Explicit bosonization of the massive Thirring model in 3+1 dimensions

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(Received 27 February 1997)

We bosonize the massive Thirring model in 3+1D for a small coupling constant and arbitrary mass. The bosonized action is explicitly obtained both in terms of a Kalb-Ramond tensor field as well as in terms of a dual vector field. An exact bosonization formula for the current is derived. The small and large mass limits of the bosonized theory are examined in both the direct and dual forms. We finally obtain the exact bosonization of the free fermion with an arbitrary mass. [S0556-2821(97)04918-7]

PACS number(s): 11.10.Lm, 11.10.Ef

The method of bosonization has proven to be a very powerful tool for investigating two-dimensional theories allowing, for instance, the obtainment of exact solutions of nonlinear theories such as quantum electrodynamics and the sine-Gordon model [1]. A lot of effort has been spent in order to generalize this method to higher dimensions [2,3], but explicit results are only available in 2+1D [4–8].

In the present work we aim to provide an explicit bosonization of the massive Thirring model (MTM) in 3+1D. Contrary to earlier approaches, our analysis is valid for arbitrary mass and coupling. Using the functional methods developed in [2], we show that the MTM can be bosonized either in terms of a second rank pseudotensor Kalb-Ramond gauge field or in terms of a dual vector gauge field. An exact bosonization formula for the current is derived both in terms of the tensor and vector fields. We then perform a small coupling expansion, obtaining thereby concrete expressions for the bosonized Lagrangian and its dual for arbitrary mass. The small and large mass limits are analyzed in detail. The leading contribution in the small mass limit behaves as a Proca theory, either in the tensor or vector cases. In the large mass limit, however, the leading contribution is nonlocal. This is to be compared with the corresponding analysis in 2+1D [4–8], where the roles of the two limits are reversed.

We also consider the case of a free fermion with an arbitrary mass as a limiting situation of the MTM when the coupling constant vanishes. In this case, we get exact explicit results for the bosonized Lagrangian and its dual. Interestingly, the tensor field Lagrangian that appears in the small mass limit has been shown to be connected with QED [9].

Let us consider the master Lagrangian [2]

$$\mathcal{L}_M = -\tfrac{1}{6} F_{\mu\nu\alpha}F^{\mu\nu\alpha} + \overline{\psi}(i\partial\!\!\!/ - m - \lambda B\!\!\!/)\psi + \epsilon^{\mu\nu\alpha\beta}B_\mu \partial_\nu A_{\alpha\beta} \tag{1}$$

where $A_{\alpha\beta}$ is the second rank antisymmetric Kalb-Ramond tensor field and

$$F^{\mu\nu\alpha} = \partial^{[\mu}A^{\nu\alpha]} \tag{2}$$

is the corresponding field intensity tensor. B_μ is an external vector field and λ is its coupling constant to the fermions. Note that λ has dimension of mass inverse. This Lagrangian is invariant under independent gauge transformations on the vector and tensor fields. Consider the Euclidean generating functional in the presence of external sources

$$Z = \int DB_\mu DA_{\alpha\beta} D\psi D\overline{\psi} \exp\left\{ -\int d^4z \left[\mathcal{L}_M + \epsilon^{\mu\nu\alpha\beta}\partial_\nu A_{\alpha\beta} J_\mu + B_\mu K^\mu \right] \right\} \delta[\partial_\mu B^\mu]\delta[\partial_\alpha A^{\alpha\beta}], \tag{3}$$

where K_μ must be conserved in order to preserve gauge invariance and the delta functions are for fixing the gauge. Upon integration over B_μ and $A_{\alpha\beta}$ [2], we obtain the MTM as the resulting theory:

$$Z = \int D\psi D\overline{\psi} \exp\left\{ -\int d^4z \left[\overline{\psi}(i\partial\!\!\!/ - m)\psi - \frac{\lambda^2}{4} j^\mu j_\mu + \lambda j_\mu \left(J^\mu + \frac{K^\mu}{2} \right) \right] \right\}, \tag{4}$$

where $j^\mu = \overline{\psi}\gamma^\mu\psi$ is the fermionic current. By comparing the generating functionals (3) and (4), we can make the identifications

$$\lambda j^\mu = \epsilon^{\mu\nu\alpha\beta}\partial_\nu A_{\alpha\beta} = 2B_\mu . \tag{5}$$

The above results are valid for arbitrary values of λ and m. Note that the antisymmetric Kalb-Ramond field must be a pseudotensor. This behavior under parity transformations is a general feature of the bosonized fields in any dimension and follows from the current bosonization formulas.

Let us perform now the fermionic integration in Eq. (3). Observe that even though λ is dimensionful, we are just dealing with the familiar fermionic determinant in the presence of the external field $\lambda\ B_\mu$ which has the usual dimension. Hence, only one-loop graphs will contribute to this determinant. In the small λ approximation, the leading order contribution is a two-legs graph. We therefore obtain the effective action

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$$S_{\rm eff}=\int d^4z\Big(-\tfrac{1}{6}F_{\mu\alpha\beta}F^{\mu\alpha\beta}+\epsilon^{\mu\nu\alpha\beta}B_\mu\partial_\nu A_{\alpha\beta}+\tfrac{1}{2}B_\mu\Pi^{\mu\nu}B_\nu$$
$$+\epsilon^{\mu\nu\alpha\beta}\partial_\nu A_{\alpha\beta}J_\mu\Big), \tag{6}$$

where $\Pi^{\mu\nu}$ is the lowest order contribution to the vacuum polarization tensor of QED, which in Euclidean momentum space is given by [10]

$$\Pi^{\mu\nu}(k)=(k^2\delta^{\mu\nu}-k^\mu k^\nu)\Pi(k^2),$$

where

$$\Pi(k^2)=-\frac{\lambda_R^2}{12\pi^2}\left\{\frac{1}{3}+2\left(1-\frac{2m^2}{k^2}\right)\left(\frac{1}{2}x\,\ln\frac{x+1}{x-1}-1\right)\right\} \tag{7}$$

in which $x=(1+4m^2/k^2)^{1/2}$. In the above expression, the renormalized coupling constant λ_R is given, in lowest order, by

$$\lambda_R^2=\left(1-\frac{\lambda_R^2}{12\pi^2}\ln\,\Lambda^2\right)\lambda^2, \tag{8}$$

where Λ is an ultraviolet cutoff. Notice that in the effective action (6) we have set the external source $K^\mu=0$ because after the identification (5), the use of two sources would be superfluous.

From the effective action (6), we can obtain the bosonized theory (or its dual) by integrating either over B_μ or $A_{\alpha\beta}$. The quadratic B_μ integration can be made in a straightforward manner, giving the result

$$Z=\int DA_{\alpha\beta}\delta(\partial_\alpha A^{\alpha\beta})\exp\Big(-\int d^4z\{-\tfrac{1}{6}F_{\mu\alpha\beta}F^{\mu\alpha\beta}$$
$$-\tfrac{1}{3}F_{\mu\alpha\beta}[G(z-z')]F^{\mu\alpha\beta}+\epsilon^{\mu\nu\alpha\beta}\partial_\nu A_{\alpha\beta}J_\mu\}\Big), \tag{9}$$

where the Fourier transform of $G(z-z')$ is

$$G(k)=\frac{1}{k^2\Pi(k^2)}. \tag{10}$$

The exponent in the above integrand is the bosonized theory corresponding to the MTM for *arbitrary* mass and small λ. This is one of the central results of our paper.

Let us investigate now the small and large mass limits of the bosonized theory. From Eq. (7) and Eq. (10) a straightforward computation yields, respectively,

$$G(k)\underset{m\to 0}{\sim}\frac{36\pi^2}{5\lambda_R^2k^2}\left[1+O\left(\frac{m^2}{k^2}\right)\right],$$

$$G(k)\underset{m\to\infty}{\sim}-\frac{48\pi^2m^2}{\lambda_R^2k^4}\left[1+O\left(\frac{k^2}{m^2}\right)\right]. \tag{11}$$

Observe that in the small mass limit, because of the gauge fixing constraint, the leading term is a mass term for the Kalb-Ramond field. In the large mass limit, the leading term is already nonlocal. This is a consequence of the nonconstant behavior of the vacuum polarization tensor of QED which must vanish for small k, in such a way that the Coulomb potential has vanishingly small corrections at large distances. Should the vacuum polarization tensor have a constant behavior for large mass (small k) we would also have a Proca type Lagrangian for the bosonized theory. This may happen in higher dimensions [2,3].

Let us point out that here we have a similar situation to the three-dimensional case where the leading behavior in the small and large mass limits give different expressions [4–8], but only the role is reversed because the local form is obtained there in the large mass case.

Now we can get the dual version of the bosonized theory by starting from Eq. (6) and performing the quadratic integration over the Kalb-Ramond field. The result is [2]

$$Z=\int DG_\mu\,\exp\Big(-\int d^4z\{\tfrac{1}{2}G_{\mu\nu}[\Pi(z-z')]G^{\mu\nu}$$
$$+G^\mu G_\mu+2G_\mu J^\mu\}\Big), \tag{12}$$

where $G_{\mu\nu}=\partial_\mu G_\nu-\partial_\nu G_\mu$ and $\Pi(z-z')$ is the inverse Fourier transform of $\Pi(k^2)$. Observe that the first term came from the fermionic integration and was not involved in the integration over the Kalb-Ramond field.

The small and large m limits of the above expression can be obtained trivially from the expressions in Eq. (11). Observe that in the small mass limit, the leading contribution yields a Proca Lagrangian. It is interesting to see that in this limit both the original and dual bosonized Lagrangians are of the Proca type. The fact that the dual of a vector Proca theory is a Kalb-Ramond Proca theory in arbitrary dimension has been observed in [2].

Let us consider now the free fermion field with an arbitrary mass. This can be obtained by taking the limit $\lambda_R\to 0$, in which case the fermionic integration becomes exact. From Eq. (11) we can see that in this case only the second term in the exponent in Eq. (9) is relevant and the bosonic action corresponding to a free massive Dirac fermion in $3+1$D is given by

$$\int d^4z\,\overline{\psi(i\partial\!\!\!/-m)}\psi\leftrightarrow-\frac{1}{3}\int d^4zF_{\mu\alpha\beta}[\tilde{G}(z-z')]F^{\mu\alpha\beta}, \tag{13}$$

where we have rescaled the Kalb-Ramond field as $A_{\mu\nu}\to\lambda_R^{-1}A_{\mu\nu}$ and therefore

$$\tilde{G}(z)\equiv\lambda_R^2G(z). \tag{14}$$

It is interesting to note that the leading contribution to the small mass limit of Eq. (13) reproduces the theory studied in [9] in connection to QED, where G is proportional to $1/\Box$, whereas in the large mass limit G will be proportional to $1/(\Box)^2$.

A very important remark now is in order. Observe that according to the identity (5) the Thirring interaction corresponds to the first term in Eq. (9). We have just seen, on the other hand, that the free part of the fermion Lagrangian can be exactly bosonized by Eq. (13). One could then be tempted to exactly bosonize the full MTM by just adding the two pieces as it occurs in $1+1$D. This however is not true here as our computation clearly shows that only in the $\lambda_R\to 0$

limit we can bosonize the free massive fermion Lagrangian as Eq. (13) and ignore the higher-order insertions in the fermionic determinant. A consequence of this observation is that in spite of the fact that the linear current bosonization formula (5) is always valid, the bosonization formula for the free fermion Lagrangian depends on the interacting part of the theory it is embedded on and only in the free case it is given by Eq. (13). This is the most remarkable point of departure from the usual bosonization scheme in $1+1$D.

There is an important point which needs further clarification. This is related to the fact that even though the MTM is a nonrenormalizable theory in $3+1$D its bosonized version in the small λ_R limit is a generalized free theory. This may have some implications concerning the interpretation of a nonrenormalizable theory which we still do not completely understand and plan to investigate in the future.

We conclude by remarking that the results obtained here open a broad field of research in the bosonization of theories in the physical dimension of $3+1$. One could devise, for instance, the use of the master Lagrangian given in [2] in order to obtain explicit bosonization formulas for QED. Also one could explore the concrete bosonized theories obtained here through the operator formulation developed in $2+1$D [4] for the direct bosonization of the fermion field operator.

Both authors were partially supported by CNPq-Brazilian National Research Council. RB is very grateful to the Instituto de Física-UFRJ for the kind hospitality.

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