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Cosmic microwave background anisotropies from second order gravitational perturbations

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This paper presents a complete analysis of the effects of second order gravitational perturbations on cosmic microwave background anisotropies, taking explicitly into account scalar, vector and tensor modes. We also consider the second order perturbations of the metric itself obtaining them, for a universe dominated by a collisionless fluid, in the Poisson gauge, by transforming the known results in the synchronous gauge. We discuss the resulting second order anisotropies in the Poisson gauge, and analyze the possible relevance of the different terms. We expect that, in the simplest scenarios for structure formation, the main effect comes from the gravitational lensing by scalar perturbations that is known to give a few percent contribution to the anisotropies at small angular scales. [S0556-2821(97)02918-4]

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I. INTRODUCTION

The increasing number of measurements of cosmic microwave background (CMB) anisotropies in the last years and the very ambitious observational programs for the future generation of detectors makes us hope that the angular spectrum of the anisotropies will be known with great accuracy within the next decade. This fact has stimulated theoretical efforts to obtain more precise predictions for the anisotropies produced in the different structure formation models, and it is expected that future observations will be very helpful in distinguishing among them and in putting constraints on the cosmological parameters.

Most of these theoretical computations involve numerical or semianalytic solutions of the linearized Boltzmann equation. Nonlinear gravitational effects on the anisotropies have been computed for some particular processes, such as the gravitational lensing from density perturbations [1–6] and the Rees-Sciama effect [7–12] (which is second order in a flat matter-dominated universe, as the gravitational potential is constant to first order). It has been shown that the effect of the gravitational lensing by density perturbations is to smooth the so-called Doppler or acoustic peaks in the angular spectrum at high ℓ, and it is thus relevant for the analysis of the small angle observations [13]. On the other hand, the nonlinear Rees-Sciama effect is in most cases expected to be much smaller than the first order anisotropies [14,15] unless early reionization substantially erases the first order anisotropies.

In a recent paper, Pyne and Carroll [16] have presented a nice framework for a complete computation of second and higher order gravitational perturbations of the CMB. Their algorithm essentially involves computing the redshift experienced by the photons during their travel from the last scattering to the observer in terms of their perturbed geodesics and then obtaining the perturbed geodesics up to the required order. The study of second order anisotropies is relevant because they can produce a non-negligible contribution compared to the first order ones, due to the long distances involved in the problem. The reason is that several second order terms include integrals of the metric perturbations along the photons path that can enhance small effects as the photons travel from the last scattering surface. Moreover, second order effects are also important because they give the primary contribution to some statistical measures of the anisotropies that are vanishing for the linear contribution, as, for example, the three-point function of temperature anisotropies [17–19]. In any case, it is important to know the magnitude of the second order effects as they contribute to the theoretical error of linear anisotropy calculations.

In this paper, we apply the formalism proposed by Pyne and Carroll to the computation of the full second order anisotropies in the Poisson gauge. We first present a computation of the second order anisotropies that generalizes the results of Ref. [16], in that we consider the motion of the observer and the emitter, we explicitly include the second order perturbations of the metric itself, and we take into account scalar, vector, and tensor modes. We then consider the Poisson gauge, which, in the case of scalar first order perturbations, reduces to the longitudinal gauge. We obtain the second order perturbed metric for a dust-dominated universe in the Poisson gauge explicitly, and then discuss the second order anisotropies for this particular case.

Throughout this paper Greek indices $\mu,\nu,\ldots$ take values from 0 to 3, and Latin ones $i,j,\ldots$ from 1 to 3. We take, for the metric, signature $+2$; units are such that $c=1$.

II. TEMPERATURE ANISOTROPIES

The quantity of interest is the angular variation of the temperature measured by an observer.

We consider a perturbed flat Robertson-Walker spacetime and use conformal time η [$d\eta \equiv dt/a(t)$, where $a(t)$ is the scale factor of the universe]. We can write the line element as

$$ds^2 \simeq a^2(\eta)\left(g^{(0)}_{\mu\nu}+g^{(1)}_{\mu\nu}+\frac{1}{2}g^{(2)}_{\mu\nu}+\cdots\right)dx^\mu dx^\nu, \quad (2.1)$$

where the term between parentheses is the conformally transformed metric $(g_{\mu\nu})$, $g^{(0)}_{\mu\nu}$ is the background Minkowski metric, and $g^{(1)}_{\mu\nu}$ and $g^{(2)}_{\mu\nu}$ are the first and second order perturbations, respectively.

Photons travel along null geodesics $x^{\mu}(\lambda)$, which we parametrize with λ in the conformal metric, connecting the observer, at coordinates $x^{\mu}_{\mathcal{O}}=(\eta_{\mathcal{O}},\mathbf{x}_{\mathcal{O}})$, to the emitting hypersurface, which we take at constant conformal time $\eta_{\mathcal{E}}$. This hypersurface can be taken as the last scattering surface, at redshift z_{LS}. At larger redshifts the hydrogen is ionized and Compton scattering off electrons (linked to photons by electromagnetic interactions) couples photons to baryons. At z_{LS}, hydrogen recombines and photons can start their travel. We assume that thermal radiation with temperature $T_{\mathcal{E}}(\mathbf{p},\hat{\mathbf{d}})$ is emitted by every point with coordinates p_i in this hypersurface. This temperature depends also on the direction of emission described by the vector $\hat{\mathbf{d}}$, normalized to unity in the background. The different photon paths are specified by the direction from which they arrive at $\mathcal{O}$, specified by a vector $\hat{\mathbf{e}}$ normalized to unity in the background. This direction fixes the point $\mathbf{p}$ and the direction $\hat{\mathbf{d}}$ at emission.

If the CMB has a blackbody distribution and the photons suffer a redshift z during their travel from the emitter $\mathcal{E}$ to the observer $\mathcal{O}$, the emitted frequency $\omega_{\mathcal{E}}$ and the observed one $\omega_{\mathcal{O}}$ are related by $\omega_{\mathcal{O}}=\omega_{\mathcal{E}}/(1+z)$. Since the occupation number per frequency mode is conserved, the corresponding photon temperatures are related by $T_{\mathcal{O}}=T_{\mathcal{E}}/(1+z)$. The anisotropies detected by an observer are due to inhomogeneities in the temperature at emission and to the different redshift suffered by photons coming from different directions. We will compute this quantity up to second order in gravitational perturbations.

The temperature measured by an observer at $\mathcal{O}$ can be written as

$$T_{\mathcal{O}}(\mathbf{x}_{\mathcal{O}},\hat{\mathbf{e}})=\frac{\omega_{\mathcal{O}}}{\omega_{\mathcal{E}}}T_{\mathcal{E}}(\mathbf{p},\hat{\mathbf{d}}),\tag{2.2}$$

with $\omega=-g_{\mu\nu}U^{\mu}k^{\nu}$, where U^{μ} is the four-velocity of the observer or emitter and $k^{\nu}=dx^{\nu}/d\lambda$ is the wave vector of the photon in the conformal metric, tangent to the null geodesic $x^{\nu}(\lambda)$, followed by the photon from the emission to the observation point. In fact, we will propagate photons back from the observation point to the emission surface. We thus need to obtain $\omega_{\mathcal{E}}$, $\mathbf{p}$, and $\hat{\mathbf{d}}$ for a given initial set of values $\mathbf{x}_{\mathcal{O}}$, $\hat{\mathbf{e}}$, and $\omega_{\mathcal{O}}$. The resulting quantities are functions of the photon path and wave vector, which we expand in series of the metric perturbations $g^{(r)}_{\mu\nu}$ and their derivatives as

$$x^{\mu}(\lambda)=x^{(0)\mu}(\lambda)+x^{(1)\mu}(\lambda)+x^{(2)\mu}(\lambda)+\cdots,$$

$$k^{\mu}(\lambda)=k^{(0)\mu}(\lambda)+k^{(1)\mu}(\lambda)+k^{(2)\mu}(\lambda)+\cdots.\tag{2.3}$$

Contrary to the assumptions of Ref. [16], we are not taking the observer and emitter comoving with the total fluid of the universe. In this way we keep track, to first order, of the dipole due to the observer's motion and of the Doppler effect due to the emitter's motion, which are otherwise lost. In addition to these effects, to second order, we also take into account cross terms involving the velocities and other sources of anisotropy.

We can expand the four-velocity as

$$U^{\mu}=\frac{1}{a}\left(\delta^{\mu}_{0}+v^{(1)\mu}+\frac{1}{2}v^{(2)\mu}+\cdots\right).\tag{2.4}$$

This is subject to the normalization condition $U^{\mu}U_{\mu}=-1$.

It is also useful to write the perturbed spatially flat conformal metric as

$$g_{00}=-(1+2\psi^{(1)}+\psi^{(2)}+\cdots),\tag{2.5}$$

$$g_{0i}=z^{(1)}_{i}+\frac{1}{2}z^{(2)}_{i}+\cdots,\tag{2.6}$$

$$g_{ij}=(1-2\phi^{(1)}-\phi^{(2)})\delta_{ij}+\chi^{(1)}_{ij}+\frac{1}{2}\chi^{(2)}_{ij}+\cdots,\tag{2.7}$$

where[1] $\chi^{(r)i}_{i}=0$ and the functions $\psi^{(r)}$, $z^{(r)}_{i}$, $\phi^{(r)}$, and $\chi^{(r)}_{ij}$ represent the rth order perturbation of the metric.

The normalization condition for the velocity fixes the time component $v^{(r)0}$ in terms of the lapse perturbation $\psi^{(r)}$. For the first and second order perturbations we obtain

$$v^{(1)0}=-\psi^{(1)},\tag{2.8}$$

$$v^{(2)0}=-\psi^{(2)}+3(\psi^{(1)})^{2}+2z^{(1)}_{i}v^{(1)i}+v^{(1)}_{i}v^{(1)i}.\tag{2.9}$$

In order to obtain the variation in the sky of the observed temperature up to second order, according to Eq. (2.2), we need to expand $\omega_{\mathcal{O}}$ and $\omega_{\mathcal{E}}$ up to second order in gravitational perturbations,

$$\omega=\omega^{(0)}(1+\tilde{\omega}^{(1)}+\tilde{\omega}^{(2)}+\cdots),\tag{2.10}$$

and also to expand the temperature at emission:

$$T_{\mathcal{E}}(\mathbf{p},\hat{\mathbf{d}})=T^{(0)}_{\mathcal{E}}(1+\tau(\mathbf{p},\hat{\mathbf{d}})).\tag{2.11}$$

We will not perform a full expansion of $\tau(\mathbf{p},\hat{\mathbf{d}})$, as a calculation of this quantity would be beyond the aim of this paper. We will instead assume that it is known for a given model and compute the additional effect of gravity along the photons path. We also have to take into account that the point $\mathbf{p}$ and direction $\hat{\mathbf{d}}$ at emission need to be expanded in the expression of $\tau(\mathbf{p},\hat{\mathbf{d}})$ as $\mathbf{p}=\mathbf{p}^{(0)}+\mathbf{p}^{(1)}+\cdots$, and $\hat{\mathbf{d}}=\mathbf{d}^{(0)}+\mathbf{d}^{(1)}+\cdots$. Performing these expansions in Eq. (2.2) we obtain [16]

[1] Indices are raised and lowered using δ^{ij} and δ_{ij}, respectively.

$$T_{\mathcal{O}}(\mathbf{x}_{\mathcal{O}},\hat{\mathbf{e}})=\frac{\omega_{\mathcal{O}}^{(0)}}{\omega_{\mathcal{E}}^{(0)}}T_{\mathcal{E}}^{(0)}\Bigg[1+(\tilde{\omega}_{\mathcal{O}}^{(1)}-\tilde{\omega}_{\mathcal{E}}^{(1)}+\tau)+\Bigg(\tilde{\omega}_{\mathcal{O}}^{(2)}-\tilde{\omega}_{\mathcal{E}}^{(2)}+(\tilde{\omega}_{\mathcal{E}}^{(1)})^2-\tilde{\omega}_{\mathcal{O}}^{(1)}\tilde{\omega}_{\mathcal{E}}^{(1)}+\tilde{\omega}_{\mathcal{O}}^{(1)}\tau-\tilde{\omega}_{\mathcal{E}}^{(1)}\tau+p^{(1)i}\frac{\partial\tau}{\partial x^i}+d^{(1)i}\frac{\partial\tau}{\partial d^i}\Bigg)+\cdots\Bigg], \tag{2.12}$$

where τ and its spatial derivatives have to be evaluated at $(\mathbf{p}^{(0)},\mathbf{d}^{(0)})$. The first factor gives the mean temperature at the observation point $T_{\mathcal{O}}^{(0)}\equiv T_{\mathcal{E}}^{(0)}\omega_{\mathcal{O}}^{(0)}/\omega_{\mathcal{E}}^{(0)}$, and the round brackets inside the term in square brackets define the first and second order perturbations $\delta T^{(1)}$ and $\delta T^{(2)}$ that we are looking for.

To compute them, we will use the same background geodesics as in Ref. [16],

$$x^{(0)\mu}=(\lambda,(\lambda_{\mathcal{O}}-\lambda)e^i),\qquad k^{(0)\mu}=(1,-e^i), \tag{2.13}$$

and boundary conditions at the origin:

$$x^{(1)\mu}(\lambda_{\mathcal{O}})=x^{(2)\mu}(\lambda_{\mathcal{O}})=0,\qquad k^{(1)i}(\lambda_{\mathcal{O}})=k^{(2)i}(\lambda_{\mathcal{O}})=0. \tag{2.14}$$

The condition that the wave vector be null fixes the value of $k^{(1)0}(\lambda_{\mathcal{O}})$ and $k^{(2)0}(\lambda_{\mathcal{O}})$. We will only need $k^{(1)0}(\lambda_{\mathcal{O}})$ explicitly:

$$k^{(1)0}(\lambda_{\mathcal{O}})=-\psi_{\mathcal{O}}^{(1)}-z_{\mathcal{O}}^{(1)i}e_i-\phi_{\mathcal{O}}^{(1)}+\frac{1}{2}\chi_{\mathcal{O}}^{(1)ij}e_ie_j\,. \tag{2.15}$$

Using the metric, four-velocity, and wave vector expansions we can obtain the quantities in the expansion of ω:

$$\omega^{(0)}=a^{-1},\qquad \tilde{\omega}^{(1)}=k^{(1)0}+\psi^{(1)}+v_i^{(1)}e^i+z_i^{(1)}e^i, \tag{2.16}$$

$$\tilde{\omega}^{(2)}=k^{(2)0}+\frac{1}{2}\psi^{(2)}+\frac{1}{2}z_i^{(2)}e^i+\frac{1}{2}v_i^{(2)}e^i-\frac{1}{2}(\psi^{(1)})^2+\frac{1}{2}v_i^{(1)}v^{(1)i}+k^{(1)0}\psi^{(1)}-v_i^{(1)}k^{(1)i}-z_i^{(1)}k^{(1)i}-\psi^{(1)}z_i^{(1)}e^i-2\phi^{(1)}v_i^{(1)}e^i+\chi_{ij}^{(1)}e^iv^{(1)j}+\frac{dk^{(1)0}}{d\lambda}\Delta\lambda+(\psi_{,j}^{(1)}+z_{i,j}^{(1)}e^i+v_{i,j}^{(1)}e^i)p^{(1)j},$$

where $\Delta\lambda$ is the difference in affine parameter between the points where the background and first order geodesics intersect the $\eta=\eta_{\mathcal{E}}$ hypersurface, and is given by $\Delta\lambda=-x^{(1)0}+\cdots$. It can also be seen [16] that $p^{(1)i}=x^{(1)i}+x^{(1)0}e^i$ and

$$d^{(1)i}=e^i-\frac{e^i-k^{(1)i}}{|e^i-k^{(1)i}|}. \tag{2.17}$$

Finally, we obtain, for the first order temperature anisotropy,

$$\delta T^{(1)}=\tilde{\omega}_{\mathcal{O}}^{(1)}-\tilde{\omega}_{\mathcal{E}}^{(1)}+\tau=-\phi_{\mathcal{O}}^{(1)}+\frac{1}{2}\chi_{\mathcal{O}}^{(1)ij}e_ie_j+v_{\mathcal{O}}^{(1)i}e_i-k_{\mathcal{E}}^{(1)0}-v_{\mathcal{E}}^{(1)i}e_i-z_{\mathcal{E}}^{(1)i}e_i-\psi_{\mathcal{E}}^{(1)}+\tau \tag{2.18}$$

and, for the second order one,

$$\delta T^{(2)}=\left(k^{(2)0}+\frac{1}{2}\psi^{(2)}+\frac{1}{2}v_i^{(2)}e^i+\frac{1}{2}z_i^{(2)}e^i-\frac{1}{2}(\psi^{(1)})^2+\frac{1}{2}v_i^{(1)}v^{(1)i}+k^{(1)0}\psi^{(1)}-\psi^{(1)}z_i^{(1)}e^i-2\phi^{(1)}v_i^{(1)}e^i+\chi_{ij}^{(1)}e^iv^{(1)j}\right)\Bigg|_{\mathcal{E}}^{\mathcal{O}}+(v_{\mathcal{E}i}^{(1)}+z_{\mathcal{E}i}^{(1)})k_{\mathcal{E}}^{(1)i}+\frac{dk^{(1)0}}{d\lambda}\Bigg|_{\mathcal{E}}x_{\mathcal{E}}^{(1)0}-(\psi_{,j}^{(1)}+z_{i,j}^{(1)}e^i+v_{i,j}^{(1)}e^i+\tau_{,j})_{\mathcal{E}}(x^{(1)j}+x^{(1)0}e^j)_{\mathcal{E}}+\frac{\partial\tau}{\partial d^i}\Bigg|_{\mathcal{E}}d^{(1)i}-(k^{(1)0}+v_i^{(1)}e^i+z_i^{(1)}e^i+\psi^{(1)}-\tau)_{\mathcal{E}}(k^{(1)0}+v_i^{(1)}e^i+z_i^{(1)}e^i+\psi^{(1)})\big|_{\mathcal{E}}^{\mathcal{O}}. \tag{2.19}$$

The next step is to obtain the null geodesics up to second order; in particular, we need to compute the quantities $k^{(2)0}$, $k^{(1)\mu}$, and $x^{(1)\mu}$ to substitute in Eqs. (2.18) and (2.19). This problem has been solved for a general perturbed spacetime in Ref. [16] using the geodesic expansion introduced by Pyne and Birkinshaw [20]. Following their method, we obtain, for perturbations around a flat Robertson-Walker background in any gauge, that the first order wave vector is given by

$$k^{(1)0}(\lambda_{\mathcal{E}})=\psi_{\mathcal{O}}^{(1)}-\phi_{\mathcal{O}}^{(1)}+\frac{1}{2}\chi_{\mathcal{O}}^{(1)ij}e_ie_j-2\psi_{\mathcal{E}}^{(1)}-z_{\mathcal{E}}^{(1)i}e_i+I_1(\lambda_{\mathcal{E}}), \tag{2.20}$$

with

$$I_1(\lambda_{\mathcal{E}})=\int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}}d\lambda\, A^{(1)\prime}, \tag{2.21}$$

where $A^{(1)}\equiv\psi^{(1)}+\phi^{(1)}+z_i^{(1)}e^i-\frac{1}{2}\chi_{ij}^{(1)}e^ie^j$, and

$$k^{(1)i}(\lambda_{\mathcal{E}})=2\phi_{\mathcal{O}}^{(1)}e^i+z_{\mathcal{O}}^{(1)i}-\chi_{\mathcal{O}}^{(1)ij}e_j-2\phi_{\mathcal{E}}^{(1)}e^i-z_{\mathcal{E}}^{(1)i}+\chi_{\mathcal{E}}^{(1)ij}e_j-I_1^i(\lambda_{\mathcal{E}}), \tag{2.22}$$

with

$$I_1^i(\lambda_{\mathcal{E}})=\int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}}d\lambda A^{(1),i}. \tag{2.23}$$

For the first order geodesics, we obtain

$$x^{(1)0}(\lambda_{\mathcal{E}})=(\lambda_{\mathcal{E}}-\lambda_{\mathcal{O}})\left[\psi_{\mathcal{O}}^{(1)}-\phi_{\mathcal{O}}^{(1)}+\frac{1}{2}\chi_{\mathcal{O}}^{(1)ij}e_ie_j\right]+\int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}}d\lambda[-2\psi^{(1)}-z_i^{(1)}e^i+(\lambda_{\mathcal{E}}-\lambda)A^{(1)\prime}],$$

$$x^{(1)i}(\lambda_{\mathcal{E}})=(\lambda_{\mathcal{E}}-\lambda_{\mathcal{O}})[2\phi_{\mathcal{O}}^{(1)}e^i+z_{\mathcal{O}}^{(1)i}-\chi_{\mathcal{O}}^{(1)ij}e_j]-\int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}}d\lambda[2\phi^{(1)}e^i+z^{(1)i}-\chi^{(1)ij}e_j+(\lambda_{\mathcal{E}}-\lambda)A^{(1),i}]. \tag{2.24}$$

For the second order, we need only the difference between the wave vector at emission and observation:

$$\begin{aligned}k_{\mathcal{E}}^{(2)0}-k_{\mathcal{O}}^{(2)0}=&\psi_{\mathcal{O}}^{(2)}-\psi_{\mathcal{E}}^{(2)}-\frac{1}{2}z_{\mathcal{E}}^{(2)i}e_i+\frac{1}{2}z_{\mathcal{O}}^{(2)i}e_i+2\psi_{\mathcal{O}}^{(1)}k_{\mathcal{O}}^{(1)0}-2\psi_{\mathcal{E}}^{(1)}k_{\mathcal{E}}^{(1)0}\\&-(2x^{(1)i}\psi_{,i}^{(1)}+2x^{(1)0}\psi^{(1)\prime}-z^{(1)i}k_i^{(1)}+x^{(1)0}z_i^{(1)\prime}e^i+x^{(1)i}z_{j,i}^{(1)}e^j)_{\mathcal{E}}+I_2(\lambda_{\mathcal{E}}),\end{aligned} \tag{2.25}$$

with

$$I_2(\lambda_{\mathcal{E}})=\int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}}d\lambda\left[\frac{1}{2}A^{(2)\prime}-(z_i^{(1)\prime}-\chi_{ij}^{(1)\prime}e^j)(k^{(1)i}+e^ik^{(1)0})+2k^{(1)0}A^{(1)\prime}+2\phi^{(1)\prime}A^{(1)}+x^{(1)0}A^{(1)\prime\prime}+x^{(1)i}A_{,i}^{(1)\prime}\right], \tag{2.26}$$

where $A^{(2)}\equiv\psi^{(2)}+\phi^{(2)}+z_i^{(2)}e^i-\frac{1}{2}\chi_{ij}^{(2)}e^ie^j$.

We can now write the temperature anisotropy in terms of the metric perturbations. Replacing Eq. (2.20) into Eq. (2.18) we obtain, for first order,

$$\delta T^{(1)}=\psi_{\mathcal{E}}^{(1)}-\psi_{\mathcal{O}}^{(1)}+v_{\mathcal{O}}^{(1)i}e_i-v_{\mathcal{E}}^{(1)i}e_i+\tau-I_1(\lambda_{\mathcal{E}}). \tag{2.27}$$

This is a general expression, valid in any gauge, that takes into account scalar, vector, and tensor perturbations. It includes the effect of intrinsic anisotropies in the last scattering surface, dipole due to the observer's motion, Doppler effect from the emitter's motion, and gravitational redshift of the photons. It is equivalent to the well-known result originally obtained by Sachs and Wolfe [21]. It can be seen that the full expression is gauge invariant up to a monopole term; the relative contributions from the intrinsic, Doppler, and gravitational redshift contributions are, however, gauge dependent.

Analogously, for second order, we obtain

$$\begin{aligned}\delta T^{(2)}=&\frac{1}{2}\psi_{\mathcal{E}}^{(2)}-\frac{1}{2}\psi_{\mathcal{O}}^{(2)}+\frac{3}{2}(\psi_{\mathcal{O}}^{(1)})^2-\frac{1}{2}(\psi_{\mathcal{E}}^{(1)})^2-I_2(\lambda_{\mathcal{E}})-v_{\mathcal{E}}^{(1)i}e_i\psi_{\mathcal{E}}^{(1)}\\&+[I_1(\lambda_{\mathcal{E}})+v_{\mathcal{E}}^{(1)i}e_i]\left(2\psi_{\mathcal{O}}^{(1)}-\phi_{\mathcal{O}}^{(1)}+\frac{1}{2}\chi_{\mathcal{O}}^{(1)ij}e_ie_j-v_{\mathcal{O}}^{(1)i}e_i-\psi_{\mathcal{E}}^{(1)}-\tau+v_{\mathcal{E}}^{(1)i}e_i+I_1(\lambda_{\mathcal{E}})\right)\\&+x_{\mathcal{E}}^{(1)0}A_{\mathcal{E}}^{(1)\prime}+(x_{\mathcal{E}}^{(1)j}+x_{\mathcal{E}}^{(1)0}e^j)(\psi_{,j}^{(1)}-v_{i,j}^{(1)}e^i+\tau_{,j})_{\mathcal{E}}+v_{\mathcal{O}}^{(1)i}\left(\frac{1}{2}v_{\mathcal{O}i}^{(1)}-2\phi_{\mathcal{O}}^{(1)}e_i+\chi_{\mathcal{O}ij}^{(1)}e^j\right)\\&-\frac{1}{2}v_{\mathcal{E}i}^{(1)}v_{\mathcal{E}}^{(1)i}+\psi_{\mathcal{E}}^{(1)}\tau+\frac{\partial\tau}{\partial d^i}d^{(1)i}-\psi_{\mathcal{O}}^{(1)}(\psi_{\mathcal{E}}^{(1)}+\tau)-v_{\mathcal{O}}^{(1)i}e_i\left(\psi_{\mathcal{O}}^{(1)}-\phi_{\mathcal{O}}^{(1)}+\frac{1}{2}\chi_{\mathcal{O}}^{(1)kj}e_ke_j-\tau-\psi_{\mathcal{E}}^{(1)}\right)\\&+v_{\mathcal{E}i}^{(1)}[-z_{\mathcal{E}}^{(1)i}+z_{\mathcal{O}}^{(1)i}+2\phi_{\mathcal{O}}^{(1)}e^i-\chi_{\mathcal{O}}^{(1)ij}e_j-I_1^i(\lambda_{\mathcal{E}})].\end{aligned} \tag{2.28}$$

This is also a general expression that is valid in any gauge and takes into account scalar, vector, and tensor perturbations. It also includes the effects of the motion of the observer and the emitter. In the previous expression we have dropped the terms proportional to $v^{(2)i}$ as this computation is not aimed at obtaining v^i at the emission or observation points, but assumes that they are known quantities.

To proceed further with the computation, we need to know the initial values and the evolution of the perturbations. To solve this it is necessary to fix a gauge. There are different possibilities: The synchronous gauge ($\psi^{(r)}=z^{(r)}=0$) turns out to be convenient for many calculations and has been widely used for linear anisotropy computations. Another choice is the Poisson gauge ($z_i^{(r),i}=\chi_{ij}{}^{(r),j}=0$), recently discussed by Bertschinger [22], which in the case of scalar perturbations reduces to the longitudinal gauge. The latter gauge, in which $z_i^{(r)}=\chi_{ij}^{(r)}=0$, has become very popular, because the evolution equations are most similar to the Newtonian ones, and thus closest to our classical intuition. All second order temperature anisotropy calculations have been performed in this gauge. Since the vector and tensor modes are set to zero by hand, the longitudinal gauge should not be used to study perturbations beyond the linear regime: This is because in the nonlinear case the scalar, vector, and tensor modes are dynamically coupled and vector and tensor modes cannot be set to zero arbitrarily. This could be a problem when studying the Rees-Sciama effect that explicitly involves nonlinearities in the metric perturbations; we will come to this point in Sec. IV. We will use the Poisson gauge, which overcomes the above limitation of the longitudinal gauge, while keeping all its advantages in terms of physical interpretation of the results.

III. SECOND ORDER PERTURBATIONS IN GENERAL RELATIVITY

We consider the gravitational instability of irrotational collisionless matter in a flat Robertson-Walker background up to second order. Different approaches to this problem have been proposed. The first solution of the second order relativistic equations has been obtained, in the synchronous gauge, in a pioneering work by Tomita [23]. Matarrese, Pantano, and Saez [24,25] obtained the leading order terms of the expansion, using a different method, based on the so-called fluid-flow approach. Salopek, Stewart, and Croudace [26] used a gradient expansion technique to obtain second order metric perturbations; an intrinsic limitation of their method is, however, that nonlocal terms, such as the nonlinear tensor modes, are lost. Russ *et al.* [27] recently rederived the metric perturbations to second order in the synchronous gauge, using a tetrad formalism. We are interested here in obtaining the solution in the Poisson gauge. Instead of perturbing the Einstein equations in this gauge and then solving them, we will transform the solution known in the synchronous gauge to the Poisson one, using the second order gauge transformation recently developed in Ref. [28] (for more details see Ref. [29]).

Up to this point we have been completely general in the inclusion of scalar, vector, and tensor modes. Now, in order to give a more quantitative insight on the relevance of the different contributions, we will make some restrictions. First, we will neglect vector modes at the linear order. The fact that they have decreasing amplitude and that they are not generated in inflationary theories makes us expect that their role will not be relevant, at least if inflation was the mechanism for primordial fluctuation generation. We will, however, keep track of the second order vector modes generated by the coupling with scalar modes. Second, we will neglect the effect of linear tensor modes as sources for second order metric perturbations: Because of the graviton free streaming inside the horizon, this is a very reasonable approximation. We will then only consider the second order perturbations generated by linear scalar perturbations. These are expected to give the dominant contribution, as vector and tensor modes have decaying amplitude. The solutions given in Refs. [23,27] and hence those obtained in this section apply to this restricted case. For the computation of the CMB anisotropies we will, however, keep the contribution of the linear tensor modes in $\delta T^{(1)}$ and $\delta T^{(2)}$ everywhere, except as source for $g_{\mu\nu}^{(2)}$. The contribution of tensor modes can in fact be comparable to the scalar contribution to $\delta T^{(1)}$ at large scales in many inflationary models [30].

The first order solution to the perturbed Einstein equations in the synchronous gauge is given by (see, e.g., Ref. [31])

$$\psi_S^{(1)}=z_S^{(1)}=0,$$

$$\phi_S^{(1)}=\frac{5}{3}\varphi+\frac{\eta^2}{18}\nabla^2\varphi, \tag{3.1}$$

$$\chi_{S ij}^{(1)}=-\frac{\eta^2}{3}\left(\varphi_{,ij}-\frac{1}{3}\delta_{ij}\nabla^2\varphi\right)+\chi_{ij}^{\top(1)},$$

where $\varphi=\varphi(\mathbf{x})$ is the initial peculiar gravitational potential. $\chi_{ij}^{\top(1)}$ is the tensor (transverse and traceless) contribution, which can be written as

$$\chi_{ij}^{\top(1)}(\mathbf{x},\eta)=\frac{1}{(2\pi)^3}\int d^3\mathbf{k}\exp(i\mathbf{k}\cdot\mathbf{x})\chi_\sigma^{(1)}(\mathbf{k},\eta)\epsilon_{ij}^\sigma(\hat{\mathbf{k}}), \tag{3.2}$$

where $\epsilon_{ij}^\sigma(\hat{\mathbf{k}})$ is the polarization tensor, with σ ranging over the polarization components $+,\times$, and $\chi_\sigma^{(1)}(\mathbf{k},\eta)$ is the amplitude. Its time evolution during the matter-dominated era can be represented as

$$\chi_\sigma^{(1)}(\mathbf{k},\eta)\approx A(k)a_\sigma(\mathbf{k})\left(\frac{3j_1(k\eta)}{k\eta}\right), \tag{3.3}$$

where $a_\sigma(\mathbf{k})$ is a zero mean random variable with autocorrelation function $\langle a_\sigma(\mathbf{k})a_{\sigma'}(\mathbf{k}')\rangle=(2\pi)^3k^{-3}\delta^3(\mathbf{k}+\mathbf{k}')\delta_{\sigma\sigma'}$. The spectrum of the gravitational wave background depends on the processes by which it was generated, and, for example, in most inflationary models, $A(k)$ is nearly scale invariant and proportional to the Hubble constant during inflation.

The second order perturbations are given by [23,27,29]

$$\psi_S^{(2)}=z_S^{(2)}=0,$$

$$\phi_S^{(2)}=\frac{\eta^4}{252}\left(-\frac{10}{3}\varphi^{,ki}\varphi_{,ki}+(\nabla^2\varphi)^2\right)+\frac{5}{18}\eta^2\left(\varphi^{,k}\varphi_{,k}+\frac{4}{3}\varphi\nabla^2\varphi\right), \tag{3.4}$$

$$\chi_{Sij}^{(2)}=\frac{\eta^4}{126}\left(19\varphi_{,i}^{,k}\varphi_{,kj}-12\varphi_{,ij}\nabla^2\varphi+4(\nabla^2\varphi)^2\delta_{ij}-\frac{19}{3}\varphi^{,kl}\varphi_{,kl}\delta_{ij}\right)+\frac{5}{9}\eta^2\left(-6\varphi_{,i}\varphi_{,j}-4\varphi\varphi_{,ij}+2\varphi^{,k}\varphi_{,k}\delta_{ij}+\frac{4}{3}\varphi\nabla^2\varphi\delta_{ij}\right)+\pi_{Sij},$$

where the traceless and transverse contribution π_{Sij} satisfies the inhomogeneous wave equation

$$\pi''_{Sij}+\frac{4}{\eta}\pi'_{Sij}-\nabla^2\pi_{Sij}=-\frac{\eta^4}{21}\nabla^2 S_{ij}, \tag{3.5}$$

with

$$S_{ij}=\nabla^2\Psi_0\delta_{ij}+\Psi_{0,ij}+2(\varphi_{,ij}\nabla^2\varphi-\varphi_{,ik}\varphi_{,j}^{,k}), \tag{3.6}$$

where

$$\nabla^2\Psi_0=-\frac{1}{2}((\nabla^2\varphi)^2-\varphi_{,ik}\varphi^{,ik}). \tag{3.7}$$

This equation can be solved using the Green method; we obtain for π_{Sij} that

$$\pi_{Sij}=\frac{\eta^4}{21}S_{ij}+\frac{4}{3}\eta^2 T_{ij}(\mathbf{x})+\tilde{\pi}_{ij}(\mathbf{x},\eta), \tag{3.8}$$

where $\nabla^2 T_{ij}=S_{ij}$ and the remaining piece $\tilde{\pi}_{ij}(\mathbf{x},\eta)$, accounting for a term that is constant in time and another one that oscillates with decreasing amplitude, can be written as

$$\tilde{\pi}_{ij}(\mathbf{x},\eta)=\frac{1}{(2\pi)^3}\int d^3\mathbf{k}\exp(i\mathbf{k}\cdot\mathbf{x})\frac{40}{k^4}S_{ij}(\mathbf{k})\times\left(\frac{1}{3}+\frac{\cos(k\eta)}{(k\eta)^2}-\frac{\sin(k\eta)}{(k\eta)^3}\right), \tag{3.9}$$

with $S_{ij}(\mathbf{k})=\int d^3\mathbf{x}\exp(-i\mathbf{k}\cdot\mathbf{x})S_{ij}(\mathbf{x})$.

The gauge transformation is determined to each order by a four-vector $\xi^{(r)\mu}$ that we split as $\xi^{(r)0}=\alpha^{(r)}$ and $\xi^{(r)i}=\partial^i\beta^{(r)}+d^{(r)i}$, with $\partial_i d^{(r)i}=0$. In Ref. [28] the vectors $\xi^{(1)\mu}$ and $\xi^{(2)\mu}$, describing the gauge transformation from the synchronous to the Poisson gauge, have been explicitly obtained in terms of the synchronous metric perturbations $g_{S\mu\nu}^{(1)}$ and $g_{S\mu\nu}^{(2)}$. Using the metric perturbations in the synchronous gauge presented above, we can write the first order gauge transformation as

$$\alpha^{(1)}=\frac{\eta}{3}\varphi,$$

$$\beta^{(1)}=\frac{\eta^2}{6}\varphi, \tag{3.10}$$

and $d^{(1)i}=0$, in the absence of vector modes in the initial conditions. For second order, we obtain

$$\alpha^{(2)}=-\frac{2}{21}\eta^3\Psi_0+\eta\left(\frac{10}{9}\varphi^2+4\Theta_0\right),$$

$$\beta^{(2)}=\eta^4\left(\frac{1}{72}\varphi^{,i}\varphi_{,i}-\frac{1}{42}\Psi_0\right)+\frac{\eta^2}{3}\left(\frac{7}{2}\varphi^2+6\Theta_0\right), \tag{3.11}$$

where $\nabla^2\Theta_0=\Psi_0-\frac{1}{3}\varphi^{,i}\varphi_{,i}$ and

$$\nabla^2 d_j^{(2)}=\eta^2\left(-\frac{4}{3}\varphi_{,j}\nabla^2\varphi+\frac{4}{3}\varphi^{,i}\varphi_{,ij}-\frac{8}{3}\Psi_{0,j}\right). \tag{3.12}$$

We can now compute the metric perturbations in the Poisson gauge using the transformation rules of Ref. [28]. For the first order, we obtain

$$\psi_P^{(1)}=\phi_P^{(1)}=\varphi,$$

$$\chi_{Pij}^{(1)}=\chi_{ij}^{\top(1)}. \tag{3.13}$$

These equations show the well-known result for scalar perturbations in the longitudinal gauge and the gauge invariance for tensor modes at the linear level. For the second order, we obtain

$$\psi_P^{(2)}=\eta^2\left(\frac{1}{6}\varphi^{,i}\varphi_{,i}-\frac{10}{21}\Psi_0\right)+\frac{16}{3}\varphi^2+12\Theta_0,$$

$$\phi_P^{(2)}=\eta^2\left(\frac{1}{6}\varphi^{,i}\varphi_{,i}-\frac{10}{21}\Psi_0\right)+\frac{4}{3}\varphi^2-8\Theta_0, \tag{3.14}$$

$$\nabla^2 z_P^{(2)i}=-\frac{8}{3}\eta(\varphi^{,i}\nabla^2\varphi-\varphi^{,ij}\varphi_{,j}+2\Psi_0^{,i}),$$

$$\chi_{Pij}^{(2)}=\tilde{\pi}_{ij}.$$

Note that the resulting expressions for ψ_P and ϕ_P can be recovered, except for the subleading time-independent terms by taking the weak-field limit of Einstein's theory (e.g., Ref. [32]) and then expanding in powers of the perturbation amplitude; this is basically the method employed in previous second order computations of the Rees-Sciama effect. Also interesting is the way in which the tensor modes appear in this gauge: The transformation from the synchronous to the Poisson gauge has in fact dropped the Newtonian and post-Newtonian contributions, whose physical interpretation in terms of gravitational waves is highly nontrivial (see the discussion in Ref. [33]); what remains is a wavelike piece plus a constant term which has no effects on $\delta T^{(2)}$.

IV. POISSON GAUGE ANISOTROPIES

Let us start by discussing the first order anisotropies that are described by Eq. (2.27). The first term $\psi_{\mathcal{E}}^{(1)}$ represents the contribution from the gravitational redshift of the photons due to the difference in gravitational potential between the emission and observation points. $\psi_{\mathcal{O}}^{(1)}$ only contributes to the monopole and can be neglected. The term $v_{\mathcal{O}}^{(1)i}e_i$ is the dipole due to the motion of the observer. The term $v_{\mathcal{E}}^{(1)i}e_i$ accounts for the Doppler effect due to the velocity of the photon-baryon fluid at recombination and contributes to the acoustic peaks. The term τ describes the intrinsic anisotropies in the photon temperature and is highly model dependent. For example, for adiabatic perturbations, in which all the components (baryons, photons, dark matter) have a constant number density ratio, the photon energy density, and thus the temperature, varies proportionally to the potential fluctuations (at scales larger than the Jeans length). It can be seen that in this case

$$\tau = \frac{1}{4}\frac{\delta\rho_\gamma}{\rho_\gamma}\bigg|_{\mathcal{E}} = \frac{1}{3}\frac{\delta\rho_T}{\rho_T}\mathcal{E} = -\frac{2}{3}\phi_{\mathcal{E}}^{(1)}.$$

It is the combination of this term and the first one that gives the standard result for adiabatic perturbations at large angular scales, $\delta T = \frac{1}{3}\varphi$. At small scales, τ gives the main contribution to the acoustic peaks. We have not intended here to include a computation of τ and $v_{\mathcal{E}}^{(1)i}$ that would involve solving the linearized transport equation for the photons that is coupled to the fluid evolution equations for the cold dark matter component and the baryons, the Boltzmann equation for the neutrino distribution, and the Einstein equations for the metric perturbations. This problem has been treated and solved numerically by several authors (see, e.g., [34–40]). We assume that τ and $v_{\mathcal{E}}^{(1)i}$ are known for a given model, and compute the additional anisotropy generated by the metric perturbations along the photon path up to second order.

Finally, the contribution to $\delta T^{(1)}$ from the last term, $I_1(\lambda_{\mathcal{E}})$ [called the integrated Sachs-Wolfe effect and given by Eq. (2.21)], represents the additional gravitational redshift due to the time variation of the metric during the photon travel. As in the linear regime the scalar potentials ϕ and ψ are constant in time for a flat matter-dominated universe, their contribution vanishes (they will however give a nonvanishing contribution in the nonlinear regime). The contribution of tensor perturbations to the temperature anisotropies arises exclusively from $I_1(\lambda_{\mathcal{E}})$ at linear order. In many inflationary models, in which besides the usual scalar perturbations also a background of gravitational waves is produced, their contribution to the CMB anisotropies can be comparable to that of scalar perturbations at large scales. The contribution from the observer's motion term is of order 10^{-3}, while the remaining part contributes for an order 10^{-5}.

We can now discuss the second order anisotropies that are given by Eq. (2.28). The first term, given by $\psi_{\mathcal{E}}^{(2)}$, represents the gravitational redshift of the photons due to the second order metric perturbations, and is much smaller than its first order equivalent. Then, there are several terms involving products of two of the terms contributing to $\delta T^{(1)}$; these are all very small compared to $\delta T^{(1)}$ (at least three orders of magnitude smaller) and can safely be neglected. Also the term $(\partial\tau/\partial d^i)d^{(1)i}$ is the product of two small quantities and can be neglected.

Then, there is the term $(x_{\mathcal{E}}^{(1)j} + x_{\mathcal{E}}^{(1)0}e^j)(\psi_{,j}^{(1)} - v_{i,j}^{(1)}e^i + \tau_{,j}^{(1)})_{\mathcal{E}}$, which can be split into a piece proportional to

$$\begin{aligned} x_\perp^{(1)j}(\lambda_{\mathcal{E}}) &\equiv (\delta^{ij} - e^i e^j)x_i^{(1)}(\lambda_{\mathcal{E}}) \\ &= (\lambda_{\mathcal{E}} - \lambda_{\mathcal{O}})(-\chi_{\mathcal{O}}^{\top(1)jk}e_k + \chi_{\mathcal{O}}^{\top(1)ik}e_k e_i e^j) \\ &\quad + \int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}} d\lambda (\chi^{\top(1)jk}e_k - \chi^{\top(1)ik}e_k e_i e^j) \\ &\quad - \int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}} d\lambda (\lambda_{\mathcal{E}} - \lambda)\bigg(2\varphi^{,j} - 2\varphi_{,i}e^i e^j \\ &\quad - \frac{1}{2}\chi_{kl}^{\top(1),j}e^k e^l + \frac{1}{2}\chi_{kl,i}^{\top(1)}e^k e^l e^i e^j\bigg), \end{aligned} \tag{4.1}$$

and another piece proportional to

$$\begin{aligned} (x_\parallel^{(1)j} + x_0^{(1)}e^j)_{\mathcal{E}} &\equiv e^j(x_i^{(1)}e^i + x_0^{(1)})_{\mathcal{E}} \\ &= -e^j \int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}} d\lambda \bigg(2\varphi - \frac{1}{2}\chi_{kl}^{\top(1)}e^k e^l\bigg). \end{aligned} \tag{4.2}$$

The first piece describes the effect of the gravitational lensing on the photons as they travel from the last scattering surface to the observer. The transverse displacement $x_\perp^{(1)j}$ includes the usual contribution from scalar perturbations (φ), which has been considered in some previous studies [13,6] and has an observable effect on small angular scales, and a new contribution due to the gravitational wave background $(\chi_{ij}^{\top(1)})$ acting as a source of lensing. The second piece is due to the time delay effect of the lenses that changes the spacelike distance to the intersection of the photon path with the last scattering surface. The scalar part of this term is expected to be suppressed with respect to the gravitational lensing term due to the spatial derivative of φ that appears in Eq. (4.1). The gravitational wave part is probably of the same order of magnitude as its gravitational lensing counterpart. The term $x_{\mathcal{E}}^{(1)0}A_{\mathcal{E}}^{(1)\prime}$ is similar in form to the gravitational lensing and time delay terms: It arises due to the difference in affine parameter along the perturbed and background geodesics. As it involves time derivatives of the metric perturbations, the contribution from scalar terms will be small, but the tensor contribution is expected to be larger.

The next term to consider couples the velocity of the photon-baryon fluid with the perturbation to the photon wave vector at emission and is given by $v_{\mathcal{E}i}^{(1)}I_1^i(\lambda_{\mathcal{E}})$. Comparing it with the gravitational lensing contribution, we expect some reduction because the Doppler contribution to the first order anisotropies is smaller than the other first order terms (although of the same order of magnitude) and some enlargement because of the factor $(\lambda_{\mathcal{E}} - \lambda)$ of difference between the $I_1^i(\lambda_{\mathcal{E}})$ and $x_\perp^{(1)j}$ expressions. A more careful quantitative estimate, which would require a choice of the particular structure formation model of interest, is beyond the aim of this paper.

Finally, we have the term $I_2(\lambda_{\mathcal{E}})$, which is given by Eq. (2.26) and is an integral of several terms. The first one accounts for the Rees-Sciama effect, given by

$$\delta T_{\rm RS}=\frac{1}{2}\int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}}d\lambda\left(\psi^{(2)'}+\phi^{(2)'}+z_i^{(2)'}e^i-\frac{1}{2}\chi_{ij}^{(2)'}e^ie^j\right). \tag{4.3}$$

We can use the second order perturbations of the metric obtained in Sec. III to compute it. The contribution from the scalar perturbations $\psi^{(2)}$ and $\phi^{(2)}$ is given by

$$\delta T_{\rm RS}=\int_{\eta_{\mathcal{O}}}^{\eta_{\mathcal{E}}}d\eta\,\eta\left(\frac{1}{3}\varphi^{,i}\varphi_{,i}-\frac{20}{21}\Psi_0\right), \tag{4.4}$$

where the terms inside the brackets have to be evaluated along the background geodesic parametrized by $\lambda=\eta$. This piece coincides with that considered in some previous studies of the Rees-Sciama effect [8–10]. The resulting anisotropies turn out to be between one and two orders of magnitude smaller that the first order ones. The contribution from the vector and tensor modes can be obtained by substituting $z_i^{(2)}$ and $\chi_{ij}^{(2)}$ from Eq. (3.14) into Eq. (4.3). Let us estimate their magnitudes compared to that of the scalar piece. The integrand for the vector piece is $z_i^{(2)'}e^i\sim k\varphi^2$, while the scalar one is $\psi^{(2)'}\sim\eta k^2\varphi^2$. Thus, $z_i^{(2)'}e^i\sim\psi^{(2)'}/(\eta k)\sim\psi^{(2)'}(aH/k)$ and the vector contribution is suppressed with respect to the scalar one, as the wavelengths of interest are smaller than the Hubble radius. This estimate is similar to the one obtained in Ref. [14]. In the same way, the integrand for the tensor piece is $\chi_{ij}^{(2)'}e^ie^j\sim k\varphi^2/(k\eta)^2\sim\psi^{(2)'}/(k\eta)^3\sim\psi^{(2)'}(aH/k)^3$. Hence, also the tensor contribution is much suppressed with respect to the scalar one.

The integrand of the second term contributing to $I_2(\lambda_{\mathcal{E}})$ is $\chi_{ij}^{\top(1)'}e^j(k^{(1)i}+e^ik^{(1)0})$; it represents a correction to the anisotropies generated by linear gravitational waves, due to the perturbation of the photon wave vector. The piece containing $k^{(1)0}$ is expected to be smaller than the other one; the largest contribution can arise from the term

$$-\int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}}d\lambda\,\chi_{ij}^{\top(1)'}e^j\int_{\lambda_{\mathcal{O}}}^{\lambda}d\lambda' A^{(1),i}.$$

The last four terms in $I_2(\lambda_{\mathcal{E}})$ will have a small contribution coming from scalar perturbations as they involve time derivatives of the potentials ψ and ϕ that are constant at linear order. The contribution coming from gravitational waves is expected to be larger, in particular the last two ones

$$\int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}}d\lambda(x^{(1)0}\chi_{ij}^{\top(1)''}e^ie^j+x^{(1)k}\chi_{ij,k}^{\top(1)'}e^ie^j).$$

These can be interpreted as the gravitational lensing and time delay effects acting on the anisotropies generated by the linear gravitational wave background.

V. CONCLUSIONS

We have computed the anisotropies in the CMB radiation up to second order perturbations in the metric around a flat Robertson-Walker spacetime. This calculation generalizes the results of Ref. [16] in that we have taken into account the velocity of the emitter and the observer, we have considered scalar, vector, and tensor perturbations, and we have explicitly included the second order perturbations of the metric. We have obtained these second order metric perturbations for a universe filled with a collisionless fluid in the Poisson gauge, by performing a second order gauge transformation of the synchronous gauge solutions, which have already been studied in some detail in the literature.

Using these results, we have discussed the relevance of the second order contributions to the anisotropies in the Poisson gauge. The most relevant expected contribution is due to the gravitational lensing of photons due to density perturbations, which has already been the subject of several studies. We have shown that also a gravitational wave background acts as a source of lensing for the CMB photons. This effect is much smaller than the scalar one for a gravitational wave background with spectral index $n_T=0$ as generated during an inflationary period. Other sources of gravitational waves with more power than the inflationary ones at small scales may give a larger contribution through this effect. Other contributions include the time delay effect of scalar and tensor lensing, a coupling of the velocity at emission with the perturbed photon wave vector, and a second order perturbation to the integrated Sachs-Wolfe piece. This term includes the well-known Rees-Sciama effect, which has been widely studied for the time variation of the scalar gravitational potential. Using the second order perturbed metric in the Poisson gauge obtained in Sec. III, we have shown that the additional contributions to the anisotropies arising from the vector and tensor modes induced by linear scalar perturbations are expected to be suppressed with respect to the scalar one. We have also pointed out the existence of two more terms that are corrections to the anisotropies generated by the linear gravitational wave background, due to the perturbation of the photon wave vector and to the lensing and time delay effects on gravitational wave anisotropies; these can give a relevant contribution to the integrated term. These contributions deserve a more detailed quantitative analysis.

Although in the light of the present analysis we do not expect that the second order gravitational effects will give a major contribution to the anisotropies at any scale, it is interesting to know if they could be detected by the planned high accuracy satellite observations. The gravitational lensing by scalar perturbations is known to give a few percent effect in some structure formation models and thus will be relevant if the expected 1% sensitivity is achieved. The amplitude of the second order terms is also important because they contribute to the theoretical error of the anisotropy computations that will be used to determine the cosmological parameters from the measured multipoles.

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