Amplitude analysis of reactions $\pi^- p \to \eta\pi^- p$ and $\pi^- p \to \eta\pi^0 n$ measured on a polarized target and the exotic 1^{-+} meson

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Recently several experimental groups analyzed data on $\pi^- p \to \eta\pi^- p$ and $\pi^- p \to \eta\pi^0 n$ reactions with an exotic 1^{-+} P wave and found conflicting evidence for an exotic meson $I=11^{-+}(1405)$. High statistics data on these reactions are presently being analyzed by the BNL E852 Collaboration. All these analyses are based on the crucial assumption that the production amplitudes do not depend on nucleon spin. This assumption is in sharp conflict with the results of measurements of $\pi^- p \to \pi^-\pi^+ n$, $\pi^+ n \to \pi^+\pi^- p$, and $K^+ n \to K^+\pi^- p$ on polarized targets at CERN, which find a strong dependence of production amplitudes on the nucleon spin. To ascertain the existence of an exotic meson $1^{-+}(1405)$, it is necessary to perform a model-independent amplitude analysis of reactions $\pi^- p \to \eta\pi^- p$ and $\pi^- p \to \eta\pi^0 n$. We demonstrate that measurements of these reactions on transversely polarized targets enable the required model-independent amplitude analysis without the assumption that production amplitudes are independent of nucleon spin. Two variants of the Monte Carlo method are proposed for finding the amplitudes and their errors. We suggest that high statistics measurements of the reactions $\pi^- p \to \eta\pi^- p$ and $\pi^- p \to \eta\pi^0 n$ be made on polarized targets at BNL and at Protvino IHEP and that model-independent amplitude analyses of these polarized data be performed to advance hadron spectroscopy on the level of spin-dependent production amplitudes. [S0556-2821(97)04219-7]

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I. INTRODUCTION

The search for meson states with non-$q\bar{q}$ quantum numbers such as $J^{PC}=0^{+-},1^{-+},2^{+-},\ldots$ has attracted much attention in recent years. Of special importance are the reactions $\pi^- p \to \eta\pi^- p$, $\pi^- p \to \eta\pi^0 n$, and $\pi^- p \to \eta\eta' n$. In these reactions the dimeson system is produced predominantly in spin states $J=0$ (S wave), $J=1$ (P wave), and $J=2$ (D wave) for masses up 2.6 GeV. It is the P wave that is of special interest as it carries exotic quantum numbers $I=1J^{PC}=1^{-+}$ for reactions $\pi^- p \to \eta\pi^- p$ and $\pi^- p \to \eta\pi^0 n$ and $I=0J^{PC}=1^{-+}$ for $\pi^- p \to \eta\eta' n$.

Measurements of $\pi^- p \to \eta\pi^0 n$ at 100 GeV/c by the GAMS Collaboration [1] found large forward-backward asymmetry with pronounced features at around 1300 MeV. Similar forward-backward asymmetry was found in measurements of $\pi^- p \to \eta\pi^- p$ at 6.3 GeV/c by the KEK E-179 Collaboration [2,3]. The higher statistics measurement of $\pi^- p \to \eta\pi^- p$ and $\pi^- p \to \eta\pi^0 n$ reactions at 18 GeV/c by the BNL E-852 Collaboration [4] confirmed significant forward-backward asymmetry in the data beginning at an invariant mass of about 1.2 GeV in both reactions. The behavior of the asymmetry suggests the presence of a large exotic P wave interfering with the dominant D wave with its $a_2(1320)$ resonance. The question arises whether there is a resonant production of the $\eta\pi^-$ or $\eta\pi^0$ state in the exotic P wave. The reliable determination of the existence of an exotic resonance in a 1^{-+} P wave requires a model-independent amplitude analysis of the data.

The reactions $\pi^- p \to \eta\pi^- p$, $\pi^- p \to \eta\pi^0 n$, and $\pi^- p \to \eta\eta' n$ are described by 14 spin-dependent production amplitudes: two S-wave amplitudes S_n, six P-wave amplitudes P_n^0, P_n^-, and P_n^+, and six D-wave amplitudes D_n^0, D_n^-, and D_n^+, where $n=0,1$ is the nucleon helicity flip $n=|\lambda_p-\lambda_n|$. The amplitudes S_n, P_n^0, and D_n^0 describe the production with dimeson helicity $\lambda=0$ and correspond to unnatural exchange. The amplitudes P_n^-, D_n^- and P_n^+, D_n^+ describe production with a dimeson helicity ± 1 and correspond to unnatural and natural exchanges, respectively.

All previous amplitude analyses of reactions $\pi^- p \to \eta\pi^- p$, $\pi^- p \to \eta\pi^0 n$, and $\pi^- p \to \eta\eta' n$ on unpolarized targets are model dependent. They use a very strong simplifying assumption that the production amplitudes do not depend on nucleon spin [5–7]. The purpose of this assumption is to reduce the number of amplitudes by half and thus to enable the amplitude analysis of unpolarized moments measured in these reactions to proceed. These analyses simply ignore the nucleon helicity flip index n.

Using such an enabling assumption, the different collaborations found the exotic $I=11^{-+}$ meson, but in different amplitudes. The GAMS Collaboration reported a $1^{-+}(1405)$ state with a width of 180 MeV [1] observed only in the amplitude $|P^0|^2$. The KEK E-179 Collaboration [2,3] found a $|P^-|^2$ nonresonating, but found a resonance $1^{-+}(1323)$ with a width of 143 MeV in the amplitude $|P^+|^2$ and possibly in $|P^0|^2$. The VES Collaboration [8] measured $\pi^- p \to \eta\pi^- p$ and $\pi^- p \to \eta'\pi^- p$ at 37 GeV/c at IHEP Protvino and found a possible $1^{-+}(1400)$ state only in the amplitude $|P^+|^2$. Amplitude analysis of the BNL E-852 Collaboration higher statistics data at 18 GeV/c is in progress, but it also uses the simplifying assumption that production amplitudes do not depend on nucleon spin. All these analyses are subjected to an eightfold ambiguity and in Refs.

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[2, 3, 8] all eight solutions are presented.

For completeness we note that the GAMS Collaboration measured the reaction $\pi^- p \to \eta\eta' n$ at 38 GeV/c [9] and found evidence for a new state $X(1920)$. The unusual production and decay properties could be understood if $X(1920)$ had a non-$q\bar{q}$ structure, being either a 0^{++} or 2^{++} glueball or a $I=01^{-+}$ exotic meson. Unfortunately, the low statistics does not allow even a model-dependent amplitude analysis.

The simplifying assumption that the production amplitudes do not depend on nucleon spin is not necessary in measurements on polarized targets. In 1978, Lutz and Rybicki showed [10] that measurements of the reactions $\pi N \to \pi^+\pi^- N$ and $KN \to K\pi N$ on a polarized target yield enough observables that model-independent amplitude analysis is possible, determining the spin-dependent production amplitudes. The measurement of these reactions is of special interest to hadron spectroscopy because they permit one to study the spin dependence of resonance production directly on the level of spin-dependent production amplitudes. Several such measurements were done at the CERN Proton Synchrotron.

The high statistics measurement of $\pi^- p \to \pi^-\pi^+ n$ at 17.2 GeV/c on an unpolarized target [11] was later repeated with a transversely polarized target at the same energy [12–17]. Model-independent amplitude analyses were performed for various intervals of dimeson mass at small momentum transfers $-t=0.005-0.2\ (\mathrm{GeV}/c)^2$ [12–15] and over a large interval of momentum transfer $-t=0.2-1.0\ (\mathrm{GeV}/c)^2$ [16,17].

Additional information was provided by the measurement of $\pi^+ n \to \pi^+\pi^- p$ and $K^+ n \to K^+\pi^- p$ reactions on a polarized deuteron target at 5.98 and 11.85 GeV/c [18,19]. The data allowed one to study the t evolution of mass dependence of moduli of amplitudes [20]. Detailed amplitude analyses [21,22] determined the mass dependence of amplitudes at larger momentum transfers $-t=0.2-0.4\ (\mathrm{GeV}/c)^2$.

The crucial finding of all these measurements was the evidence for a strong dependence of production amplitudes on nucleon spin. The process of resonance production is very closely related to nucleon transversity or the nucleon spin component in the direction perpendicular to the production plane. For instance, in $\pi^- p \to \pi^-\pi^+ n$ at small t and dipion masses below 1000 MeV, all amplitudes with recoil nucleon transversity ''down'' are smaller than transversity ''up'' amplitudes, irrespective of dimeson spin and helicity. In particular, the S-wave amplitude with recoil nucleon transversity up is found to resonate at 750 MeV in both solutions [23–25] irrespective of the method of amplitude analysis [25], while the S-wave amplitude with recoil nucleon transversity down is nonresonating and large in both solutions. It is important to stress that the discovery of the narrow scalar state $\sigma(750)$ in $\pi^- p \to \pi^-\pi^+ n$ and $\pi^+ n \to \pi^+\pi^- p$ [24,25] was possible only because these reactions were measured on polarized targets that allowed the model-independent determination of the spin-dependent production amplitudes.

The assumption that production amplitudes in $\pi^- p \to \eta\pi^- p$ and $\pi^- p \to \eta\pi^0 n$ do not depend on nucleon spin contradicts all that we have learned from the measurements of $\pi N \to \pi^+\pi^- N$ on polarized targets at CERN. Applied to the reactions $\pi^- p \to \pi^-\pi^+ n$ and $\pi^+ n \to \pi^+\pi^- p$, the assumption has observable consequences that can be tested directly in measurements on polarized targets. In a previous paper [26] we have shown how all these consequences are in contradiction with the CERN polarized data on $\pi N_\uparrow \to \pi^+\pi^- N$ and $K^+ n_\uparrow \to K^+\pi^- p$ (see Figs. 1 and 2 of Ref. [26]). We must conclude that the CERN polarized data invalidate the assumption that production amplitudes do not depend on nucleon spin. Consequently, some of the results of analyses of $\pi^- p \to \eta\pi^- p$ and $\pi^- p \to \eta\pi^0 n$ may not be reliable.

The question of reliability of amplitude analyses based on the assumption of the independence of production amplitudes on nucleon spin is of special importance to searches for exotic resonances such as $1^{-+}(1405)$ in $\pi^- p \to \eta\pi^- p$ and $\pi^- p \to \eta\pi^0 n$ reactions or confirmation of the narrow $\sigma(750)$ state in $\pi^- p \to \pi^0\pi^0 n$ reaction. Only a model independent analysis will resolve questions concerning the existence of such resonances that are not seen in the integrated mass spectrum but only on the level of spin dependent production amplitudes.

In a previous paper [26] we have shown how measurements of $\pi^- p \to \pi^0\pi^0 n$ on a polarized targets allow a model-independent amplitude analysis of this reaction (and $\pi^- p \to \eta\eta n$). Using the results of Lutz and Rybicki [10], we show in this work that measurements of $\pi^- p \to \eta\pi^- p$ and $\pi^- p \to \eta\pi^0 n$ on a polarized target again allow a model-independent determination of moduli of all production amplitudes and cosines of certain independent relative phases. We find an eightfold ambiguity, which is the same situation as in model-dependent analyses of unpolarized data. We propose that high statistics measurements of $\pi^- p \to \eta\pi^- p$ and $\pi^- p \to \eta\pi^0 n$ be made at Brookhaven Multiparticle Spectrometer and at IHEP Protvino in conjunction with measurements of $\pi^- p \to \pi^0\pi^0 n$ reaction on polarized target.

The paper is organized as follows. In Sec. II we review our basic notation and definitions of observables and amplitudes. In Sec. III we present the expressions for unpolarized and polarized moments in terms of amplitudes. In Sec. IV we discuss the method of the model-independent amplitude analysis of data on $\pi^- p \to \eta\pi^- p$ and $\pi^- p \to \eta\pi^0 n$ on polarized target. In Secs. V and VI we describe two variants of the Monte Carlo method for finding the amplitudes and their errors. The paper closes in Sec. VII, where we present a summary and our proposals.

II. BASIC FORMALISM

The kinematical variables that describe the reactions $\pi^- p \to \eta\pi^- p$ and $\pi^- p \to \eta\pi^0 n$ on a polarized proton target at rest are $s,t,m,\theta,\phi,\psi,\delta$, where s is the center-of-mass system (c.m.s.) energy squared, t is four-momentum transfer to the nucleon squared, and m is the invariant mass of the $\eta\pi$ system. The angles θ,ϕ describe the direction of η in the $\eta\pi^-$ or $\eta\pi^0$ rest frame. The angle ψ is the angle between the direction of target transverse polarization and the normal $\vec{n}$ to the scattering plane (Fig. 1). The direction of normal $\vec{n}$ is defined according to Basel convention by $\vec{p}_\pi \times \vec{p}_{\eta\pi}$, where $\vec{p}_\pi$ and $\vec{p}_{\eta\pi}$ are the incident and dimeson momenta in the target proton rest frame. The angle δ is the angle between the direction of target polarization vector and its transverse component (Fig. 1). The analysis is usually carried out in the t-channel helicity frame for the $\eta\pi$ dimeson system. The

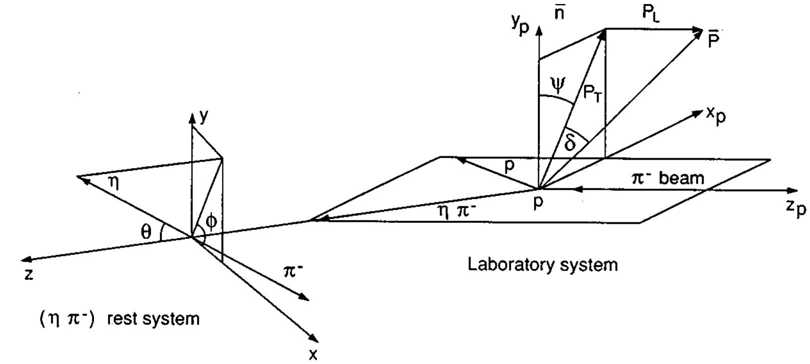

FIG. 1. Definition of the coordinate system used to describe the target polarization $\vec{P}$ and the decay of the dimeson $\eta\pi^-$ system.

helicities of initial and final nucleons are always defined in the s-channel helicity frame.

When the polarization of the recoil nucleon is not measured, the unnormalized angular distribution of $\eta\pi^-$ or $\eta\pi^0$ production on polarized protons at rest at fixed s, m, and t can be written [10] as

$$I(\Omega,\psi,\delta)=I_U(\Omega)+P_T\cos\psi I_C(\Omega)+P_T\sin\psi I_S(\Omega)+P_L I_L(\Omega), \tag{2.1}$$

where $P_T=P\cos\delta$ and $P_L=P\sin\delta$ are the transverse and longitudinal components of target polarization $\vec{P}$ with respect to the incident momentum (Fig. 1). In the data analysis of angular distribution of the dimeson system, it is convenient to use expansions of the angular distributions in terms of spherical harmonics. In the notation of Lutz and Rybicki [10] we have

$$\begin{aligned} I_U(\Omega)&=\sum_{L,M} t_M^L \mathrm{Re} Y_M^L(\Omega),\\ I_C(\Omega)&=\sum_{L,M} p_M^L \mathrm{Re} Y_M^L(\Omega),\\ I_S(\Omega)&=\sum_{L,M} r_M^L \mathrm{Im} Y_M^L(\Omega),\\ I_L(\Omega)&=\sum_{L,M} q_M^L \mathrm{Im} Y_M^L(\Omega). \end{aligned} \tag{2.2}$$

The moments t_M^L are unpolarized and are measured in experiments on unpolarized targets. Experiments with transversely polarized targets measure transverse polarized moments p_M^L and r_M^L, but not the longitudinal polarized moments q_M^L. More details on these observables are given in Refs. [10, 26].

The reaction $\pi^- p \to \eta\pi^- p$ (or $\pi^- p \to \eta\pi^0 n$) is described by production amplitude $H_{\lambda_n,0\lambda_p}(s,t,m,\theta,\phi)$, where λ_p and λ_n are the helicities of the proton and neutron, respectively. The production amplitudes can be expressed in terms of production amplitudes corresponding to definite dimeson spin J and helicity λ using an angular expansion

$$H_{\lambda_n,0\lambda_p}=\sum_{J=0}^{\infty}\sum_{\lambda=-J}^{+J}(2J+1)^{1/2}H^J_{\lambda\lambda_n,0\lambda_p}(s,t,m)d^J_{\lambda 0}(\theta)e^{i\lambda\phi}. \tag{2.3}$$

In the following we will consider only S-wave ($J=0$), P-wave ($J=1$), and D-wave ($J=2$) amplitudes. Since the experimental moments with $M>2$ vanish, we will restrict the dimeson helicity λ only to values $\lambda=0$ and $\lambda=\pm 1$.

The amplitudes $H^J_{\lambda\lambda_n,0\lambda_p}(s,t,m)$ can be expressed in terms of nucleon helicity amplitudes with definite t-channel exchange naturality. The nucleon s-channel helicity amplitudes describing the production of $\eta\pi^-$ (or $\eta\pi^0$) system in the S-, P-, and D-wave states are

$$\begin{aligned} &H^0_{0+,0+}=S_0,\quad H^0_{0+,0-}=S_1 \quad \text{for } 0^-\frac{1}{2}^+\to 0^+\frac{1}{2}^+,\\ &H^1_{0+,0+}=P^0_0,\quad H^1_{0+,0-}=P^0_1 \quad \text{for } 0^-\frac{1}{2}^+\to 1^-\frac{1}{2}^+,\\ &H^1_{\pm1+,0+}=\frac{P_0^+\pm P_0^-}{\sqrt{2}},\quad H^1_{\pm1+,0-}=\frac{P_1^+\pm P_1^-}{\sqrt{2}},\\ &H^2_{0+,0+}=D^0_0,\quad H^2_{0+,0-}=D^0_1 \quad \text{for } 0^-\frac{1}{2}^+\to 2^+\frac{1}{2}^+,\\ &H^2_{\pm1+,0+}=\frac{D_0^+\pm D_0^-}{\sqrt{2}},\quad H^2_{\pm1+,0-}=\frac{D_1^+\pm D_1^+}{\sqrt{2}}. \end{aligned} \tag{2.4}$$

At large s, the amplitudes $S_n, P_n^0, P_n^-, D_n^0, D_n^-$, $n=0,1$, are dominated by the unnatural exchanges. The amplitudes P_n^+, D_n^+, $n=0,1$, are dominated by natural exchanges. The index $n=|\lambda_p-\lambda_n|$ is the nucleon helicity flip.

The observables measured in experiments on transversely polarized targets are most simply related to nucleon transversity amplitudes of definite naturality [10,19,27]. With $k=1/\sqrt{2}$, they are defined as

$$\begin{aligned} S&=k(S_0+iS_1),\quad \overline{S}=k(S_0-iS_1);\\ P^0&=k(P^0_0+iP^0_1),\quad \overline{P^0}=k(P^0_0-iP^0_1);\\ P^-&=k(P^-_0+iP^-_1),\quad \overline{P^-}=k(P^-_0-iP^-_1);\\ P^+&=k(P^+_0-iP^+_1),\quad \overline{P^+}=k(P^+_0+iP^+_1);\\ D^0&=k(D^0_0+iD^0_1),\quad \overline{D^0}=k(D^0_0-iD^0_1);\\ D^-&=k(D^-_0+iD^-_1),\quad \overline{D^-}=k(D^-_0-iD^-_1); \end{aligned} \tag{2.5}$$

$$D^{+}=k(D_{0}^{+}-iD_{1}^{+}),\quad \overline{D^{+}}=k(D_{0}^{+}+iD_{1}^{+}).$$

The nucleon helicity and nucleon transversity amplitudes differ in the quantization axis for the nucleon spin. The transversity amplitudes $S,P^0,P^-,P^+,D^0,D^-,D^+$ $(\overline{S},\overline{P^0},\overline{P^-},\overline{P^+},\overline{D^0},\overline{D^-},\overline{D^+})$ describe the production of $\eta\pi$ state with the recoil nucleon spin antiparallel or down (parallel or up) relative to the normal $\vec{n}$ to the production plane. The amplitudes are normalized such that the reaction cross section $\Sigma=d^2\sigma/dmdt$,

$$\begin{aligned}\Sigma=&|S|^2+|\overline{S}|^2+|P^0|^2+|\overline{P^0}|^2+|P^-|^2+|\overline{P^-}|^2+|P^+|^2\\&+|\overline{P^+}|^2+|D^0|^2+|\overline{D^0}|^2+|D^-|^2+|\overline{D^-}|^2+|D^+|^2\\&+|\overline{D^+}|^2.\end{aligned}\tag{2.6}$$

Amplitude analysis is usually carried out with normalized amplitudes for which the sum (2.6) is equal to 1. Then for each amplitude $A=S,\ldots,D^+$ we have

$$0<|A|^2<1,\quad 0<|\overline{A}|^2<1.\tag{2.7}$$

The unnormalized amplitudes are simply $|A|^2\Sigma$ and $|\overline{A}|^2\Sigma$, $A=S,\ldots,D^+$. It is these unnormalized moduli squared that reveal the existence of resonances that cannot be seen in the spin-averaged cross section $d^2\sigma/dmdt$.

III. OBSERVABLES IN TERMS OF AMPLITUDES

It is useful to express the moments t_M^L and p_M^L in terms of quantities that do not depend explicitly on whether we use nucleon helicity or nucleon transversity amplitudes. The required quantities are spin-averaged partial-wave intensity

$$I_A=|A|^2+|\overline{A}|^2=|A_0|^2+|A_1|^2\tag{3.1}$$

and partial-wave polarization

$$P_A=|A|^2-|\overline{A}|^2=2\epsilon_A\mathrm{Im}(A_0A_1^*),\tag{3.2}$$

where $\epsilon_A=+1$ for $A=S,P^0,P^-,D^0,D^-$ and $\epsilon_A=-1$ for $A=P^+,D^+$. We also need spin-averaged interference terms

$$R(AB)=\mathrm{Re}(AB^*+\overline{AB^*})=\mathrm{Re}(A_0B_0+\epsilon_A\epsilon_BA_1B_1),\tag{3.3}$$

$$Q(AB)=\mathrm{Re}(AB^*-\overline{AB^*})=\mathrm{Re}(\epsilon_BA_0B_1^*-\epsilon_AA_1B_0^*).\tag{3.4}$$

Then moments t_M^L are expressed in terms of intensities I_A and interference terms $R(AB)$. The moments p_M^L are expressed in terms of polarizations P_A and interference terms $Q(AB)$. The moments r_M^L are interferences between the natural and unnatural exchange amplitudes. To describe moments r_M^L, it is useful to introduce notation

$$N(AP^+)=\mathrm{Re}(AP^{+*}-\overline{AP^{+*}}),\tag{3.5}$$

$$N(AD^+)=\mathrm{Re}(AD^{+*}-\overline{AD^{+*}}),$$

where $A=S,P^0,P^-,D^0,D^-$.

Using the results of the Lutz and Rybicki [10], we obtain the following expressions for moments with $c=\sqrt{4\pi}$: unpolarized moments t_M^L

$$ct_0^0=I_S+I_{P^0}+I_{P^-}+I_{P^+}+I_{D^0}+I_{D^-}+I_{D^+},$$

$$\begin{aligned}ct_0^1=&2R(SP^0)+\frac{4}{\sqrt{5}}R(P^0D^0)\\&+2\sqrt{\frac{3}{5}}\,[R(P^-D^-)+R(P^+D^+)],\end{aligned}$$

$$ct_1^1=2\sqrt{2}R(SP^-)+2\sqrt{\frac{6}{5}}\,R(P^0D^-)-2\sqrt{\frac{2}{5}}\,R(P^-D^0),$$

$$\begin{aligned}ct_0^2=&\frac{2}{\sqrt{5}}I_{P^0}-\frac{1}{\sqrt{5}}(I_{P^-}+I_{P^+})+2R(SD^0)+\frac{2}{7}\sqrt{5}I_{D^0}\\&+\frac{\sqrt{5}}{7}(I_{D^-}+I_{D^+}),\end{aligned}$$

$$ct_1^2=2\sqrt{\frac{6}{5}}\,R(P^0P^-)+2\sqrt{2}R(SD^-)+\frac{2\sqrt{10}}{7}\,R(D^0D^-),$$

$$ct_2^2=\sqrt{\frac{6}{5}}\,(I_{P^-}-I_{P^+})+\frac{\sqrt{30}}{7}(I_{D^-}-I_{D^+}),\tag{3.6}$$

$$ct_0^3=6\sqrt{\frac{3}{35}}\,R(P^0D^0)-\frac{6}{\sqrt{35}}[R(P^-D^-)+R(P^+D^+)],$$

$$ct_1^3=8\sqrt{\frac{3}{35}}\,R(P^0D^-)+\frac{12}{\sqrt{35}}\,R(P^-D^0),$$

$$ct_2^3=2\sqrt{\frac{6}{7}}\,[R(P^-D^-)-R(P^+D^+)],$$

$$ct_0^4=\frac{6}{7}I_{D^0}-\frac{4}{7}(I_{D^-}+I_{D^+}),$$

$$ct_1^4=\frac{4}{7}\sqrt{15}R(D^0D^-),$$

$$ct_2^4=\frac{2\sqrt{10}}{7}\,(I_{D^-}-I_{D^+});$$

polarized moments p_M^L:

$$cp_0^0=P_S+P_{P^0}+P_{P^-}-P_{P^+}+P_{D^0}+P_{D^-}-P_{D^+},$$

$$\begin{aligned}cp_0^1=&2Q(SP^0)+\frac{4}{\sqrt{5}}Q(P^0D^0)\\&+2\sqrt{\frac{3}{5}}\,[Q(P^-D^-)-Q(P^+D^+)],\end{aligned}$$

$$cp_1^1 = 2\sqrt{2}Q(SP^-) + 2\sqrt{\frac{6}{5}}\,Q(P^0D^-) - 2\sqrt{\frac{2}{5}}\,Q(P^-D^0),$$

$$cp_0^2 = \frac{2}{\sqrt{5}}P_{P^0} - \frac{1}{\sqrt{5}}(P_{P^-} - P_{P^+}) + 2Q(SD^0) + \frac{2\sqrt{5}}{7}\,P_{D^0} + \frac{\sqrt{5}}{7}(P_{D^-} - P_{D^+}),$$

$$cp_1^2 = 2\sqrt{\frac{6}{5}}\,Q(P^0P^-) + 2\sqrt{2}Q(SD^-) + \frac{2\sqrt{10}}{7}\,Q(D^0D^-),$$

$$cp_2^2 = \sqrt{\frac{6}{5}}\,(P_{P^-} + P_{P^+}) + \frac{\sqrt{30}}{7}(P_{D^-} + P_{D^+}), \tag{3.7}$$

$$cp_0^3 = 6\sqrt{\frac{3}{35}}\,Q(P^0D^0) - \frac{6}{\sqrt{35}}[Q(P^-D^-) - Q(P^+D^+)],$$

$$cp_1^3 = 8\sqrt{\frac{3}{35}}\,Q(P^0D^-) + \frac{12}{\sqrt{35}}\,Q(P^-D^0),$$

$$cp_2^3 = 2\sqrt{\frac{6}{7}}\,[Q(P^-D^-) + Q(P^+D^+)],$$

$$cp_0^4 = \frac{6}{7}P_{D^0} - \frac{4}{7}(P_{D^-} - P_{D^+}),$$

$$cp_0^4 = \frac{4}{7}\,\sqrt{15}Q(D^0D^-),$$

$$cp_2^4 = \frac{2\sqrt{10}}{7}\,(P_{D^-} + P_{D^+});$$

and polarized moments r_M^L:

$$cr_1^1 = -2\sqrt{2}N(SP^+) - 2\sqrt{\frac{2}{5}}\,N(D^0P^+) - 2\sqrt{\frac{6}{5}}\,N(P^0D^+),$$

$$cr_1^2 = -2\sqrt{\frac{6}{5}}\,N(P^0P^+) - 2\sqrt{2}N(SD^+) - \frac{2\sqrt{10}}{7}\,N(D^0D^+),$$

$$cr_2^2 = -2\sqrt{\frac{6}{5}}\,N(P^-P^+) - \frac{2\sqrt{30}}{7}\,N(D^-D^+),$$

$$cr_1^3 = +\frac{12}{\sqrt{35}}\,N(D^0P^+) - 8\sqrt{\frac{3}{35}}\,N(P^0D^+), \tag{3.8}$$

$$cr_2^3 = -2\sqrt{\frac{6}{7}}\,N(D^-P^+) - 2\sqrt{\frac{6}{7}}\,N(P^-D^+),$$

$$cr_1^4 = -\frac{4}{7}\,\sqrt{15}N(D^0D^+),$$

$$cr_2^4 = -\frac{4}{7}\,\sqrt{10}N(D^-D^+).$$

IV. MODEL-INDEPENDENT AMPLITUDE ANALYSIS

Our starting point is the observation of symmetry in the relations for moments t_M^L and p_M^L. We find that we get p_M^L from t_M^L by replacing intensities I_A by polarizations $\epsilon_A P_A$, $\epsilon = +1$ for $A = S, P^0, P^-, D^0, D^-$ and $\epsilon_A = -1$ for $A = P^+, D^+$, and by replacing the interference terms $R(AB) \rightarrow Q(AB)$ for unnatural exchange amplitudes and $R(P^+D^+) \rightarrow -Q(P^+D^+)$ for natural exchange amplitudes. To solve the system of equations t_M^L and p_M^L it will be useful to work with transversity amplitudes. Then the definitions (3.1)–(3.4) suggest the construction of two sets of equations corresponding to the sum and difference of the moments t_M^L and p_M^L. In this way we get two independent sets of equations for amplitudes of opposite transversity.

The first set of new observables reads

$$a_1 = \frac{c}{2}(t_0^0 + p_0^0) = |S|^2 + |P^0|^2 + |P^-|^2 + |\overline{P^+}|^2 + |D^0|^2 + |D^-|^2 + |\overline{D^+}|^2,$$

$$a_2 = \frac{c}{2}(t_0^1 + p_0^1) = 2\mathrm{Re}(SP^{0*}) + \frac{4}{\sqrt{5}}\mathrm{Re}(P^0D^{0*}) + 2\sqrt{\frac{3}{5}}[\mathrm{Re}(P^-D^{-*}) + \mathrm{Re}(\overline{P^+}\,\overline{D^+}{*})],$$

$$a_3 = \frac{c}{2}(t_1^1 + p_1^1) = 2\sqrt{2}\mathrm{Re}(SP^{-*}) + 2\sqrt{\frac{6}{5}}\mathrm{Re}(P^0D^{-*}) - 2\sqrt{\frac{2}{5}}\mathrm{Re}(P^-D^{0*}),$$

$$a_4 = \frac{c}{2}(t_0^2 + p_0^2) = \frac{2}{\sqrt{5}}|P^0|^2 - \frac{1}{\sqrt{5}}(|P^-|^2 + |\overline{P^+}|^2) + 2\mathrm{Re}(SD^{0*}) + \frac{2\sqrt{5}}{7}\,|D^0|^2 + \frac{\sqrt{5}}{7}\,(|D^-|^2 + |\overline{D^+}|^2),$$

$$a_5 = \frac{c}{2}(t_1^2 + p_1^2) = 2\sqrt{\frac{6}{5}}\mathrm{Re}(P^0P^{-*}) + 2\sqrt{2}\mathrm{Re}(SD^{-*}) + \frac{2\sqrt{10}}{7}\mathrm{Re}(D^0D^{-*}),$$

$$a_6=\frac{c}{2}(t_2^2+p_2^2)$$
$$=\sqrt{\frac{6}{5}}\,(|P^-|^2-|\overline{P^+}|^2)$$
$$+\frac{\sqrt{30}}{7}\,(|D^-|^2-|\overline{D^+}|^2), \tag{4.1}$$

$$a_7=\frac{c}{2}(t_0^3+p_0^3)$$
$$=6\sqrt{\frac{3}{35}}\mathrm{Re}(P^0D^0*)$$
$$-\frac{6}{\sqrt{35}}[\mathrm{Re}(P^-D^-*)+\mathrm{Re}(\overline{P^+}\,\overline{D^+}*)],$$

$$a_8=\frac{c}{2}(t_1^3+p_1^3)$$
$$=8\sqrt{\frac{3}{35}}\mathrm{Re}(P^0D^-*)+\frac{12}{\sqrt{35}}\mathrm{Re}(P^-D^0*),$$

$$a_9=\frac{c}{2}(t_2^3+p_2^3)$$
$$=2\sqrt{\frac{6}{7}}[\mathrm{Re}(P^-D^-*)-\mathrm{Re}(\overline{P^+}\,\overline{D^+}*)],$$

$$a_{10}=\frac{c}{2}(t_0^4+p_0^4)$$
$$=\frac{6}{7}|D^0|^2-\frac{4}{7}(|D^-|^2+|\overline{D^+}|^2),$$

$$a_{11}=\frac{c}{2}(t_1^4+p_1^4)=\frac{4}{7}\sqrt{15}\mathrm{Re}(D^0D^-*),$$

$$a_{12}=\frac{c}{2}(t_2^4+p_2^4)=\frac{2\sqrt{10}}{7}(|D^-|^2-|\overline{D^+}|^2).$$

The first set of equations (4.1) involves seven moduli

$$|S|,|P^0|,|P^-|,|\overline{P^+}|,|D^0|,|D^-|,|\overline{D^+}|, \tag{4.2}$$

and ten cosines of relative phases between unnatural exchange amplitudes

$$\cos(\gamma_{SP^0}),\cos(\gamma_{SP^-}),\cos(\gamma_{SD^0}),\cos(\gamma_{SD^-}), \tag{4.3}$$

$$\cos(\gamma_{P^0P^-}),\cos(\gamma_{P^0D^0}),\cos(\gamma_{P^0D^-}), \tag{4.4}$$

$$\cos(\gamma_{P^-D^0}),\cos(\gamma_{P^-D^-}),\cos(\gamma_{D^0D^-}), \tag{4.5}$$

and one cosine of relative phase between natural exchange amplitudes

$$\cos(\overline{\gamma_{P^+D^+}}). \tag{4.6}$$

The second set of observables $\overline{a_i}$, $i=1,2,\ldots,12$, corresponding to the differences of moments t_M^L and p_M^L involves the same moduli and cosines as the first set but for amplitudes of opposite transversity: seven moduli

$$|\overline{S}|,|\overline{P^0}|,|\overline{P^-}|,|P^+|,|\overline{D^0}|,|\overline{D^-}|,|D^+|, \tag{4.7}$$

ten cosines of relative phases between unnatural exchange amplitudes

$$\cos(\overline{\gamma_{SP^0}}),\cos(\overline{\gamma_{SP^-}}),\cos(\overline{\gamma_{SD^0}}),\cos(\overline{\gamma_{SD^-}}), \tag{4.8}$$

$$\cos(\overline{\gamma_{P^0P^-}}),\cos(\overline{\gamma_{P^0D^0}}),\cos(\overline{\gamma_{P^0D^-}}),$$
$$\cos(\overline{\gamma_{P^-D^0}}),\cos(\overline{\gamma_{P^-D^-}}),\cos(\overline{\gamma_{D^0D^-}}), \tag{4.9}$$

and one cosine of relative phase between natural exchange amplitudes

$$\cos(\gamma_{P^+D^+}). \tag{4.10}$$

We will now show that the cosines (4.4) and (4.5) can be expressed in terms of the cosines (4.3). For instance, we can write

$$\gamma_{P^0P^-}=\phi_{P^0}-\phi_{P^-}$$
$$=(\phi_S-\phi_{P^-})-(\phi_S-\phi_{P^0})=$$
$$\gamma_{SP^-}-\gamma_{SP^0}. \tag{4.11}$$

Then

$$\cos(\gamma_{P^0P^-})=\cos(\gamma_{SP^0})\cos(\gamma_{SP^-})+\sin(\gamma_{SP^0})\sin(\gamma_{SP^-}). \tag{4.12}$$

Since the signs of the sines $\sin(\gamma_{SP^0})$ and $\sin(\gamma_{SP^-})$ are not known, we write

$$\sin(\gamma_{SP^0})=\epsilon_{SP^0}|\sin(\gamma_{SP^0})|,$$
$$\sin(\gamma_{SP^-})=\epsilon_{SP^-}|\sin(\gamma_{SP^-})|. \tag{4.13}$$

Hence

$$\cos(\gamma_{P^0P^-})=\cos(\gamma_{SP^0})\cos(\gamma_{SP^-})$$
$$+\epsilon_{P^0P^-}\sqrt{(1-\cos^2\gamma_{SP^0})(1-\cos^2\gamma_{SP^0})}, \tag{4.14}$$

where $\epsilon_{P^0P^-}=\pm1$ is the sign ambiguity. The remaining cosines in Eqs. (4.4) and (4.5) can be written in the form similar to Eq. (4.14) with their own sign ambiguities. The sign ambiguities of cosines (4.4) and (4.5) can be expressed in terms of sign ambiguities corresponding to the sines $\sin(\gamma_{SP^0})$, $\sin(\gamma_{SP^-})$, $\sin(\gamma_{SD^0})$, and $\sin(\gamma_{SD^-})$. We can write

$$\epsilon_{P^0P^-}=\epsilon_{SP^0}\epsilon_{SP^-},$$
$$\epsilon_{P^0D^0}=\epsilon_{SP^0}\epsilon_{SD^0}, \tag{4.15}$$
$$\epsilon_{P^0D^-}=\epsilon_{SP^0}\epsilon_{SD^-},$$

$$\epsilon_{P^-D^0}=\epsilon_{SP^-}\epsilon_{SD^0}, \tag{4.16}$$
$$\epsilon_{P^-D^-}=\epsilon_{SP^-}\epsilon_{SD^-},$$
$$\epsilon_{D^0D^-}=\epsilon_{SD^0}\epsilon_{SD^-}.$$

The reversal of all signs ϵ_{SP^0}, ϵ_{SP^-}, ϵ_{SD^0}, and ϵ_{SD^-} yields the same signs in Eqs. (4.15) and (4.16). The sign ambiguities (4.16) are not independent. They are uniquely determined by the sign ambiguities (4.15). Only sign ambiguities (4.15) are independent and there are eight, sign combinations in Eq. (4.15). The following table lists all eight allowed sets of sign ambiguities of cosines (4.4) and (4.5):

	1	2	3	4	5	5	7	8
$\epsilon_{P^0P^-}$	+	−	+	+	−	−	+	−
$\epsilon_{P^0D^0}$	+	+	−	+	−	+	−	−
$\epsilon_{P^0D^-}$	+	+	+	−	+	−	−	−
$\epsilon_{P^-D^0}$	+	−	−	+	+	−	−	+
$\epsilon_{P^-D^-}$	+	−	+	−	−	+	−	+
$\epsilon_{D^0D^-}$	+	+	−	−	−	−	+	+

Using expressions like (4.14) for cosines (4.4) and (4.5), the number of unknowns is reduced to 12. With each choice of sign ambiguity from the above table we have a set of 12 equations for 12 unknown that can be solved numerically by the χ^2 method or by Monte Carlo methods described in the Secs. V and VI below. Of course, there is an eightfold ambiguity and we obtain eight solutions for moduli (4.2) and cosines (4.3) and (4.6) in each (m,t) bin. Since each solution is uniquely labeled by the choice of sign ambiguities, there is no problem linking solutions in neighboring (m,t) bins. Similarly, we obtain eight solutions for moduli (4.7) and cosines (4.8) and (4.11) from the second set of equations $\overline{a_i}$, $i=1,2,\ldots,12$.

The eight solutions from the first set of equations a_i, $i=1,2,\ldots,12$, are independent of the eight solutions obtained from the second set of equations $\overline{a_i}$, $i=1,2,\ldots,12$. Consequently, there will be a 64-fold ambiguity in the partial wave intensities, which we can write

$$I_A(i,j)=|A(i)|^2+|\overline{A(j)}|^2, \quad i,j=1,2,\ldots.8, \quad (4.17)$$

where $A=S,P^0,P^-,P^+,D^0,D^-,D^+$. As in the case of amplitude analysis of $\pi^- p_\uparrow \rightarrow \pi^-\pi^+ n$ at 17.2 GeV/c [12–17], the unpolarized moments t^L_M should come from measurements on unpolarized targets.

V. INVERSE MONTE CARLO METHOD

In the usual (direct) Monte Carlo method [24] the normalized amplitudes are analytically expressed in terms of normalized moments. The moments are then randomly varied within their errors, and for each such selection new amplitudes are calculated. If physical values are obtained for the amplitudes, they are retained. Amplitudes with unphysical values are rejected. The distributions of physical values of amplitudes define their range (error) and average values. The method has the advantage that it retains the identity of different analytical solutions where ambiguities exist. Unfortunately, Eqs. (4.1) cannot be solved analytically.

In the inverse Monte Carlo method we make use of the fact that the normalized moments are expressed in terms of normalized amplitudes with moduli $|A|<1$ and cosines of relative phases $-1\leqslant\cos\gamma\leqslant+1$. We can randomly vary the values of moduli and cosines within these ranges and see if the right-hand sides of Eqs. (4.1) fall into the error range of all observables $a_1,a_2,\ldots,a_{12}$. If they do, the values of moduli and cosines are retained and collected. Otherwise the selection is rejected. In each (m,t) bin we thus obtain a distribution of values for each modulus and cosine from which we calculate the average value and from its range the asymmetric error bars for each amplitude. The same procedure is applied to the second set of equations for amplitudes of opposite transversity and for each selection of sign ambiguities. The solutions of the two sets are not entirely independent because the normalized moduli must satisfy the condition (2.6) with $\Sigma\equiv1$. This means that we can use Monte Carlo to select randomly 13 moduli and use Eq. (2.6) with $\Sigma\equiv1$ to calculate the 14th modulus (say $|\overline{D^+}|^2$). However, this calculated modulus must still satisfy the condition $0<|A|^2<1$.

We will refer to this method of finding a solution and its errors for amplitudes as the inverse Monte Carlo method. We note that this method can be applied also to find a solution for amplitudes in the reaction $\pi^- p\rightarrow\pi^0\pi^0 n$ when the G wave is included [see Eq. (6.1) of Ref. [26]].

VI. MULTISTAGE INVERSE MONTE CARLO METHOD

Instead of selecting random values of the moduli and independent cosines all at the same time, we can proceed in stages, taking into account the structure of Eqs. (4.1). However, the aim will be the same: to find in each (m,t) bin a distribution of values for each modulus and independent cosine and to calculate from this distribution the average value and asymmetric error bars of the amplitudes.

In the first stage we select moduli consistent with the four equations in (4.1) that contain only moduli. We start with

$$b_{12}=\frac{7}{2\sqrt{10}}a_{12}=|D^-|^2-|\overline{D^+}|^2. \quad (6.1)$$

The values of b_{12} lie in the range from $b_{12}(\min)=b_{12}-\Delta b_{12}$ to $b_{12}(\max)=b_{12}+\Delta b_{12}$. Obviously

$$b_{12}(\min)+|\overline{D^+}|^2\leqslant|D^-|^2\leqslant b_{12}(\max)+|\overline{D^+}|^2. \quad (6.2)$$

We select $|\overline{D^+}|^2$ such that

$$0<|\overline{D^+}|^2<1, \quad (6.3)$$

$$b_{12}(\min)+|\overline{D^+}|^2<1,$$

$$b_{12}(\max)+|\overline{D^+}|^2>0.$$

Then we select $|D^-|^2$ by the Monte Carlo method such that two conditions are satisfied. The first condition is

$$|D^-|^2+|\overline{D^+}|^2<1. \quad (6.4)$$

The second condition are the inequalities (6.2). If these conditions on $|D^-|^2$ and $|\overline{D^+}|^2$ are satisfied, we go to a_{10} to select $|D^0|^2$. Define

$$b_{10}=\frac{7}{2}a_{10}=3|D^0|^2-2(|D^-|^2+|\overline{D^+}|^2). \quad (6.5)$$

We select $|D^0|^2$ by the Monte Carlo method and verify the conditions

$$|D^0|^2 < 1 - |D^-|^2 - |\overline{D^+}|^2, \tag{6.6}$$

$$b_{10}(\min) + 2(|D^-|^2 + |\overline{D^+}|^2) \leq |D^0|^2 \leq b_{10}(\max) + 2(|D^-|^2 + |\overline{D^+}|^2). \tag{6.7}$$

Notice that Eq. (6.7) provides an additional condition on $|D^-|^2$ and $|\overline{D^+}|^2$:

$$\begin{aligned} b_{10}(\min) + 2(|D^-|^2 + |\overline{D^+}|^2) &< 1,\\ b_{10}(\max) + 2(|D^-|^2 + |\overline{D^+}|^2) &> 0. \end{aligned} \tag{6.8}$$

With $|D^0|$ and $|D^-|$ now fixed we obtain additional constraint from a_{11}. Define

$$b_{11} = \frac{7}{4\sqrt{15}} a_{11} = |D^0||D^-| \cos(\phi_{D^0 D^-}). \tag{6.9}$$

Hence

$$-|D^0||D^-| \leq b_{11} \leq +|D^0||D^-|. \tag{6.10}$$

But b_{11} is in the range

$$b_{11}(\min) \leq b_{11} \leq b_{11}(\max). \tag{6.11}$$

To obtain a physical value of $\cos(\phi_{D^0 D^-})$ the inequalities (6.10) and (6.11) must overlap. If no overlap exists, we have to go back and select new $|D^0|$ and $|D^-|$ until we get a nonempty overlap.

Next we go to a_6. We first select by Monte Carlo calculation $|\overline{P^+}|^2$ and $|P^-|^2$ such that

$$|\overline{P^+}|^2 < 1 - |D^0|^2 - |D^-|^2 - |\overline{D^+}|^2, \tag{6.12}$$

$$|P^-|^2 < 1 - |\overline{P^+}|^2 - |D^0|^2 - |D^-|^2 - |\overline{D^+}|^2, \tag{6.13}$$

and then verify the consistency conditions

$$a_6(\min) - \frac{\sqrt{30}}{7}(|D^-|^2 - |\overline{D^+}|^2) \leq \sqrt{\frac{6}{5}}(|P^-|^2 - |\overline{P^+}|^2) \leq a_6(\max) - \frac{\sqrt{30}}{7}(|D^-|^2 - |\overline{D^+}|^2). \tag{6.14}$$

If the bounds (6.14) are not satisfied we use Monte Carlo calculation to select new $|\overline{P^+}|^2$ and $|P^-|^2$. If these bounds are satisfied we select $|P^0|^2$ and $|S|^2$ by Monte Carlo calculation such that

$$\begin{aligned} 0 &< |P^0|^2 < 1 - |P^-|^2 - |\overline{P^+}|^2 - |D^0|^2 - |D^-|^2 - |\overline{D^+}|^2,\\ 0 &< |S|^2 < 1 - |P^0|^2 - |P^-|^2 - |\overline{P^+}|^2 - |D^0|^2 - |D^-|^2 - |\overline{D^+}|^2. \end{aligned} \tag{6.15}$$

The sum

$$|S|^2 + |P^0|^2 + |P^-|^2 + |\overline{P^+}|^2 + |D^0|^2 + |D^-|^2 + |\overline{D^+}|^2 \tag{6.16}$$

must be within the error range of a_1. If this bound is violated, we select another $|S|^2$ and $|P^0|^2$ or even $|P^-|^2$ and $|\overline{P^+}|^2$, and if necessary also $|D^0|^2$, $|D^-|^2$, and $|\overline{D^+}|^2$.

In the next stage we determine the five independent cosines in Eqs. (4.1) defined as

$$\begin{aligned} c_1 &= \cos(\gamma_{SP^0}), \quad c_2 = \cos(\gamma_{SP^-}),\\ c_3 &= \cos(\gamma_{SD^0}), \quad c_4 = \cos(\gamma_{SD^-}),\\ c_5 &= \cos(\gamma_{\overline{P^+}\,\overline{D^+}}). \end{aligned} \tag{6.17}$$

We also define, for $i,j = 1,2,3,4$,

$$c_{ij}(\varepsilon) = c_i c_j + \varepsilon_{ij}\sqrt{(1-c_i^2)(1-c_j^2)}, \tag{6.18}$$

where ε_{ij} is the sign ambiguity. In the following a selection of one of the eight sign ambiguities (see the table of Sec. IV) will be understood and we will write c_{ij} instead of $c_{ij}(\varepsilon)$. It is convenient to organize the relevant equations in the following way:

$$a_4 = 2|S||D^0|c_3 + \Delta, \tag{6.19}$$

$$a_{11} = \frac{4}{7}\sqrt{15}|D^0||D^-|c_{34}, \tag{6.20}$$

$$a_3 = 2\sqrt{2}|S||P^-|c_2 + 2\sqrt{\frac{6}{5}}|P^0||D^-|c_{14} - \sqrt{\frac{2}{5}}|P^-||D^0|c_{23}, \tag{6.21}$$

$$a_5 = 2\sqrt{2}|S||D^-|c_4 + 2\sqrt{\frac{6}{5}}|P^0||P^-|c_{12} + \frac{2\sqrt{10}}{7}|D^0||D^-|c_{34}, \tag{6.22}$$

$$a_8 = 8\sqrt{\frac{3}{35}}|P^0||D^-|c_{14} + \frac{12}{\sqrt{35}}|P^-||D^0|c_{23}, \tag{6.23}$$

$$a_9 = 2\sqrt{\frac{6}{7}}\{|P^-||D^-|c_{24} - |\overline{P^+}||\overline{D^+}|c_5\}, \tag{6.24}$$

$$a_2 = 2|S||P^0|c_1 + \frac{4}{\sqrt{5}}|P^0||D^0|c_{13} + 2\sqrt{\frac{3}{5}}\{|P^-||D^-|c_{24} + |\overline{P^+}||\overline{D^+}|c_5\}, \tag{6.25}$$

$$a_7 = \frac{6}{\sqrt{35}}\{\sqrt{3}|P^0||D^0|c_{13} - |P^-||D^-|c_{24} - |\overline{P^+}||\overline{D^+}|c_5\}. \tag{6.26}$$

In Eq. (6.18) Δ is the fixed term of moduli in a_4.

To proceed, we first select c_3 such that the right-hand side (rhs) of Eq. (6.19) is in the error range of a_4, $a_4(\mathrm{min}) \leqslant a_4 \leqslant a_4(\mathrm{max})$. Then we select c_4 such that the rhs of Eq. (6.20) is in the error range of a_{11}. Next we make a Monte Carlo selection of c_1 and c_2 such that the rhs of Eqs. (6.21), (6.22), and (6.23) are within the error range of a_3, a_5, and a_8, respectively. With c_1, c_2, c_3, and c_4 thus fixed we seek by Monte Carlo calculation the value of c_5 with the rhs of Eq. (6.24) within error range of a_9. Finally, we verify the consistency of $c_1, \ldots, c_5$ with Eqs. (6.25) and (6.26). If an inconsistency is found, first we try new selections of c_5. If this does not work, we go back to new selections of c_1 and c_2 in Eqs. (6.21)–(6.23). If inconsistencies persist, we try new selections of c_3 and c_4 in Eqs. (6.19) and (6.20).

A similar procedure is applied to the second set of equations for amplitudes of opposite transversity. In this approach the normalization condition (2.6) with $\Sigma \equiv 1$ may not be satisfied exactly by the average values of the moduli squared.

Multistage inverse Monte Carlo method involves obviously more programming than the simple inverse Monte Carlo method. Only practical experience can tell which method is preferable.

VII. SUMMARY

Measurements of $\pi^- p \to \pi^- \pi^+ n$, $\pi^+ n \to \pi^+ \pi^- p$, and $K^+ n \to K^+ \pi^- p$ on polarized targets at CERN found evidence for a strong dependence of pion production amplitudes on nucleon spin. This evidence invalidates the assumption [5,6] that production amplitudes in $\pi^- p \to \eta\pi^- p$ and $\pi^- p \to \eta\pi^0 n$ reactions do not depend on nucleon spin. The amplitude analyses of these reactions based on the assumption of the independence of production amplitudes on nucleon spin are thus insufficient and are likely to be unreliable. To ascertain the existence of exotic resonance $1^{-+}(1405)$ and study its properties, a reliable, model-independent amplitude analysis is required. Nucleon spin is not only relevant to the dynamics of production processes. It also allows the model-independent determination of spin-dependent production amplitudes from measurements of $\pi^- p \to \eta\pi^- p$ and $\pi^- p \to \eta\pi^0 n$ on polarized targets, as we have shown. Our major assumption was that moments with $M > 2$ do not contribute to the angular distributions. This may not be true at large momentum transfers. In this case one has to use the formalism developed by Sakrejda [16], which takes into account the helicities $\lambda = \pm 2$ of the D wave.

The importance of the nucleon-spin dependence of amplitudes is not just a mathematical possibility; it is an experimental fact that has been firmly established by the CERN measurements on polarized targets [12–22]. Once the amplitudes have been reconstructed from the data on a polarized target, their dependence on nucleon spin should be examined by checking their dependence not only on the invariant mass m but also on the variables t and s. We point out that in formation experiments at low energies, if a three-body isobar decay of the resonance could be studied, then it is possible to incorporate the nucleon spin dependence of amplitudes without a polarized target. Of course, this is not possible in production experiments at higher energies.

Instruments shape research and determine which discoveries are made. Polarized targets have proven themselves to be valuable and important tools of discovery. We propose that high statistics measurements of the reactions $\pi^- p \to \eta\pi^- p$ and $\pi^- p \to \eta\pi^0 n$ be made on polarized targets at the BNL Multiparticle Spectrometer and at the IHEP in Protvino, in conjunction with high statistics measurements of $\pi^- p \to \pi^0 \pi^0 n$ on polarized targets. Such experiments will also be feasible for the recently proposed Japanese Hadron Project (JHP). When built, JHP will be a high-intensity 50-GeV proton accelerator complex with high-quality pion, kaon and antiproton secondary beams [28]. The availability of such secondary beams will make JHP an ideal facility for hadron spectroscopy using polarized targets in a search for new resonant states at the level of spin-dependent production amplitudes.

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