Infrared finite electron propagator

Emili Bagan*
Physics Department, Bldg. 510A, Brookhaven National Laboratory, Upton, New York 11973

Martin Lavelle†
Grup de Física Teòrica and IFAE, Edificio Cn, Universitat Autònoma de Barcelona, E-08193 Bellaterra (Barcelona), Spain

David McMullan‡
School of Mathematics and Statistics, University of Plymouth, Drake Circus, Plymouth, Devon PL4 8AA, United Kingdom

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We investigate the properties of a dressed electron which reduces, in a particular class of gauges, to the usual fermion. A one-loop calculation of the propagator is presented. We show explicitly that an infrared finite, multiplicative, mass shell renormalization is possible for this dressed electron, or, equivalently, for the usual fermion in the above-mentioned gauges. The results are in complete accord with previous conjectures. [S0556-2821(97)01918-8]

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I. INTRODUCTION

A fundamental question in gauge theories is the following: What is the correct description of an asymptotic field? In an Abelian theory this problem takes on its most pristine form, and the obstacle to adopting the naive in-out identification of asymptotically free fields is clearly identified with the infrared divergences associated with the masslessness of the gauge fields. As such, we will restrict ourselves in this paper to quantum electrodynamics (QED), i.e., a nonconfining, Abelian gauge theory where the gauge symmetry is unbroken. After presenting and heuristically motivating an ansatz for a charged particle (henceforth an ‘‘electron’’) in QED, this description will be put to a highly nontrivial test: we will calculate the one-loop propagator and show that it is infrared finite in a suitable (and previously predicted) mass-shell renormalization scheme. Another interpretation of this result is that we have found a new class of gauges, parametrized by a vector **v**, where the usual fermion propagator is infrared finite.

Mass-shell renormalization of the electron propagator is hindered in most gauges by the appearance of infrared divergences (see, e.g., p. 410 of Ref. [1]) although the position of the pole is itself gauge independent [2,3]. It is well known that these infrared problems are a consequence of the difficulties in defining the physical asymptotic fields correctly. In the confining theory of quantum chromodynamics (QCD) this is self-evidently a highly nontrivial problem, but even in our paradigm theory, perturbative QED, no satisfactory answer has yet been given to this question. It is understood that the masslessness of the photon means that the electromagnetic interaction falls off too slowly for us to just ignore it and replace the physical electron by a bare fermion. The coherent state technique [4], where one adds soft photons, has been developed to deal with these divergences. For a summary of the usual approaches we refer to Supplement 4 of Ref. [5]. Despite this understanding of the root of the infrared problem, it does not seem that a full description of charged states in gauge theories exists. The coherent state approach has not, for example, been carried through for the strong interaction. However, even for QED, previous work on dressing electrons seems somewhat *ad hoc* and prescriptive in nature. In what follows we will stress the systematic and predictive nature of the approach we advocate.

There are certain general properties to be found in any description of an electron: it must be nonlocal [6–8] and it must be noncovariant [8–10]. Both these things follow from the gauge symmetry of QED. Nonlocality can be simply shown to follow from demanding that Gauss' law holds on a physical, gauge-invariant state, a more rigorous proof is contained in Ref. [6]. The noncovariance of such a description is a result of the difficulties in reconciling Lorentz and gauge symmetries in the charged sector (see Sec. 8 of Ref. [8]). At the naivest level these requirements amount to the need to dress a charge with an electromagnetic ‘‘cloud,’’ whose exact form depends upon the position and velocity of the charge. The neglect of such a dressing when one uses a bare fermion as an asymptotic field is equivalent to switching off the coupling which is clearly unphysical and this in fact underlies the infrared problem.

Although these divergences may, however, be, essentially, ignored in calculations of scattering processes in QED it is clear that a better understanding of their origins and of how to describe physical charged states is highly desirable. An understanding of bound states cannot come from switching off the coupling, even asymptotically, and insight into how to dress the constituent charges of, e.g., positronium would, we feel, be of great practical value. Furthermore, in QCD, which is worse affected by such infrared problems, the asymptotic region is really the short-distance regime [11] and so an understanding of the dressings associated with color charges will yield valuable information about the glu-

*Current and permanent address: IFAE, Universitat Autònoma de Barcelona. Electronic address: iftebag@cc.uab.es
†After 1st October at Univ. of Plymouth.
‡Electronic address: d.mcmullan@plymouth.ac.uk

ons and sea quarks in hadrons—our present lack of understanding of this structure being revealed most glaringly in the so-called proton spin crisis (see, e.g., Ref. [12]). We remark that dressings underlie Cornwall's pinch technique [13] and also recall here the long-suggested connection between the severe infrared divergences of QCD and the confinement phenomenon [14–16]. Further attempts to construct gauge-invariant descriptions of quarks and gluons may be found in Refs. [17–19].

How should we now dress our electron? We expect to surround the charge with a cloud and, since the dressed particle should correspond to a physical state, we expect our expression to be gauge invariant. Many years ago Dirac presented such a formula [20]:

$$\psi_f(x)=\exp\left(-ie\int d^4z f^\mu(x-z)A_\mu(z)\right)\psi(x), \qquad (1)$$

where f_μ is a field-independent function obeying

$$\partial_\mu f^\mu(w)=\delta^{(4)}(w), \qquad (2)$$

and we note that the sign of e used in this paper is the opposite to that of Bjorken and Drell. It may be straightforwardly seen that this is gauge invariant. It is also visibly non-local and, depending upon the choice of f^μ, can be non-covariant. Several authors have employed this formula (see, e.g., Refs. [21–25]) to study the construction of physical states. The stability of such dressings around static charges in QED was considered in Ref. [26].

We now note that there is a gauge in which the argument of the exponential in (1) vanishes: $f^\mu(x)A_\mu(x)=0$, we will call such gauges "dressing gauges." This connection between a specific type of gauge fixing and the dressing for a charged state is quite general and explained in more detail in Ref. [8]. This simple observation, however, has an important consequence for us: we expect that if one dresses the charge correctly no infrared problem will arise. We now see that working in the dressing gauge should also permit an infrared finite mass-shell renormalization if the dressing is a physical one. In the light of the known general structures associated with any construction of an electron, we still need to make the form of f_μ, and hence the particular dressing gauge, precise.

Our first restriction is to limit the form of the nonlocality of the cloud. In Ref. [8] it was argued that one must avoid nonlocality in time otherwise there would be no natural prescription for the identification of asymptotic fields for the far distant past and future. One can, in principal, have a dressing that is local in time outside some bounded interval of time. However, for the class of dressings we are interested in here, we restrict the dressing to a particular time slice, i.e., we assume that $f_0=0$. This specification notwithstanding, we still have a great deal of freedom in our choice of the three f_i components.

The next step is to recall that Dirac (see Ref. [20] and Sec. 80 of Ref. [27]) suggested using the following form for the f_i:

$$\psi_c(x)=\exp\left(-ie\,\frac{\partial_i A_i}{\nabla^2}(x)\right)\psi(x), \qquad (3)$$

where the action of ∇^{-2} is understood as

$$\frac{1}{\nabla^2}g(x_0,\mathbf{x})=-\frac{1}{4\pi}\int d^3z\,\frac{g(x_0,\mathbf{z})}{|\mathbf{x}-\mathbf{z}|}. \qquad (4)$$

It is clear that this is a special case of Eq. (1) and is hence gauge invariant. The dressing gauge here is the familiar Coulomb gauge. The appealing feature of this choice of dressing is that the commutators of the electric and magnetic fields with Eq. (3) yield just the electric and magnetic fields we expect of a static charge. Using the canonical equal-time commutator, $[E_i(\mathbf{x}),A_j(\mathbf{y})]=i\delta_{ij}\delta(\mathbf{x}-\mathbf{y})$, one finds, for example, that taking an eigenstate $|\epsilon\rangle$ of the electric field operator, with eigenvalue ϵ_i, and adding a dressed fermion (3) to the system then

$$E_i(\mathbf{x})\psi_c(\mathbf{y})|\epsilon\rangle=\left(\epsilon_i(\mathbf{x})-\frac{e}{4\pi}\frac{\mathbf{x}_i-\mathbf{y}_i}{|\mathbf{x}-\mathbf{y}|^3}\right)\psi_c(\mathbf{y})|\epsilon\rangle, \qquad (5)$$

This means that it is natural to interpret this dressed, gauge-invariant fermion as describing a static charge.

It might be now be argued that this last argument, based as it is on the free-field canonical commutation relations and hence completely ignoring renormalization, may not hold in the full theory. For this reason two of us recently [8] considered the one-loop propagator of the dressed charge (3) in a general covariant gauge and in Coulomb gauge.[1] The results demonstrated that a multiplicative, infrared finite, mass-shell renormalization of the propagator was possible. It was, however, only possible at the static mass-shell point, $p=(m,0,0,0)$—which is of course in complete accord with the above interpretation of this dressing.

Although this result is highly attractive and sheds new light on the infrared finiteness of the Coulomb gauge, it covers in some sense only "one point" in a space of dressings. In Ref. [8] a gauge-invariant description of a dressed charge moving with some constant velocity, which reduces in the static limit to Eq. (3), was presented (see Sec. II below for the specific form of this dressing). It was there conjectured that the propagator of this dressed electron would be infrared finite if the correct (moving) renormalization point on the mass shell was used. In a recent work [29] we demonstrated that, in the small velocity limit, a multiplicative renormalization of this ansatz was possible. No new infrared divergences arose, but it was clear that this could be the case when terms of order $\mathbf{v}^2$ were retained in the dressing. In this paper we will consider the dressed propagator for an arbitrary velocity and verify the conjectures of Ref. [8]. The usual electron propagator will, in other words, be shown to be infrared finite in a class of gauges depending upon a free parameter (the three-vector, $\mathbf{v}$).

After this introduction, the rest of this paper is structured as follows. In Sec. II we discuss the exact form of the dressing we use and the equivalent (dressing) gauge. We also

[1]It may appear that the above description is local in Coulomb gauge, recall, however, that in that gauge we must use Dirac brackets and the bracket between the fermion and the electric field is nonlocal. See Refs. [28,8] for details.

describe the renormalization of the fermion propagator in different gauges in QED. Section III, the heart of the paper, is devoted to the explicit regularization and renormalization of the propagator. Here we obtain the promised result that an infrared finite mass-shell renormalization is possible. In Sec. IV a discussion of our results is presented. The Appendix devoted to the integrals we have required concludes this work.

II. THE DRESSING, THE GAUGE, AND THE SELF-ENERGY

The dressed electron which we will work with in this paper has the form [8,29]

$$\psi_v=\exp\left(\frac{ie}{4\pi}\,\gamma\int d^3z\,\frac{\gamma^{-2}\partial_1A_1(x^0,\mathbf{z})+\partial_2A_2(x^0,\mathbf{z})+\partial_3A_3(x^0,\mathbf{z})-v^1E_1(x^0,\mathbf{z})}{[(x_1-z_1)^2\gamma^2+(x_2-z_2)^2+(x_3-z_3)^2]^{1/2}}\right)\psi(x), \tag{6}$$

where $\gamma=1/\sqrt{1-\mathbf{v}^2}$ and $\mathbf{v}=(v^1,0,0)$. We propose it for the following reasons: it is gauge invariant and its commutators with the electric and magnetic fields are such that

$$\mathbf{E}(x)=-\frac{e}{4\pi}\,\gamma\,\frac{\mathbf{x}-\mathbf{y}}{[(x_1-y_1)^2\gamma^2+(x_2-y_2)^2+(x_3-y_3)^2]^{3/2}}, \tag{7}$$

and

$$\mathbf{B}(x)=\mathbf{v}\times\mathbf{E}(x), \tag{8}$$

which one may recognize as the correct electric and magnetic fields for a charge moving with constant velocity, $\mathbf{v}$, along the x_1 axis (see, e.g., Chap. 19 of Ref. [30]). This expression is analogous to Eq. (3) and indeed reduces to it for $\mathbf{v}\to 0$. In the nonrelativistic case this dressed electron reduces to

$$\psi_v(x)=\exp\left(-ie\,\frac{\partial_jA_j+v_iE_i}{\nabla^2}\right)\psi(x). \tag{9}$$

The renormalization of the propagator of this field at order e^2 and first order in $\mathbf{v}$ is to be found in Ref. [29]. Before computing the propagator of ψ_v, we will now briefly discuss its complex relation with that of the static dressed electron, ψ_c.

It is important to first note that the form of the dressing appropriate to the moving electron (6) does not follow from a naive boost to the dressing for the static electron (3). This is a concrete manifestation of the fact [8–10] that Lorentz transformations cannot be implemented unitarily on charged fields. As such, it is not possible to argue that the good infrared properties found in the static case can be simply boosted up to the moving dressing. Given the surprising nature of this fact, it is helpful to show how such a boost must act on such a charged field and hence make clear why it is not now a unitary mapping.

We recall that as a four-vector, the potential $A_\mu(x)$ transforms under a Lorentz transformation $x\to x'=\Lambda x$ as $A'_\mu(x)=UA_\mu(x)U^{-1}$ where U is the appropriate unitary operator and $A'_\mu(x)=\Lambda^\nu_\mu A_\nu(x')$. Under a boost with velocity, $\mathbf{v}$, in the x^1 direction we find that the dressing gauge appropriate to the static charge becomes

$$\partial_iA_i(x)\to\gamma^2(\gamma^{-2}\partial_1A_1+\partial_2A_2+\partial_3A_3-v^1E_1)(x')+\mathbf{v}^2\gamma^2(\partial_1A_1-\partial_0A_0-\partial_2A_2-\partial_3A_3)(x'). \tag{10}$$

From Eq. (6) we see that the first term in this expression is the dressing gauge for the moving charge and so the second term here obstructs the identification of the dressing gauge that we need for our nonstatic charge. Since we know that we can construct the dressing directly from the gauge, this exemplifies the fact that on charged states the Lorentz transformations are not implemented by the unitary mapping, U. However, as argued in Refs. [31,8], a gauge-covariant implementation of the Lorentz transformations can be constructed by combining the above unitary transformation with a field-dependent gauge transformation. Thus, to transform the static dressing to the boosted one we take $A_\mu(x)\to\widetilde{A}_\mu(x)$, where

$$\widetilde{A}_\mu(x)=A'_\mu(x)+\partial_\mu\Theta(x), \tag{11}$$

and

$$\Theta(x)=\frac{\mathbf{v}^2\gamma^2}{4\pi}\int d^3z\,\frac{(\partial_1A_1-\partial_0A_0-\partial_2A_2-\partial_3A_3)(x'_0,\mathbf{z}')}{|\mathbf{x}-\mathbf{z}|}, \tag{12}$$

where the point $(x'_0,\mathbf{z}')$ in the integrand is the boost applied to $(x_0,\mathbf{z})$.

Having constructed the dressing gauge, and hence the dressing for a moving charge, we now need to address the quantum field theoretic aspects of this approach. Given the obvious importance of gauge invariance to us we will work in a gauge-invariant regularization scheme, viz. dimensional regularization. In consequence we may drop tadpoles, and we will do this consistently below. As a result we can reexpress the dressed fermion as

$$\psi_v=\left(1-\frac{ie}{4\pi}\,\gamma\int d^3z\,\frac{\gamma^{-2}\partial_1A_1(x^0,\mathbf{z})+\partial_2A_2(x^0,\mathbf{z})+\partial_3A_3(x^0,\mathbf{z})-v^1E_1(x^0,\mathbf{z})}{[(x_1-z_1)^2\gamma^2+(x_2-z_2)^2+(x_3-z_3)^2]^{1/2}}\right)\psi(x), \tag{13}$$

since the e^2 terms we so neglect will just yield tadpoles in the one-loop calculation at hand. This last equation can be rewritten as

$$\psi_v(x)=\left\{1-ie\,\frac{\gamma^{-2}\partial_1 A_1+\partial_2 A_2+\partial_3 A_3+v^1[\partial_0 A_1-\partial_1 A_0]}{\gamma^{-2}\partial_1^2+\partial_2^2+\partial_3^2}+O(e^2)\right\}\psi(x), \tag{14}$$

where we have employed the standard identity

$$\left(\frac{\partial^2}{\partial\xi_1^2}+\frac{\partial^2}{\partial\xi_2^2}+\frac{\partial^2}{\partial\xi_3^2}\right)\left(-\frac{1}{4\pi}\right)\frac{1}{\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}}=\delta(\xi_1)\,\delta(\xi_2)\,\delta(\xi_3), \tag{15}$$

which under the change of variables, $\xi_i\rightarrow\gamma x_i$, can be rewritten as

$$\left(\frac{1}{\gamma^2}\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)\left(-\frac{1}{4\pi}\right)\times\frac{\gamma}{\sqrt{\gamma^2 x_1^2+x_2^2+x_3^2}}=\delta(x_1)\,\delta(x_2)\,\delta(x_3), \tag{16}$$

from which Eq. (14) follows. It proved in practice convenient to further reexpress Eq. (14) in a more covariant looking fashion as

$$\psi_v(x)=\left\{1+ie\,\frac{\mathcal{G}^\mu A_\mu(x)}{\partial^2-(\eta\cdot\partial)^2+(v\cdot\partial)^2}+O(e^2)\right\}\psi(x), \tag{17}$$

where

$$\mathcal{G}_\mu=[(\eta+v)_\mu(\eta-v)_\nu-g_{\mu\nu}]\partial^\nu, \tag{18}$$

and we have introduced the vectors, $\eta^\mu=(1,0,0,0)$ and $v^\mu=(0,v^1,0,0)\equiv(0,\mathbf{v})$ from which the relations $v\cdot\eta=0$, $\eta^2=1$ and $v^2=-\mathbf{v}^2$ follow immediately. We stress that v is *not* the four-velocity, $u^\mu=\gamma(1,\mathbf{v})=\gamma(\eta+v)^\mu$.

We may calculate the gauge-invariant, one-loop propagator of $\psi_v(x)$ in one of two ways. One may either work in an arbitrary Lorentz gauge or one may perform the calculation in the dressing gauge. For an arbitrary v^1 the dressing gauge is now

$$\gamma^{-2}\partial_1 A_1+\partial_2 A_2+\partial_3 A_3+v^1[\partial_0 A_1-\partial_1 A_0]=\mathcal{G}^\mu A_\mu=0, \tag{19}$$

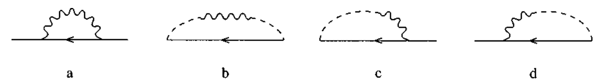
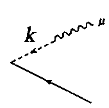


FIG. 1. The diagrams which yield the one-loop dressed propagator. In the appropriate dressing gauge only (a) contributes. In a general gauge all the diagrams must be evaluated. The dashed lines indicate the projection of the photon propagator from the ($\mathbf{v}$-dependent) vertices in the dressing (see the above Feynman rules).

and the free-photon propagator in this gauge has the form

$$D^v_{\mu\nu}=\frac{1}{k^2}\left\{-g_{\mu\nu}+\frac{(1-\xi)k^2-[k\cdot(\eta-v)]^2\gamma^{-2}}{[k^2-(k\cdot\eta)^2+(k\cdot v)^2]^2}k_\mu k_\nu-\frac{k\cdot(\eta-v)}{k^2-(k\cdot\eta)^2+(k\cdot v)^2}[k_\mu(\eta+v)_\nu+(\eta+v)_\mu k_\nu]\right\}, \tag{20}$$

where ξ is a gauge parameter which we set to zero in what follows; this ensures, $\mathcal{G}^\mu D_{\mu\nu}=0$. Even then this is really a class of gauges parametrized by $\mathbf{v}$, which flows into the Coulomb gauge for $\mathbf{v}\rightarrow 0$. We are not aware of any previous work with such gauges. The Feynman rule for the extra vertex from the dressing is



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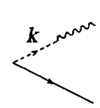

$$=-ek^\rho\,\frac{(\eta+v)_\mu(\eta-v)_\rho-g_{\mu\rho}}{k^2-(k\cdot\eta)^2+(k\cdot v)^2}.$$

With these rules we can calculate the dressed propagator in an arbitrary gauge.

At order e^2 we have as well as the usual interaction vertex, contributions from the expansion in the coupling of the dressing. These effects mean that, even if we work in a covariant gauge, the integrand of the sum of all the Feynman diagrams is noncovariant. We have checked explicitly that, after discarding tadpoles, the same total integrand is found in both an arbitrary Lorentz gauge (i.e., it is independent of the Lorentz gauge parameter) and in the dressing gauge, Eq. (19). In a general gauge one has to take all of the diagrams of Fig. 1 into account, while in the gauge (19) only Fig. 1(a) appears.

The result for the self-energy is (in $D=2\omega$ dimensions)

$$-i\Sigma=e^2\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\left\{\frac{1}{k^2}\frac{1}{(p-k)^2-m^2}[2(\omega-1)\not{p}-2\omega m-2(\omega-1)\not{k}]+\frac{1}{k^2}\frac{1}{(p-k)^2-m^2}[-2(\not{p}-m)]\right.$$
$$+\frac{1}{(p-k)^2-m^2}\frac{1}{k^2-(k\cdot\eta)^2+(k\cdot v)^2}[2(\not{p}-m)+(\not{\eta}+\not{v})k\cdot(\eta-v)]+\frac{1}{(p-k)^2-m^2}\frac{1}{[k^2-(k\cdot\eta)^2+(k\cdot v)^2]^2}$$
$$\times(\not{p}-m)[\gamma^{-2}(k\cdot\eta-k\cdot v)^2-k^2]+\frac{1}{k^2}\frac{1}{(p-k)^2-m^2}\frac{1}{k^2-(k\cdot\eta)^2+(k\cdot v)^2}[-(p^2-m^2)(\not{\eta}+\not{v})k\cdot(\eta-v)$$
$$\left.-2\not{k}k\cdot(\eta-v)p\cdot(\eta+v)]+\frac{1}{k^2}\frac{1}{(p-k)^2-m^2}\frac{1}{[k^2-(k\cdot\eta)^2+(k\cdot v)^2]^2}(p^2-m^2)\not{k}[k^2-\gamma^{-2}(k\cdot\eta-k\cdot v)^2]\right\}. \tag{21}$$

Actually this is the self-energy in the dressing gauge. In covariant gauges we must include all the diagrams of Fig. 1 and so it is more natural there to consider the whole propagator. For simplicity we will use the self-energy henceforth. The detailed renormalization of this will be presented in the next section.

III. DIVERGENCES AND RENORMALIZATION

In this section we will first recall some facts about the mass-shell renormalization of the usual fermion propagator and set up our conventions. We will then give the results of our calculations for the renormalization constants.

A. Setting things up

To renormalize the electron propagator one requires two different renormalizations: a mass shift ($m\to m-\delta m$) and a fermion wave-function renormalization. The first of these is known to be gauge independent and in noncovariant gauges, such as Coulomb gauge, it is independent of the exact choice of mass shell point (i.e., it is the same for all choices of p_0 and $\mathbf{p}$ which are on shell) [8]. Based upon our experience with the renormalization of the dressed electron (9), where we only retained terms of first order in $\mathbf{v}$, we will use the following multiplicative, matrix renormalization for the fermion:

$$\psi\to\sqrt{Z_2}\,\exp\left\{-i\,\frac{Z'}{Z_2}\,\sigma_{\mu\nu}\eta^\mu v^\nu\right\}\psi, \tag{22}$$

which is reminiscent of a naive Lorentz boost upon a fermion. At lowest order we can recast this as

$$\psi\to\left(\sqrt{Z_2}\mathbf{I}+\frac{Z'}{\sqrt{Z_2}}\,\not{\eta}\not{v}\right)\psi. \tag{23}$$

In the small $\mathbf{v}$ limit such a multiplicative renormalization was found to be possible [29]. These relations define our three renormalization constants. The counterterms in the self-energy can thus be seen to be (with $Z_2=1+\delta Z_2$)

$$-i\Sigma^{\text{counter}}=\delta Z_2(\not{p}-m)+2iZ'(p\cdot\eta\not{v}-p\cdot v\not{\eta})+i\,\delta m. \tag{24}$$

For a multiplicative renormalization to be possible, the ultraviolet divergences have to also have this form for arbitrary values of p^2, $p\cdot\eta$, $p\cdot v$, and v^2. We find the following such ultraviolet divergences (see the Appendix for a discussion of how to perform the integrations):

$$-i\Sigma^{\text{UV}}=i\,\frac{\alpha}{4\pi}\,\frac{1}{2-\omega}\left\{-3m+(\not{p}-m)[-3-2\chi(\mathbf{v})]+2(p\cdot v\,\not{\eta}-p\cdot\eta\not{v})\left[\frac{1}{\mathbf{v}^2}+\frac{1+\mathbf{v}^2}{2\mathbf{v}^2}\,\chi(\mathbf{v})\right]\right\}, \tag{25}$$

where $\alpha=(m^2)^{\omega-2}(e^2/4\pi)$ and we have introduced the definition

$$\chi(\mathbf{v})=\frac{1}{|\mathbf{v}|}\ln\frac{1-|\mathbf{v}|}{1+|\mathbf{v}|}. \tag{26}$$

This displays the need for our matrix multiplication renormalization. We note that the UV divergences are *local* in the external momentum, but *nonlocal* in the velocity v.

It is clear that after performing the integrals in Eq. (21) the (renormalized) self-energy including loops and counterterms will have the general form

$$-i\Sigma=m\alpha+\not{p}\beta+p\cdot\eta\not{\eta}\delta+m\not{v}\epsilon, \tag{27}$$

where $\alpha,...,\epsilon$ are functions depending upon p^2, $p\cdot\eta$, $p\cdot v$, and v^2. Our choice of renormalization scheme is to insist that the on-shell form of the renormalized propagator is just the tree-level one: i.e., there should be a pole at the physical mass, m, and this should have residue unity. Since the propagator is noncovariant we must specify for which point on the mass shell we will require this. Our interpretation of this propagator as corresponding to a dressed electron with velocity given by $\mathbf{v}$ leads us to choose the point

$$p=m\gamma(1,v^1,0,0)=m\gamma(\eta+v). \tag{28}$$

The conjecture of Refs. [8,29] is that the so-renormalized propagator will be infrared finite.

To find the mass shift renormalization constant, δm, we use the mass-shell condition that there is a pole at m. This implies that the renormalized self-energy must obey

$$\widetilde{\alpha}+\widetilde{\beta}+\frac{(p\cdot\eta)^2}{m^2}\,\widetilde{\delta}+\frac{p\cdot v}{m}\,\widetilde{\epsilon}=0. \tag{29}$$

Here the tildes signify that we put the momentum p^2 on shell in the self-energy (propagator): $p^2=m^2$. Note that the counterterms, Z_2 and Z', from Eq. (24) do not enter in (29) since this is on shell and so just δm will now be determined. As stated above, the mass shift is gauge parameter independent in covariant gauges and it has been seen to be independent of the exact choice of mass shell point in both the Coulomb gauge [8] and in the renormalization of the slowly moving dressed charge [29]. We therefore expect that Eq. (29) will hold for *any* point on the mass shell and this will provide a check on our calculations of the functions $\alpha,\ldots,\epsilon$.

In this notation we may write the Taylor expansion of the propagator in (p^2-m^2) as

$$\begin{aligned} iS_v=i\,\frac{\not p+m}{p^2-m^2}&-\frac{1}{p^2-m^2}\,\{(2m^2\widetilde{\Delta}+\widetilde{\beta})\not p\\ &+(2m^2\widetilde{\Delta}+\widetilde{\alpha}+2\widetilde{\beta})m-p\cdot\eta\,\not\eta\,\widetilde{\delta}-m\not v\,\widetilde{\epsilon}\}\\ &+O[(p^2-m^2)^0], \end{aligned} \tag{30}$$

where

$$\begin{aligned}\widetilde{\Delta}(p\cdot\eta,p\cdot v,v^2)=&\left(\frac{\partial\alpha}{\partial p^2}+\frac{\partial\beta}{\partial p^2}+\frac{(p\cdot\eta)^2}{m^2}\,\frac{\partial\delta}{\partial p^2}\right.\\ &\left.\left.+\frac{p\cdot v}{m}\,\frac{\partial\epsilon}{\partial p^2}\right)\right|_{p^2=m^2}.\end{aligned} \tag{31}$$

Note that the infrared divergences that arise are contained in the function, Δ. Clearly we will now require the second term in Eq. (30) to vanish at our renormalization point. Requiring that the coefficients of m, $\not\eta$, and $\not v$ all so vanish at our physically motivated mass-shell condition (28) gives us three independent equations, which we choose to write as

$$\begin{aligned} 2m^2\overline{\Delta}+\overline{\beta}-\overline{\delta}&=0,\\ \gamma(2m^2\overline{\Delta}+\overline{\beta})-\overline{\epsilon}&=0,\\ 2m^2\overline{\Delta}+\overline{\alpha}+2\overline{\beta}&=0, \end{aligned} \tag{32}$$

where the bars denote that the functions are now evaluated at $p=\gamma m(\eta+v)$.

Since we confidently expect the mass shift to be fixed by Eq. (29) above, we seem to have three equations [i.e., Eq. (32)] and two unknowns (δZ_2 and Z') and one might worry that perhaps no solution exists. However, we can rapidly see that no such problem exists for our choice of mass-shell point. If we now explicitly separate out the contributions of the δZ_2 and Z' counterterms to the self-energy from the rest (and give what is left, i.e., those coming from the loop integrations and the mass shift counterterm, a subscript L) then we find that Eq. (32) can be rewritten as

$$\begin{aligned} i\,\delta Z_2-2\mathbf{v}^2 iZ'&=\overline{\delta_L}-\overline{\beta_L}-2m^2\overline{\Delta},\\ i\,\delta Z_2-2iZ'&=\gamma^{-1}\overline{\epsilon_L}-\overline{\beta_L}-2m^2\overline{\Delta},\\ i\,\delta Z_2&=-\overline{\alpha_L}-2\overline{\beta_L}-2m^2\overline{\Delta}. \end{aligned} \tag{33}$$

We point out that $\Delta=\Delta_L$, i.e., no counterterms appear in Δ. This set of equations has a solution if

$$\gamma^2\overline{\delta_L}+\overline{\alpha_L}+\overline{\beta_L}-\gamma\mathbf{v}^2\overline{\epsilon_L}=0, \tag{34}$$

and we recognize that this is nothing else but Eq. (29) at the physical renormalization point (28). We therefore have the following two equations which determine our counterterms:

$$\begin{aligned} Z'&=\frac{1}{2i}\,[\gamma^2\overline{\delta_L}-\gamma\overline{\epsilon_L}],\\ \delta Z_2&=-\frac{1}{i}\,[\overline{\alpha_L}+2\overline{\beta_L}+2m^2\overline{\Delta}]. \end{aligned} \tag{35}$$

B. The renormalization constants

The calculation of the self-energy and the counterterms is a laborious task.[2] A discussion of the necessary integrations may be found in the Appendix. Here we will quote the relevant results. For Eq. (29) we obtained

$$\begin{aligned} \widetilde{\alpha}+\widetilde{\beta}+\frac{(p\cdot\eta)^2}{m^2}\,\widetilde{\delta}+\frac{p\cdot v}{m}\,\widetilde{\epsilon}=&-i\,\frac{\alpha}{4\pi}\left(\frac{3}{\hat{\epsilon}}+4\right)m+i\,\delta m+i\,\frac{\alpha}{4\pi}\left\{\frac{p\cdot(\eta+v)}{m^2}\,[p\cdot(\eta-v)\widetilde{I}_2^g+p\cdot\eta\widetilde{I}_2^\eta+\mathbf{v}^2p\cdot v\widetilde{I}_2^v]\right\}\\ &-2i\,\frac{\alpha}{4\pi}\,p\cdot(\eta+v)\left\{\frac{p\cdot(\eta-v)}{m^2}\left(\frac{1}{2}\,\widetilde{I}_2^p+\widetilde{\mathcal{I}}_3^g\right)+\frac{p\cdot\eta}{m^2}\left(\frac{1}{2}\,\widetilde{I}_2^\eta+\widetilde{\mathcal{I}}_3^\eta\right)+\frac{\mathbf{v}^2p\cdot v}{m^2}\left(\frac{1}{2}\,\widetilde{I}_2^v+\widetilde{\mathcal{I}}_3^v\right)\right.\\ &+p\cdot(\eta-v)\widetilde{I}_3^{pp}+\frac{(p\cdot\eta)^3}{m^2}\,\widetilde{I}_3^{\eta\eta}+\mathbf{v}^2\,\frac{(p\cdot v)^3}{m^2}\,\widetilde{I}_3^{vv}+p\cdot\eta\left[1+\frac{p\cdot\eta}{m^2}\,p\cdot(\eta-v)\right]\widetilde{I}_3^{p\eta}\\ &\left.+p\cdot v\left[\mathbf{v}^2+\frac{p\cdot v}{m^2}\,p\cdot(\eta-v)\right]\widetilde{I}_3^{pv}+\frac{p\cdot\eta p\cdot v}{m^2}\,(p\cdot v+\mathbf{v}^2p\cdot\eta)\widetilde{I}_3^{\eta v}\right\}, \end{aligned} \tag{36}$$

we refer to the Appendix for the exact meaning of the additional notation here. Recall that only the mass shift counterterm appears in Eq. (36). The first term on the RHS here arises from the first term on the RHS of Eq. (21) which is the integrand

[2] Both MATHEMATICA and REDUCE were used.

of the self-energy in Feynman gauge. The gauge invariance of δm means that this is the correct answer. We need to see that the other terms all cancel on shell no matter what exact on-shell point is employed. Using Eqs. (A22) and (A23) from the Appendix, we can see that they do and that we obtain the standard result

$$\delta m = m \frac{\alpha}{4\pi}\left(\frac{3}{\hat{\epsilon}}+4\right), \tag{37}$$

where $1/\hat{\epsilon}=1/(2-\omega)-\gamma_E+\ln 4\pi$.

To verify that the infrared singularities cancel we should consider $\overline{\Delta}$, which we recall is where they arise. We find the following terms containing infrared divergences:

$$m^2\overline{\Delta}_{\rm IR}=i\frac{\alpha}{4\pi}\int_0^1 du\, u^{2\omega=5}\left(-2+2\int_0^1\frac{dx}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2x}}[1+\mathbf{v}^2-2\mathbf{v}^2x]-(1-\mathbf{v}^2)\int_0^1\frac{dx}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2x}}\,x\,\frac{3+\mathbf{v}^2-2\mathbf{v}^2x}{2(1-\mathbf{v}^2x)}\right), \tag{38}$$

where the subscript IR signifies that only the infrared singular terms have been retained. The first term comes from the covariant part of the self-energy and the others have a noncovariant origin. We find it remarkable and highly gratifying, that the sum of the integrals over x gives just $+2$ and so we see that there is no infrared divergence in the dressed propagator.

Since this is the main result of this paper let us stress that we do not see any *a priori* reason why these divergences should cancel—other than our original motivation. It is certainly *not* the case that they cancel for any point on the mass shell. We have verified this by changing the relative sign of the vector v between the dressing (6) and the choice of mass-shell point, (29). The infrared divergences did not then cancel. This shows the great sensitivity of the calculation.

For completeness we now give the full expressions for Z_2 and Z'. We found

$$Z_2=1+\frac{\alpha}{4\pi}\left\{\frac{1}{\hat{\epsilon}}[3+2\chi(\mathbf{v})]-4(1-\mathbf{v}^2)\chi(\mathbf{v})-4\kappa(\mathbf{v})\right\}, \tag{39}$$

and

$$Z'=\frac{\alpha}{4\pi}\left\{\frac{1}{\hat{\epsilon}}\left[\frac{1}{\mathbf{v}^2}+\frac{1+\mathbf{v}^2}{2\mathbf{v}^2}\chi(\mathbf{v})\right]-\frac{1}{\mathbf{v}^2}(1-\mathbf{v}^2)\chi(\mathbf{v}) -\frac{1+\mathbf{v}^2}{\mathbf{v}^2}\kappa(\mathbf{v})\right\}, \tag{40}$$

where

$$\kappa(\mathbf{v})=\frac{1}{|\mathbf{v}|}[L_2(|\mathbf{v}|)-L_2(-|\mathbf{v}|)], \tag{41}$$

where L_2 is the dilogarithm ($L_2(x)=-\int_0^x dt/t\ \ln[1-t]$). In the small $\mathbf{v}$ limit these reduce to the expressions we found in Ref. [29], which in turn reduce to the Coulomb gauge result [32,8] for $\mathbf{v}\to 0$. We have also checked that these agreements hold for the results for the individual functions, $\alpha,\ldots,\epsilon$. (Although to compare with the results of Ref. [29] for infrared divergent terms, one needs to make the translation: $1/\hat{\epsilon}\to\ln\lambda^2/m^2$, where λ is a small photon mass.) These limits provide a further check upon our results.

IV. CONCLUSIONS

We have seen that the electron propagator is infrared finite in the class of gauges (19) if a suitable on-shell condition is used. This calculation may also be understood as the calculation of a dressed propagator in a general gauge. The renormalization procedure was completely standard except for the matrix nature of the fermion wave-function renormalization. This was introduced in Ref. [29] and appears rather natural given the subtleties concerning boosting charged states. We stress again that the cancellation of the various infrared divergences that appear in the individual terms is not fortuitous but has been predicted in Refs. [8, 29]. We believe that this is compelling evidence that the description of an asymptotic electron which we employ has a firm physical basis. Using Ref. [33] it may be seen that the soft divergences will exponentiate and so we expect these results to hold at all orders. We also stress that we have calculated the wave-function renormalization constants explicitly and that they may be used to find S-matrix elements involving incoming and outgoing dressed charges.

Our requirement of the particular renormalization point used in this paper makes it clear that gauge invariance alone does not provide an infrared finite propagator. We have tried to stress here the need for an understanding of what meaning (if any) a gauge-invariant dressed field possesses. The dressings we have studied correspond to velocity eigenstates. Other types of dressings should, we feel, also be constructed and investigated.

As far as the further applications of the dressed fields of this paper are concerned, the extension of this approach to the electron-photon vertex functions is the obvious next step. If the momentum transfer is nonzero the incoming and outgoing electrons will have different velocities and should accordingly be differently dressed, we therefore do not expect the infrared divergences present in the usual, undressed vertex to cancel in any particular gauge, since no gauge condition would remove all the dressings. However, if we keep the dressings we expect the dressed vertex to be infrared finite in any gauge if the appropriate mass-shell conditions for the fermions are chosen. These calculations will be presented elsewhere.

As far as QCD is concerned, it is clearly harder to construct gauge-invariant descriptions of charges. In perturbation theory, dressings for quarks and gluons have been constructed and shown to give a gauge-independent meaning to the concept of color charges [34]. It has also been seen that there is an obstruction to dressing colour charges nonperturbatively [35]. A proof of this, a treatment of perturbative dressings for quarks and gluons in QCD and a full discussion of the implications of these matters is to be found in Ref. [8]. We also refer to Refs. [17–19]. For theories where the gauge symmetry is spontaneously broken, dressings may be constructed in the Higgs sector [36]. Perturbative and nonperturbative studies of dressed, non-Abelian Green's functions have, we feel, many practical applications.

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APPENDIX: ABOUT THE INTEGRALS

A treatment of integrals required for calculations in Coulomb gauge may be found in Ref. [32]. The integrals considered here are related to that discussion, but are more general in that an extra vector is involved in our case.

1. General formulas

We need the generic integral

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k^2-2k\cdot p-M^2)^\alpha} \frac{k_{\mu_1}\cdots k_{\mu_n}}{[k^2-(k\cdot\eta)^2+(k\cdot v)^2]^\beta}, \tag{A1}$$

where the second factor in the denominator reflects the structure of the gauge boson propagator, Eq. (20). We first go to Euclidean space and exponentiate the denominators using

$$X^{-\alpha}=\frac{1}{\Gamma(\alpha)}\int_0^\infty dy\, y^{\alpha-1}e^{-Xy}. \tag{A2}$$

Then we make use of

$$\int_{\mathrm{Euc}} \frac{d^{2\omega}k}{(2\pi)^{2\omega}}\, e^{-[k^t\mathcal{M}k-J^tk]}=\frac{\pi^\omega}{(2\pi)^{2\omega}}\frac{1}{\sqrt{\det\mathcal{M}}}\, e^{(1/4)J^t\mathcal{M}^{-1}J}. \tag{A3}$$

In our case the $(2\omega)\times(2\omega)$ matrix $\mathcal{M}$ is

$$\mathcal{M}_{\mu\nu}=(y+z)\delta_{\mu\nu}-z\eta_\mu\eta_\nu-zv_\mu v_\nu, \tag{A4}$$

where y and z are the Feynman parameters used to exponentiate the two denominators in our generic integral. Similarly in our case

$$J_\mu=2yp_\mu. \tag{A5}$$

To go from scalar integrals to vector or tensor ones, we simply have to take derivatives according to the recipe

$$k_\mu\to\frac{1}{2y}\frac{\partial}{\partial p^\mu}. \tag{A6}$$

Upon changing the variables

$$y=(1-x)t,\quad z=xt;\Rightarrow dydz=t\,dx\,dt;\quad x\in[0,1];$$
$$t\in[0,\infty] \tag{A7}$$

we get

$$\det\mathcal{M}=t^{2\omega}(1-x)(1-\mathbf{v}^2x), \tag{A8}$$

and so, back in Minkowski space, we have

$$A_{\mu\nu}\equiv[\mathcal{M}^{-1}]_{\mu\nu}=g_{\mu\nu}+\frac{x}{1-x}\eta_\mu\eta_\nu-\frac{x}{1-\mathbf{v}^2x}v_\mu v_\nu. \tag{A9}$$

One finally thus obtains

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k^2-2k\cdot p-M^2)^\alpha} \frac{B}{[k^2-(k\cdot\eta)^2+(k\cdot v)^2]^\beta}=\frac{(-1)^{\alpha+\beta}}{(2\pi)^{2\omega}}\frac{i\pi^\omega}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1\frac{dx}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2x}}(1-x)^{\alpha-1}x^{\beta-1}C, \tag{A10}$$

where various pairs of B's and C's are related as follows:

$$B=1,\quad C=\frac{\Gamma(\alpha+\beta-\omega)}{\Delta_g^{\alpha+\beta-\omega}},$$

$$B=k_\mu,\quad C=(1-x)\frac{\Gamma(\alpha+\beta-\omega)}{\Delta_g^{\alpha+\beta-\omega}}(Ap)_\mu,$$

$$B=k_\mu k_\nu,\quad C=(1-x)^2(Ap)_\mu(Ap)_\nu\frac{\Gamma(\alpha+\beta-\omega)}{\Delta_g^{\alpha+\beta-\omega}}-\frac{1}{2}A_{\mu\nu}\frac{\Gamma(\alpha+\beta-1-\omega)}{\Delta_g^{\alpha+\beta-1-\omega}}, \tag{A11}$$

and lastly

$$B=k_\mu k_\nu k_\rho, \quad C=(1-x)^3(Ap)_\mu(Ap)_\nu(Ap)_\rho \frac{\Gamma(\alpha+\beta-\omega)}{\Delta_g^{\alpha+\beta-\omega}}-\frac{(1-x)}{2}[A_{\mu\nu}(Ap)_\rho+A_{\mu\rho}(Ap)_\nu$$
$$+A_{\nu\rho}(Ap)_\mu]\frac{\Gamma(\alpha+\beta-1-\omega)}{\Delta_g^{\alpha+\beta-1-\omega}}, \tag{A12}$$

where we have further introduced the notation

$$\Delta_g=(1-x)[(1-x)p_\mu p_\nu A^{\mu\nu}+M^2]. \tag{A13}$$

We also use the relation

$$\frac{1}{k^2}\frac{1}{(p-k)^2-m^2}=\int_0^1 \frac{du}{[k^2-2uk\cdot p-u(m^2-p^2)]^2}, \tag{A14}$$

to, where necessary, combine the two covariant denominators coming from the fermion propagator and the vector boson propagator.

For integrals with one or two covariant denominator structures Δ_g takes on different forms. For an integral with one covariant and one noncovariant denominator term (so two structures in total) we have, for Δ_g,

$$\Delta_2=(1-x)(\Pi+m^2-p^2), \quad \Pi=(1-x)p^2+x(p\cdot\eta)^2-\frac{(1-x)x}{1-\mathbf{v}^2x}(p\cdot v)^2. \tag{A15}$$

If we have two noncovariant structures and one noncovariant term in the denominator, then we have, for Δ_g,

$$\Delta_3=u(1-x)\{u\Pi+m^2-p^2\}, \tag{A16}$$

the similarity between these last two equations indicates the utility of this notation.

2. The on-shell integrals needed for the mass shift

To compute the mass shift, we need to know the following integrals for $p^2=m^2$ and arbitrary $p\cdot\eta,\ p\cdot v,\ v$:

$$\frac{16\pi^2}{i(m^2)^{\omega-2}}\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{1}{(p-k)^2-m^2}\frac{k_\mu}{k^2-(k\cdot\eta)^2+(k\cdot v)^2}=I_2^p p_\mu+p\cdot\eta I_2^\eta\eta_\mu+p\cdot v I_2^v v_\mu \tag{A17}$$

where we define, for on-shell momentum p,

$$\widetilde{I}_2^p=\int_0^1 \frac{dx}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2x}}(1-x)\left[\frac{1}{\hat{\epsilon}}-\ln\frac{\widetilde{\Delta}_2}{m^2}\right],$$
$$\widetilde{I}_2^\eta=\int_0^1 \frac{dx}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2x}}x\left[\frac{1}{\hat{\epsilon}}-\ln\frac{\widetilde{\Delta}_2}{m^2}\right], \tag{A18}$$
$$\widetilde{I}_2^v=\int_0^1 \frac{dx}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2x}}\frac{-x(1-x)}{1-\mathbf{v}^2x}\left[\frac{1}{\hat{\epsilon}}-\ln\frac{\widetilde{\Delta}_2}{m^2}\right],$$

and, as in the main body of the paper, a tilde signifies that the function is evaluated on an arbitrary point on the mass shell, $p^2=m^2$.

We also need the integrals

$$\frac{16\pi^2}{i(m^2)^{\omega-2}}\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{1}{k^2}\frac{1}{(p-k)^2-m^2}\frac{k_\mu k_\nu}{k^2-(k\cdot\eta)^2+(k\cdot v)^2}=I_3^g g_{\mu\nu}+I_3^\eta\eta_\mu\eta_\nu+I_3^v v_\mu v_\nu+I_3^{pp}p_\mu p_\nu+(p\cdot\eta)^2 I_3^{\eta\eta}\eta_\mu\eta_\nu$$
$$+(p\cdot v)^2 I_3^{vv}v_\mu v_\nu+p\cdot\eta I_3^{p\eta}(p_\mu\eta_\nu+\eta_\mu p_\nu)+p\cdot v I_3^{pv}(p_\mu v_\nu$$
$$+v_\mu p_\nu)+p\cdot\eta p\cdot v I_3^{\eta v}(v_\mu\eta_\nu+\eta_\mu v_\nu), \tag{A19}$$

where

$$I_3^g = \frac{1}{2} I_2^p - \int_0^1 du \ \ln u \int_0^1 \frac{dx}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2 x}} (1-x) = \frac{1}{2} I_2^p + \mathcal{I}_3^g ,$$

$$I_3^\eta = \frac{1}{2} I_2^\eta - \int_0^1 du \ \ln u \int_0^1 \frac{dx}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2 x}} x = \frac{1}{2} I_2^\eta + \mathcal{I}_3^\eta , \tag{A20}$$

$$I_3^v = \frac{1}{2} I_2^v - \int_0^1 du \ \ln u \int_0^1 \frac{dx}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2 x}} \frac{-x(1-x)}{1-\mathbf{v}^2 x} = \frac{1}{2} I_2^v + \mathcal{I}_3^v ,$$

and we see that the u integral is just -1; similarly for on-shell p we have

$$\widetilde{I}_3^{pp} = \int_0^1 du \int_0^1 \frac{dx}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2 x}} \frac{-(1-x)^2}{\widetilde{\Pi}},$$

$$\widetilde{I}_3^{\eta\eta} = \int_0^1 du \int_0^1 \frac{dx}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2 x}} \frac{-x^2}{\widetilde{\Pi}},$$

$$\widetilde{I}_3^{vv} = \int_0^1 du \int_0^1 \frac{dx}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2 x}} \frac{-x^2(1-x)^2}{(1-\mathbf{v}^2 x)^2} \frac{1}{\widetilde{\Pi}},$$

$$\widetilde{I}_3^{p\eta} = \int_0^1 du \int_0^1 \frac{dx}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2 x}} \frac{-x(1-x)}{\widetilde{\Pi}}, \tag{A21}$$

$$\widetilde{I}_3^{pv} = \int_0^1 du \int_0^1 \frac{dx}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2 x}} \frac{x(1-x)^2}{1-\mathbf{v}^2 x} \frac{1}{\widetilde{\Pi}},$$

$$\widetilde{I}_3^{\eta v} = \int_0^1 du \int_0^1 \frac{dx}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2 x}} \frac{x^2(1-x)}{1-\mathbf{v}^2 x} \frac{1}{\widetilde{\Pi}},$$

and the trivial u integral just yields 1. It takes some algebra to show that

$$p\cdot(\eta-v)\widetilde{I}_3^{pp} + \frac{(p\cdot\eta)^3}{m^2}\widetilde{I}_3^{\eta\eta} + \mathbf{v}^2 \frac{(p\cdot v)^3}{m^2}\widetilde{I}_3^{vv} + p\cdot\eta\left[1+\frac{p\cdot\eta}{m^2} p\cdot(\eta-v)\right]\widetilde{I}_3^{p\eta} + p\cdot v\left[\mathbf{v}^2 + \frac{p\cdot v}{m^2} p\cdot(\eta-v)\right]\widetilde{I}_3^{pv}$$
$$+ \frac{p\cdot\eta\, p\cdot v}{m^2}(p\cdot v + \mathbf{v}^2 p\cdot\eta)\widetilde{I}_3^{\eta v} = -\frac{1}{m^2}\int_0^1 \frac{dx}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2 x}}\left\{p\cdot\eta - p\cdot v\,\frac{1-x}{1-\mathbf{v}^2 x}\right\}, \tag{A22}$$

and similarly that

$$\frac{p\cdot(\eta-v)}{m^2}\widetilde{\mathcal{I}}_3^g + \frac{p\cdot\eta}{m^2}\widetilde{\mathcal{I}}_3^\eta + \mathbf{v}^2 \frac{p\cdot v}{m^2}\widetilde{\mathcal{I}}_3^v = \frac{1}{m^2}\int_0^1 \frac{dx}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2 x}}\left\{p\cdot\eta - p\cdot v\,\frac{1-x}{1-\mathbf{v}^2 x}\right\}, \tag{A23}$$

but armed with these results we may easily obtain the standard result for the mass shift, as given by Eq. (36).

3. An example

We now round off this appendix by showing how the above general discussion may be applied to compute a particular noncovariant integral. Consider therefore

$$\frac{1}{(m^2)^{\omega-2}}\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(p-k)^2-m^2} \frac{1}{k^2-(k\cdot\eta)^2+(k\cdot v)^2} = \frac{i\pi^\omega}{(2\pi)^{2\omega}}\int_0^1 \frac{dx}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2 x}} \frac{\Gamma(2-\omega)}{(\Delta_2/m^2)^{2-\omega}}, \tag{A24}$$

where Δ_2 is given in Eq. (A15). This relation follows from Eq. (A10). We now expand this in $\varepsilon=(2-\omega)$ and obtain

$$\frac{i}{16\pi^2}\int_0^1 \frac{dx}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2x}}\left\{\frac{1}{\hat{\epsilon}}-\ln\,\Delta_2\right\}+O(\epsilon). \tag{A25}$$

The change of variables, $x=(1-t^2)/(1-\mathbf{v}^2t^2)$, is now useful. The integral coefficient of the pole in ε can then be reexpressed as

$$\int_0^1 \frac{dx}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2x}}=2\int_0^1 dt\,\frac{1}{1-\mathbf{v}^2t^2}=\frac{1}{|\mathbf{v}|}\ln\frac{1-|\mathbf{v}|}{1+|\mathbf{v}|}\equiv-\chi(\mathbf{v}), \tag{A26}$$

where we recall the definition of χ from Eq. (26).

The second integral in Eq. (A25) depends on p. We will not calculate it for an arbitrary p, but rather in a Taylor expansion around the correct, physical pole for the dressing we use. Again employing the notation that bars over functions signify that they are evaluated at $p=m\gamma(\eta+v)$, we find

$$\overline{\Pi}=\frac{m^2}{1-\mathbf{v}^2x},\quad \overline{\Delta}_2=(1-x)\,\frac{m^2}{1-\mathbf{v}^2x},\quad \frac{\partial}{\partial p^2}\,\Delta_2\bigg|_{p=m\gamma(\eta+v)}=-x(1-x). \tag{A27}$$

Thus we obtain

$$\int_0^1 dx\,\frac{\ln(\Delta_2/m^2)}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2x}}=\int_0^1 dx\,\frac{\ln(1-x)-\ln(1-\mathbf{v}^2x)}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2x}}-\frac{p^2-m^2}{m^2}\int_0^1 dx\,\frac{x(1-\mathbf{v}^2x)}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2x}}. \tag{A28}$$

Repeating the transformation of variables, these two integrals yield, respectively,

$$\int_0^1 dx\,\frac{\ln(1-x)-\ln(1-\mathbf{v}^2x)}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2x}}=2\int_0^1 dt\,\frac{\ln\,t^2}{1-\mathbf{v}^2t^2}=\frac{2}{|\mathbf{v}|}\,[L_2(-|\mathbf{v}|)-L_2(|\mathbf{v}|)]\equiv-2\kappa(\mathbf{v}), \tag{A29}$$

and

$$\int_0^1 dx\,\frac{x(1-\mathbf{v}^2x)}{\sqrt{1-x}\sqrt{1-\mathbf{v}^2x}}=\frac{3}{4}-\frac{1}{4\mathbf{v}^2}-\frac{(1-\mathbf{v}^2)(1+3\mathbf{v}^2)}{8\mathbf{v}^2}\,\chi(\mathbf{v}). \tag{A30}$$

Putting everything together we obtain for our exemplary integral

$$\frac{1}{(m^2)^{\omega-2}}\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{1}{(p-k)^2-m^2}\,\frac{1}{k^2-(k\cdot\eta)^2+(k\cdot v)^2}=\frac{i}{16\pi^2}\left\{-\chi(\mathbf{v})\,\frac{1}{\hat{\epsilon}}+2\kappa(\mathbf{v})+\frac{p^2-m^2}{m^2}\left[\frac{3}{4}-\frac{1}{4\mathbf{v}^2}-\frac{(1-\mathbf{v}^2)(1+3\mathbf{v}^2)}{8\mathbf{v}^2}\,\chi(\mathbf{v})\right]\right\}+O[(p^2-m^2)^2]. \tag{A31}$$

In the limit $\mathbf{v}\to 0$ this correctly yields

$$\frac{i}{16\pi^2}\left\{\frac{2}{\hat{\epsilon}}+4+\frac{p^2-m^2}{m^2}\,\frac{4}{3}\right\}. \tag{A32}$$

Very similar manipulations yield the other integrals we require.

Finally we should also mention that various consistency relations between integrals have been checked [e.g., replacing a factor of $(k\cdot\eta)^2$ in a numerator by $k^2+\mathbf{k}^2$ and performing the two resulting integrals separately] and seen to hold.

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