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$N=1$, $D=3$ superanyons, osp(2|2), and the deformed Heisenberg algebra

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We introduce an $N=1$ supersymmetric generalization of the mechanical system describing a particle with fractional spin in $D=1+2$ dimensions and being classically equivalent to the formulation based on the Dirac monopole two-form. The model introduced possesses hidden invariance under the $N=2$ Poincaré supergroup with a central charge saturating the *BPS* bound. At the classical level the model admits a Hamiltonian formulation with two first class constraints on the phase space $T^*(\mathbb{R}^{1,2})\times\mathcal{L}^{1|1}$, where the Kähler supermanifold $\mathcal{L}^{1|1}\cong \mathrm{OSp}(2|2)/\mathrm{U}(1|1)$ is a minimal superextension of the Lobachevsky plane. The model is quantized by combining the geometric quantization on $\mathcal{L}^{1|1}$ and the Dirac quantization with respect to the first class constraints. The constructed quantum theory describes a supersymmetric doublet of fractional spin particles. The space of quantum superparticle states with a fixed momentum is embedded into the Fock space of a deformed bosonic oscillator. [S0556-2821(97)07716-3]

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I. INTRODUCTION

Anyons [1], being particles with fractional spin and statistics [2,3] in (1+2)-dimensional space-time, are not a purely group-theoretical concept, but they may originate, for instance, from the field theory in the presence of Chern-Simons field [4–7]. Several physical phenomena such as the fractional Hall effect [8–10] and high-T_c superconductivity [11] can be explained by means of this concept.

We observe today a considerable interest in the study of point-particle models of anyons [12–23], which is generated by a possibility to derive a field theory for anyons by quantizing a classical mechanical system in $D=1+2$ dimensions. The most efficient method known to realize the quantum anyon states is to use the fields transforming in unitary irreducible representations of the universal covering group of $\overline{\mathrm{SO}^{\uparrow}(1,2)}\cong\overline{\mathrm{SU}(1,1)}$ [12–15,22–26]. These representations are infinite-dimensional and, hence, an infinite set of equations are required to single out one independent physical component. Although various versions of such equations have already been suggested (Refs. [12–15,22–26]), the problem remains open to realize them in a form that admits anyon self-interaction.

A convenient formulation of free field equations for fractional spin particles was suggested in Ref. [15]. In their approach, both the mass-shell constraint and the spin-fixing condition (which are imposed as independent equations in other models [12–14,16,20,22,23]) originate as integrability conditions for the field equations of motion. This is achieved by making use of the well known realization of so(1,2) as the Lie algebra of quadratic polynomials of the creation and annihilation operators of the harmonic oscillator. As a consequence, only the particles with spins $(2n+1)/4$, $n=0,1,2,...$ (called semions) appear in the spectrum of the model [15]. Recently, it has been recognized [27] that in order to extend the semion construction [15] to the case of arbitrary fractional spin particles one should make use of the deformed Heisenberg algebra (DHA) (see [28,27] and references therein) and the superalgebra osp(2|2). Thereby the one-particle anyon states can be realized in the $\mathbb{Z}_2$-graded Fock space of the deformed oscillator, where the grading is induced by the Klein operator being one of the DHA generators. These results show the actual relationship between DHA and the fractional spin concept.

In this article we demonstrate that the DHA naturally originates in the quantum supersymmetric theory of anyons.

We introduce $N=1$, $D=3$ super Poincaré invariant action for a massive fractional spin superparticle living in $\mathbb{R}^{3|2}\times\mathcal{L}$, where $\mathbb{R}^{3|2}$ denotes the $N=1$, $D=3$ flat superspace and $\mathcal{L}$ is a Lobachevsky plane. This mechanical system is a minimal supersymmetric extension of the special anyon model suggested in [23]. Our interest in the latter is due to the fact that the model proves to be classically equivalent to the formulation based on the monopolelike symplectic two-form [17–20] and, hence, allows interaction to arbitrary background fields. On the other hand, it can be treated as a reduction of the $D=(1+3)$-dimensional massive spinning particle model developed in [29].

By construction, the model under consideration is manifestly $N=1$ supersymmetric. But it turns out to possess hidden invariance with respect to $N=2$ Poincaré supergroup with a central charge saturating the Bogomol'nyi-Prasad-Sommerfield (BPS) bound (see, for instance, [30]) on the mass shell. As is known, this condition on central charge corresponds to shortening of $N=2$ massive supermultiplets. The appearance of $N=2$ supersymmetry has a remarkable counterpart in Hamilton formulation of the theory. Namely, the dynamics is restricted to a surface of second class constraints in such a way that it takes the form of the mechanics on the phase space $T^*(\mathbb{R}^{1,2})\times\mathcal{L}^{1|1}$, where the Kähler supermanifold $\mathcal{L}^{1|1}=\mathrm{OSp}(2|2)/\mathrm{U}(1|1)$ (of complex dimension 1+1) is a minimal superextension of the Lobachevsky plane. It is the ''inner'' supermanifold $\mathcal{L}^{1|1}$, which supplies the particle with a superspin degree of freedom. OSp(2|2) emerges as the group of all the superholomorphic canonical transformations on $\mathcal{L}^{1|1}$.

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The $N=2$ Poincaré superalgebra with central charge and the superalgebra osp(2|2) prove to be closely related in this construction both at the classical and quantum level. Let us comment on this crucial point in more detail. In Hamilton approach the dynamics is described by one first class and six second class constraints. The second class constraints have a complicated nonlinear structure that makes it practically impossible to perform straightforwardly the Dirac canonical quantization (probably, it is the reason why superanyon models have not been quantized until now). Our solution to the problem is as follows. We first reduce the dynamics with respect to four second class constraints arriving to the phase space $T^*(\mathbb{R}^{1,2})\times\mathcal{L}^{1|1}$. As a consequence, the superalgebra osp(2|2) is naturally realized in terms of the nonlinear Poisson bracket. Special structure of the reduced phase space makes it possible to apply the Berezin-Kostant quantization method [31,32] for the inner phase space. This method has been recently extended to the supermanifold $\mathcal{L}^{1|1}$ [33–35]. On $T^*(\mathbb{R}^{1,2})\times\mathcal{L}^{1|1}$, the rest constraints (one of first class and two of second class) are equivalent to two first class constraints. In quantum theory, these constraints are imposed to annihilate physical states, and it proves to be equivalent to the requirement of the Poincaré superalgebra to be consistent quantum mechanically. Thus, combining the geometric quantization on $\mathcal{L}^{1|1}$ for the second class constraints and the Dirac quantization with respect to the first class ones, we can quantize the superparticle with arbitrary (fixed) fractional superspin. Short massive representations of the $N=2$ Poincaré superalgebra with central charge are realized on the fields transforming in atypical unitary representations of osp(2|2). Moreover, the known connection between unitary representations of osp(2|2) and the DHA makes possible an alternative elegant realization of the superanyon doublet in the Fock space of the deformed bosonic oscillator.

This article is organized as follows. The anyon model on the configuration space $\mathbb{R}^{1,2}\times\mathcal{L}$ and its quanitzation is considered in Sec. II. In Sec. III we analyze the $N=1, D=3$ superanyon model. The global symmetries of the model and the structure of the reduced phase space are studied in detail. Section IV is devoted to the quantization of the superanyon model. Summary and concluding remarks are given in Sec. V. In Appendix A we collect the conventions used throughout the paper. In Appendix B we describe the realization of the Lobachevsky plane as a homogeneous space of the Lorentz group.

II. ANYON MODEL ON $\mathbb{R}^{1,2}\times\mathcal{L}$

As a starting point for supersymmetric extension, let us consider a model of the fractional spin particle suggested in [23]. The configuration space of the model $\mathbb{R}^{1,2}\times\mathcal{L}$, where $\mathcal{L}\cong\mathrm{SU}(1,1)/\mathrm{U}(1)$ denotes a Lobachevsky plane, is a homogeneous space of the $D=3$ Poincaré group. The model is described by the action functional

$$S=\int d\tau\, L, \quad L=m(\dot{x},n)+is\frac{\bar{z}\dot{z}-\dot{\bar{z}}z}{\zeta}, \tag{1}$$

where

$$n_a\equiv\frac{\zeta_a}{\zeta}=-\left(\frac{1+z\bar{z}}{1-z\bar{z}},\ \frac{z+\bar{z}}{1-z\bar{z}},\ i\frac{z-\bar{z}}{1-z\bar{z}}\right), \quad n^2\equiv-1.$$

Here x^a and $z,\bar{z}$ are coordinates[1] on $\mathbb{R}^{1,2}$ and $\mathcal{L}$, respectively; ζ_a and ζ are defined by relations (B5) and (B7) in Appendix B; m and s denote the mass and spin of the particle. The model possesses global invariance with respect to the Poincaré group. Infinitesimal Poincaré transformations (with f^a and ω^a parameters of translations and Lorentz transformations) read

$$\delta x^a=f^a, \quad \delta z=\delta\bar{z}=0,$$

$$\delta x^a=\epsilon^{abc}x_b\omega_c, \quad \delta z=-i(\omega,\xi), \quad \delta\bar{z}=i(\omega,\bar{\xi}), \tag{2}$$

where the vectorlike objects ξ_a, $\bar{\xi}_a$ are defined by Eq. (B6). The Lagrangian (1) is manifestly translation invariant, whereas the Lorentz transformations change it by a total derivative:

$$\delta L=-\frac{s}{2}\frac{d}{d\tau}\left(\frac{\partial}{\partial z}\xi_a+\frac{\partial}{\partial\bar{z}}\bar{\xi}_a\right)\omega^a. \tag{3}$$

To verify Eq. (3), one should know that, by virtue of Eqs. (B3), (B5), and (B7), n_a transforms as a three-vector, hence the first term in the action functional is manifestly Poincaré invariant. As to the second term, it can be written as $s\int\Sigma_0$, with the one-form Σ_0 being a solution of the equation $d\Sigma_{a0}=\Omega_0$, for the Lorentz invariant Kähler two-form

$$\Omega_0=-2i\frac{dz\wedge d\bar{z}}{\zeta^2} \tag{4}$$

associated to the Lobachevsky plane. The invariance of Ω_0 implies that Σ_0 may get exact contributions under Lorentz transformations. It is Σ_0 which contributes to the total derivative in the right-hand side of Eq. (3).

The global symmetries related to the Poincaré group generate all the independent Noether currents of the model. It is worth pointing out the existence of another global space-time symmetry of the action functional

$$\delta x^a=-\rho n^a, \quad \delta z=0, \tag{5}$$

where ρ is a constant parameter. This rather unusual transformation commutes with the Poincaré ones, and the associated Noether current is trivial. To clarify this fact, consider the equations of motion. Accounting for the identities $\dot{z}\equiv(\xi,\dot{n})$, $\dot{\bar{z}}\equiv(\bar{\xi},\dot{n})$, the Euler-Lagrange equations read

$$\dot{n}^a=0,\ (\dot{x},\xi)=-\frac{is}{m}\dot{z}=0,\ (\dot{x},\bar{\xi})=\frac{is}{m}\dot{\bar{z}}=0,$$

so, in particular, n^a is a conserved vector. Therefore transformation (5) reduces to special space-time translations on-shell.

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[1]The Lobachevsky space $\mathcal{L}$ is realized as the unit disc of complex plane, $|z|<1$.

Since L is a first-order homogeneous function of velocities, the action remains invariant under world-line reparametrizations of the form

$$\delta_\epsilon x^a = \dot{x}^a \epsilon(\tau), \quad \delta_\epsilon z = \dot{z}\epsilon(\tau), \quad \delta_\epsilon \bar{z} = \dot{\bar{z}}\epsilon(\tau), \tag{6}$$

with the parameter $\epsilon(\tau)$ being arbitrary modulo standard boundary conditions.

Remarkable features of the model become transparent in the Hamiltonian formalism. All the relations defining canonical momenta conjugate to x^a, z, $\bar{z}$ constitute the set of primary constraints:

$$T_a = p_a - m n_a \approx 0, \tag{7}$$

$$T = p_z - is\frac{\bar{z}}{\zeta} \approx 0, \ \ \bar{T} = p_{\bar{z}} + is\frac{z}{\zeta} \approx 0. \tag{8}$$

The Hamiltonian is a linear combination of these constraints. There are no secondary constraints and Eqs. (7) and (8) describe the complete set of constraints in the model. The matrix of (canonical) Poisson brackets of the constraints (7) and (8) turns out to have the rank equal to four, which is a maximal possible value for antisymmetric 5×5 matrix. Hence, we have four second class constraints and one of the first class.

It is expedient for further consideration to reduce the dynamics to the surface of the constraints (8) removing momenta p_z, $p_{\bar{z}}$. For $s\neq0$ the corresponding Dirac brackets are denoted by $\{\,,\}^*$ and have the form

$$\{x^a, p_b\}^* = \delta^a{}_b\,, \ \ \{z, \bar{z}\}^* = -\frac{i}{2s}(1 - z\bar{z})^2, \tag{9}$$

the rest brackets between the variables equal to zero. The reduced phase space obtained in this way is seen to be isomorphic to the product of two symplectic manifolds, $T^*(\mathbb{R}^{1,2})\times\mathcal{L}$, where $\mathcal{L}$ has a standard nonlinear symplectic structure of the Lobachevsky plane [31,36].

Let us discuss the physical observables[2] of the model. First, consider the Hamiltonian generators of the Poincaré transformations (2). For the energy-momentum vector $\mathcal{P}_a$ and the angular momentum vector $\mathcal{J}_a$, one has

$$\mathcal{P}_a = p_a\,, \ \ \mathcal{J}_a = \epsilon_{abc} x^b p^c + J_a\,, \tag{10}$$

where J_a denotes the spin momentum vector

$$J_a = i\xi_a p_z - \frac{s}{2}\partial\xi_a - i\bar{\xi}_a p_{\bar{z}} - \frac{s}{2}\bar{\partial}\,\bar{\xi}_a = -s n_a\,. \tag{11}$$

Here we have accounted for the constraints (8). With respect to Poisson bracket (9), the functions (10) generate the Poincaré algebra iso(1,2), whereas the spin generators (11) span internal Lorentz algebra so(1,2) related to the automorphism group of the Lobachevsky plane. The phase-space Casimir functions $\mathcal{P}^a\mathcal{P}_a = p^2$ and $\mathcal{P}^a\mathcal{J}_a = -s(p,n)$ reduce to constants in virtue of the constraints (7)

$$T^{(1)} = p^2 + m^2 \approx 0, \tag{12a}$$

$$T^{(2)} = (p,n) + m \approx 0. \tag{12b}$$

One can also verify that functions of the Poincaré generators exhaust all the physical observables in the model. Therefore the model must describe the irreducible dynamics of $D=3$ particle with mass m and spin s. In addition, the particle energy p^0 is positive, as a consequence of Eq. (7).

Remarkably, the mixed first and second class constraints (7) proves to be equivalent to the first class constraints (12). This immediately follows from the decomposition

$$p_a \equiv 2\frac{(p,\xi)}{\zeta^2}\bar{\xi}_a + 2\frac{(p,\bar{\xi})}{\bar{\zeta}^2}\xi_a - (p,n)n_a\,, \tag{13}$$

which is true for arbitrary three-vector p_a, in virtue of Eq. (B8). Really, the constraints (7) imply $(p,\xi)=(p,\bar{\xi})=0$, hence Eqs. (12) are fulfilled. On the other hand, by squaring Eq. (13) one gets

$$4\left|\frac{(p,\xi)}{\zeta}\right|^2 \equiv p^2 + (p,n)^2. \tag{14}$$

Thus, the constraints (12) imply $(p,\xi)=(p,\bar{\xi})=0$. Moreover, it is readily seen that either of the two sets of constraints defines the same set of physical observables. Hence, the set of three constraints (7) (among which there are two of second class and one of first class) are equivalent to the pair of first class constraints (2). The above observation is important for quantization below.

On the mass shell (12a), Eq. (7) can be treated as a parametrization of the mass hyberboloid by local complex coordinates z, $\bar{z}$. This means, however, that we can rewrite the two-form (4) in the way

$$\Omega_0 = \frac{1}{2}\,\frac{\epsilon^{abc}p_a dp_b \wedge dp_c}{(-p^2)^{3/2}}\,, \tag{15}$$

that is, as a Dirac monopole two-form. Consequently, our model proves to be a reformulation of the well known anyon models based on the monopolelike two-form [17–20]. This fact can be alternatively established by deriving the Dirac brackets (to be denoted below by $\{\,,\}^{**}$) associated to the pair of second class constraints $(p,\xi)=0$, $(p,\bar{\xi})=0$. These brackets have the explicit structure

$$\{x^a, x^b\}^{**} = s\frac{\epsilon^{abc}p_c}{(-p^2)^{3/2}}\,, \quad \{x^a, p_b\}^{**} = \delta^a{}_b\,,$$

$$\{p_a, p_b\}^{**} = 0 \tag{16}$$

and reproduce the Poisson brackets for the mentioned particle models.

In a sense the "minimal" approach based on the two-form (15) and Poisson bracket (16) appears to be very natural. In particular, it is well adapted for the introduction of consistent coupling to external fields [17–20] and for consideration of self-interacting anyons [21]. However, the realization of quantization scheme in terms of "nonlocalizable"

[2]Physical observables are understood as phase space functions weakly commuting with the first class constraints.

coordinates has become a difficult problem in view of the complicated structure of the Poisson brackets for coordinates. Moreover, it is hardly possible in this approach to introduce ''localizable'' coordinates without loss of manifest covariance [22]. To the contrary, the formulation on the extended phase space $T^*(\mathbb{R}^{1,2})\times\mathcal{L}$ admits a natural quantization scheme we are going to describe.

To quantize the model, we shall make use of the following prominent features of the model. First, all physical observables, which are phase space functions commuting with the first class constraints, are actually functions of the Poincaré generators (10) only. Thus the quantization problem is to construct an appropriate realization for the unitary representations of the Poincaré group. Classically, the Poincaré generators (10) in the phase space of the model are split into two pieces, one of which includes only space-time variables and another corresponds to the internal space $\mathcal{L}$. It is the latter part of the generators which is relevant for nontrivial spin values. Second, one can observe that the spin part (11) of $\mathcal{J}_a$ coincides with the covariant Berezin symbols of the group $\overline{\mathrm{SU}(1,1)}$ on the Lobachevsky plane [31,36]. In view of all the features mentioned, it seems sensible to combine the Dirac canonical quantization for the Minkowski degrees of freedom with a geometric quantization for spin.

We realize the Hilbert space of one-particle anyon states of mass m and spin $s>0$ as a space of functions $F(p,\bar{z})$, F: $\mathbb{R}^{1,2}\times\mathcal{L}\rightarrow\mathbb{C}$ to be antiholomorphic[3] on the Lobachevsky plane (that is, antiholomorphic in the unit disk of $\mathbb{C}$, $|z|<1$). The operator realization of the classical Poincaré generators $\mathcal{P}_a$ and $\mathcal{J}_a$ [Eq. (10)] reads

$$\hat{\mathcal{P}}_a=p_a\,,\quad \hat{\mathcal{J}}_a=-i\epsilon_{abc}p^b\frac{\partial}{\partial p_c}+\hat{J}^s_a\,,$$
$$\hat{J}^s_a=-\bar{\xi}_a\bar{\partial}-s\overline{\partial\xi_a}\,, \tag{17}$$

where $\bar{\partial}\equiv\partial/\partial\bar{z}$. The generators (17) are Hermitian with respect to the following inner product:

$$\langle F|G\rangle=(2s-1)\int d^3p\int_{\mathcal{L}}\frac{dzd\bar{z}}{2\pi i}\zeta^{2s-2}\overline{F(p,\bar{z})}G(p,\bar{z}). \tag{18}$$

To complete the quantization, we impose operator counterparts of the first-class constraints (12) on the physical states F^{phys}:

$$(p^2+m^2)F^{\mathrm{phys}}(p,\bar{z})=0,$$
$$[(p,\hat{J}^s)-ms]F^{\mathrm{phys}}(p,\bar{z})=0. \tag{19}$$

This construction corresponds to the well known realization (see, e.g., [12,25,22]) of the $D=3$ Poincaré group representations of mass m and spin $s>0$ in terms of infinite-component fields transforming by an appropriate irreducible unitary representation of discrete series of D^s_+ of the group $\overline{\mathrm{SU}(1,1)}$ bounded below [36–38,25]. The components $F_n(x)$, $n=0,1,2,...$ of the fields are obtained by the series expansion of our wave function in $|n\rangle\equiv[\Gamma(2s+n)/\Gamma(n+1)\Gamma(2s)]^{1/2}\bar{z}^n$. Finally let us note that the case of $s<0$ can be treated in a similar way by the use of the representation of $\overline{\mathrm{SU}(1,1)}$ being to the other discrete series D^{-s}_- bounded above.

III. SUPERPARTICLE DYNAMICS ON $T^*(\mathbb{R}^{1,2})\times\mathcal{L}^{1|1}$

The simplest way to obtain a supersymmetic generalization of the model described is to extend the configuration space to a supermanifold $\mathbb{R}^{3|2}\times\mathcal{L}$, where the Grassmann sector is parameterized by an anticommuting Majorana spinor[4] θ^α, and to substitute $\dot{x}^a$ in the action by $\Pi^a=\dot{x}^a-i(\sigma^a)_{\alpha\beta}\theta^\alpha\dot{\theta}^\beta$. Then, one results with the $N=1$, $D=3$ superanyon theory[5] with the action functional

$$S=\int d\tau L,\quad L=m(\Pi,n)+is\frac{\bar{z}\dot{z}-\dot{\bar{z}}z}{\zeta}. \tag{20}$$

By construction, the model possesses global symmetry with respect to the $N=1$ Poincaré supergroup, and the corresponding infinitesimal transformations read

$$\delta x^a=f^a,\quad \delta z=0,\quad \delta\theta^\alpha=0,$$
$$\delta x^a=i\epsilon^\alpha(\sigma^a)_{\alpha\beta}\theta^\beta,\quad \delta z=0,\quad \delta\theta^\alpha=\epsilon^\alpha, \tag{21}$$
$$\delta x^a=\epsilon^{abc}x_b\omega_c\,,\quad \delta z=-i(\omega,\xi),\quad \delta\theta^\alpha=\frac{i}{2}\omega^\alpha{}_\beta\theta^\beta.$$

Here, f^a, $\omega^a=(\omega^a\delta_a)^\alpha{}_\beta$, and ϵ^α are the parameters of translations. Lorentz, and supersymmetry transformations, respectively. Similarly to the nonsupersymmetric model (1), Lorentz transformations change the Lagrangian (20) by total derivatives.

Along with the dynamical symmetries (21), the theory possesses several invariances which do not lead to new independent Noether currents. Such global symmetries are described by the transformations

$$\delta x^a=-\rho n^a,\quad \delta z=0,\quad \delta\theta^\alpha=0,$$
$$\delta x^a=0,\quad \delta z=0,\quad \delta\theta^\alpha=-2i\mu n^\alpha{}_\beta\theta_\beta, \tag{22}$$
$$\delta x^a=-\eta^\alpha(\sigma^a)_{\alpha\beta}n^\beta{}_\gamma\theta^\gamma,\quad \delta z=0,\quad \delta\theta^\alpha=-in^\alpha{}_\beta\eta^\beta,$$

where ρ and μ are bosonic infinitesimal parameters and η^α Grassmann ones, $n^\alpha{}_\beta\equiv(n^a\sigma_a)^\alpha{}_\beta$ is constructed in terms of z, $\bar{z}$ as in Eq. (1). The transformations (21) and (22) turn out to generate a closed superalgebra off the mass-shell. To analyze the structure of that superalgebra, it is convenient to pass to the Hamiltonian formalism,

[3] This particular realization is useful, since it provides the correspondence principle for Eqs. (9), (10), and (11) and gives the proper energy sign.

[4] The reality conditions for spinors in the SU(1,1) formalism are described in Appendix A.

[5] The case of extended supersymmetry $N>1$, deserves special treatment and will be considered elsewhere.

Introducing the momenta conjugate to x^a, z ,$\bar{z}$, θ^α and defining the canonical Poisson brackets

$$\{x^a,p_b\}=\delta^a{}_b\,,\quad \{z,p_z\}=\{\bar{z},p_{\bar{z}}\}=1,\ \{\theta^\alpha,\pi_\beta\}=\delta^\alpha{}_\beta,$$

we observe that the model contains the following set of constraints:

$$T_a=p_a-mn_a\approx 0, \tag{23}$$

$$T_\alpha=\pi_\alpha+imn_{\alpha\beta}\theta^\beta\approx 0, \tag{24a}$$

$$T=p_z-is\frac{\bar{z}}{\zeta}\approx 0,\ \bar{T}=p_{\bar{z}}+is\frac{z}{\zeta}\approx 0, \tag{24b}$$

which involve six constraints of the second class and one of the first class. As it is obvious, the first class constraint generates world-line reparametrizations and thus the physical Hamiltonian is zero. The Hamiltonian generators of the super Poincaré transformations (21) look like

$$\mathcal{P}_a=p_a\,,\quad \mathcal{J}_a=\epsilon_{abc}x^bp^c+J_a\,,\quad \mathcal{Q}^1_\alpha=ip_{\alpha\beta}\theta^\beta-\pi_\alpha\,, \tag{25}$$

where

$$J_a=-\frac{i}{2}(\sigma_a)_{\alpha\beta}\theta^\alpha\pi^\beta+i\xi_ap_z-\frac{s}{2}\partial\xi_a-i\bar{\xi}_ap_{\bar{z}}-\frac{s}{2}\overline{\partial\xi_a}. \tag{26}$$

further, the generators of transformations (22) have the form

$$\mathcal{Z}=-(p,n),\quad \mathcal{K}=2in_{\alpha\beta}\theta^\alpha\pi^\beta,$$
$$\mathcal{Q}^2_\alpha=-p_{\alpha\beta}n^\beta{}_\gamma\theta^\gamma+in_\alpha{}^\beta\pi_\beta. \tag{27}$$

The generators (25) and (27) prove to satisfy the (anti) commutation relations

$$\{\mathcal{J}_a,\mathcal{J}_b\}=\epsilon_{abc}\mathcal{J}^c,\quad \{\mathcal{J}_a,\mathcal{P}_b\}=\epsilon_{abc}\mathcal{P}^c,$$

$$\{\mathcal{J}_a,\mathcal{Q}^I_\alpha\}=\frac{i}{2}(\sigma_a)_{\alpha\beta}\mathcal{Q}^{I\beta},\quad \{\mathcal{Q}^I_\alpha,\mathcal{K}\}=2\epsilon^{IJ}\mathcal{Q}^J_\alpha, \tag{28}$$

$$\{\mathcal{Q}^I_\alpha,\mathcal{Q}^J_\beta\}=-2i\delta^{IJ}p_{\alpha\beta}-2\epsilon^{IJ}\epsilon_{\alpha\beta}\mathcal{Z},$$

the rest brackets being equal to zero, where I, $J=1,2$, $\epsilon^{IJ}=-\epsilon^{IJ}$, $\epsilon^{01}=1$. What we have obtained is $N=2$ Poincaré superalgebra with a central charge described by $\mathcal{Z}$ and U(1) isotopic charge $\mathcal{K}$ acting on the internal index of $\mathcal{Q}_\alpha{}^I$. The functions (25) generate $N=1$ subalgebra.

Let us discuss in more detail the system of constraints (23) and (24) which are different from that defined by Eqs. (7) and (8) by the presence of fermionic constraints (24a). The latter can be rewritten in a more familiar, for superparticle models, form

$$T'_\alpha=\pi_\alpha+ip_{\alpha\beta}\theta^\beta\approx 0 \tag{29}$$

on the surface of constraints (23). We prefer, however, to use the original representation (24a) in which the fermionic constraints do not involve the space-time variables and admit an interesting geometric interpretation related to the reduction (for $s>0$, $m>0$) to the surface of second class constraints (24). To explain this interpretation, consider the respective Dirac brackets:

$$\{z,\bar{z}\}^*=-\frac{i\zeta^2}{2s}\left(1+\frac{1}{2}\frac{\theta\bar{\theta}}{\zeta}\right),$$

$$\{\theta^\alpha,\theta^\beta\}^*=-n^{\alpha\beta}\frac{i}{2m}\left(1-\frac{1}{2}\frac{\theta\bar{\theta}}{\zeta}\right), \tag{30}$$

$$\{z,\theta^\alpha\}^*=\frac{i}{2\sqrt{2ms}}z^\alpha\theta,\quad \{\bar{z},\theta^\alpha\}^*=\frac{i}{2\sqrt{2ms}}\bar{z}^\alpha\bar{\theta},$$

while the rest Dirac brackets involving the space-time variables keep their canonical form, i.e., they vanish except $\{x^a,p_b\}^*=\delta^a{}_b$. Here we denote

$$\sqrt{\frac{s}{2m}}\,\theta\equiv z_\alpha\theta^\alpha=z\theta^0-\theta^1,\quad \sqrt{\frac{s}{2m}}\bar{\theta}\equiv\bar{z}_\alpha\theta^\alpha=\theta^0-\bar{z}\theta^1; \tag{31}$$

the twistorlike variables z^α, $\bar{z}^\alpha$ are defined in Appendix B. The complex Grassmann variable θ is in a one-to-one correspondence with Majorana spinor θ^α and, together with its complex conjugate $\bar{\theta}$, can be used to parametrize the odd sector of the constrained surface. From Eqs. (30) one deduces

$$\{\theta,\bar{\theta}\}^*=-\frac{i\zeta}{s}\left(1+\frac{1}{2}\frac{z\bar{z}\bar{\theta}\theta}{\zeta}\right),$$
$$\{z,\bar{\theta}\}^*=\frac{i\zeta}{2s}z\bar{\theta},\quad \{\bar{z},\theta\}^*=-\frac{i\zeta}{2s}\bar{z}\theta. \tag{32}$$

Equations (30) and (32) mean that the symplectic structure on the reduced phase space is induced by the two-superform

$$\Lambda=dp_a\wedge dx^a+s\Omega,$$

where

$$\Omega=2i\left(1-\frac{1+z\bar{z}}{2\zeta}\bar{\theta}\theta\right)\frac{dz\wedge d\bar{z}}{\zeta^2}+i\left(\frac{d\theta\wedge d\bar{\theta}}{\zeta}-\frac{\bar{z}\theta}{\zeta^2}dz\wedge d\bar{\theta}-\frac{z\bar{\theta}}{\zeta^2}d\bar{z}\wedge d\theta\right). \tag{33}$$

We follow Berezin's conventions for superforms [31] (see Appendix A). It is noticeable that Ω can be represented as

$$\Omega=i\left(dz\frac{\partial}{\partial z}+d\theta\frac{\vec{\partial}}{\partial\theta}\right)\wedge\left(d\bar{z}\frac{\partial}{\partial\bar{z}}+d\bar{\theta}\frac{\vec{\partial}}{\partial\bar{\theta}}\right)\Phi, \tag{34}$$

where

$$\Phi(z,\bar{z},\theta,\bar{\theta})=-2\ln\left(\zeta+\frac{1}{2}\theta\bar{\theta}\right), \tag{35}$$

so, we conclude that Ω and, hence, Λ are closed, $d\Lambda=d\Omega=0$.

The above consideration shows that the reduced phase space has the structure of direct product of symplectic spaces $T^*(\mathbb{R}^{1,2})\times\mathcal{L}^{1|1}$, $\mathcal{L}^{1|1}=\mathcal{L}\times\mathbb{C}^{0|1}$ being a complex supermanifold (of dimension $1+1$) parametrized by the complex even z and odd θ coordinates. The symplectic structure on $\mathcal{L}^{1|1}$ is determined by the closed nondegenerate superform Ω which is in fact a Kähler superform, in accordance with Eq. (34), and the corresponding superpotential reads as in Eq. (35). This Kähler supermanifold has been introduced in Refs. [34,35] as coadjoint orbit of simplest orthosymplectic supergroups [degenerate orbit of OSp(2|2) and a regular orbit of OSp(1|2)] and termed *superunit disk*. Therefore, $\mathcal{L}^{1|1}$ is a homogeneous space [35] of the supergroup OSp(2|2), $\mathcal{L}^{1|1}=\mathrm{OSp}(2|2)/\mathrm{U}(1|1)$ [hence, it can also be realized in the manner $\mathcal{L}^{1|1}=\mathrm{OSp}(1|2)/\mathrm{U}(1)$].

OSp(2|2) turns out to be the group of all *canonical* (with respect to Ω) *superholomorphic transformations* on $\mathcal{L}^{1|1}$. Infinitesimally, these transformations look like

$$\delta z=-i\omega^a\xi_a-\epsilon_\alpha z^\alpha\theta,\quad \delta\theta=-\frac{i}{2}\omega^a\frac{\partial}{\partial z}\xi_a\theta-i\mu\theta+2\overline{\epsilon_\alpha}z^\alpha,$$

where ω^α, μ are bosonic real parameters and ϵ_α fermionic complex ones. The functions

$$J_a=-sn_a\left(1-\frac{1}{2}\frac{\theta\bar\theta}{\xi}\right),\quad B=-s\left(1+\frac{1}{2}\frac{\theta\bar\theta}{\zeta}\right),$$
$$\theta^\alpha=\sqrt{\frac{s}{2m}}\frac{z^\alpha\theta-\overline{z^\alpha}\theta}{\zeta},\quad \pi_\alpha=i\sqrt{\frac{ms}{2}}\frac{z_\alpha\bar\theta+\overline{z_\alpha}\theta}{\zeta}\tag{36}$$

serve as the corresponding (real) generators of OSp(2|2), and their algebra, with respect to the Dirac brackets, reads

$$\{J_a,J_b\}^*=\epsilon_{abc}J^c,$$
$$\{J_a,\theta^\alpha\}^*=\frac{i}{2}(\sigma_a)^\alpha{}_\beta\theta^\beta,\quad \{J_a,\pi_\alpha\}^*=-\frac{i}{2}(\sigma_a)_\alpha{}^\beta\pi_\beta,$$
$$\{\theta^\alpha,B\}^*=\frac{1}{2m}\pi^\alpha,\quad \{\pi_\alpha,B\}^*=-\frac{m}{2}\theta_\alpha,\tag{37}$$
$$\{\theta^\alpha,\theta^\beta\}^*=\frac{i}{2ms}J^{\alpha\beta},\quad \{\pi_\alpha,\pi_\beta\}^*=\frac{im}{2s}J_{\alpha\beta},$$
$$\{\theta^\alpha,\pi_\beta\}^*=-\frac{1}{2s}\delta^\alpha{}_\beta B,\quad \{J_a,B\}^*=0.$$

The generators J_α and θ^α (or π_α) form a superalgebra osp(1|2).

Let us note that the role of OSp(2|2) for the superparticle model (20) is similar to the internal Lorentz group $\mathrm{SU}(1,1)/\mathbb{Z}_2$, whose action is defined on $\mathcal{L}$ only, in the particle model of Sec. II. Really, in accordance with Eqs. (30)–(33) the reduced phase space [the surface of constraints (24)] of the superparticle is isomorphic to $T^*(\mathbb{R}^{1,2})\times\mathcal{L}^{1|1}$, whereas its particle counterpart is $T^*(\mathbb{R}^{1,2})\times\mathcal{L}$. OSp(2|2) [respectively, $\mathrm{SU}(1,1)/\mathbb{Z}_2$] leaves invariant the Kähler two-superform Ω (33) on $\mathcal{L}^{1|1}$ [respectively, the Kähler two-form Ω_0 on $\mathcal{L}$]. We introduce the one-form Σ_0, $d\Sigma_0=\Omega_0$, into the action functional [the second term in Eq. (1)]. Σ_0 changes at most by total derivatives under the $\mathrm{SU}(1,1)/\mathbb{Z}_2$ transformations. Let us now rewrite the action functional (20) in the form

$$S=\int\left(mn_a dx^a-i\left[mn_{\alpha\beta}\theta^\alpha d\theta^\beta-s\frac{\overline{zd}z-zd\bar z}{\zeta}\right]\right).$$

It is easy to verify that the term in the square brackets is related to a one-superform Σ such that $d\Sigma=\Omega$. Thus, Σ changes at most by exact contributions under the OSp(2|2) transformations.

It should be emphasized that neither OSp(2|2) nor its nonsupersymmetric analogue $\mathrm{SU}(1,1)/\mathbb{Z}_2$ (the internal Lorentz group) originate as symmetry (super) groups of the corresponding mechanical systems. The true symmetry (super) groups of the models (1) and (20) are the Poincaré group and its $N=1$ superextension, respectively, which exhaust all global invariance transformations giving rise to independent Noether currents. However, the internal Lorentz algebra so(1,2) and its superextension osp(2|2) naturally appear in the Hamilton approach as building blocks of the (super) Poincaré generators. Really, we have seen that the Poincaré generators (10) in $T^*(\mathbb{R}^{1,2})\times\mathcal{L}$ consist of two sectors, one of which is associated with the space-time coordinates and momenta and the second coincides with the so(1,2) generators (11). A similar phenomenon takes place in the superparticle model. It is apparent that on the constrained surface (24) the generators of the Poincaré supergroup become phase-space functions depending on x^a, p_a and OSp(2|2) generators (36). This observation will be of primary importance when quantizing the model in the following section.

In spite of the strong analogy mentioned between the particle and superparticle models, there is an essential difference in realization of the global symmetry groups in the reduced phase space. The action of the Poincaré group is obviously well defined on $T^*(\mathbb{R}^{1,2})\times\mathcal{L}$. At the same time, supersymmetry cannot be globally realized on $T^*(\mathbb{R}^{1,2})\times\mathcal{L}^{1|1}$ and restores only on the surface of the rest constraints (23). Straightforward calculations of (anti) commutation relations of the generators (25), (27), with respect to the Dirac brackets, show that all the brackets (28) remain intact in the strong sense except $\{\mathcal{Q}^I_\alpha,\mathcal{Q}^J_\beta\}^*$ and $\{\mathcal{Z},\mathcal{Q}^I_\alpha\}^*$. The latter can be presented in the manner

$$\{\mathcal{Q}^I_\alpha,\mathcal{Q}^J_\beta\}^*=-2i\delta^{IJ}p_{\alpha\beta}-2\epsilon^{IJ}\epsilon_{\alpha\beta}\mathcal{Z}+(p^2+m^2)c^{(1)IJ}_{\alpha\beta}+((p,n)+m)c^{(2)IJ}_{\alpha\beta},$$
$$\{\mathcal{Z},\mathcal{Q}^I_\alpha\}^*=(p^2+m^2)c^{(1)I}_\alpha+((p,n)+m)c^{(2)I}_\alpha,$$

where $c^{(\cdot)IJ}_{\alpha\beta}$, $c^{(\cdot)I}_\alpha$ are some functions on $T^*(\mathbb{R}^{1,2})\times\mathcal{L}^{1|1}$, whose explicit expressions are rather cumbersome and not important here. Hence the Poincaré superalgebra restores only on the surface of constraints (23). Let us discuss this point in more detail.

Similarly to the constraints structure in the anyon model of Sec. II, Eq. (23) describes two second class and one first class constraints which are equivalent to the pair of first class constraints (12). The latter can be used to evaluate the Casimir functions $C_1=\mathcal{P}^a\mathcal{P}_a$ and $C_2=\mathcal{P}^a\mathcal{J}_a+\frac{1}{8}\mathcal{Q}^{I\alpha}\mathcal{Q}^I_\alpha-\frac{1}{4}\mathcal{Z}\mathcal{K}$

of $N=2$ Poincaré superalgebra, which turns out to conserve identically on the total constraint surface. Then we find that the model describes a superparticle with mass m, superspin s, central charge $\mathcal{Z}=m$, and positive energy $p^0>0$.

Relation $\mathcal{Z}=m$ corresponds to saturating the BPS bound $m\geqslant|\mathcal{Z}|$ for massive multiplets in extended supersymmetry. The specific feature of such a choice is multiplet-shortening through central charges [30]. This is the case $m=|\mathcal{Z}|$ when a massive supermultiplet contains the same number of particles as a massless one. Such massive multiplets are called hypermultiplets [30]. In the case of $N=2, D=3$ Poincaré superalgebra, a massive multiplet (superparticle) of superspin s describes a quartet of particles with spins $(s,s+\frac{1}{2},s+\frac{1}{2},s+1)$ for $m>|\mathcal{Z}|$ and a doublet $(s,s+\frac{1}{2})$ for $m=|\mathcal{L}|$. We conclude that our model describes a massive $N=2$ hypermultiplet of superspin s or, in other words, a supersymmetric doublet of anyons with spins s and $s+\frac{1}{2}$.

Because of the relation $\mathcal{Z}=m$, not all Hamiltonian generators (25) and (27) of the $N=2$ Poincaré superalgebra are functionally independent, when restricted to the total constraint surface (23), (24), but only their $N=1$ subset (25). The rest generators can be expressed as follows:

$$\mathcal{Q}^2_\alpha=-\frac{i}{m}p_\alpha{}^\beta\mathcal{Q}^1_\beta,\quad \mathcal{K}=-\frac{1}{2m}\mathcal{Q}^{1\alpha}\mathcal{Q}^1_\alpha,\quad \mathcal{Z}=-\frac{p^2}{m}=m \tag{38}$$

on the full constraint surface. Moreover, any physical observable proves to be a function of the $N=1$ super Poincaré generators (25) only.

Equation (38) shows that the hidden $N=2$ supersymmetry (22) can be treated as an artifact of the embedding of $N=2$ Poincaré superalgebra into the universal enveloping algebra of the $N=1$ one. The transformations (22) present themselves special linear combinations of the $N=1$ transformations (21) with the coefficients depending on the on-shell conserved quantities.

Concluding this section we consider the reduction to the surface of the rest second class constraints $(p,\xi)=0$, $(p,\bar{\xi})=0$. The reduced phase space is originated from the symplectic two-superform

$$\Lambda=dp_a\wedge dx^a+s\Omega,$$

$$\Omega=\frac{1}{2}\,\frac{\epsilon^{abc}p_a dp_b\wedge dp_c}{(-p^2)^{3/2}}+\frac{im}{s\sqrt{-p^2}}\left(\eta_{ab}-\frac{p_a p_b}{p^2}\right)\times\theta^\alpha(\sigma^a)_{\alpha\beta}dp^b\wedge d\theta^\beta-\frac{im}{s\sqrt{-p^2}}p_{\alpha\beta}d\theta^\alpha\wedge d\theta^\beta. \tag{39}$$

The respective nonvanishing Dirac brackets are

$$\{x^a,x^b\}^{**}=s\frac{\epsilon^{abc}p_c}{(-p^2)^{3/2}}\left(1-\frac{m}{2s}\theta^\alpha\theta_\alpha\right),\quad \{x^a,p_b\}^{**}=\delta^a{}_b,$$

$$\{x^a,\theta^\alpha\}^{**}=-\frac{i}{2}\,\frac{\epsilon^{abc}p_b(\sigma_c)^\alpha{}_\beta\theta^\beta}{p^2}, \tag{40}$$

$$\{\theta^\alpha,\theta^\beta\}^{**}=-\frac{i}{2m}\,\frac{p^{\alpha\beta}}{(-p^2)^{1/2}}.$$

Thus we result in $N=1$ superextension of the minimal anyon model with monopolelike two-form (15). The superparticle dynamics on the reduced phase superspace is subject to mass-shell (12a) only and the Hamiltonian reduces to

$$H=\frac{1}{2}e(\tau)(p^2+m^2), \tag{41}$$

where $e(\tau)$ is a Lagrange multiplier. Because of the complicated nonlinear structure of Dirac brackets (40), it is a nontrivial problem to obtain their Hilbert space operator realization. That is why we choose another course to quantize this model.

IV. QUANTIZATION OF THE SUPERANYON MODEL

The quantization scheme of Sec. II, which was applied to the anyon model with phase space $T^*(\mathbb{R}^{1,2})\times\mathcal{L}$, consists of combining the Dirac canonical quantization for the space-time degrees of freedom with the geometric quantization for the curved inner subspace. The efficiency of such an approach originated from the facts that (i) the phase space is a product of two symplectic spaces; (ii) the algebra of classical physical observables is spanned by functions of the Poincaré generators; (iii) the spin part of the Lorentz generators coincides with Berezin's symbols for generators of the unitary representations $D^{|s|}_\pm$ of $\overline{\mathrm{SU}(1,1)}$. These features have natural generalizations in the supersymmetric case, so the quantization scheme remains powerful too.

We have seen that the superanyon dynamics can be formulated, upon the reduction with respect to the second class constraints (24), on the phase space $T^*(\mathbb{R}^{1,2})\times\mathcal{L}^{1|1}$ which is a product of two symplectic (super) manifolds. Similarly to the nonsupersymmetric case, all the classical observables are functions of the $N=1$ super Poincaré generators (25). On $T^*(\mathbb{R}^{1,2})\times\mathcal{L}^{1|1}$, the generators (25) are constructed in terms of the space-time variables x^a, p^a and osp(2|2) generators (36). The crucial point is that the osp(2|2) generators prove to coincide with Berezin's symbols of generators of an irreducible positive-weight representation of the superalgebra[6] osp(2|2) on superunit disk $\mathcal{L}^{1|1}$ [33–35]. That is why the mentioned quantization procedure is well suited to the superanyon model. Begin this procedure with a brief exercise in the geometric quantization on the superunit disk.

Atypical unitary representations of the superalgebra osp(2|2) can be realized in a $\mathbb{Z}_2$-graded space $\mathcal{O}_s$ of antiholomorphic superfunctions over $\mathcal{L}^{1|1}$ of the form

$$f(\overline{z,\theta})=f_0(\bar{z})+\sqrt{s}\,\bar{\theta}f_1(\bar{z}),\quad s>0, \tag{42}$$

where $f_{0,1}:\ \mathcal{L}\to\mathbb{C}$ are ordinary antiholomorphic functions on the Lobachevsky plane. A function $f\in\mathcal{O}_s$ is said to be even if $f_1(\bar{z})=0$ and odd if $f_0(\bar{z})=0$. The action of Hamiltonian generators (36) in $T^*(\mathbb{R}^{1,2})\times\mathcal{L}^{1|1}$ can be lifted to the unitary representation in $\mathcal{O}_s$ by the use of geometric quantization

[6]Strictly speaking, we deal with so-called atypical representations of osp(2|2) [35].

method [31,32]. The straightforward computations lead to the following expressions for osp(2|2) generators (see [35] for details):

$$\hat{J}_a=-\overline{\xi_a\partial}-\partial\overline{\xi_a}\left(s+\frac{1}{2}\theta\frac{\vec{\partial}}{\partial\theta}\right),\quad \hat{B}=-s+\frac{1}{2}\theta\frac{\vec{\partial}}{\partial\theta},\tag{43}$$

$$\sqrt{2ms}\,\hat{\theta}^\alpha=\frac{1}{2}\theta[\overline{z^\alpha}\partial+2s(\partial\overline{z^\alpha})]-\overline{z^\alpha}\frac{\vec{\partial}}{\partial\theta},$$

$$\sqrt{\frac{2s}{m}}\hat{\pi}_\alpha=\frac{i}{2}\theta[\overline{z_\alpha}\partial+2s(\partial\overline{z_\alpha})]+i\overline{z_\alpha}\frac{\vec{\partial}}{\partial\theta}.$$

The (anti) commutation relations for $\hat{J}_a$, $\hat{B}$, $\hat{\theta}^\alpha$, and $\hat{\pi}_\alpha$ follow from Eqs. (37) by replacing $\{\ ,\ \}^*\rightarrow 1/i[\ ,\ ]_\mp$ (anticommutator for two odd operators and commutator in the rest cases). It is the representation $\mathbb{D}_+^s$ of osp(2|2) of positive weight s, which is realized by operators (43). With respect to the subalgebra su(1,1), the representation is decomposed into a sum of two irreducible unitary representations of discrete series $\mathbb{D}_+^s=D_+^s\oplus D_+^{s+1/2}$. The even (odd) component of $f\in\mathcal{O}_s$ transforms by representation $D_+^s(D_+^{s+1/2})$.

The geometric quantization method on $\mathcal{L}^{1|1}$ implies that the representation space is equipped with the Hermitian two-form

$$\langle f|g\rangle_{\mathcal{L}^{1|1}}^s=\int_{\mathcal{L}^{1|1}}d\mu(z,\overline{z},\theta,\overline{\theta})e^{-s\Phi(z,\overline{z},\theta,\overline{\theta})}\overline{f(z,\theta)}g(\overline{z},\overline{\theta}),\tag{44}$$

where $f,g\in\mathcal{O}_s$, $\Phi(z,\overline{z},\theta,\overline{\theta})$ is the Kähler superpotential (35), and $d\mu(z,\overline{z},\theta,\overline{\theta})$ is a Liouville supermeasure on $\mathcal{L}^{1|1}$. Taking into account the definition of the closed two-superform (33), $\Omega\equiv dr^A\Omega_{A\overline{B}}d\overline{r}^B$, $dr^A\equiv(dz,d\theta)$, $d\overline{r}^A\equiv(d\overline{z},d\overline{\theta})$, one can calulate the supermeasure explicitly [33,24]:

$$d\mu(z,\overline{z},\theta,\overline{\theta})=-s\,\mathrm{det}\|\Omega_{A\overline{B}}\|\frac{dzd\overline{z}}{2\pi i}d\theta d\overline{\theta}$$

$$=-2\left(1-\frac{1}{2}\frac{\theta\overline{\theta}}{\zeta}\right)\frac{dzd\overline{z}}{2\pi i}\frac{d\theta d\overline{\theta}}{\zeta}.\tag{45}$$

Accounting for Eqs. (35), (42), and (45), we integrate over the Grassmann variables in Eq. (44). Thus, the Hermitian form turns into

$$\langle f|g\rangle_{\mathcal{L}^{1|1}}^s=\langle f_0|g_0\rangle_{\mathcal{L}}^s+\langle f_1|g_1\rangle_{\mathcal{L}}^{s+1/2},\tag{46}$$

where $\langle\cdot|\cdot\rangle_{\mathcal{L}}^l$ is the inner product for the representation space of D_+^l

$$\langle\varphi|\chi\rangle_{\mathcal{L}}^l=(2l-1)\int_{|z|>1}\frac{dzd\overline{z}}{2\pi i}\zeta^{2l-2}\overline{\varphi(z)}\chi(\overline{z}).$$

It is a matter of direct verification to prove that the generators (43) realize the irreducible unitary representation of osp(2|2).

Now we are in a position to construct the Hilbert space of the superanyon states. The space $\mathcal{H}$ of wave functions chosen in the form

$$F(p,\overline{z},\overline{\theta})=F_0(p,\overline{z})+\sqrt{s}\,\overline{\theta}F_1(p,\overline{z})\tag{47}$$

is naturally $\mathbb{Z}_2$ graded. The operator analogues for the classical observables (25) are defined by

$$\hat{\mathcal{J}}_a=-i\epsilon_{abc}p^b\frac{\partial}{\partial p_c}+\hat{\mathcal{J}}_a,\quad \hat{\mathcal{P}}_a=p_a,\quad \hat{\mathcal{Q}}_\alpha^1=ip_{\alpha\beta}\hat{\theta}^\beta-\hat{\pi}_\alpha.\tag{48}$$

Owing to Eq. (38), the operator extensions for Eq. (27) can be chosen in the manner

$$\hat{\mathcal{Q}}_\alpha^2=-\frac{i}{m}p_{\alpha\beta}\hat{\pi}^\beta-m\hat{\theta}_\alpha,\quad \hat{\mathcal{K}}=1-2\theta\frac{\vec{\partial}}{\partial\theta},\quad \hat{\mathcal{Z}}=m.\tag{49}$$

Now, it is crucial to find the conditions, under which the operators (48) and (49) realize a representation of the $N=2$ Poincaré superalgebra with central charge. Straightforward calculations show that the operators (48) and (49) satisfy almost all algebraic relations (28) but

$$\begin{aligned}[\hat{\mathcal{Q}}_\alpha^I,\hat{\mathcal{Q}}_\beta^J]_+={}&2\delta^{IJ}p_{\alpha\beta}-2im\epsilon^{IJ}\epsilon_{\alpha\beta}\\&-\frac{1}{8ms}(p^2+m^2)\\&\times[4\delta^{IJ}\hat{J}_{\alpha\beta}+i\epsilon^{IJ}\epsilon_{\alpha\beta}(4s-1+\hat{\mathcal{K}})]\\&+\frac{1}{4ms}[4(p,\hat{J})+m(\hat{\mathcal{K}}-4s-1)]\\&\times(\delta^{IJ}p_{\alpha\beta}-im\epsilon^{IJ}\epsilon_{\alpha\beta}).\end{aligned}\tag{50}$$

Hence we conclude that the operators (48) and (49) form the superalgebra provided the wave functions are subject to the equations

$$\begin{aligned}(p^2+m^2)F(p,\overline{z},\overline{\theta})&=0,\\ [4(p,\hat{J})+m\hat{\mathcal{K}}]F(p,\overline{z},\overline{\theta})&=m(4s+1)F(p,\overline{z},\overline{\theta}).\end{aligned}\tag{51}$$

These equations turn out to be *super Poincaré covariant*. Moreover, the solutions of Eq. (51) describe the superanyon doublet with the mass m and the superspin $s>0$. Accounting for (47) the equations (51) are reduced to

$$(p^2+m^2)F_0(p,\overline{z})=0,\ \ (p^2+m^2)F_1(p,z)=0,$$

$$(p,\hat{J}^s)F_0(p,\overline{z})=msF_0(p,\overline{z}),$$

$$(p,\hat{J}^{s+1/2})F_1(p,\overline{z})=m(s+\tfrac{1}{2})F_1(p,\overline{z}),$$

where $\hat{J}^l=-\overline{\xi_a\partial}-l\overline{\partial\xi_a}$, $l=s,\ s+1/2$. Comparing these equations with Eqs. (17) and (19), one observes that the even component of wave function $F(p,\overline{z},\overline{\theta})$ describes the particle with spin s, whereas the odd one describes the particle with spin $s+\frac{1}{2}$.

Finally, the space $\mathcal{H}^{m,s}$ of solutions to Eq. (51) is naturally endowed with unique, modulo normalization, super Poincaré and osp(2|2) invariant inner product. It looks like

$$(F,G)=\int\frac{d\vec{p}}{p^0}\langle F|G\rangle^s_{\mathcal{L}^{1|1}},\quad p^0=\sqrt{\vec{p}^2+m^2}>0, \tag{52}$$

where $\langle F|G\rangle^s_{\mathcal{L}^{1|1}}$ denotes the Hermitian form (46), $p^a\equiv(p^0,\vec{p})$. The generators (48) realize the unitary irreducible representation of the Poincaré superalgebra of mass m and superspin $s>0$ in the space $\mathcal{H}^{m,s}$. The case of $s<0$ can be treated in a similar way using the doublet of representations $D_-^{-s}\oplus D_-^{-s-1/2}$.

It is remarkable that the construction proposed admits another interpretation which is not related directly to geometric quantization. It turns out that the odd operators $\hat{\pi}_\alpha$ (or $\hat{\theta}^\alpha$), defined by Eqs. (43), together with the U(1)-charge $\mathcal{K}$ realize a representation of the deformed Heisenberg algebra (DHA) [28,27]. This follows from the identities

$$[\hat{\pi}_\alpha,\hat{\pi}_\beta]_-=\frac{m}{8s}\epsilon_{\alpha\beta}(1+\nu\hat{\mathcal{K}}),\quad [\hat{\mathcal{K}},\hat{\pi}_\alpha]_+=0,\quad \hat{\mathcal{K}}^2=1, \tag{53}$$

where

$$\nu=4s-1. \tag{54}$$

The operators $a^+=2\sqrt{2s/m}\,\hat{\pi}_1$ and $a=2\sqrt{2s/m}\,\hat{\pi}_0$ are termed creation and annihilation operators, respectively; ν is said to be deformation parameter. For $\nu=0$ (that corresponds to supersemion $s=1/4$ [15]) the operators $\hat{\pi}_\alpha$ describe the usual (undeformed) Heisenberg algebra. In the framework of the DHA $\hat{\mathcal{K}}$ is known as Klein operator.

Now, one can reformulate the quantization in terms of the deformed oscillator representation. The osp(2|2)-representation space $\mathcal{O}_s$ provides us with a realization for the Fock space of the deformed bosonic oscillator, the latter being defined as a linear space spanned by the vectors $|0\rangle$, $|n\rangle=c_n(a^+)^n|0\rangle$, $n=1,2,\ldots$ (c_n is chosen in such a way that $\langle n|n\rangle=1$). The Fock vacuum $|0\rangle$ is defined by

$$a|0\rangle=0,\quad \langle 0|0\rangle=1,\quad \hat{\mathcal{K}}|0\rangle=|0\rangle. \tag{55}$$

Since

$$a^+a|n\rangle=\left(n+\frac{\nu}{2}\left[1+(-1)^{n+1}\right]\right)|n\rangle,$$

the representation is unitary if $\nu>-1(s>0)$. The Klein operator induces the $\mathbb{Z}_2$-graded structure in the Fock space

$$\hat{\mathcal{K}}|n\rangle=(-1)^n|n\rangle. \tag{56}$$

The states $\{|2k\rangle,\ k=0,1,2,\ldots\}$ form an orthonormal basis in the even subspace, while the states $\{|2k+1\rangle,\ k=0,1,2,\ldots\}$ form the same in the odd subspace.

The osp(2|2) generators can be written in terms of the DHA as follows:

$$\hat{J}_a=-\frac{2s}{m}(\sigma_a)_{\alpha\beta}\hat{\pi}^\alpha\hat{\pi}^\beta,\quad \hat{\theta}^\alpha=\frac{i}{m}\hat{\mathcal{K}}\hat{\pi}^\alpha,\quad \hat{B}=-\frac{1}{4}\hat{\mathcal{K}}(1+\nu\hat{\mathcal{K}}). \tag{57}$$

After that the quantization procedure can be performed in the same manner we have already described. Therefore, the superanyon doublet is naturally realized in terms of the Fock space of the deformed bosonic oscillator. For a fixed momentum of the superparticle one can conceive the spin-s states live in the even subspace of the deformed Fock space and the spin- $(s+1/2)$ ones in the odd subspace.

It is worth pointing out that only the generators of supersymmetry $\hat{\mathcal{Q}}^I_\alpha$ mix even and odd quantum states. The generators of the Poincaré algebra map the even (odd) subspace of $\mathcal{H}$ onto itself and this point was used in [27] to realize the fractional spin one-particle states. The physical states $F(p,\overline{z,\theta})\in\mathcal{H}^{m,s}\subset\mathcal{H}$ were postulated to be solutions of the spinor equations

$$(p_{\alpha\beta}\hat{\pi}^\beta+\epsilon m\hat{\pi}_\alpha)F(p,\overline{z,\theta})=0,\quad \epsilon=\pm. \tag{58}$$

One gets $F_1(p,\overline{z})=0$ for the solutions of Eq. (58), while the even component $F_0(p,\overline{z})$ describes the irreducible quantum dynamics of the anyon with mass m and spin $s=\epsilon(1+\nu)/4$. It is the superanyon dynamics which makes use of all the power of the DHA construction.

Sorokin, Tkach, and Volkov [15] showed that in three dimensions the dynamics of (super)particles with (super)spin 1/4,3/4,5/4,... can be naturally described by the use of the usual undeformed oscillator representation ($\nu=0$). As one may see now the deformed Heisenberg algebra provides the description of dynamics of arbitrary fractional (super)spin (super)particles.

V. CONCLUSION

In this paper we have constructed the classical and quantum dynamics of superparticles with arbitrary fractional superspin in $D=1+2$ dimensions. Our consideration was based on the use of $N=1$ supersymmetric action functional (20) which generalizes the anyon mechanical system (1) with the Lobachevsky plane in the role of spin space. Thereby, Eq. (39) constitutes a supersymmeric generalization of the Dirac monopole two-form, which is usually used for introducing consistent couplings of $D=1+2$ particle to unconstrained background fields [17–21]. It is believed that the superextension proposed offers a way to describe $N=1$ superanyon dynamics in the presence of external superfields. Moreover, the model (20) possesses hidden invariance with respect to the $N=2$ Poincaré supergroup with the central charge whose on-shell value saturates the BPS bound and, hence, corresponds to the shortening of $N=2$ massive supermultiplets.

$N=2$ Poincaré supersymmetry is not the only hidden algebraic structure originating in the model. In Hamiltonian approach, the system is characterized by one first class and six second class constraints. By restricting the dynamics to the surface of second class constraints (23) and (24), one results in the formulation on reduced phase space $T^*(\mathbb{R}^{1,2})\times\mathcal{L}^{1|1}$, where the Kähler supermanifold $\mathcal{L}^{1|1}=\mathrm{OSp}(2|2)/\mathrm{U}(1|1)$ is the minimal superextension of the

Lobachevysky space. The supergroup OSp(2|2) is related to the symplectic structure on $\mathcal{L}^{1|1}$ as the group of all superholomorphic canonical transformations on $\mathcal{L}^{1|1}$.

Poincaré supersymmetry and OSp(2|2) are closely related to each other, both at the classical and quantum levels. More precisely, the symplectic two-form (39) $dp_a \wedge dx^a + s\Omega$ on the reduced phase space is invariant under the $N=1$ supersymmetry transformations on the mass-shell $p^2+m^2=0$, while Ω remains unchanged with respect to OSp(2|2). That is why the super Poincaré generators are built of the generators of OSp(2|2) along with the space-time coordinates and momenta.

The structure of the reduced phase space implies a natural technique to quantize the model. It consists of combining the geometric quantization on $\mathcal{L}^{1|1}$ and conventional Dirac quantization on $T^*(\mathbb{R}^{1,2})$. The $N=2$ Poincaré supersymmetry turns out to be consistent provided imposing the quantum equation of motion which single out the physical states of superparticle. Then the massive super Poincaré representation with the superspin $s>0$ and the central charge equal to the mass m is realized on the superfields transforming in the atypical representation of osp(2|2) [35], which splits, with respect to the subalgebra su(1,1) of osp(2|2), into the doublet of discrete series representations $D_+^s \oplus D_+^{s+1/2}$. Hence we obtain a direct superextension of the well studied description of fractional spin states using the representations D_+^s [12–15,22,24].

The space of superparticle states with a fixed momentum is shown to be embedded into the Fock space of the deformed quantum oscillator. The deformation parameter ν is related to the superspin by simple expression $\nu=4s-1$ $(s>0)$. This result generalizes some known constructions for anyons [27] and (super) semions [15].

We have studied the case of $N=1$ supersymmetric dynamics of anyons. It would be of interest to extend the above consideration to the case of N-extended Poincaré supersymmetry. Here it is crucial to find an adequate analogue of the spin phase space $\mathcal{L}^{1|1}$. We hope to present respective constructions elsewhere.

Note added. While this paper was being completed, we received a paper [40] in which the relationship between DHA and anyon wave equations is also discussed.

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APPENDIX A: CONVENTIONS

We define $D=1+2$ Minkowski metric η_{ab} and Levi-Civita tensor ϵ_{abc} as follows: $\eta_{ab}=\mathrm{diag}(-,+,+)$ and $\epsilon_{012}=-\epsilon^{012}=1$. Latin letters are used to denote vector indices and Greek letters for spinor ones. Due to the well-known isormorphism $\mathrm{SO}^{\uparrow}(1,2)\cong \mathrm{SU}(1,1)/\mathbb{Z}_2$, the fundamental spinor representation and its conjugate are defined by the transformation laws $\psi_\alpha \to N_\alpha{}^\beta \psi_\beta$, where $\alpha,\beta=0,1$, and $\overline{\psi}_{\dot\alpha}\equiv\overline{(\psi_\alpha)}\to \overline{N}_{\dot\alpha}{}^{\dot\beta}\overline{\psi}_{\dot\beta}$, respectively. Here $N\in \mathrm{SU}(1,1)$ and $\overline{N}$ its complex conjugate

$$\|N_\alpha{}^\beta\| = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}, \quad |a|^2-|b|^2=1. \tag{A1}$$

The spinor representations are equivalent, since SU(1,1) possesses not only invariant spinor antisymmetric metric $\epsilon_{\alpha\beta}=-\epsilon_{\beta\alpha}=-\epsilon^{\alpha\beta}$ $(\epsilon_{01}=1)$ and its conjugate, which are used for raising and lowering spinor indices by the rule $\psi_\alpha=\epsilon_{\alpha\beta}\psi^\beta$, $\psi^\alpha=\epsilon^{\alpha\beta}\psi_\beta$, but also the invariant tensor with mixed indices

$$g_{\alpha\dot\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A2}$$

that allows to convert dotted spinor indices into undotted ones in the manner $\overline{\psi}_\alpha = g_\alpha{}^{\dot\alpha}\overline{\psi}_{\dot\alpha}$, $\overline{\psi}_{\dot\alpha}=g_{\dot\alpha}{}^\alpha\overline{\psi}_\alpha$, where $g_\alpha{}^{\dot\alpha}=\epsilon^{\dot\alpha\dot\beta}g_{\alpha\dot\beta}$ and $g_{\dot\alpha}{}^\alpha=\epsilon^{\alpha\beta}g_{\beta\dot\alpha}$. This makes it possible to use undotted spinors only.

Spinors may be subject to a covariant reality condition of the form

$$\overline{\psi_\alpha}=\Delta\psi_\alpha \Leftrightarrow \overline{\psi^\alpha}=-\Delta\psi^\alpha, \quad |\Delta|=1, \tag{A3}$$

for some parameter Δ. We choose $\Delta=1$ for the odd coordinates θ^α of $N=1, D=3$ superspace.

The Dirac matrices are chosen in the form

$$(\sigma_0)_{\alpha\beta}=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\sigma_1)_{\alpha\beta}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\sigma_2)_{\alpha\beta}=\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \tag{A4}$$

$$(\sigma_a)_{\alpha\gamma}(\sigma_b)^\gamma{}_\beta = i\epsilon_{abc}(\sigma^c)_{\alpha\beta}-\eta_{ab}\epsilon_{\alpha\beta},$$

such that the matrices $(\sigma_a)_{\alpha\dot\alpha}=g_{\dot\alpha}{}^\beta(\sigma_a)_{\alpha\beta}$ are Hermitian. The double-sheeted covering map π: $\mathrm{SU}(1,1)\to \mathrm{SO}^{\uparrow}(1,2)$ mentioned is constructed with the help of the σ matrices by associating with an element $N=\|N_\alpha{}^\beta\|\in \mathrm{SU}(1,1)$ its image $\Lambda(N)=\|\Lambda^a{}_b\|\in \mathrm{SO}^{\uparrow}(1,2)$, in the connected component of the identity of the Lorentz group, defined by

$$\Lambda^b{}_a(\sigma_b)_{\alpha\dot\alpha}=N_\alpha{}^\beta \overline{N_{\dot\alpha}}{}^{\dot\beta}(\sigma_a)_{\beta\dot\beta}. \tag{A5}$$

We follow Berezin's conventions for superforms [31]. The Grassmann parity $\epsilon(\Omega)$ in a superalgebra of exterior superforms is defined by requiring that (i) the Grassmann parity of an even (odd) 0-form is equal to 0 (1); (ii) the Grassmann parity of exterior differential is equal to 1, $\epsilon(d\Omega)=\epsilon(\Omega)+1$. Thus, if r^A are coordinates on a supermanifold of parity ϵ_A, then $r^Ar^B=(-1)^{\epsilon_A\epsilon_B}r^Br^A$, $dr^Ar^B=(-1)^{\epsilon_B(\epsilon_A+1)}r^Bdr^A$, $dr^Adr^B=(-1)^{(\epsilon_A+1)(\epsilon_B+1)}dr^Bdr^A$. Finally, the Leibniz rule looks like $d(\Omega_1\Omega_2)=d(\Omega_1)\Omega_2+(-1)^{\epsilon(\Omega_1)}\Omega_1 d\Omega_2$.

APPENDIX B: LOBACHEVSKY PLANE AS A HOMOGENEOUS SPACE

Here we describe a "manifestly Lorentz-covariant" realization of Lobachevsky plane $\mathcal{L}=\mathrm{SU}(1,1)/\mathrm{U}(1)$ as a homogeneous space of $\mathrm{SO}^{\uparrow}(1,2)$. This realization is used throughout the paper. $\mathcal{L}$ is identified with a unit open disc in a complex plane, $\mathcal{L}\cong\{z\in\mathbb{C},|z|<1\}$. The proper orthochronous Lorentz group $\mathrm{SO}^{\uparrow}(1,2)\cong \mathrm{SU}(1,1)/\mathbb{Z}_2$ acts on $\mathcal{L}$ by fractional linear transformations

$$N:\ z \to z' = \frac{az-b}{\bar{a}-\bar{b}z}, \quad N \in \mathrm{SU}(1,1). \tag{B1}$$

One can rewrite Eq. (B1) in a manifestly covariant form by introducing the two-component twistorlike objects

$$z^\alpha \equiv (1,z), \quad \overline{z^\alpha} \equiv (\bar{z},1) \tag{B2}$$

transforming by the law

$$N:\ z^\alpha \to z^{\alpha\prime} = \left(\frac{\partial z'}{\partial z}\right)^{1/2} N^{-1}{}_\beta{}^\alpha, \qquad \overline{z_\alpha} \to \overline{z^{\alpha}}{}' = \left(\frac{\partial \overline{z'}}{\partial \bar{z}}\right)^{1/2} N^{-1}{}_\beta{}^\alpha \overline{z^\beta}, \tag{B3}$$

or, in infinitesimal form,

$$\delta z = \frac{i}{2}\omega_{\alpha\beta} z^\alpha z^\beta, \quad \delta\bar{z} = -\frac{i}{2}\omega_{\alpha\beta}\overline{z^\alpha z^\beta}, \tag{B4}$$

where $\omega_{\alpha\beta} \equiv (\omega^a \sigma_a)_{\alpha\beta}$ are the parameters of infinitesimal Lorentz transformations. As it is seen, each of z^α and $\overline{z^\alpha}$ transforms simultaneously as a $D=3$ Lorentz spinor and a tensor field on $\mathcal{L}$. Using z^α and $\overline{z^\alpha}$ we may construct the following vector densities

$$\zeta_a \equiv -(\sigma_a)_{\alpha\beta} z^\alpha \overline{z^\beta} = -(1+z\bar{z}, z+\bar{z}, i(z-\bar{z})), \tag{B5}$$

$$\xi_a \equiv -\frac{1}{2}(\sigma_a)_{\alpha\beta} z^\alpha z^\beta = -\frac{1}{2}(2z, 1+z^2, i(z^2-1)), \quad \overline{\xi_a} \equiv \overline{(\xi_a)}, \tag{B6}$$

and the scalar density

$$\zeta \equiv \epsilon_{\alpha\beta} z^\alpha \overline{z^\beta} = 1 - z\bar{z}, \quad \zeta^a \zeta_a = -2\xi^a \overline{\xi_a} = -\zeta^2 \tag{B7}$$

as well. The identity

$$4\frac{\xi_a \overline{\xi_b}}{\zeta^2} \equiv i\epsilon_{abc} n^c + n_a n_b + \eta_{ab}, \quad n_a \equiv \frac{\zeta_a}{\zeta} \tag{B8}$$

is useful in practice. The chief advantage of the technique described consists in the fact that z^α and $\overline{z^\alpha}$ are the only independent tensorlike fields associated with the homogeneous space structure on $\mathcal{L}$. Our treatment here follows Ref. [39] where objects like z^α were introduced on two-sphere S^2.

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