b/b_0)$ and $\chi=\sqrt{v_d b_0^d}\hat{\chi}$ as the dilaton[1] and inflaton fields, respectively. In Eq. (18), the overdots denote time derivatives and $\mathcal{D}_t$ is the covariant derivative with respect to the remnant SO($N-3-d$) gauge field $\hat{B}(t)$ in $\mathbf{R}$:

$$\mathcal{D}_t\mathbf{f}(t)=\frac{d}{dt}\mathbf{f}(t)+\hat{B}(t)\mathbf{f}(t),\quad \mathcal{D}_t\mathbf{g}(t)=\frac{d}{dt}\mathbf{g}(t)+\hat{B}(t)\mathbf{g}(t). \tag{19}$$

Notice that $f_0(t),\mathbf{f}=\{f_p\}$ represent the gauge field components in the 4-dimensional physical space-time, while $\mathbf{g}=\{g_q\}$ denotes the components in the space I^d and $\hat{B}$ is an $(N-3-d)\times(N-3-d)$ antisymmetric matrix $\hat{B}=(B_{pq})$. The potential W in Eq. (18) is given by

$$W=e^{-d\beta\psi}\left[-e^{-2\beta\psi}\frac{1}{16\pi k}\frac{d(d-1)}{4}\frac{1}{b_0^2}\right.$$
$$\left.+e^{-4\beta\psi}\frac{1}{b_0^4}\frac{d(d-1)}{8e^2}V_2(\mathbf{g})+\frac{\Lambda}{8\pi k}+U(\chi)\right]$$
$$+e^{-2\beta\psi}\frac{1}{(ab_0)^2}\frac{3d}{32e^2}(\mathbf{f}\cdot\mathbf{g})^2+e^{d\beta\psi}\frac{3}{4e^2a^4}V_1(f_0,\mathbf{f}), \tag{20}$$

where $\Lambda=v_d b_0^d\hat{\Lambda}$, $U(\chi)=v_d b_0^d\hat{U}(\hat{\chi}/\sqrt{v_d b_0^d})$ and

$$V_1(f_0,\mathbf{f})=\frac{1}{8}[(f_0^2+\mathbf{f}^2-1)^2+4f_0^2\mathbf{f}^2], \tag{21}$$

$$V_2(\mathbf{g})=\frac{1}{8}(\mathbf{g}^2-1)^2 \tag{22}$$

are related to the external and internal components of the gauge field, respectively. The variables N and $\hat{B}$ are Lagrange multipliers associated with the symmetries of the effective action (18). The lapse function N is associated with

[1]The scale factor $b(t)$ of the internal space induces a behavior similar to the case of a minimally coupled scalar field. In fact, by introducing the field ψ by $b\sim\exp\psi$, this quantity corresponds to the scalar field which appears in the harmonic expansion of the Kaluza-Klein theory.

the invariance of $S_{\rm eff}$ under arbitrary time reparametrizations, while $\hat{B}$ is connected with the local remnant SO$(N-d-3)$ gauge invariance. The equations of motion for the physical variables a, ψ, χ, f_0, $\mathbf{f}$, and $\mathbf{g}$ can be found in Ref. [4].

The canonical conjugate momenta associated with the canonical variables in model (18) are given by

$$\pi_a = -\frac{12\pi}{k}\frac{a}{N}\dot{a}, \quad \pi_\psi = 16\pi^2\frac{a^3}{N}\dot{\psi}, \quad \pi_\chi = 16\pi^2\frac{a^3}{N}\dot{\chi}, \tag{23}$$

$$\pi_{f_0} = \frac{12\pi^2}{e^2}e^{d\beta\psi}\frac{a}{N}\dot{f}_0, \quad \pi_{\mathbf{f}} = \frac{12\pi^2}{e^2}e^{d\beta\psi}\frac{a}{N}\mathcal{D}_t\mathbf{f},$$

$$\pi_{\mathbf{g}} = \frac{4\pi^2}{e^2 b_0^2}e^{-2\beta\psi}\frac{a^3}{N}\mathcal{D}_t\mathbf{g}. \tag{24}$$

For simplicity we replace[2] the variables (a,ψ,χ) by the new variables (μ,ϕ,ξ):

$$a = e^\mu\left(\frac{k}{6\pi}\right)^{1/2}, \quad \psi = \phi\left(\frac{3}{4\pi k}\right)^{1/2}, \quad \chi = \xi\left(\frac{3}{4\pi k}\right)^{1/2}. \tag{25}$$

The corresponding new conjugate momenta then read

$$\pi_\mu = -\left(\frac{2k}{3\pi}\right)^{1/2}\frac{e^{3\mu}}{N}\dot{\mu}, \quad \pi_\phi = \left(\frac{2k}{3\pi}\right)^{1/2}\frac{e^{3\mu}}{N}\dot{\phi},$$

$$\pi_\xi = \left(\frac{2k}{3\pi}\right)^{\frac{1}{2}}\frac{e^{3\mu}}{N}\dot{\xi}. \tag{26}$$

The Hamiltonian and SO$(N-3-d)$ gauge constraints are then obtained by varying Eq. (18) with respect to N and $\hat{B}$, and in terms of the momenta (26) are given by

$$-\pi_\mu^2 - e^{4\mu} + \pi_\phi^2 + \pi_\xi^2 + e^{2\mu-d\alpha\phi}\frac{e^2}{6\pi^2}[\pi_{f_0}^2 + \pi_{\mathbf{f}}{}^2] + e^{2\alpha\phi}\frac{3e^2b_0^2}{d\pi k}\pi_{\mathbf{g}}{}^2 + e^{6\mu}\left(\frac{4k}{3}\right)^2 W = 0, \tag{27}$$

$$\pi_{f_p}f_q + \pi_{g_p}g_q - \pi_{f_q}f_p - \pi_{g_q}g_p = 0, \tag{28}$$

where $\alpha = \sqrt{12/d(d+2)}$.

The canonical quantization follows by promoting the conjugate momenta into operators as

$$\pi_\mu \mapsto -i\frac{\partial}{\partial\mu}, \quad \pi_\phi \mapsto -i\frac{\partial}{\partial\phi}, \quad \pi_\xi \mapsto -i\frac{\partial}{\partial\xi},$$

$$\pi_{f_0} \mapsto -i\frac{\partial}{\partial f_0}, \quad \pi_{\mathbf{f}} \mapsto -i\frac{\partial}{\partial\mathbf{f}}, \quad \pi_{\mathbf{g}} \mapsto -i\frac{\partial}{\partial\mathbf{g}}. \tag{29}$$

The Hamiltonian constraint (27) is then quantized to yield the Wheeler-DeWitt equation:

$$\left\{\frac{\partial^2}{\partial\mu^2} - e^{4\mu} - \frac{\partial^2}{\partial\phi^2} - \frac{\partial^2}{\partial\xi^2} - e^{2\mu-d\alpha\phi}\frac{e^2}{6\pi^2}\left[\frac{\partial^2}{\partial f_0^2} + \frac{\partial^2}{\partial\mathbf{f}^2}\right] - e^{2\alpha\phi}\frac{3e^2b_0^2}{d\pi k}\frac{\partial^2}{\partial\mathbf{g}^2} + e^{6\mu}\left(\frac{4k}{3}\right)^2 W\right\}\Psi = 0, \tag{30}$$

where in the usual parametrization of the factor ordering ambiguity, $\pi_\mu^2 \mapsto -\mu^{-p}(\partial/\partial\mu)(\mu^p(\partial/\partial\mu))$, we have set $p=0$.

The richness of the effective model (18) and the corresponding Wheeler-DeWitt equation (30) is quite evident. In this reduced model the gauge field has nonvanishing time-dependent components in *both* the external and internal spaces. Moreover, we have also two time-dependent scalar fields, the dilaton and the inflaton. This contrasts with previous work in the literature, where either static magnetic monopole configurations with nonzero components only in I^d or scalar fields were present. Our model allows thus to consider several possibilities.

Aiming to study the compactification process we shall focus our analysis on the variables μ and ϕ and the contributions to the potential W from the gauge field. This choice is justifiable as it can be seen from Eq. (30) that the kinetic term for the external components of the gauge field is suppressed in an expanding Universe, while for the internal components the kinetic term is not relevant as compactifying solutions require $\mathbf{g}$ to seat at the extremum of the potential $V_2(\mathbf{g})$ [4]. In doing that, we shall keep the inflaton field frozen as it has been shown that this field does not affect the compactification process [4]. Of course, we could instead consider taking μ and χ as the physically relevant variables and freeze the remaining ones and actually models of this type have been studied in Ref. [15].

Hence, in what follows we shall restrict ourselves to the study of compactification and hence concentrate our study on the subsystem where the relevant variables are μ and ϕ. Hence, it requires solving the Wheeler-DeWitt equation (30) for the static vacuum configuration of the gauge and inflaton fields:

$$\xi = \xi^v, \quad f_0 = f_0^v, \quad \mathbf{f} = \mathbf{f}^v, \quad \mathbf{g} = \mathbf{g}^v = \mathbf{0}; \tag{31}$$

we also assume that $U(\xi^v)=0$ and that $\mathbf{f}$ and $\mathbf{g}$ are orthogonal. The notation $v_1 \equiv V_1(f_0^v,\mathbf{f}^v)$ and $v_2 \equiv V_2(\mathbf{g}^v) = \frac{1}{8}$ will be used throughout this paper. The Wheeler-DeWitt equation suitable for the study of compactification is the following:

$$\left[\frac{\partial^2}{\partial\mu^2} - \frac{\partial^2}{\partial\phi^2} + U(\mu,\phi)\right]\Psi(\mu,\phi) = 0, \tag{32}$$

where

[2]The replacement $\psi\to\phi$ and $\chi\to\xi$ is a mere rescaling, while introducing $\mu\to\ln a$ for the scale factor can bring some advantages. In fact, the minisuperaspace metric becomes then proportional to diag$(1,-1)$ with useful consequences as far as the Wheeler-DeWitt equation is concerned [19].

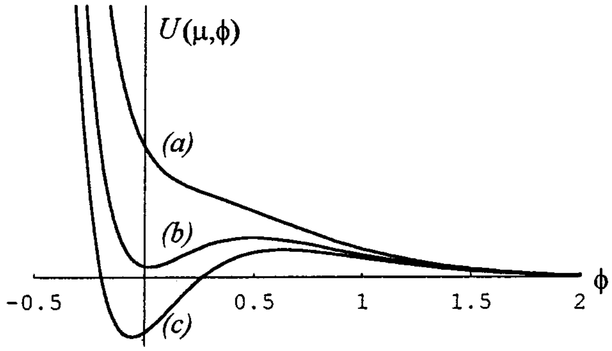

FIG. 1. Potential $U(\mu=\text{const},\phi)$ for some values of Λ and $d=6$ [(a) $\Lambda>c_2/16\pi k$, (b) $c_1/16\pi k<\Lambda<c_2/16\pi k$, (c) $\Lambda<c_1/16\pi k$].

$$U(\mu,\phi)=e^{6\mu}\left(\frac{4k}{3}\right)^2\Omega(\mu,\phi)-e^{4\mu} \tag{33}$$

and

$$\Omega(\mu,\phi)=e^{-d\alpha\phi}\left[-e^{-2\alpha\phi}\frac{1}{16\pi k}\frac{d(d-1)}{4}\frac{1}{b_0^2}+e^{-4\alpha\phi}\frac{1}{b_0^4}\frac{d(d-1)}{8e^2}v_2+\frac{\Lambda}{8\pi k}\right]+e^{d\alpha\phi-4\mu}\left(\frac{6\pi}{k}\right)^2\frac{3}{4e^2}v_1. \tag{34}$$

The scenario associated with this choice is analogous to the ones of Refs. [11–13,20], with the novel feature of taking into account the external components of the gauge field. As it will be seen, the last term in Eq. (34) is central in our model and constitutes one of the *major differences* with respect to, for instance, Ref. [11]. Indeed, it is precisely this term sets the dependence of early Universe scenarios ($\mu\ll 0$, i.e., $a\to 0$) on different values of d and v_1, brought about by the gauge field components in the 4-dimensional spacetime.

Moreover, as it will be discussed in the next section, it is the term $e^{d\alpha\phi-4\mu}(6\pi/k)^2 3/4e^2 v_1$ in Eq. (34) that establishes that the external spatial dimensions and the internal d dimensions are at the *same footing* in the early Universe prior to compactification, i.e., when $\mu\ll 0$. It is only through the expansion of the external dimensions (increase of μ) that compactification ($b\to b_0$) is achieved. Thus, it is the dynamics of the 3-dimensional physical space which induces the evolution of I^d towards compactification. Furthermore, we shall see how different values for v_1 and d do lead to different quantum scenarios, i.e., solutions of the Wheeler-DeWitt equation, whose physical features can be compared with those of Refs. [11,12].

III. SOLUTIONS WITH DYNAMICAL COMPACTIFICATION

In this section we shall establish the boundary conditions for the Wheeler-DeWitt equations (32)–(34) and obtain solutions with dynamical compactification for certain regions of the $\mu\phi$ plane. Let us first address the latter issue, i.e., the scenario for dynamical compactification in our model.

As discussed in Ref. [4], from the classical point of view, different values for the cosmological constant Λ lead to different compactifying scenarios. Indeed, for $\Lambda>c_2/16\pi k$ ($c_2=[(d+2)^2(d-1)/(d+4)]e^2/16v_2$) there are no compactifying solutions and for

$$\frac{c_1}{16\pi k}<\Lambda<\frac{c_2}{16\pi k} \tag{35}$$

[$c_1=d(d-1)e^2/16v_2$] a compactifying solution exists which is classicaly stable, but semiclassically unstable. Finally, a value of $\Lambda<c_1/16\pi k$ implies that the value of the effective 4-dimensional cosmological constant, $\Lambda^{(4)}=8\pi k\Omega(\infty,\phi)$, is negative (see Fig. 1). Since the 4-dimensional cosmological constant $\Lambda^{(4)}$ must satisfy the bound

$$|\Lambda^{(4)}|<10^{-120}\frac{1}{16\pi k}, \tag{36}$$

we are led to choose $\Lambda=c_1/16\pi k$. On the other hand, since we are interested in compactifying solutions, for which $\phi\approx 0$, we shall take Λ such that $\phi=0$ corresponds to the absolute minimum of Eq. (33). This corresponds to $b_0^2=16\pi k v_2/e^2$, and the fine-tuning [4]

$$\Lambda=\frac{d(d-1)}{16b_0^2}. \tag{37}$$

The potential (33) simplifies then to

$$U(\mu,\phi)=e^{6\mu-d\alpha\phi}\frac{2k\Lambda}{9\pi}(e^{-2\alpha\phi}-1)^2-e^{4\mu}+e^{2\mu+d\alpha\phi}\frac{3\pi}{k}\frac{v_1}{v_2}b_0^2, \tag{38}$$

and its form is shown in Fig. 2. Moreover, as can be seen from the plot of $\Omega(\mu=\text{const},\phi)$ in Fig. 3 [cf. Eq. (33)], for μ greater than a critical value μ_c, the potential $U(\mu,\phi)$ has a local maximum $\phi_{\max}$ given approximately by $e^{-2\alpha\phi_{\max}}=d/(d+4)$. This critical value arises from the last term in Eq. (38) and in a first order approximation is given by

$$a_c^4\sim e^{4\mu_c}=\frac{-B-\sqrt{B^2-4AC}}{2A}, \tag{39}$$

where

$$A=-\alpha A_1 A_2\left(\frac{2k\Lambda}{9\pi}\right)^2, \tag{40}$$

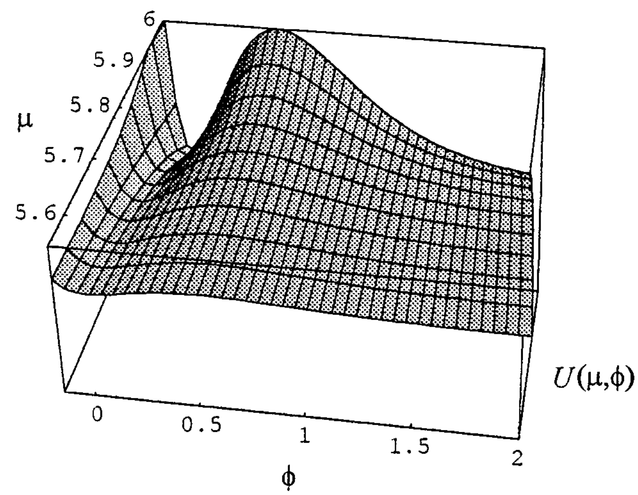

FIG. 2. Potential $U(\mu,\phi)$ for $d=6$ and large μ ($\mu>\mu_c$; see Fig. 3).

$$B=d\frac{2\Lambda}{3}\frac{v_1}{v_2}b_0^2\left[dA_1e^{2d\alpha\phi_0}\left(\frac{1}{\phi_{\max}-\phi_0}+\alpha d\right)+A_2e^{2d\alpha\phi_{\max}}\left(\frac{1}{\phi_{\max}-\phi_0}-\alpha d\right)\right],$$

$$C=\alpha d^5e^{2d\alpha\phi_0}\left(\frac{3\pi}{k}\frac{v_1}{v_2}b_0^2\right)^2,$$

with $A_1=-8e^{-2\alpha\phi_{\max}}=-8d/d+4$, $A_2=[(d+4)^2e^{-4\alpha\phi_0}-d^2]/2$, and $e^{-2\alpha\phi_0}=[(d+2)^2-\sqrt{(d+2)^4-d^2(d+4)^2}]/(d+4)^2$.

We now turn to the discussion of the boundary conditions for the Wheeler-DeWitt equation.[3] We shall use the path integral representation for the ground state of the Universe,

$$\Psi[\mu,\phi]=\int_C D\mu D\phi\exp(-S_E), \tag{41}$$

which does allow us to evaluate $\Psi(\mu,\phi)$ close to $\mu=-\infty$. In here, $S_E=-iS_{\rm eff}$ is the Euclidean action, obtained through the effective action (18) and taking $d\tau=iNdt$:

$$S_E=\int d\tau\frac{6\pi}{k}\left[-a\dot{a}^2+a^3\dot{\phi}^2-\frac{a}{4}+a^3e^{-d\alpha\phi}(e^{-2\alpha\phi}-1)^2\frac{\Lambda}{3}+\frac{1}{a}e^{d\alpha\phi}\frac{2\pi k}{e^2}v_1\right]. \tag{42}$$

To ensure that the sum C does corresponds to compact $(d+4)$-metrics we must impose conditions on $\tilde{a}(t)$ and $b(t)$ at $\tau=0$ (where τ is the Euclidean time $d\tau=iNdt$), such that the Euclidean metric

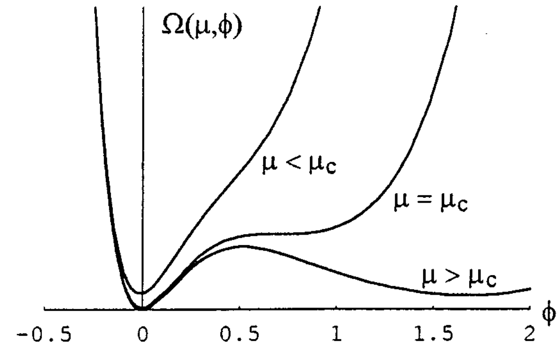

FIG. 3. Potential $\Omega(\mu=\text{const},\phi)$ for $d=6$ and some values of μ.

$$\hat{g}=d\tau^2+\tilde{a}^2(\tau)\Sigma_{i=1}^3\omega^i\omega^i+b^2(\tau)\Sigma_{m=4}^{d+3}\omega^m\omega^m \tag{43}$$

is compact. In Ref. [11] the following conditions were suggested: $\tilde{a}=0$, $b>0$, $d\tilde{a}/d\tau=1$, and $db/d\tau=0$ at $\tau=0$, which can also be inferred from the regularity of the Euclidean equations of motion [12]. Notice that physical reasons, such as the vanishing of the internal gauge field components and of the gravitational coupling in 4 dimensions, prevent the interchange of these conditions. Clearly, this approach to select the boundary conditions to the Hartle-Hawking wave function is not quite correct from the quantum point of view as it implies a simultaneous fixing of both canonical and corresponding conjugated momentum variables.

As far as our reduced model [see Eq. (18)] is concerned, consistent boundary conditions can be implemented as follows. Let us first point out that our reduced model is similar to a closed Friedmann-Robertson-Walker model with a scalar field ψ (or ϕ) [11]. Hence, our boundary conditions which are consistent with a 4-geometry closing off in a regular way and with regular field configurations: $a(0)=0$ and $(d\psi/d\tau)(0)=0$. The next step is to note that the corresponding constraint (Friedmann) equation in our model imples that $(da/d\tau)(0)=1$ [21,22]; i.e., the condition $a(0)=0$ is equivalent to $(da/d\tau)(0)=1$. In addition [6,19] $(d\psi/d\tau)(0)=0$ leads, using $\psi\sim\ln b$, to $(db/d\tau)(0)=0$ and $b(0)>0$. It is important to realize that the geometries summed over in the path integral will be closed at $\tau=0$ for the 4-dimensional physical spacetime, but generally not regular, and also that the geometries will be regular at $\tau=0$ for the extra d-dimensional space. From the constraint equation, the other condition $(da/d\tau)(0)=1$ (regularity) will hold at saddle points and, similarly, for $b(0)>0$ which will follow from the corresponding regularity of the equations of motion [12].

Thus, integrating Eq. (42) from an initial point $\tau=0$ to $\Delta\tau$, a very close point to $\tau=0$, we get

$$S_E=\int_0^{\Delta\tau}d\tau\frac{6\pi}{k}\left[-\tau-\frac{\tau}{4}+\tau^3e^{-d\alpha\phi}(e^{-2\alpha\phi}-1)^2\frac{\Lambda}{3}+\frac{1}{\tau}e^{d\alpha\phi}\frac{2\pi k}{e^2}v_1\right], \tag{44}$$

[3]A discussion on the mathematical aspects of generalizing the Hartle-Hawking no-boundary proposal to higher dimensions can be found in the Appendix.

TABLE I. Boundary conditions on $\mathcal{I}^-$ for Ψ.

	$v_1=0$	$v_1\neq 0$
on $\mathcal{I}^-$ and $\phi<0$	$\Psi=0(1)$ for $d<19$ ($d\geqslant 19$)	$\Psi=0$
on $\mathcal{I}^-$ and $\phi>0$	$\Psi=1$	$\Psi=0$

where we used $a\approx\tau$ close to $\tau=0$. Finally, by setting $a=e^{\mu}\sqrt{k/6\pi}$, the integration yields

$$S_{\mathrm{E}}=\begin{cases} -\dfrac{5}{8}e^{2\mu}+e^{4\mu-d\alpha\phi}(e^{-2\alpha\phi}-1)^2\dfrac{k\Lambda}{72\pi} & \text{for } v_1=0,\\ +\infty & \text{for } v_1\neq 0. \end{cases} \tag{45}$$

Since, with a suitable choice of the metric, we can have $\Psi=e^{-S_{\mathrm{E}}}$ near the past null infinity (see Ref. [11]), $\mathcal{I}^-$, we can easily obtain the boundary conditions. This analysis is simplified by introducing the following new variables:

$$x=e^{\mu}\sinh\phi,$$

$$y=e^{\mu}\cosh\phi, \tag{46}$$

such that the past null infinity $\mathcal{I}^-$ now corresponds to the lines $x=y$ and $x=-y$. The boundary conditions on $\mathcal{I}^-$, which are shown in Table I, can be easily obtained from Eq. (45). For all over $\mathcal{I}^-$, the normal derivative vanishes, $\partial\Psi/\partial n=0$.

Let us now further proceed with our search for solutions to the Wheeler-DeWitt equation. In this situation, one must generally begin by determining the regions where the solution is oscillatory and where it is exponential. This can be heuristically done by examining the regions where for surfaces of constant U, the minisuperspace metric $ds^2=d\mu^2-d\phi^2$ is either spacelike ($ds^2>0$) or timelike ($ds^2<0$):

In spacelike regions we can locally perform a Lorentz-type transformation to new coordinates $(\tilde{\mu},\tilde{\phi})$:

$$\tilde{\mu}=\mu\cosh\theta-\phi\sinh\theta,$$
$$\tilde{\phi}=-\mu\sinh\theta+\phi\cosh\theta, \tag{47}$$

where θ is a constant, such that the surfaces of constant U are parallel to the $\tilde{\phi}$ axis. The potential will then depend, at least locally, only on $\tilde{\mu}$ and the Wheeler-DeWitt equation can be rewritten as

$$\left[\frac{\partial^2}{\partial\tilde{\mu}^2}-\frac{\partial^2}{\partial\tilde{\phi}^2}+U(\tilde{\mu})\right]\Psi(\tilde{\mu},\tilde{\phi})=0, \tag{48}$$

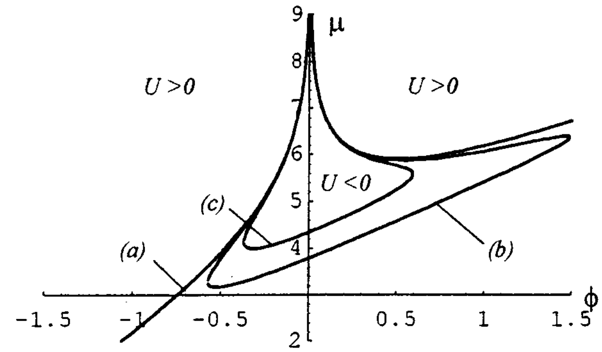

FIG. 4. $U=0$ curves in the $\mu\phi$ plane for $d=6$ and different values of the ratio v_1/v_2 [(a) $v_1/v_2=0$, (b) $v_1/v_2=\frac{1}{3}$, and (c) $v_1/v_2=1$].

and Ψ will be oscillatory if $U>0$ and exponential type if $U<0$, assuming that its dependence on $\tilde{\phi}$ is small.

Similarly, when the surfaces of constant U correpond to timelike regions of the minisuperspace metric, a Lorentz-type transformation can rotate coordinates (μ,ϕ) such that they become parallel to the $\tilde{\mu}$ axis. The potential U will then depend only on $\tilde{\phi}$, and Ψ will be exponential type for $U<0$ and oscillatory type for $U>0$, assuming now that the wave function dependence on $\tilde{\mu}$ is small. The surfaces $U=0$ depend on the relation v_1/v_2 and are given by the expression

$$e^{2\mu}=\frac{9\pi}{4k\Lambda}\frac{e^{d\alpha\phi}}{(e^{-2\alpha\phi}-1)^2}\left(1\pm\left[1-\frac{d(d-1)}{6}\frac{v_1}{v_2}\right.\right.$$
$$\left.\left.\times(e^{-2\alpha\phi}-1)^2\right]^{1/2}\right). \tag{49}$$

These surfaces (see Fig. 4) provide all points for which a Euclidean solution can be smoothly matched into a Lorentzian one, that is, $\dot{\mu}=\dot{\phi}$ (the extrinsic curvature being continuous). For $v_1/v_2=0$ we recover the result found in Ref. [11]. In order to further characterize the regions where solutions are oscillatory or exponential, we further summarize the asymptotic branches of the surface $U=0$ as follows: (i) For $v_1/v_2=0$ and $\phi\rightarrow+\infty$, $b\rightarrow+\infty$, we have $e^{2\mu}\rightarrow(9\pi/2k\Lambda)e^{d\alpha\phi}$, $\tilde{a}\rightarrow\sqrt{3/4\Lambda}$; (ii) when $\phi\rightarrow-\infty$, $b\rightarrow 0$, we have $e^{2\mu}\rightarrow(9\pi/2k\Lambda)e^{\alpha(d+4)\phi}$, $\tilde{a}\rightarrow 0$; (iii) finally, when $\phi\rightarrow 0$, $b\rightarrow b_0$, we obtain $e^{2\mu}\propto\phi^{-1}$, $\tilde{a}\rightarrow 0$; (iv) for $v_1/v_2\neq 0$ only the asymptotic branch $\phi\rightarrow 0$ survives.

However, besides the surfaces of constant U that correspond to timelike or spacelike regions, we have also to look for the curves of constant U surfaces for which the minisuperspace metric is null, $d\mu/d\phi=\pm 1$. The expression for these curves is given by $\partial U/\partial\mu=\pm\partial U/\partial\phi$, that is,

$$e^{2\mu}=\frac{9\pi}{k\Lambda}e^{d\alpha\phi}\frac{1``\pm"(1-[d(d-1)/96](v_1/v_2)(e^{-2\alpha\phi}-1)(2\mp d\alpha)\{e^{-2\alpha\phi}[6\pm\alpha(d+4)]-(6\pm d\alpha)\})^{1/2}}{(e^{-2\alpha\phi}-1)\{e^{-2\alpha\phi}[6\pm\alpha(d+4)]-(6\pm d\alpha)\}}, \tag{50}$$

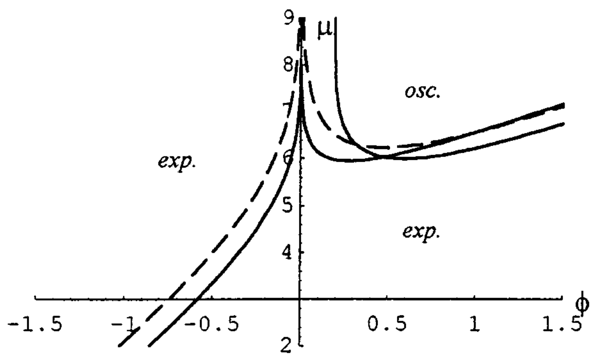

FIG. 5. $U=0$ (dashed line) and null curves (bold line) in the $\mu\phi$ plane for $d=3$ and $v_1/v_2=0$.

where the sign ''$\pm$'' is independent of the remaining ones appearing in Eq. (50). It is quite important to point out that the sign of one of the terms in Eq. (50) *depends* on the number of extra dimensions, d:

$$6-\alpha(d+4)>0 \quad \text{for } d\geqslant 4,$$

$$6-\alpha(d+4)<0 \quad \text{for } d<4. \tag{51}$$

This implies that there will be different solutions for different values of d. As far as the asymptotic branches of Eq. (50) are concerned, we have the following.

(i) For $\phi\to+\infty$, $b\to\infty$, we have the asymptotic branch

$$e^{2\mu}\to\frac{9\pi}{k\Lambda}e^{d\alpha\phi}\left(\frac{1+\sqrt{C_+}}{6-\alpha d}\right),$$

$\tilde{a}\propto\Lambda^{-1/2}$, where

$$C_\pm=1-\frac{d(d-1)}{96}\frac{v_1}{v_2}(2\mp d\alpha)(6\pm d\alpha).$$

(ii) If v_1/v_2 verifies the condition $v_1/v_2<96/[d(d-1)(2+d\alpha)(6-d\alpha)]$, then there are two other asymptotic branches

$$e^{2\mu}\to\frac{9\pi}{k\Lambda}e^{d\alpha\phi}\left(\frac{1\pm\sqrt{C_-}}{6-d\alpha}\right),$$

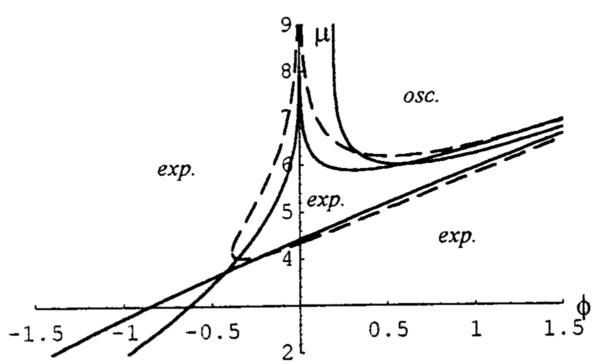

FIG. 6. $U=0$ (dashed line) and null curves (bold line) in the $\mu\phi$ plane for $d=3$ and $v_1/v_2=1$.

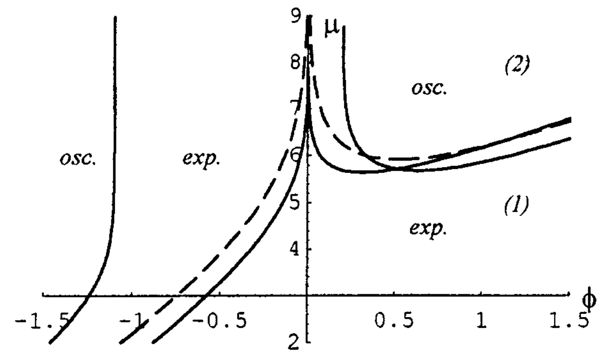

FIG. 7. $U=0$ (dashed line) and null curves (bold line) in the $\mu\phi$ plane for $d=6$ and $v_1/v_2=0$.

$\tilde{a}\propto\Lambda^{-1/2}$.

(iii) For $\phi\to-\infty$, $b\to0$, and $v_1/v_2=0$ we have

$$e^{2\mu}\to\frac{9\pi}{k\Lambda}e^{(d+4)\alpha\phi}\frac{2}{6\pm\alpha(d+4)},$$

$\tilde{a}\to0$. The lower sign branch exists only for $d\geqslant4$.

(iv) When $\phi\to-\infty$ and $v_1/v_2\neq0$, we have

$$e^{2\mu}\to\frac{9\pi}{k\Lambda}e^{(d+2)\alpha\phi}\sqrt{\frac{d(d-1)}{96}\frac{(-2\pm d\alpha)}{6\pm\alpha(d+4)}\frac{v_1}{v_2}},$$

$\tilde{a}\to0$, and the lower sign branch exists only for $d<4$.

(v) Finally, when $\phi\to0$, $b\to b_0$, then $e^{2\mu}\propto\phi^{-1}$.

(vi) There is an additional asympototic branch for $\phi\to\phi_\pm$, where $\exp(-2\alpha\phi_\pm)=(6\pm d\alpha)/[6\pm\alpha(d+4)]$, with $e^{2\mu}\approx|\phi-\phi_\pm|^{-1}$. The lower branch ϕ_- exists only for $d\geqslant4$.

In Figs. 5, 6 and Figs. 7, 8 we plot the curves $U=0$ (dashed lines) together with the ones for which $d\mu/d\phi=\pm1$ (bold lines) for cases $d=3$ and $d=6$. Notice the *difference* between the $d<4$ and the $d\geqslant4$ cases. For each region we further indicate whether Ψ is expected to be oscillatory (osc) or exponential (exp.).

In the following subsections we shall analyze in some detail different physical situations and derive the correspond-

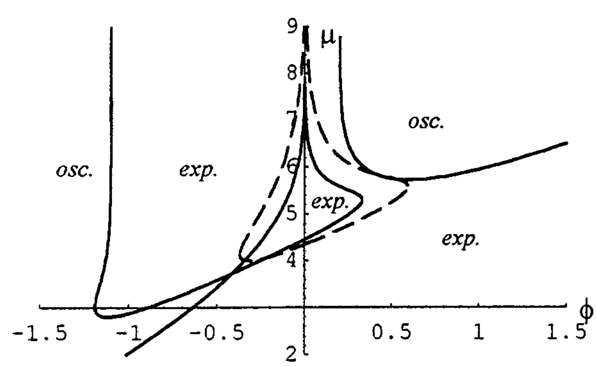

FIG. 8. $U=0$ (dashed line) and null curves (bold line) in the $\mu\phi$ plane for $d=6$ and $v_1/v_2=1$.

ing Hartle-Hawking (no-boundary) wave function. We shall employ the transformation (47), after which the Wheeler-DeWitt equation takes the general form

$$\left[\frac{\partial^2}{\partial\tilde{\mu}^2}-\frac{\partial^2}{\partial\tilde{\phi}^2}+U(\tilde{\mu},\tilde{\phi})\right]\Psi(\tilde{\mu},\tilde{\phi})=0. \tag{52}$$

We can anticipate that Secs. III D and III E contain the most interesting physical results as far as the process of compactification is concerned.

A. Wave function for $\mu>0$ and $\phi\ll 0$

This case represents the physical situation prior to the compactification process. For $\mu>0$ (i.e., $a>0$) and $\phi\ll 0$ (i.e., $b\to 0$ with $U\gg 1$) the potential (38) becomes

$$U(\mu,\phi)\approx\frac{2k\Lambda}{9\pi}e^{6\mu-(d+4)\alpha\phi}, \tag{53}$$

and we can distinguish two situations: (a) $d<4$, for which we can choose $\sinh\theta=6/\omega$ and hence $U\approx U(\tilde{\phi})=(2k\Lambda/9\pi)e^{-\omega\tilde{\phi}}$; (b) $d\geqslant 4$, for which we can choose $\cosh\theta=6/\omega$ and hence $U\approx U(\tilde{\mu})=(2k\Lambda/9\pi)e^{\omega\tilde{\mu}}$, where $\omega^2=|24(-d^2+d+8)/d(d+2)|$, with $\omega>0$.

We can now solve Eq. (52) with Eq. (53) by separation of variables to find that for $d<4$, the solution is a combination of the Bessel functions of the first kind, $I_\nu(z)$, and of the second kind, $K_\nu(z)$. For $d\geqslant 4$, we have a combination of the modified Bessel functions of the first kind, $J_\nu(z)$, and of the second kind, $Y_\nu(z)$. The study of the boundary conditions carried out above allows us to pick the appropriate Bessel function:

$$\Psi(\tilde{\mu},\tilde{\phi})\approx e^{\pm\sqrt{\epsilon}\tilde{\mu}}K_{(2/\omega)\sqrt{\epsilon}}\left[\frac{2}{\omega}\left(\frac{2k\Lambda}{9\pi}\right)^{1/2}e^{-(\omega/2)\tilde{\phi}}\right]\quad\text{for } d<4, \tag{54}$$

$$\Psi(\tilde{\mu},\tilde{\phi})\approx e^{\sqrt{\epsilon}\tilde{\phi}}J_{(2/\omega)\sqrt{\epsilon}}\left[\frac{2}{\omega}\left(\frac{2k\Lambda}{9\pi}\right)^{1/2}e^{(\omega/2)\tilde{\mu}}\right]\quad\text{for } d\geqslant 4, \tag{55}$$

$$\Psi(\tilde{\mu},\tilde{\phi})\approx J_0\left[\frac{2}{\omega}\left(\frac{2k\Lambda}{9\pi}\right)^{1/2}e^{(\omega/2)\tilde{\mu}}\right]\quad\text{for } d\geqslant 19 \text{ and } v_1=0, \tag{56}$$

where $e^{\pm\sqrt{\epsilon}\tilde{\mu}}$ means a combination of $e^{\sqrt{\epsilon}\tilde{\mu}}$ and $e^{-\sqrt{\epsilon}\tilde{\mu}}$, and ϵ is the separation constant, which is determined by matching this solution onto the solution in the adjacent region (one can also see that $\epsilon\approx 0$). In Eqs. (54)–(56) we have assumed that $\epsilon\geqslant 0$. The case $\epsilon<0$ is not consistent with the Hartle-Hawking boundary conditions for a wave function of the type $I_{\text{const}\times\sqrt{\epsilon}}(z)$. Notice that, as expected, $d<4$ implies an exponential behavior, while $d\geqslant 4$ corresponds to an oscillatory one.

B. Wave function for $\mu\gg 1$ and $\phi\gg 1$

This case corresponds to the situation where the radii of the S^3 and S^d sections are large. For $\mu\gg 1$ and $\phi\gg 1$ one has to deal with two regions separated by $\mu=(d\alpha/2)\phi$, on which different behaviors are expected. On the lower region (1) in Fig. 7 $[\mu<(d\alpha/2)\phi]$ the potential is approximately given by

$$U(\mu,\phi)\approx\begin{cases}-e^{4\mu} & \text{for } v_1/v_2=0,\\ e^{2\mu+d\alpha\phi}\dfrac{3\pi}{k}\dfrac{v_1}{v_2}b_0^2 & \text{for } v_1/v_2\neq 0.\end{cases} \tag{57}$$

For $v_1/v_2\neq 0$ we can choose $\sinh\theta=-\sqrt{(d+2)/2(d-1)}$, so that

$$U\approx U(\tilde{\phi})=e^{\bar{\omega}\tilde{\phi}}\frac{3\pi}{k}\frac{v_1}{v_2}b_0^2,$$

where $\bar{\omega}=\sqrt{8(d-1)/(d+2)}$. The solution is then

$$\Psi(\tilde{\mu},\tilde{\phi})\approx e^{\sqrt{\bar{\epsilon}}\tilde{\mu}}K_{(2/\bar{\omega})\sqrt{\bar{\epsilon}}}\left[\frac{2}{\bar{\omega}}\left(\frac{3\pi}{k}\frac{v_1}{v_2}b_0^2\right)^{1/2}e^{(\bar{\omega}/2)\tilde{\phi}}\right], \tag{58}$$

where $\bar{\epsilon}$ is the separation constant.

For $v_1/v_2=0$ the wave function is a combination of $K_0(z)$ and $I_0(z)$, with $z=\frac{1}{2}e^{2\mu}$. These solutions are, as expected, exponential type and are also valid in the region $\phi\gg 1$ and $\mu<0$.

For the other case [region (2) in Fig. 7], we have $U\approx e^{6\mu-d\alpha\phi}2k\Lambda/9\pi$. Choosing $\sinh\theta=\sqrt{d/2(d+3)}$ we get $U\approx U(\tilde{\mu})=e^{\tilde{\omega}\tilde{\mu}}2k\Lambda/9\pi$, with $\tilde{\omega}=\sqrt{24(d+3)/(d+2)}$, and Ψ is a combination of $J_\nu(z)$ and $Y_\nu(z)$, with $\nu=(2/\tilde{\omega})\sqrt{\tilde{\epsilon}}$ and

$$z=\frac{2}{\tilde{\omega}}\left(\frac{2k\Lambda}{9\pi}\right)^{1/2}e^{(\tilde{\omega}/2)\tilde{\mu}},$$

$\tilde{\epsilon}$ being a separation constant. This solution is, as expected, oscillatory.

C. Wave function for $\mu\ll 0$

This case corresponds to a 4-dimensional physical universe at a very early stage and with a generic S^d section. In the region $\mu\ll 0$ [i.e., $a(t)\to 0$] and $\phi>0$ the potential is also given by Eq. (57). For $v_1/v_2\neq 0$ we obtain

$$\Psi\approx e^{\pm\sqrt{\hat{\epsilon}}\tilde{\mu}}I_{(2/\hat{\omega})\sqrt{\hat{\epsilon}}}\left[\frac{2}{\hat{\omega}}\left(\frac{3\pi}{k}\frac{v_1}{v_2}b_0^2\right)^{1/2}e^{(\hat{\omega}/2)\tilde{\phi}}\right], \tag{59}$$

while for $v_1/v_2=0$ we have $\Psi(\mu,\phi)\approx I_0[\frac{1}{2}e^{2\mu}]$. In both cases the behavior is exponential. These solutions also apply for $\phi<0$ and $\mu<[\alpha(d+4)/2]\phi$.

For the particular situation where $\mu\ll 0$ together with $\phi\ll 0$, we further distinguish two different situations: (a) For $\mu>[(d+2)\alpha/2]\phi$ we expect a behavior similar to the one found for $\mu>0$ and $\phi\ll 0$ (see Sec. III A); (b) As for the region $[(d+2)\alpha/2]\phi<\mu<[(d+4)\alpha/2]\phi$, this is a transition region and one should expect a mixture of the previous wave functions.

D. Wave function in the neighborhood of $\phi=\phi_{\max}$

We shall now obtain approximate solutions in the neighborhood of $\phi=\phi_{\max}$ using the semiclassical approximation to the path integral (41),

$$\Psi(\mu,\phi)\approx A(\mu,\phi)e^{-S_{\mathrm{E}}(\mu,\phi)}, \tag{60}$$

where $\phi_{\max}$ is the local maximum of $U(\mu=\mathrm{const},\phi)$ and is given approximately by $e^{-2\alpha\phi_{\max}}=d/d+4$. This corresponds to the physical state of our universe where the extra d-dimensional space is at an equilibrium point, corresponding to its maximum value.

Using then the classical field equations of motion obtained from S_{eff} to integrate the Euclidean action we get, for $v_1/v_2=0$,

$$S_{\mathrm{E}}=\frac{3}{16k^2\Omega}\left\{\left[1-\left(\frac{4k}{3}\right)^2 e^{2\mu}\Omega\right]^{3/2}-1\right\}, \tag{61}$$

where the potential $\Omega(\mu,\phi)$, given by

$$\Omega(\mu,\phi)=e^{-d\alpha\phi}\frac{\Lambda}{8\pi k}(e^{-2\alpha\phi}-1)^2+e^{d\alpha\phi-4\mu}\left(\frac{27\pi b_0^2}{16k^3}\frac{v_1}{v_2}\right), \tag{62}$$

was assumed to be approximately constant near $\phi=\phi_{\max}$. Hence, in the region $U<0$,

$$\Psi\approx A(\mu,\phi)\exp\left[\frac{3}{16k^2\Omega}\right]\times\exp\left\{-\frac{3}{16k^2\Omega}\left[1-\left(\frac{4k}{3}\right)^2 e^{2\mu}\Omega\right]^{3/2}\right\}, \tag{63}$$

where the prefactor A is such that it verifies the condition $A(-\infty,\phi)=1$.

In the region $U>0$ the wave function becomes oscillatory, and the WKB procedure shows that

$$\Psi\approx B(\mu,\phi)\exp\left[\frac{3}{16k^2\Omega}\right]\times\cos\left\{\frac{3}{16k^2\Omega}\left[\left(\frac{4k}{3}\right)^2 e^{2\mu}\Omega-1\right]^{3/2}-\frac{\pi}{4}\right\}. \tag{64}$$

Replacing Eq. (64) in the Wheeler-DeWitt equation one obtains the prefactor

$$B(\mu,\phi)\approx e^{-\mu}\left[\left(\frac{4k}{3}\right)^2 e^{2\mu}\Omega-1\right]^{-1/4}. \tag{65}$$

For $v_1/v_2\neq 0$ these results are still valid for $\mu>0$. For $\mu<0$ we expect the behavior described in Sec. III C.

E. Wave function in the neighborhood of $\phi=0$ and large μ

FInally, we consider the case where the 4-dimensional physical universe is in a stage of large S^3 radius and with $b\sim b_0$. In the neighborhood of $\phi=0$, at the minimum of $U(\mu=\mathrm{const},\phi)$, we consider the dominant term of the potential for large μ:

$$U(\mu,\phi)\approx e^{6\mu-d\alpha\phi}\frac{2k\Lambda}{9\pi}(e^{-2\alpha\phi}-1)^2\approx e^{6\mu}\phi^2\frac{8\alpha^2k\Lambda}{9\pi}. \tag{66}$$

Notice that the potential vanishes for $\phi=0$ and that in Eq. (66) we exhibit the dominant term for values of ϕ around the minimum. Quadratic potentials of this kind are found in massive scalar field models [6].

We now perform a simple change of variables,

$$x=e^{3\mu},$$

$$y=e^{-2\alpha\phi}, \tag{67}$$

from which yields the Wheeler-DeWitt equation

$$\left[9x^2\frac{\partial^2}{\partial x^2}-4\alpha^2y^2\frac{\partial^2}{\partial y^2}+9x\frac{\partial}{\partial x}-4\alpha^2y\frac{\partial}{\partial y}+x^2y^{d/2}(y-1)^2\frac{2k\Lambda}{9\pi}\right]\Psi(x,y)=0. \tag{68}$$

As we are interested in the limit $x\ll 1$ and $y\approx 1$, we actually have to solve

$$\left[x^2\frac{\partial^2}{\partial x^2}+x\frac{\partial}{\partial x}+\frac{1}{9}x^2y^{d/2}(y-1)^2\frac{2k\Lambda}{9\pi}\right]\Psi(x,y)=0. \tag{69}$$

Thus, choosing

$$z=\frac{1}{3}\sqrt{\frac{2k\Lambda}{9\pi}}xy^{d/4}|y-1|,$$

one easily sees that Ψ is a combination of Bessel functions $J_0(z)$ and $Y_0(z)$, where

$$z=\sqrt{\frac{2k\Lambda}{9\pi}\frac{2\alpha}{3}}e^{3\mu}|\phi|.$$

If $\Psi\propto J_0(z)$, then, as $z\rightarrow 0$, the wave function behaves as $\Psi\approx 1-z^2/4$. If, on the other hand, Ψ also depends on $Y_0(z)$, then, as $z\rightarrow 0$, Ψ behaves asymptotically as $\Psi\approx(2/\pi)\ln(z/2)$. This behavior is depicted in Fig. 9.

According to the standard interpretational rules of quantum cosmology (see, for instance, Ref. [23]), the probabilistic interpretation of the wave function does make sense in the classical and the semiclassical regions. Therefore, as the large-μ region corresponds to a classical region, the fact that the wave function is highly peaked around $\phi=0$ means that the most probable configuration does indeed correspond to solutions with compactification for expanding external spacetime. In the next section we shall draw additional physical information concerning some of the solutions in this section.

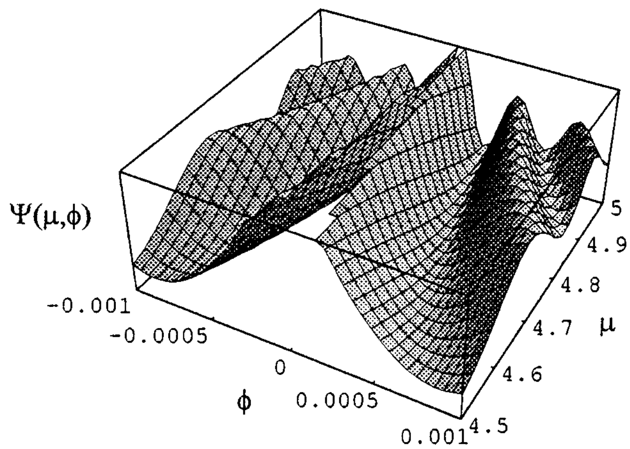

FIG. 9. Wave function in the neighborhood of $\phi=0$.

IV. INTERPRETATION OF THE WAVE FUNCTION

In order to interpret the wave function we shall use the trace of the square of the extrinsic curvature, $K^2=K_{\hat{I}\hat{J}}K^{\hat{I}\hat{J}}$, to see whether the wave function in the semiclassical limit corresponds to a Lorentzian or to a Euclidean geometry. This is justified as the Wheeler-DeWitt equation is the same from whatever metric (Lorentzian or Euclidean) one derives it. The extrinsic curvature is a measure of the variation of the normal to the hypersurfaces of constant time, and is given by

$$K_{\hat{I}\hat{J}}=N^{-1}\left(-\frac{1}{2}\frac{\partial h_{\hat{I}\hat{J}}}{\partial t}+\nabla_{\hat{J}}N_{\hat{I}}\right), \tag{70}$$

where $h_{\hat{I}\hat{J}}$ is the $(d+3)$-metrics and $N_{\hat{I}}$ are the components of the shift vector. From Eq. (12) and using Eq. (26) we obtain

$$K^2=-e^{-6\mu+d\alpha\phi}\frac{3\pi}{2k}\left[9\frac{\partial^2}{\partial\mu^2}+\left(\frac{d\alpha}{2}\right)^2\frac{\partial^2}{\partial\phi^2}+3d\alpha\frac{\partial^2}{\partial\mu\partial\phi}\right]. \tag{71}$$

Performing the Lorentz-type transformation (47) with $\sinh\theta=\sqrt{d/2(d+3)}$, and using $\tilde{\omega}=\sqrt{24(d+3)/(d+2)}$, K^2 simplifies to

$$K^2=-e^{-\tilde{\omega}\tilde{\mu}}\frac{9\pi}{k}\left(\frac{d+3}{d+2}\right)\frac{\partial^2}{\partial\tilde{\mu}^2}, \tag{72}$$

and we see that, in regions where the wave function behaves as an exponential, the quantity $K^2\Psi/\Psi$ is negative. Therefore, in the classical limit, K is imaginary and we have a Euclidean $(d+4)$-geometry. When the wave function is oscillatory, the corresponding K is real, and the $(d+4)$-geometry is Lorentzian. Note that a Lorentz geometry corresponds to a classical state of the Universe, while a Euclidean one is normally associated with a quantum or tunneling state. As shown in Figs. 7 and 8 there exist, for $d\geqslant 4$, well defined Lorentzian regions for *different* values of the ratio v_1/v_2. These regions are, however, inexistent when $d<4$ as depicted in Figs. 5 and 6.

In the oscillatory region, the wave function can be further interpreted using the WKB approximation $\Psi=\mathrm{Re}(Ce^{iS})$, where S is a rapidly varying phase and C a slowly varying prefactor. One chooses S to satisfy the classical Hamilton-Jacobi equation

$$-\left(\frac{\partial S}{\partial\mu}\right)^2+\left(\frac{\partial S}{\partial\phi}\right)^2+U(\mu,\phi)=0. \tag{73}$$

The significance of S becomes evident when operating π_μ on Ψ (for π_ϕ the procedure is analogous):

$$\pi_\mu\Psi=\left[\frac{\partial S}{\partial\mu}-i\frac{\partial}{\partial\mu}\ln C\right]\Psi. \tag{74}$$

Since in the WKB approximation we assume

$$\left|\frac{\partial S}{\partial\mu}\right|\gg\left|\frac{\partial}{\partial\mu}\ln C\right|,$$

we have

$$\pi_\mu=\frac{\partial S}{\partial\mu},\qquad \pi_\phi=\frac{\partial S}{\partial\phi}. \tag{75}$$

The wave function corresponds then to a two-parameter subset of solutions which obey Eqs. (75) and that can be regarded as providing the boundary conditions for the classical solutions. We shall now try to obtain an approximate solution for the Hamilton-Jacobi equation (73) in the region close to the space segment $U=0$ and $\phi=\phi_{\max}$ as it is there that classical trajectories start. Assuming that S is separable and that $|\partial S/\partial\mu|\gg|\partial S/\partial\phi|$ we can use a series expansion around $\phi=\phi_{\max}$ to obtain

$$S\approx\pm\frac{e^{3\mu}}{3}\left(\frac{2k\Lambda}{9\pi}\right)^{1/2}\left(\frac{d}{d+4}\right)^{d/4}\left[\frac{4}{d+4}-E\left(e^{-2\alpha\phi}-\frac{d}{d+4}\right)^2\right], \tag{76}$$

where

$$E=\frac{3}{16}\frac{(d+2)(d+4)}{d}\left\{\left[1+\frac{4}{3}\left(\frac{d+4}{d+2}\right)\right]^{1/2}-1\right\}.$$

The upper (lower) sign on Eq. (76) corresponds to a collapsing (expanding) universe. This result agrees with Eq. (64). Using Eqs. (75) and (26) we have, for the gauge $N=1$,

$$\dot{\mu}\approx\mp\left(\frac{\Lambda}{3}\right)^{1/2}\left(\frac{d}{d+4}\right)^{d/4}\left[\frac{4}{d+4}-E\left(e^{-2\alpha\phi}-\frac{d}{d+4}\right)^2\right], \tag{77}$$

$$\dot{\phi}\approx\pm\left(\frac{\Lambda}{3}\right)^{1/2}\frac{4\alpha}{3}Ee^{-2\alpha\phi}\left(e^{-2\alpha\phi}-\frac{d}{d+4}\right). \tag{78}$$

If ϕ_0, the initial value of ϕ, is close to $\phi_{\max}$, then $\dot{\phi}$ will be very small and the scale factor $a(t)$ will grow exponentially like

$$\exp\left[\left(\frac{\Lambda}{3}\right)^{1/2}\left(\frac{d}{d+4}\right)^{d/4}\frac{4}{d+4}t\right]$$

for an expanding universe. Given the fine-tuning (37) this last expression becomes

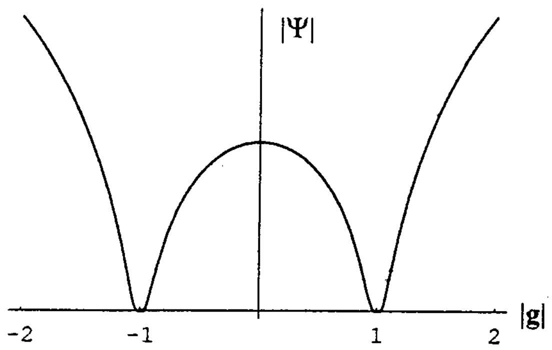

FIG. 10. Module of the wave function for $\mu>0$ and $\phi\ll 0$.

$$a(t)\approx\exp\left[\frac{1}{b_0}\left(\frac{d}{d+4}\right)^{(d+2)/4}\left(\frac{d-1}{3(d+4)}\right)^{1/2}t\right], \qquad (79)$$

which gives

$$a(t)\approx\exp\left(\frac{1}{b_0}\frac{1}{\sqrt{3}e}t\right)$$

for $d\rightarrow+\infty$.

Thus, we confirm the expectation that ϕ configurations to which the main contribution to the potential after compactification is an effective cosmological constant do correspond, in the semiclassical regime, to inflationary solutions for expanding universes.

A. Wave function for the vacuum configuration $v_2=V_2(\mathbf{g}^v)$

Throughout the previous sections we have assumed that $v_2>0$. This corresponds to the choice $\mathbf{g}=\mathbf{0}$ for the potential (22), which is obviously associated to a classically unstable situation. Nevertheless, since the wave function can be interpreted, at least in a semiclassical situation, as giving the probability of a certain configuration, one expects, for consistency, to have the wave function peaked around $\mathbf{g}=\mathbf{0}$ when unfreezing v_2 and varying $\mathbf{g}$. This means that the most probable configuration should correspond to the choice $\mathbf{g}=\mathbf{0}$.

The dependence of Ψ on v_2 can be seen fixing the value of the gauge coupling constant e and rewriting the term $2k\Lambda/9\pi$ in potential (38) as

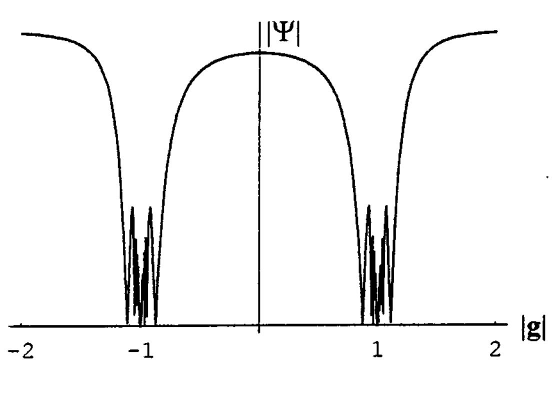

FIG. 11. Module of the wave function in the region (2) of Fig. 7.

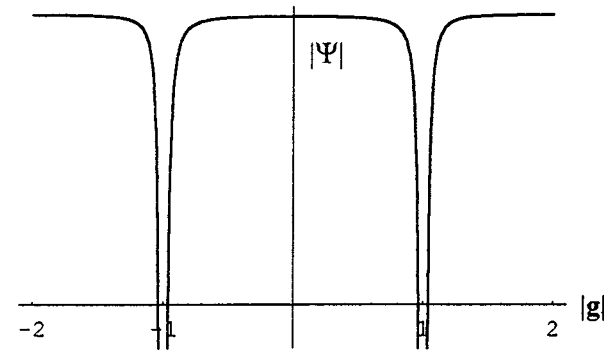

FIG. 12. Module of the wave function in the neighborhood of $\phi=\phi_{\max}$ and $\mu>0$.

$$\left(\frac{e}{12\pi}\right)^2\frac{d(d-1)}{2v_2}$$

[where Eq. (37) was used]. Furthermore, using the value of the radius of compactification, $b_0^2=16\pi k v_2/e^2$, we can see that the term

$$b_0^2\frac{v_1}{v_2}=\frac{v_1}{16\pi k}e^2$$

in Eq. (38) does not depend on v_2. We hence conclude that solutions depending on Λ will depend on v_2^{-1}.

We are only interested in regions where $\mu>0$ $(a>0)$, i.e., in regions where the probabilistic interpretation can be unambiguously used, from which implies that we have the following cases.

(a) For $\mu>0$ and $\phi\ll 0$ [i.e., $b(t)\rightarrow 0$], $\Psi\propto K_\nu(\Lambda^{1/2})$ (Fig. 10) or $\Psi\propto J_\nu(\Lambda^{1/2})$ (Fig. 11) according to the value of d [cf. wave functions (53) and (54)].

(b) For $\mu\gg 1$, $\phi\gg 1$ and $\mu>(d\alpha/2)\phi$, Ψ is a combination of $J_\nu(\Lambda^{1/2})$ and $Y_\nu(\Lambda^{1/2})$. If $\mathbf{g}$ is not too large, the behavior of Y_ν is similar to the one of J_ν (Fig. 11).

(c) For $\phi\approx\phi_{\max}$ and $\mu>0$ the wave function is given by either Eq. (63) or (64) (Fig. 12).

(d) Finally, for $\phi\approx 0$, the wave function Ψ is a combination of $J_0(\Lambda^{1/2})$ and $Y_0(\Lambda^{1/2})$, whose behavior is similar to the one depicted in Fig. 11.

Notice that when Ψ is oscillatory, the peak for $\mathbf{g}=\mathbf{0}$ will disappear for certain values of μ. Nevertheless, we have always $\Psi(|\mathbf{g}|=\pm 1)=0$. We can therefore conclude that we do observe the expected maximum of the wave function for $\mathbf{g}=\mathbf{0}$.

V. CONCLUSIONS

In this paper we have obtained solutions of the Wheeler-DeWitt equation derived from the effective model that arises from dimensionally reducing to 1 dimension the Einstein-Yang-Mills generalized Kaluza-Klein theory in $D=4+d$ dimensions. We considered an $\mathbf{R}\times S^3\times S^d$ topology and the corresponding Hartle-Hawking boundary conditions. The dimensional reduction was achieved by restricting the field configurations to be homogeneous and isotropic through coset space compactification as indicated in Secs. I and II.

This model of compactification has been proposed in Refs. [3,4]. In particular, the crucial role played by the external space components of the gauge field in order to achieve classically as well as semiclassically stable compactifications was shown in Ref. [4].

In Sec. II we have presented the most salient features of the model and set up the Hamiltonian constraint which allows us to obtain the Wheeler-DeWitt equation to study the compactification process from the quantum-mechanical point of view. Notice that in our model the gauge-field-associated angular momentum is also constrained to vanish. The richness of our effective model (18) is quite evident. In this reduced model the gauge field has *non-vanishing* time-dependent components in *both* the external and internal spaces. Moreover, we have also two time-dependent scalar fields, the dilaton and the inflaton. This contrasts with previous work in the literature, where either static magnetic monopole configurations with nonzero components only in I^d or scalar fields were present.

In Sec. III we have obtained no-boundary solutions of the Wheeler-DeWitt equation which exhibit very interesting features. The term

$$e^{d\alpha\phi-4\mu}\left(\frac{6\pi}{k}\right)^2\frac{3}{4e^2}v_1$$

in Eq. (34) establishes that the external spatial dimensions and the internal d dimensions are at the *same footing* in the early Universe prior to compactification, i.e., when $\mu \ll 0$. It is only through the expansion of the external dimensions (increase of μ) that compactification ($b \to b_0$) is achieved. Thus, it is the dynamics of the 3-dimensional physical space which induces the evolution of I^d towards compactification.

We also find that stable compactifying solutions do correspond to extrema of the wave function of the Universe, showing that the process of compactification does indeed takes place for expanding universes. Furthermore, our analysis indicates that the main properties of the Hartle-Hawking wave function do depend on the following two features. On the one hand, on a nonvanishing contribution to the potential (38) of the external physical space dimensions of the gauge field, a feature already found in the classical analysis of Ref. [4], on the other hand, also on the number d of internal space dimensions. In the case we set the contribution of the external space dimensions of the gauge field to the potential (38) to vanish, we find that we recover the main aspects of the discussion of Ref. [11], where compactification was discussed in the framework of an Einstein-Maxwell model with a magnetic monopole configuration whose gauge (Maxwell) field contribution was nonvanishing only for the internal space. The same can be said about Ref. [12], where a stable compactification was achieved through the nonvanishing contribution of the internal components of a $(D-4)$th rank antisymmetric tensor field. Finally, we also find that for expanding models, inflationary solutions can be predicted, as shown in Sec. IV, if in the semiclassical regime the potential is essentially given by an effective cosmological constant.

ACKNOWLEDGMENTS

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APPENDIX A: HARTLE-HAWKING PROPOSAL AND ITS GENERALIZATION TO HIGHER SPACETIME DIMENSIONS

For clarification purposes, let us briefly outline here the main features of the Hartle-Hawking proposal [5] and its generalization to higher spacetime dimensions [11] (see also Refs. [12,13,20,24]). In quantum cosmology it is assumed that the quantum state of a $D=4$ universe is described by a wave function $\Psi[h_{ij},\Phi]$, which is a functional of the spatial 3-metric, h_{ij}, and matter fields generically denoted by Φ on a compact 3-dimensional hypersurface Σ. The hypersurface Σ is then regarded as the boundary of a compact 4-manifold $\mathcal{M}^4$ on which the 4-metric $g_{\mu\nu}$ and the matter fields Φ are regular. The metric $g_{\mu\nu}$ and the fields Φ coincide with h_{ij} and Φ_0 on Σ and the wave function is then defined through the path integral over 4-metrics, ${}^4g$, and matter fields:

$$\Psi[h_{ij},\Phi_0]=\int_{\mathcal{C}}D[{}^4g]D[\Phi]\exp(-S_{\mathrm{E}}[{}^4g,\Phi]), \quad \text{(A1)}$$

where S_{E} is the Euclidean action and $\mathcal{C}$ is the class of 4-metrics $g_{\mu\nu}$ and regular fields Φ defined on Euclidean compact manifolds M^4 and with *no other* boundary than Σ. An extension of the Hartle-Hawking proposal for universes with $D>4$ dimensions was first discussed in Ref. [11]. Let us summarize it, mentioning some of its difficulties and comparing it with the $D=4$ case.

In $D=4$ the theory of cobordism [24] guarantees that for all compact 3-surfaces there *always* exists a compact 4-dimensional manifold such that S^3 is the *only* boundary, or equivalently, all 3-dimensional *compact* hypersurfaces are cobordant to zero [24]. Let us now consider the case for $D>4$. In these D-dimensional models, the wave function would be a functional of the $(D-1)$ spatial metric h_{IJ} and matter fields Φ on a $(D-1)$-hypersurface Σ_{D-1} and is defined as the result of performing a path integral over all compact D-metrics and regular matter fields on M^D that match h_{IJ} and the matter fields on Σ_{D-1}.

Let us then start by assuming that the $(D-1)$-surface Σ_{D-1} does not possess any disconnected parts [11]. Would there always be a D-dimensional manifold $\mathcal{M}^D$ such that Σ_{D-1} is the only boundary? In higher-dimensional manifolds however, this is not guaranteed. There exist compact $(D-1)$-hypersurfaces Σ_{D-1} for which there is no compact D-dimensional manifold such that Σ_{D-1} is the only boundary. This seems to indicate that in $D>4$ dimensions there are configurations which cannot be attained by the sum over histories in the path integral. The wave function for such configurations would therefore be zero. In Ref. [11] this situation was be circumvented so as to obtain nonzero wave functions for such configurations, namely, by dropping the assumption that the $(D-1)$-surface Σ_{D-1} does not possess any disconnected parts.

As described in [11], if one assumes that the hypersurfaces Σ_{D-1} consist of any number $n>1$ of disconnected parts $\Sigma^{(n)}_{D-1}$, then one finds that the path integral for this disconnected configuration involves terms of two types. The first type consists of disconnected D-manifolds, each disconnected part of which closes off the $\Sigma^{(n)}_{D-1}$ surfaces separately. These will exist only if each of the $\Sigma^{(n)}_{D-1}$ are cobordant to zero, but this may not always be the case. There will indeed be a second type of term which consists of connected D-manifolds which just plainly joins some of the $\Sigma^{(n)}_{D-1}$ together. This second type of manifold will *always* exist in *any* number of dimensions, providing the $\Sigma^{(n)}_{D-1}$ are similar topologically, i.e., have the same *characteristic numbers* [24]. The wave function of any $\Sigma^{(1)}_{D-1}$ surface which is not cobordant to zero would be different from zero and obtained by assuming the existence of other surfaces of suitable topology and then summing over all compact D-manifolds which join these surfaces together. Thus, given a compact $(D-1)$-hypersurface Σ_{D-1} which is not cobordant to zero, a nonzero amplitude could be obtained by *assuming* it possesses disconnected parts.

However, the above considerations for disconnected pieces and generic Σ_{D-1} surfaces would spoil the Hartle-Hawking prescription since the manifold would have more than one boundary. In other words, the general extension above discussed would imply a description in terms of propagation between such generic Σ_{D-1} surfaces. The wave function would then depend on every piece and not on a single one as advocated in [11]. Nevertheless, if we restrict ourselves, as we do in the present paper, to the case of a truncated model with a global topology given by a product of a 3-dimensional manifold to a d-dimensional one, then the spacelike sections always form a boundary of a D-dimensional manifold with no other boundaries [20]. Since hypersurfaces $S^3\times S^d$ are always cobordant to zero, it implies that for spacetimes with topology $\mathbf{R}\times S^3\times S^d$ the Hartle-Hawking proposal can be always implemented, and thus we can consider the original no-boundary proposal in our study.

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