Compact three-dimensional QED with a θ term and axionic confining strings

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We discuss three-dimensional compact QED with a θ term due to an axionic field. The variational gauge-invariant functional is considered and it is shown that the ground-state energy is independent of θ in a leading approximation. The mass gap of the axionic field is found to be dependent upon θ, the mass gap of the photon field, and the scalar potential. The vacuum expectation of the Wilson loop is shown to be independent of θ in a leading approximation, to obey the area law, and to lead to confinement. We also briefly discuss the properties of axionic confining strings. [S0556-2821(97)05018-2]

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I. INTRODUCTION

Within quantum field theory one can add to any Lagrangian of a field with a nontrivial topology a gauge- and Lorentz-invariant ''θ term.'' Such terms take the form of the topological charge of the theory with a coefficient of θ, the Fourier transform variable of the winding number, suitably normalized. These terms arise naturally in any theory where there are topologically nontrivial solutions of the classical equations of motion, i.e., instantons. For example, see Ref. [1] and the references therein. The θ term cannot be traced in a perturbation theory because it has no effect upon the classical equations of motion. In this paper we shall consider the effect of the θ term in compact $(2+1)$-dimensional QED.

Much work has been carried out upon such theories and their associated phenomena. $(2+1)$-dimensional QED was studied by Polyakov [2], for zero θ, who found a mechanism of confinement due to nonperturbative effects caused by monopoles. This has led to its consideration as a simpler (Abelian) model with many of the same characteristics as QCD. Compact three-dimensional QED (QED_3) with a nonzero θ was considered in a path-integral approach by Vergeles [3] who found that the long-range interactions of the monopoles and the antimonopoles were responsible for the suppression of the θ dependence of the vacuum energy of the theory as a subleading term in volume (in the limit of large volume). This is in striking contrast to QCD, where the θ dependence of the vacuum energy in the absence of massless quarks is firmly established.

Vergeles used the intuitive approach of a partition function of a gas of $N+Q$ monopoles and N antimonopoles, with screening in the bulk and all excess monopoles deposited upon the boundary, and summed over N and Q in the limit of large volume. He found that the θ dependence of the free energy increased as $V^{1/3}$ but, for large volume, that this is suppressed as subleading by the θ-independent part of the free energy which increased with V.

The work of Vergeles was later generalized by Samuel who proposed that, in a $(3+1)$-dimensional Yang-Mills theory, if long-ranged interactions between instantons are present the θ parameter may relax to zero [4]. Zhitnitsky [5], also discussed these problems in $(2+1)$-dimensional QED and $(3+1)$-dimensional gluodynamics. Both of these models possess long-range interactions of topological charges but Zhitnitsky found that, because the pseudoparticles of $(3+1)$-dimensional gluodynamics possess an additional quantum number (apart from the topological charge), only in QED_3 does the physics not depend on θ.

Recently, Polyakov has used the compact U(1) theory as a model to construct the new type of strings, the so-called confining strings [6]. It is of natural interest to ask the question of how the θ term will reveal itself in this new string theory. In [6], Polyakov proposed the hypothesis that all gauge theories are equivalent to a certain nonstandard string theory, where different gauge groups are accounted for by the weights ascribed to the world sheets of different topologies. The string ansatz was established in the case of the Abelian gauge groups and conjectures were made concerning the non-Abelian generalization. The theories were considered in the absence of any θ terms, however, and we find that our proposed low-energy Lagrangian for QED_3 with a θ term reproduces the results of [6] with a θ-dependent, but subleading in momentum cutoff, modification to the photon mass and a θ-dependent shift of the minimal surface. This modification of the photon mass is in exact agreement with the results of our variational calculation.

We follow the gauge-invariant functional variational method of [7] to approach the problem of QED_3 with a θ term. Initially, the aim was to repeat the calculations of [7] with nonzero θ. Through the Hamiltonian formalism, however, it became obvious that, to introduce the θ term in QED_3 and to be able to solve the theory at the space-time boundary, one has to introduce an extra degree of freedom. A scalar field must be added whose mass is a free parameter of the theory. There are two very natural ways to see the form that the Lagrangian including a θ term should take.

The first way is to see that U(1) singular monopole solutions can be obtained from the original Georgi-Glashow model by breaking the internal symmetry down from SU(2) to U(1) with a triplet of scalar fields [2]. The new U(1) theory is compact because U(1) is a subgroup of a compact SU(2) internal symmetry group. The masses of the charged vector bosons are a UV cutoff for this new theory but the

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mass of the third scalar field is independent of this cutoff and may even be made zero in the Bogomol'nyi-Prasad-Sommerfield (BPS) limit. Thus, the low-energy U(1) theory has two degrees of freedom; the photon and the scalar fields. Without the θ term there is no interaction between the scalar and the vector field. In the original SU(2) theory it is natural to have the scalar and vector fields coupled within the θ term and this form is preserved when the internal symmetry is broken to yield the desired $(2+1)$-dimensional U(1) theory. The term we find is the same as that proposed by Affleck, Harvey, and Witten for the SU(2) Georgi-Glashow model with fermions [8]. In the limit of a very massive scalar field, our model reduces to the purely gauge field model studied by Polyakov and Vergeles, but it is in the BPS limit (zero potential but nonzero vacuum expectation value for the scalar field) that the scalar sector of our model becomes most interesting.

The second way to obtain the same Lagrangian is to consider the compactification of a U(1) theory from $3+1$ to $2+1$ dimensions. The component of the vector field in the direction of the compactified spatial dimension becomes the scalar field and, although the coupling constants in the $(3+1)$-dimensional U(1) theory are dimensionless, the source of the mass dimensions of the coupling constants in the $(2+1)$-dimensional theory becomes apparent.

Naively, one may assume that the addition of such a θ term would make the dynamics of the fields dependent upon θ. We have found that the dynamically generated mass of the photon is independent of θ in the limit of large UV cutoff momentum in QED_3 with a scalar field and, in agreement with [3], that the vacuum energy of the photon is independent of θ in this limit. We have also found that the scalar field has a dynamically generated mass, the square of which has two terms; one is proportional to θ^2 and the other is a bare mass term involving the coefficient of the scalar potential. In the BPS limit the coefficient of the scalar potential is set to zero and the scalar mass becomes proportional to θ. The prediction of a dynamically generated scalar mass proportional to θ is a new result and one that could be verified through a lattice simulation. The θ dependence of the scalar vacuum energy is also greatly suppressed. In the BPS limit it is suppressed in terms of $O(z^{3/2})$ where $z\sim\exp[-\Lambda]$ and Λ is the UV cutoff of our theory. In the limit of a very massive scalar field the scalar, and hence the total, vacuum energy is independent of θ with all θ dependence suppressed in subleading terms of the scalar mass. This is in agreement with [3]. We have also calculated the string tension and show that, independently of θ, this obeys the area law and so leads to confinement as in the simple case of purely gauge field QED_3 [7].

In the next section we shall describe a simple, one-dimensional model which has many of the same characteristics as more complex theories, such as QED_3. This section is intended to have a pedagogical role as the study of this simple model gives some intuition as to the correct formalism required to write down the θ dependence of the variational ansatz. We are unaware of any other instances in the literature of a term similar to the θ term we use and so we shall give some motivation for it in Sec. III, where we shall also explain the form of our Hamiltonian. In Sec. IV we give our ansatz for the wave functional. In Sec. V we shall perform the variational calculation; examining the θ dependence of the vacuum energies, the masses of the theory, and the string tension. In Sec. VI, the modification of the Polyakov model of the confining string, which we shall call the axionic string, will be considered and comparisons made to other recent papers on this specific topic.

II. A SIMPLE ONE-DIMENSIONAL MODEL

To develop some intuition for the formalism required in order to write down the θ dependence in the variational ansatz for the vacuum wave functional of more complex models, such as QED_3 and QCD, we shall first observe some results from a well-known and much simpler one-dimensional model. The formalism of this section will be modified slightly for the more sophisticated case of QED_3 but it is hoped that the reader will gain some insight from this simple example. This model has many of the same characteristics as more complex theories, such as a θ term, and is defined by

$$S^{\theta}=\int dt\left[\frac{1}{2}(\dot{\phi})^2+\theta\dot{\phi}-\lambda\cos\phi\right]. \tag{2.1}$$

A few results are immediately apparent from the path-integral approach. The first is that the minima of the theory occur at $\phi=(2n+1)\pi, n=0,\pm1,\pm2,\ldots$. In analogy with the Georgi-Glashow model with zero scalar potential (but with a nonzero vacuum expectation of the scalar field which is necessary to define the nontrivial topology of the theory), we shall continue to study the model in the limit of zero λ. Second, the second term is a total derivative which, after explicit evaluation of the integral, counts the solitons of the system and gives $2\pi n$ where n is the winding number of the field configuration. Hence, this θ term is completely analogous to those of more complex theories and affords the symmetry $\theta\rightarrow\theta+1$. Third, we note that Eq. (2.1) is invariant under $\phi\rightarrow\phi+2\pi$, which is analogous to the gauge invariance of more complex theories.

We shall proceed with the Hamiltonian formalism, having established the similarity between this simple theory, with its analogues of gauge invariance and a θ term with periodicity in θ, and more complex theories. In particular, we shall see how the periodicity of θ, which is explicit in the path-integral approach, is displayed within the Hamiltonian formalism.

The Hamiltonian of the system is

$$H^{\theta}=\frac{1}{2}(\pi_{\tilde{\phi}}-\theta)^2, \tag{2.2}$$

where

$$\phi=2\pi n+\tilde{\phi}. \tag{2.3}$$

First, we see that the θ term has been absorbed into a modification of the canonical momentum of the free field $\tilde{\phi}$ which is connected to the field ϕ in momentum space by $\phi(k)=2\pi n\delta(k)+\tilde{\phi}(k)$. Second, the Hamiltonian is independent of n and so is explicitly invariant under the transformation $n\rightarrow n+1$. The invariance of the Hamiltonian under

$\theta \to \theta+1$, however, is unclear. We note here that the free field $\tilde{\phi}$ fluctuates about its vacuum expectation value of zero which is centered at one of the minima of ϕ.

We require the solution of the Hamiltonian equation of the field ϕ. This will be obtained as a modification of the solution of the Hamilton equation for the field $\tilde{\phi}$,

$$H^{\theta}\psi = E^{\theta}\psi, \tag{2.4}$$

where we initially postulate the form $\psi = \psi^{\theta}\psi^{0}$, with ψ^{0} satisfying $H^{0}\psi^{0} = E^{0}\psi^{0}$. For this simple model ψ^{0} is known, but for more complex theories (e.g., QCD) it is not. We therefore have $\psi^{0} = \exp[im\tilde{\phi}]$, $E^{0} = \frac{1}{2}m^{2}$. When $\theta = 0$, $m = 0, \pm 1, \pm 2, \ldots$ but for nonzero θ this is no longer correct. We will see that it is the coefficient of the field ϕ in the exponential of ψ that must be an integer for nonzero θ. We make the further requirement that

$$H^{\theta}\psi^{\theta} = \psi^{\theta}H^{0}. \tag{2.5}$$

In this case the solution of Eq. (2.5) is $\psi^{\theta} = \exp[i\theta\tilde{\phi}]$. After relabeling $p = \theta + m$ where $p = 0, \pm 1, \pm 2, \ldots$ we have the solution for the field $\tilde{\phi}$:

$$\psi = \exp[ip\tilde{\phi}], \quad E^{\theta} = \frac{1}{2}(p-\theta)^{2}. \tag{2.6}$$

We note that in the case $\theta = 0$ the wave functional immediately reduces to the correct form and that m becomes an integer.

Rewriting in terms of ϕ, the form of the Hamiltonian equation and, due to its periodicity, the wave functional are unchanged. We obtain the solution $\psi = \exp[ip\phi]$ with energy $E^{\theta} = \frac{1}{2}(p-\theta)^{2}$.

We note that ψ is invariant under $\phi \to \phi + 2\pi$. It is clear from the θ dependence of the energy that although each energy level is shifted, the entire energy spectrum of the theory is invariant under $\theta \to \theta + 1$. We note also that the ground state is not simply the minimum of one quadratic but a continuous function linking the minima of the quadratics centered at $\phi = (2n+1)\pi, n = 0, \pm 1, \pm 2, \ldots$. Although the separate consideration here of the field at zero momentum resulted in no modification of the form of the vacuum wave functional, the extra detail is included because the analogous procedure in Sec. IV does result in a modified form of the wave functional initially deduced for the scalar field of zero vacuum expectation value.

We propose the same formalism for more complex theories; an explicitly gauge-invariant but θ-dependent Hamiltonian with the solution $\psi = \psi^{\theta}\psi^{0}$, where ψ^{θ} contains all the θ dependence of the wave functional, $H^{\theta}\psi^{\theta} = \psi^{\theta}H^{0}$, and ψ is gauge invariant. ψ^{θ} and ψ^{0} cannot be individually gauge invariant without making the theory trivially independent of θ, which is well known not to be the general case (e.g., QCD).

III. THE LAGRANGIAN AND HAMILTONIAN OF QED$_3$ WITH A θ TERM

In the next two subsections we shall show how the Lagrangian with a θ term for a $(2+1)$-dimensional U(1) theory can be obtained from breaking the internal symmetry of the SU(2) 't Hooft–Polyakov monopole and from the compactification of a $(3+1)$-dimensional U(1) theory. In the third subsection we shall discuss the Hamiltonian of the system. But we shall first give our motivation for extending the model to include a scalar field. We follow the motivation for the topological charge in [3] to write a θ term in the Lagrangian for purely gauge field QED$_3$:

$$L = -\frac{1}{4}F_{\mu\nu}^{2} + \theta g \epsilon_{\mu\nu\lambda}\partial_{\mu}F_{\nu\lambda}. \tag{3.1}$$

In the absence of any monopoles, the θ term in this Lagrangian is identically zero. The gauge group of the theory is compact, however, and the θ term is nonzero due to the noncommutability of derivatives acting upon the singular component of the vector field. In other words, if we define $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$, it is clear that in the continuum limit $\epsilon_{\mu\nu\lambda}\partial_{\mu}F^{\nu\lambda} = 0$ and that monopoles are singular configurations for which it is hard to write down any continuum dynamics in the Hamiltonian formalism.

As usual, the θ term may be treated as a total derivative and written as

$$\theta g \epsilon_{\mu\nu\lambda}\int dS_{\mu}F_{\nu\lambda}. \tag{3.2}$$

Through the Hamiltonian formalism we, therefore, obtain

$$H = \frac{1}{2}\int d^{2}x\{[(E_{i} + \theta g \epsilon_{ij}\delta(S_{j} - x)]^{2} + B^{2}\}, \tag{3.3}$$

where

$$E_{i} = -i\frac{\delta}{\delta A_{i}}, \qquad B = \epsilon_{ij}\partial_{i}A_{j}. \tag{3.4}$$

Hence the Hamiltonian is unchanged and the theory can be solved except at the limit of spatial infinity. We shall say no more than the difficulties of treating the θ term other than as a total derivative with the immediate limitations suggest that there is an absence of a physical field upon which the derivative can act. This merely hints that the theory is incomplete as it is.

We shall quickly show that this theory is independent of θ except at spatial infinity. Given this limitation we, therefore, ignore the δ function and write Eq. (3.3) as

$$H = \frac{1}{2}\int d^{2}x[E_{i}^{2} + b^{2}]. \tag{3.5}$$

This is identical to the Hamiltonian of [7]. B is replaced by b to ensure invariance of the Hamiltonian under large gauge transformations as explained in Sec. III C.

We require the action of a vortex creation operator upon the trial wave functional to be

$$V\Psi = \exp[i\theta]\Psi. \tag{3.6}$$

From [7] we know

$$V(x)B(y)=B(y)V(x)+2\pi\delta^2(x-y)V(x),$$

$$\Psi^0=\int D\chi\exp\left(-\frac{1}{2g^2}A_i^\chi G^{-1}A_i^\chi\right), \tag{3.7}$$

$$V\Psi^0=\Psi^0,$$

where $B=\epsilon_{ij}\partial_i A_j$. Equation (3.6) is, therefore, satisfied by

$$\Psi=\exp\left(\frac{i\theta}{2\pi}\oint_C dl_i\epsilon_{ij}A_j\right)\int D\chi\exp\left(-\frac{1}{2g^2}A_i^\chi G^{-1}A_i^\chi\right). \tag{3.8}$$

The contour of the integral in the θ-dependent phase is taken about the spatial plane and so all θ dependence is seen to reside only at spatial infinity.

A. Breaking the internal symmetry of the 't Hooft–Polyakov monopole

It may seem a step in the wrong direction but increasing the complexity of the theory by including a scalar field resolves this problem. The SU(2) 't Hooft–Polyakov monopole has been extensively studied (for a comprehensive treatment see [9]) and so we shall only discuss the salient points. The Lagrangian for the SU(2) non-Abelian vector and scalar fields is

$$L=-\frac{1}{4g^2}F_{\mu\nu}^aF_{\mu\nu}^a-\frac{1}{2}D_\mu\phi^aD_\mu\phi^a-V(\phi^a\phi^a),$$

$$F_{\mu\nu}^a\equiv\partial_\mu A_\nu^a-\partial_\nu A_\mu^a+\epsilon^{abc}A_\mu^bA_\nu^c, \tag{3.9}$$

$$D_\mu\phi^a\equiv\partial_\mu\phi^a+\epsilon^{abc}A_\mu^b\phi^c,$$

$$V(\phi^a\phi^a)\equiv\frac{\lambda}{2}(\phi^a\phi^a-\eta^2)^2.$$

The symmetry of the theory can be broken from SU(2) to U(1) by choosing ϕ^a to point in a specific direction in isospace (e.g., $\phi^a=\phi^3$). After breaking the symmetry, there is one neutral vector field parallel to the direction of ϕ^a (the photon) and two charged vector fields orthogonal to ϕ^a in isospace (the $W^\pm$ bosons). We shall consider the low-energy spectrum only and hence the Lagrangian of the photon and the component of the scalar triplet in the chosen direction of isospace. This scalar field has a mass proportional to $\lambda^{1/2}$ and a nonzero vacuum expectation value.

To define the θ term we shall consider the gauge-invariant tensor $F_{\mu\nu}$ introduced by 't Hooft,

$$F_{\mu\nu}\equiv F_{\mu\nu}^a\hat{\phi}^a-\frac{1}{g}\epsilon^{abc}\hat{\phi}^aD_\mu\hat{\phi}^bD_\nu\hat{\phi}^c. \tag{3.10}$$

Upon the breaking of $SU(2)\rightarrow U(1)$, by choosing $\phi^a=\phi^3$, $F_{\mu\nu}$ reduces to give the tensor of electromagnetism, $F_{\mu\nu}^3=\partial_\mu A_\nu^3-\partial_\nu A_\mu^3$. The interpretation of $F_{\mu\nu}$ is of magnetic flux density. Therefore, the magnetic charge of the unbroken theory, which is topologically invariant and proportional to the topological charge or winding number, is given by

$$q=\frac{1}{8\pi}\int d^3x\,\epsilon_{\mu\nu\lambda}\partial_\mu F_{\nu\lambda}. \tag{3.11}$$

The essential part of the θ term in this theory is, therefore, q. All constants may be absorbed into θ as it is a freely varying parameter. Using $\epsilon_{\mu\nu\lambda}\hat{\phi}^a\partial_\mu F_{\nu\lambda}^a=0$, we can therefore write the θ term of the U(1) theory as

$$\frac{\theta}{\eta}\epsilon_{\mu\nu\lambda}\partial_\mu\phi^3F_{\nu\lambda}^3, \tag{3.12}$$

where the derivative only acts upon the scalar field and η is the value of the scalar field for which the potential is a minimum. The scalar field must behave such that at spatial infinity $\phi^3=\eta$ but within that limit it can fluctuate. This is exactly the topological term suggested by Affleck, Harvey, and Witten [8], with the internal symmetry broken from SU(2) to U(1).

We finally propose the low-energy U(1) Lagrangian with a θ term to be

$$L^\theta=-\frac{1}{4g^2}(F_{\mu\nu}^3)^2-\frac{1}{2}(\partial_\mu\phi^3)^2-\frac{\lambda}{2}[(\phi^3)^2-\eta^2]^2$$
$$-\frac{\theta}{\eta}\epsilon_{\mu\nu\lambda}\partial_\mu\phi^3F_{\nu\lambda}^3. \tag{3.13}$$

In principle, one can make the scalar field heavy so it will not affect the low-energy dynamics and only appear in the θ term of the theory. Let us also note that in the BPS limit $(\lambda=0)$ of the original SU(2) theory the scalar field will be massless.

B. Compactification of U(1) theory from $3+1$ to $2+1$ dimensions

We shall show that in the BPS limit $(\lambda=0)$ exactly the same form of L^θ as above is obtained by compactifying a U(1) theory with a θ term from $3+1$ to $2+1$ dimensions. In this case there is no explicit consideration of a scalar potential but the vacuum expectation value of the scalar field is taken to be $\eta\neq 0$. The action in $3+1$ dimensions is

$$S=\int d^4x\left[-\frac{1}{4e^2}F_{\mu\nu}^2-\frac{1}{4}\theta\epsilon_{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}\right], \tag{3.14}$$

where summation over repeated indices is over $\mu,\nu,\lambda,\rho=0,1,2,3$. θ and e are dimensionless parameters.

Compactification of the third spatial dimension yields

$$S=\int d^3x\,dR\left[-\frac{1}{4e^2}F_{\mu\nu}^2-\frac{1}{2e^2}(\partial_\mu A_3)^2-\theta\epsilon_{\mu\nu\lambda}\partial_\mu A_3F_{\nu\lambda}\right], \tag{3.15}$$

where now summation of repeated indices is over $\mu,\nu,\lambda=0,1,2$. R is the radius of compactification and the $2+1$ theory is obtained in the limit of small R. The dependence of A_μ and A_3 upon the compactified coordinates x_3 can be gauged away giving, after integration,

$$S=\int d^3x\left[-\frac{1}{4g^2}F_{\mu\nu}^2-\frac{\eta^2}{2}(\partial_\mu\phi)^2-\theta\epsilon_{\lambda\mu\nu}\partial_\lambda\phi F_{\mu\nu}\right], \tag{3.16}$$

where $g^2=e^2\Lambda$, $\eta^2=\Lambda/e^2$, $\Lambda\phi=A_3$, $\Lambda=1/R$, and Λ is the UV cutoff. The origin of the mass dimensions of the coupling constants in the $(2+1)$-dimensional U(1) theory is now clear. Given the transformation $\phi^3=\eta(1+\phi)$, the Lagrangian above is easily shown to be the same as Eq. (3.13) in the BPS limit.

C. The Hamiltonian

The Hamiltonian is, therefore,

$$H^\theta=H_A^\theta+H_{\phi^3}^\theta,$$

$$H_A^\theta=\frac{1}{2}\int d^2x\left[g^2\left(E_{A_i}+\frac{2\theta}{\eta}\epsilon_{ji}\partial_j\phi^3\right)^2+\frac{1}{g^2}b^2\right], \tag{3.17}$$

$$H_{\phi^3}=\frac{1}{2}\int d^2x\left[\left(\pi_{\phi^3}-2\frac{\theta}{\eta}b\right)^2+(\partial_i\phi^3)^2+\eta^2\lambda(\phi^3)^2\right],$$

where terms of $O(\phi^3)$ and greater have been omitted because we are interested only in the limit of small λ. The Hamiltonian for the scalar sector can be written in terms of the field ϕ,

$$H_\phi^\theta=\frac{1}{2}\int d^2x\left[\frac{1}{\eta^2}(\pi_\phi-2\theta b)^2+\eta^2(\partial_i\phi)^2+4\eta^4\lambda\phi^2\right], \tag{3.18}$$

but information about the physical field ϕ^3 at zero momentum is lost.

As in the case of the simple one-dimensional model of Sec. II, the periodicity of θ is not explicit through the Hamiltonian formalism. In both the one-dimensional case and in QED_3, θ is easily seen to be periodic in the path-integral formalism with the fundamental domains of $-\frac{1}{2}<\theta<\frac{1}{2}$ for the one-dimensional model and $-1/2q<\theta<1/2q$ for QED_3. In the one-dimensional case it became apparent that, under the transformation $\theta\to\theta+1$, it is only the entire spectrum of solutions which is invariant and each separate energy level is not. But in our model of QED_3 we are considering only the low-energy Hamiltonian with an ansatz only for the vacuum or ground state of the system. Therefore, we do not expect this solution, but rather the entire energy spectrum, to be invariant under the transformation $\theta\to\theta+1/q$ and we restrict θ to its fundamental domain throughout the rest of this calculation.

In the compact theory pointlike vortices with quantized magnetic flux $2\pi n$ cannot be detected by any measurement. Within the Hamiltonian formalism, this means that the creation operator of a pointlike vortex must be indistinguishable from the unit operator. The operator $V(x)$ creates such a vortex; it generates a large transformation which belongs to the compact gauge group and must, therefore, act trivially upon all physical states:

$$V(x)=\exp\left\{i\int d^2y\frac{\epsilon_{ij}(x-y)_j}{(x-y)^2}E_i(y)\right\}. \tag{3.19}$$

We therefore write b in the Hamiltonian to ensure its invariance under the action of $V(x)$ where b is the singlet part of B and $B=\epsilon_{ij}\partial_iA_j$, as in [7]. So, if P is the projection operator upon the whole compact gauge group, we can write formally $b^2=PB^2P$. B does not commute with $V(x)$ but b does.

Gauss's law will also be satisfied by these operators. It is given by

$$\exp\left\{i\int d^2x\,\partial_i\lambda(x)E_i(x)\right\}|\Psi\rangle=|\Psi\rangle, \tag{3.20}$$

where λ here is a regular function.

IV. THE VARIATIONAL ANSATZ FOR THE VACUUM WAVE FUNCTIONAL

In this paper we require the vacuum wave functional in order to calculate the energy, or vacuum expectation value, of the Hamiltonian. Through the functional variational technique, we calculate the expectation value of the energy with our ansatz for the wave functional and minimize this with respect to the propagators to find the forms of the masses, propagators, and the vacuum energy of the theory. Hence, the form of our initial ansatz is of vital importance.

We work in analogy with Sec. II but adopt a slightly modified condition for Ψ^θ. We construct our wave functional to be gauge invariant and have nontrivial θ dependence in the following way. $\Psi^\theta[A_i^n,\phi^3]$ contains all the θ dependence of $\Psi[A_i,\phi^3]=\Sigma_n\Psi^\theta[A_i^n,\phi^3]\Psi^0[A_i^n,\phi^3]$, where the sum over n is the sum over large gauge transformations. $\Psi[A_i,\phi^3]$ is the solution of the equation

$$H^\theta\Psi[A_i,\phi^3]=E^\theta\Psi[A_i,\phi^3], \tag{4.1}$$

such that

$$\begin{aligned}E^\theta=&\int DA_iD\phi^3\sum_{n',n''}\Psi^{0*}[A_i^{n'},\phi^3]\\&\times\Psi^{\theta*}[A_i^{n'},\phi^3]H^\theta\Psi^\theta[A_i^{n''},\phi^3]\Psi^0[A_i^{n''},\phi^3]\\=&\int DA_iD\phi^3\sum_{n',n''}\Psi^{\theta*}[A_i^{n'},\phi^3]\\&\times\Psi^\theta[A_i^{n''},\phi^3]\Psi^{0*}[A_i^{n'},\phi^3]H^0\Psi^0[A_i^{n''},\phi^3],\end{aligned} \tag{4.2}$$

where H^0 corresponds to the Hamiltonian of the system with $\theta=0$. Since the components of Ψ cannot be individually gauge invariant without making the theory trivially θ independent, we have written explicitly the summation over large gauge transformations. We note here that each sector of the $\theta=0$ energy picks up a θ-dependent phase factor. From Sec. III C we know the form of H^θ and H^0. We shall first of all write the wave functional in terms of ϕ and then transform it to depend upon ϕ^3, where $\phi^3=\eta(1+\phi)$, which can be considered as just a modification of the field at zero momentum.

From a previous calculation [7], an ansatz is known for the gauge sector of the theory with $\theta=0$ and which, within the variational framework, can be used to reproduce all the known results of dynamical mass generation, Polyakov scaling, and nonzero string tension. H_ϕ^0 is just the Hamiltonian of a free, massive scalar field which has the solution

$$\exp\left\{-\frac{\eta^2}{2}\int\frac{d^2k}{(2\pi)^2}(k^2+m^2)^{1/2}\phi(k)\phi(-k)\right\}.$$

The combination of these two Gaussian factors gives the wave functional

$$\Psi^0[A_i,\phi]=\int D\chi\exp\left[-\frac{1}{2g^2}(A_i-\partial_i\chi)G^{-1}(A_i-\partial_i\chi)-\frac{\eta^2}{2}\phi K^{-1}\phi\right], \tag{4.3}$$

where G and K are, respectively, the propagators of the vector and scalar fields. They are parameters of the functional variational technique and so have no explicit form at this stage. Equation (4.3) is satisfied by

$$\Psi^\theta[A_i^n,\phi]=\int D\tilde{\chi}\exp[2i\theta\epsilon_{ji}\phi\partial_j(A_i-\partial_i\chi)]=\exp[2i\theta\epsilon_{ji}\phi\partial_j(A_i-\partial_i\chi_\nu)]. \tag{4.4}$$

The phase function $\chi(x)$ is parametrized as

$$\chi(x)=\tilde{\chi}(x)+\chi_\nu(x), \tag{4.5}$$

where $\tilde{\chi}$ is a smooth function and $\chi_\nu(x)$ contains all the discontinuities and can be written as

$$\chi_\nu=\sum_{\alpha=1}^{n_+}\theta(x-x_\alpha)-\sum_{\beta=1}^{n_-}\theta(x-x_\beta), \tag{4.6}$$

where $\theta(x-x_\alpha)$ is a polar angle on a plane centered at x_α. The functional measure can be written as

$$\int D\chi=\int D\tilde{\chi}\sum_{n_+=0}^{\infty}\sum_{n_-=0}^{\infty}\frac{1}{n_+!n_-!}\times\prod_{\alpha=1}^{n_+}\prod_{\beta=1}^{n_-}\int d^2x_\alpha d^2x_\beta\Lambda^4, \tag{4.7}$$

with the explicit UV momentum cutoff Λ.

We adopt the following notation for convenience:

$$A_i^\chi=A_i(x)-\partial_i\chi(x) \tag{4.8}$$

and, for a matrix $M(x-y)$,

$$A_iMA_i=\int d^2xd^2yA_i(x)M(x-y)A_i(y). \tag{4.9}$$

Using this notation, we can write down a gauge-invariant ansatz for the vacuum wave functional in terms of the field ϕ, with nontrivial θ dependence, which satisfies the above formalism:

$$\Psi[A_i,\phi]=\int D\chi_\nu D\tilde{\chi}\exp[-2i\theta\phi\epsilon_{ij}\partial_iA_j^{\chi_\nu}]\times\exp\left[-\frac{1}{2g^2}A_i^\chi G^{-1}A_i^\chi-\frac{\eta^2}{2}\phi K^{-1}\phi\right]. \tag{4.10}$$

We now need to modify this wave functional to write it in terms of the field ϕ^3, to ensure that the extra information about the field of zero momentum is not lost. We write

$$\Psi[A_i,\phi^3]=\int D\chi\exp\left[\frac{2i\theta}{\eta}(\phi^3-\eta)\epsilon_{ji}\partial_jA_i^{\chi_\nu}\right]\times\exp\left[-\frac{1}{2g^2}A_i^\chi G^{-1}A_i^\chi-\frac{1}{2}(\phi^3-\eta)K^{-1}(\phi^3-\eta)\right]. \tag{4.11}$$

It can be seen clearly from consideration of the vacuum expectation of the Hamiltonian above that this ansatz reduces to the desired form in the case of $\theta=0$ and will reproduce all the results of [7].

V. THE VARIATIONAL CALCULATION

The expectation value of any operator $O(A_i,\phi)$ in the wave functional (4.11) is

$$\begin{aligned}\langle O(A_i,\phi)\rangle=Z^{-1}\int DA_iD\phi^3D\tilde{\chi}'D\chi_\nu'D\tilde{\chi}''D\chi_\nu''&\times\exp\left[-\frac{2i\theta}{\eta}(\phi^3-\eta)\epsilon_{ij}\partial_i(A_j-\partial_j\chi_\nu')\right]\\&\times\exp\left[-\frac{1}{2g^2}A_i^{\chi'}G^{-1}A_i^{\chi'}-\frac{1}{2}(\phi^3-\eta)\times K^{-1}(\phi^3-\eta)\right]O(A_i,\phi)\exp\left[\frac{2i\theta}{\eta}(\phi^3-\eta)\times\epsilon_{ij}\partial_i(A_j-\partial_j\chi_\nu'')\right]\\&\times\exp\left[-\frac{1}{2g^2}A_i^{\chi''}G^{-1}A_i^{\chi''}-\frac{1}{2}(\phi^3-\eta)K^{-1}(\phi^3-\eta)\right].\end{aligned} \tag{5.1}$$

If $O(A_i)$ is explicitly gauge invariant we may shift the integration variable $A_i^{\chi''}\to A_i$. With the redefinition $\chi=\chi'-\chi''$ and $\zeta=\chi'+\chi''$, and similarly for $\tilde{\chi}$ and χ_ν, the expectation value reduces to

$$\langle O(A_i,\phi)\rangle = Z^{-1}\int DA_i D\phi^3 D\tilde{\zeta} D\tilde{\chi} D\zeta_\nu D\chi_\nu \times\exp\left[-\frac{2i\theta}{\eta}(\phi^3-\eta)\epsilon_{ij}\partial_i(A_j-\partial_j\chi_\nu)\right] \times\exp\left[-\frac{1}{2g^2}A_i^\chi G^{-1}A_i^\chi - \frac{1}{2}(\phi^3-\eta)K^{-1}\times(\phi^3-\eta)\right]O(A_i,\phi) \times\exp\left[\frac{2i\theta}{\eta}(\phi^3-\eta)\epsilon_{ij}\partial_i A_j\right] \times\exp\left[-\frac{1}{2g^2}A_i G^{-1}A_i - \frac{1}{2}(\phi^3-\eta)K^{-1}\times(\phi^3-\eta)\right]. \tag{5.2}$$

The integration over $D\zeta = D\tilde{\zeta} D\zeta_\nu$ just gives the volume of the gauge group and so cancels with the denominator.

A. Calculation of the energy density

First we shall evaluate Z:

$$Z=\int DA_i D\phi^3 D\tilde{\chi} D\chi_\nu \exp\left[\frac{2i\theta}{\eta}(\phi^3-\eta)\epsilon_{ij}\partial_i\partial_j\chi_\nu - \frac{1}{g^2}A_i G^{-1}A_i - (\phi^3-\eta)K^{-1}(\phi^3-\eta) + \frac{1}{g^2}\partial_i\chi G^{-1}A_i - \frac{1}{2g^2}\partial_i\chi G^{-1}\partial_i\chi\right]. \tag{5.3}$$

By completing the squares with the two changes of variable,

$$\phi^3 \to \phi'^3 = \phi^3 - \frac{i\theta}{\eta}\epsilon_{ij}\partial_i\partial_j\chi_\nu K = \phi^3 - \frac{2i\pi\theta}{\eta}\rho K,$$

$$A_i \to A_i' = A_i - \frac{1}{2}\partial_i\chi, \tag{5.4}$$

and omitting the dashes on the new variables, one obtains

$$Z = Z_a Z_{\phi^3} Z_\chi Z_\nu,$$

$$Z_a = \int DA_i \exp\left[-\frac{1}{g^2}A_i G^{-1}A_i\right] = \det[g^2\pi G],$$

$$Z_{\phi^3} = \int D\phi^3 \exp[-(\phi^3-\eta)K^{-1}(\phi^3-\eta)] = \det[\pi K]^{1/2},$$

$$Z_\chi = \int D\tilde{\chi}\exp\left[-\frac{1}{4g^2}\partial_i\tilde{\chi}G^{-1}\partial_i\tilde{\chi}\right] = \det\left[4g^2\pi\frac{1}{\partial^2}G\right]^{1/2},$$

$$Z_\nu = \int D\chi_\nu \exp\left[-\frac{1}{4g^2}\partial_i\chi_\nu G^{-1}\partial_i\chi_\nu - \frac{\theta^2}{\eta^2}\epsilon_{ij}\partial_i\partial_j\chi_\nu K\epsilon_{ij}\partial_i\partial_j\chi_\nu\right]. \tag{5.5}$$

Details of the derivative transformation—used here and in evaluation of the following Gaussian integrals over the singular function χ_ν—that establishes the connection between derivatives of χ_ν and the distribution function of its singularities or vortices ρ are given in Appendix A. Any singularities in $\phi(x)$ remaining after the change of variables are taken to contribute an infinite action and so are ignored. The singularities in χ_ν, however, cannot be ignored.

To evaluate Z_ν we shall write it as a partition function of a gas of vortices and use the standard trick of [2,10]

$$Z_v = \sum_{n_+,n_-=0}^{\infty}\prod_{\alpha=1}^{n_+}\prod_{\beta=1}^{n_-}\int d^2x_\alpha d^2x_\beta z^{n_++n_-} \times\exp\left\{-\frac{1}{4g^2}\left[\sum_{\alpha,\alpha'}D(x_\alpha-x_{\alpha'}) + \sum_{\beta,\beta'}D(x_\beta-x_{\beta'}) - \sum_{\alpha,\beta}D(x_\alpha-x_\beta)\right]\right\}, \tag{5.6}$$

where the vortex-vortex interaction potential $D(x)$ and the vortex fugacity z are given by

$$D(x) = 8\pi^2\int\frac{d^2k}{(2\pi)^2}\left[k^{-2}G^{-1}(k) + \frac{4\theta^2 g^2}{\eta^2}K(k)\right]\cos(kx),$$

$$z = \Lambda^2\exp\left\{-\frac{1}{8g^2}D(0)\right\}. \tag{5.7}$$

We expect the UV behavior of $G(k)$ and $K(k)$ at large momentum to be the same as in the free theory $[G(k)\to k^{-1}, K(k)\to k^{-1}]$. The vortex fugacity is the smallest variable in the theory, $z \ll g^2 \ll \Lambda$, where, in the limit of weak coupling,

$$z = \Lambda^2\exp\left\{-\frac{\pi}{2}\frac{\Lambda}{g^2}\left(1+\frac{4\theta^2 g^2}{\eta^2}\right)\right\} = \Lambda^2\exp\left\{-\frac{\pi}{2}\left(\frac{\Lambda}{g^2}+\frac{4\theta^2 g^2}{\Lambda}\right)\right\}, \tag{5.8}$$

where we have identified $\Lambda = g\eta$ from the compactification of the U(1) theory from $3+1\to 2+1$ dimensions or, alternatively, from the masses of the charged vector bosons of the theory.

We will need to calculate correlation functions of the vortex density and so, following [7], we write the vortex density as

$$\rho(x) = \sum_{\alpha,\beta}\delta(x-x_\alpha) - \delta(x-x_\beta). \tag{5.9}$$

Introducing a source term the exponential factor including the vortex fugacity in Eq. (5.7) can be written as

$$\Lambda^{2(n_+ + n_-)}\int D\chi \exp\{-2g^2\chi D^{-1}\chi + i\rho\chi + i\rho J\} \tag{5.10}$$

and the sum over the number of vortices and antivortices gives

$$Z_v = \int D\chi \exp\left\{-2g^2(\chi - J)D^{-1}(\chi - J) + \int_x 2\Lambda^2 \cos\chi(x)\right\}\Bigg|_{J=0}. \tag{5.11}$$

Calculating the functional derivatives with respect to the source term yields

$$\langle\rho(x)\rho(y)\rangle = 4g^2 D^{-1}(x-y) - 16g^4\langle D^{-1}\chi(x)D^{-1}\chi(y)\rangle. \tag{5.12}$$

The propagator of χ is easily calculated. First, the cosine potential is rewritten in the normal-ordered form

$$\cos\chi = \frac{z}{\Lambda^2}:\cos\chi:. \tag{5.13}$$

Therefore, to first order in z, the propagator of χ is

$$\int d^2x e^{ikx}\langle\chi(x)\chi(0)\rangle = \frac{1}{4g^2 D^{-1}(k) + 2z} = \frac{D(k)}{4g^2} - z\frac{D^2(k)}{8g^4} + o(z^2). \tag{5.14}$$

To first order, the correlator of the vortex densities is then

$$C(k) = \int d^2x e^{ikx}\langle\rho(x)\rho(0)\rangle = 2z + o(z^2) \tag{5.15}$$

as in [7] but with a θ-dependent modification to z.

Now, we can calculate the expectation value of the Hamiltonian:

$$\begin{aligned}\langle H^\theta\rangle = \langle H_A^\theta\rangle + \langle H_{\phi^3}^\theta\rangle = Z^{-1}\int DA_i D\phi D\tilde{\chi} D\chi_\nu \\ \times\exp\left[\frac{2i\theta}{\eta}(\phi^3-\eta)\epsilon_{ij}\partial_i\partial_j\chi_\nu\right]\exp\left[-\frac{1}{2g^2}A_i^\chi G^{-1}A_i^\chi \right.\\ \left. -\frac{1}{2}(\phi^3-\eta)K^{-1}(\phi^3-\eta)\right][H_A^0 + H_{\phi^3}^0] \\ \times\exp\left[-\frac{1}{2g^2}A_i G^{-1}A_i - \frac{1}{2}(\phi^3-\eta)K^{-1}(\phi^3-\eta)\right].\end{aligned} \tag{5.16}$$

First, we shall consider the purely gauge field sector:

$$\begin{aligned}\left\langle\frac{g^2}{2}\int d^2x[E_{A_i} + 2\theta\epsilon_{ji}\partial_j(\phi^3-\eta)]^2\right\rangle \\ = \frac{g^2}{2}Z^{-1}\int DA_i D\phi^3 D\tilde{\chi} D\chi_\nu\left[\frac{2}{g^2}\mathrm{Tr}G^{-1} - \frac{1}{g^2}A_i G^{-2}A_i\right] \\ \times\exp\left[\frac{2i\theta}{\eta}(\phi^3-\eta)\epsilon_{ij}\partial_i\partial_j\chi_\nu - \frac{1}{g^2}A_i G^{-1}A_i \right.\\ -(\phi^3-\eta)K^{-1}(\phi^3-\eta) + \frac{1}{g^2}\partial_i\chi G^{-1}A_i \\ \left. -\frac{1}{2g^2}\partial_i\chi G^{-1}\partial_i\chi\right].\end{aligned} \tag{5.17}$$

Completing the squares and performing the functional integrations gives

$$\begin{aligned}\frac{1}{V}\left\langle\frac{g^2}{2}\int d^2x[E_{A_i} + 2\theta\epsilon_{ji}\partial_j(\phi^3-\eta)]^2\right\rangle \\ = \frac{1}{2}\int\frac{d^2k}{(2\pi)^2}\left[\frac{1}{2}G^{-1}(k) - \frac{\pi^2}{g^2}k^{-2}C(k)G^{-2}\right] \\ = \frac{1}{4}\int\frac{d^2k}{(2\pi)^2}\left[G^{-1}(k) - \frac{4\pi^2}{g^2}zk^{-2}G^{-2}(k)\right].\end{aligned} \tag{5.18}$$

The Gaussian integral over χ_ν is transformed into a correlation function of ρ. This procedure is given in more detail in Appendix A.

Now for the magnetic term. In every gauge-invariant state $\langle b^2\rangle = \langle B^2\rangle$ by definition. We will, therefore, calculate $\langle B^2\rangle$. But since it is not itself gauge invariant some care is needed with the integrals over χ and ζ:

$$\begin{aligned}\left\langle\frac{1}{2g^2}b^2\right\rangle = Z^{-1}\int DAD\phi^3 D\tilde{\chi}' D\chi_\nu' D\tilde{\chi}'' D\chi_\nu''\frac{1}{2g^2} \\ \times[\epsilon_{ij}\partial_i A_j]^2\exp\left[\frac{2i\theta}{\eta}(\phi^3-\eta)\epsilon_{ij}\partial_i\partial_j(\chi_\nu' - \chi_\nu'') \right.\\ -\frac{1}{2g^2}A_i^{\chi'}G^{-1}A_i^{\chi'} - \frac{1}{2g^2}A_i^{\chi''}G^{-1}A_i^{\chi''} \\ \left. -(\phi^3-\eta)K^{-1}(\phi^3-\eta)\right] \\ = Z^{-1}\int DA_i D\phi^3 D\chi D\zeta\frac{1}{2g^2}\left[\epsilon_{ij}\partial_i\left\{A_j + \frac{1}{2}\partial_j \right.\right.\\ \left.\left. \times(\zeta-\chi)\right\}\right]^2\exp\left[\frac{2i\theta}{\eta}(\phi^3-\eta)\epsilon_{ij}\partial_i\partial_j\chi_\nu \right.\\ -\frac{1}{2g^2}A_i^\chi G^{-1}A_i^\chi - \frac{1}{2g^2}A_i G^{-1}A_i \\ \left. -(\phi^3-\eta)K^{-1}(\phi^3-\eta)\right].\end{aligned} \tag{5.19}$$

We have again used $\chi=\chi'-\chi''$ and $\zeta=\chi'+\chi''$. The linear term in ζ disappears due to the symmetry of its measure. The term quadratic in ζ is independent of G, K, and θ and so contributes nothing of interest to either the energy or the minimization equations. After completing the squares we obtain

$$\frac{1}{V}\left\langle \frac{1}{2g^2}b^2\right\rangle = Z_a^{-1}\int DA_iD\phi^3D\chi\frac{1}{2g^2}[\epsilon_{ij}\partial_iA_j]^2 \times\exp\left[-\frac{1}{g^2}A_iG^{-1}A_i\right]=\frac{1}{4}\int\frac{d^2k}{(2\pi)^2}k^2G(k). \tag{5.20}$$

So, for the purely gauge field sector, we obtain

$$\frac{1}{V}\langle H_A^\theta\rangle=\frac{1}{4}\int\frac{d^2k}{(2\pi)^2}\left[G^{-1}(k)+k^2G(k) -\frac{4\pi^2}{g^2}zk^{-2}G^{-2}(k)\right]. \tag{5.21}$$

This is of the same form as [7] but with the modified expression for z.

Following the procedure above we calculate the vacuum expectation value of the Hamiltonian of the scalar field:

$$\begin{aligned}\frac{1}{V}\left\langle\frac{1}{2}\int d^2x\left(\pi_{\phi^3}-\frac{2\theta}{\eta}b\right)^2\right\rangle &=\frac{1}{2}Z^{-1}\int DA_iD\phi^3D\tilde{\chi}D\chi_\nu[K^{-1}-(\phi^3-\eta)K^{-2}\\&\times(\phi^3-\eta)]\exp\left[\frac{2i\theta}{\eta}(\phi^3-\eta)\epsilon_{ij}\partial_i\partial_j\chi_\nu-\frac{1}{g^2}A_iG^{-1}A_i\right.\\&-(\phi^3-\eta)K^{-1}(\phi^3-\eta)+\frac{1}{g^2}\partial_i\chi G^{-1}A_i-\frac{1}{2g^2}\\&\left.\times\partial_i\chi G^{-1}\partial_i\chi\right]\\&=\int\frac{d^2k}{(2\pi)^2}\left[\frac{1}{4}K^{-1}(k)+\frac{2\pi^2\theta^2}{\eta^2}C(k)\right]\\&=\int\frac{d^2k}{(2\pi)^2}\left[\frac{1}{4}K^{-1}(k)+\frac{4\pi^2\theta^2}{\eta^2}z\right].\end{aligned} \tag{5.22}$$

Similarly, the terms quadratic in ϕ give

$$\frac{1}{V}\left\langle\frac{1}{2}\int d^2x(\partial_i\phi^3)^2\right\rangle =\frac{1}{4}\int\frac{d^2k}{(2\pi)^2}\left[k^2K(k)-\frac{16\pi^2\theta^2}{\eta^2}zk^2K^2(k)\right],$$

$$\frac{1}{V}\left\langle\frac{1}{2}\int d^2\lambda\,\eta^2(\phi^3-\eta)^2\right\rangle =\frac{1}{4}\int\frac{d^2k}{(2\pi)^2}[\lambda\eta^2K(k)-16\lambda\pi^2\theta^2zK^2(k)]. \tag{5.23}$$

Therefore, for the scalar field we obtain

$$\frac{1}{V}\langle H_{\phi^3}^\theta\rangle=\frac{1}{4}\int\frac{d^2k}{(2\pi)^2}\left\{K^{-1}(k)+\frac{16\pi^2\theta^2}{\eta^2}z +\left[K(k)-\frac{16\pi^2\theta^2}{\eta^2}zK^2(k)\right](k^2+\lambda\eta^2)\right\}. \tag{5.24}$$

B. Minimization of the vacuum energy density

Details of the functional minimization of the energy density with respect to the vector field propagator $G(k)$ and the scalar field propagator $K(k)$ are given in Appendix B.

We obtain the simple minimization equations

$$\begin{aligned}0=\frac{1}{4}[k^2-G^{-2}(k)]&-\frac{\pi^4}{g^4}k^{-2}G^{-2}(k)z\int\frac{d^2p}{(2\pi)^2}p^{-2}G^{-2}(p)\\&+\frac{4\theta^2\pi^4}{\eta^2g^2}k^{-2}G^{-2}(k)z\int\frac{d^2p}{(2\pi)^2}[1-K^2(p)(p^2+\lambda\eta^2)],\end{aligned} \tag{5.25}$$

with the solution

$$G^{-2}(k)=\frac{k^4}{k^2+m^2},$$

$$m^2=\frac{4\pi^4}{g^4}z\left[\int\frac{d^2p}{(2\pi)^2}p^{-2}G^{-2}(p) -\frac{4\theta^2g^2}{\eta^2}\int\frac{d^2p}{(2\pi)^2}[1-K^2(p)(p^2+8\lambda\eta^2)]\right] \tag{5.26}$$

and

$$0=\frac{1}{4}\left\{\frac{16\pi^4\theta^2}{g^2\eta^2}z\int\frac{d^2p}{(2\pi)^2}\left[p^{-2}G^{-2}(p)-\frac{4\theta^2g^2}{\eta^2}[1-K^2(p)\times(p^2+\lambda\eta^2)]\right]+(k^2+\lambda\eta^2)-K^{-2}(k)\right\}, \tag{5.27}$$

with the solution

$$K^{-2}(k)=k^2+m_\phi^2+m_\theta^2,$$

$$m_\phi^2=\lambda\eta^2, \tag{5.28}$$

$$m_\theta^2=\frac{16\pi^4\theta^2}{g^2\eta^2}z\int\frac{d^2p}{(2\pi)^2}\bigg[p^{-2}G^{-2}(p)-\frac{4\theta^2g^2}{\eta^2}[1-K^2(p)\times(p^2+\lambda\eta^2)]\bigg]=\frac{4\theta^2g^2}{\eta^2}m^2.$$

Explicit evaluation of the photon mass gives

$$m^2=\frac{\pi^3}{g^4}z\left[\Lambda^2-m^2\ln\left(\frac{\Lambda^2+m^2}{m^2}\right)-\frac{4\theta^2g^2}{\eta^2}(m_\phi^2+m_\theta^2)\ln\left(\frac{\Lambda^2+m_\phi^2+m_\theta^2}{m_\phi^2+m_\theta^2}\right)\right]. \quad (5.29)$$

The θ dependence of z has been shown to be subleading in terms of Λ, Eq. (5.8). In the limit of a large UV cutoff momentum Λ, all the θ dependence of z is, therefore, suppressed in subleading terms of Λ. In the BPS limit ($\lambda=0$), the θ dependence in Eq. (5.29) only occurs at $O(z^2)$ and so is suppressed further.

In the BPS limit we can write, to first order in z,

$$m^2=\frac{\pi^3}{g^4}\Lambda^2 z,$$

$$m_\phi^2=0, \quad (5.30)$$

$$m_\theta^2=4\frac{\theta^2g^2}{\eta^2}m^2.$$

We should note here that m^2 is in agreement with [7] to first order in z.

C. Evaluation of the vacuum energy density

Now, as we have the forms of the propagators and the masses for the fields we can consider the θ dependence of the vacuum expectation value of the Hamiltonian. The vacuum energy densities of the gauge and scalar sectors evaluated in the limit of a large UV cutoff Λ are

$$\frac{1}{V}\langle H_A^\theta\rangle=\frac{1}{8\pi}\left[\frac{2}{3}\Lambda^3+\frac{1}{3}m^3\right]-\int\frac{d^2k}{(2\pi)^2}\frac{\pi^2}{g^2}zk^{-2}G^{-2}(k),$$

$$\frac{1}{V}\langle H_{\phi^3}^\theta\rangle=\frac{1}{8\pi}\left\{\frac{2}{3}\Lambda^3+\frac{1}{3}\left(\lambda\eta^2+\frac{4\theta^2g^2}{\eta^2}m^2\right)^{3/2}+\lambda\eta^2\left[\Lambda-\left(\lambda\eta^2+\frac{4\theta^2g^2}{\eta^2}m^2\right)^{1/2}\right]\right\}+\int\frac{d^2k}{(2\pi)^2}\frac{4\pi^2\theta^2}{\eta^2}z[1-K^2(k)(k^2+\lambda\eta^2)]. \quad (5.31)$$

The last term in each expression combines to give the exact form of m^2. It has been shown that the θ dependence of z is a subleading term in Λ, Eq. (5.8). Therefore, the gauge sector of the theory is manifestly independent of θ. The scalar sector does have an explicit dependence upon θ but it is always suppressed by an order of z. It is interesting to note, however, that it is in the BPS limit ($\lambda=0$) and in the limit of a very massive scalar field that the θ dependence of the total vacuum energy density is most greatly suppressed. In the BPS limit the total vacuum energy density is

$$\frac{1}{V}\langle H^\theta\rangle=\frac{1}{8\pi}\left[\frac{4}{3}\Lambda^3+\frac{1}{3}\left(1+\frac{8\theta^3g^3}{\eta^3}\right)m^3\right]-\frac{1}{4}\frac{g^2}{\pi^2}m^2, \quad (5.32)$$

where the θ dependence of the scalar sector is suppressed in terms of $O(z^{3/2})$. In the limit of a very massive scalar field (λ becomes large) the vacuum energy density is

$$\frac{1}{V}\langle H^\theta\rangle=\frac{1}{8\pi}\left[\frac{4}{3}\Lambda^3+\frac{1}{3}m^3-\frac{2}{3}(\lambda\eta^2)^{3/2}+\lambda\eta^2\Lambda\right]-\frac{1}{4}\frac{g^2}{\pi^2}m^2. \quad (5.33)$$

In the limit of a very massive scalar field, the only θ dependence of the scalar sector is in the modified form of z, which we have already shown to be subleading in terms of Λ. So in this limit we recover the exact result of Vergeles.

D. Expectation value of the Wilson loop

Finally, we can calculate the expectation value of the Wilson loop, as in [7], to see how the θ dependence affects confinement:

$$W_C=\left\langle\exp\left(il\oint_C A_i dx_i\right)\right\rangle=\left\langle\exp\left(il\int_S BdS\right)\right\rangle, \quad (5.34)$$

where l is an arbitrary integer and the integral is over the area S bounded by the loop C. We have written B rather than b, since this exponential operator is invariant under transformations $B(x)\to B(x)+2\pi$, generated by the vortex operator:

$$W_C=Z^{-1}\int DA_iD\phi D\tilde{\chi}D\chi_\nu\exp\bigg[\frac{2i\theta}{\eta}(\phi^3-\eta)\epsilon_{ij}\partial_i\partial_j\chi_\nu-\frac{1}{g^2}A_iG^{-1}A_i+\frac{1}{g^2}\partial_i\chi G^{-1}A_i-\frac{1}{2g^2}\partial_i\chi G^{-1}\partial_i\chi-(\phi^3-\eta)K^{-1}(\phi^3-\eta)+il\int_S BdS\bigg]. \quad (5.35)$$

After completing the squares,

$$W_C=W_0W_\nu, \quad (5.36)$$

where

$$W_0 = Z_a^{-1}\int DA_i \exp\left[-\frac{1}{g^2}A_i G^{-1}A_i + il\int_S BdS\right],$$

$$W_\nu = Z_\nu^{-1}\int D\chi_\nu \exp\left[-\frac{1}{4g^2}\partial_i\chi_\nu\left(G^{-1}+\frac{4\theta^2 g^2}{\eta^2}\partial^2 K\right)\partial_i\chi_\nu - \frac{il}{2}\int_S dS\epsilon_{ij}\partial_i\partial_j\chi_\nu dx_i\right]. \quad (5.37)$$

In a weak coupling W_0 becomes

$$W_0 = \exp\left\{-\frac{l^2}{2}\int_{x,y}\langle B(x)B(y)\rangle d^2x d^2y\right\} = \exp\left[-\frac{l^2 g^2}{4}mS\right] \quad (5.38)$$

in the limit $k\to 0$. This term is independent of θ and gives the string tension $\sigma = (l^2 g^2/4)m$.

W_ν differs from unity only for odd l, for which it can be calculated,

$$W_\upsilon = \left\langle \exp\left(i\pi\int_S \rho(x)d^2x\right)\right\rangle = \int D\chi \exp\left(-2g^2\chi D^{-1}\chi + \int_x 2\Lambda^2\cos[\chi(x)-\alpha(x)]\right), \quad (5.39)$$

where $\alpha(x)$ is zero outside and π inside the loop. Following the normal ordering prescription for a scalar field given in Sec. V A, and noting that the solution to the classical equations which contributes to the leading order result is $\chi(x)=0$, we obtain the solution $W_\nu = \exp[-2zS]$. As in [7] this is a subleading correction to the string tension ($2z \ll \sigma$) where the θ dependence in z (and hence also in the factor $z^{1/2}$ in m) is greatly suppressed as a subleading term in Λ.

VI. AXIONIC CONFINING STRINGS

Polyakov showed that purely gauge field compact QED$_3$ is equivalent to a nonstandard string theory [6]. We shall show in this section that our proposed Lagrangian for low-energy compact QED$_3$ with a scalar field and a θ term gives rise to the same nonstandard string theory but with a θ-dependent modification of the mass of the photon. This modification of the mass is a subleading term in the UV momentum cutoff and is as predicted in Eq. (5.8).

We shall first give a brief review of the relevant details from [6]. The Wilson loop calculated in compact QED$_3$ is

$$W(C) = \int DA_\mu \exp\left[-S(A) + i\oint dx_\mu A_\mu\right],$$

$$S(A) = \frac{1}{4g^2}\int d^3x F_{\mu\nu}^2. \quad (6.1)$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In the calculation of $W(C)$ one must include the monopole configurations of the vector field. As a result, Eq. (6.2) has the representation $W(C) = W_0(C)W_M(C)$ where the first factor comes from the Gaussian integration over the vector field and the second factor is from the contribution of the pointlike monopoles. As in the instanton gas calculations [2], the contribution of one monopole at point x is considered first:

$$W_M^1(x,C) \propto \exp\left(-\frac{a}{g^2} + i\eta(x,C)\right),$$

$$\eta(x,C) = \oint_C dy_\mu A_\mu^{(\mathrm{mon})}(x-y) = \int_{\Sigma_C} d^2\sigma_\mu(y)\frac{(x-y)}{|x-y|^3}, \quad (6.2)$$

where $\eta(x,C)$ is the solid angle formed by the point x and the contour C. Σ_C is an arbitrary surface bounded by the contour C. $a = M_W\epsilon(\lambda/g^2)$ where $M_W = g\eta = \Lambda$ as stated in Sec. III. For $\lambda=0$, $\epsilon(\lambda/g^2) = 4\pi$. Summation over all possible monopole configurations leads to the scalar field theory

$$W_M(C) \propto \int D\phi \exp\left[-g^2\int d^3x\left\{\frac{1}{2}(\partial\phi)^2 + m^2 \times[1-\cos(\phi+\eta)]\right\}\right], \quad (6.3)$$

with $m^2 \propto \exp[-a/g^2]$. Rewriting this theory in terms of an effective action by introducing a rank 2 antisymmetric tensor field B, Polyakov suggested a new type of strings, which he called confining strings. Let us consider the axionic confining strings; the strings in our theory with an extra scalar (axionic) field coupled to the photon field in a θ term.

We shall show that the proposed low-energy theory for QED$_3$ with a scalar field and a θ term is the equivalent of Eq. (6.3) with a modification of the photon mass m. Working in direct analogy with the above, we calculate the Wilson loop:

$$W(C) = \int DA_\mu \exp\left[-S(A,\phi^3) + i\oint dx_\mu A_\mu\right],$$

$$S(A,\phi^3) = \int d^3x\left[\frac{1}{4g^2}F_{\mu\nu}^2 + \frac{1}{2}(\partial_\mu\phi^3)^2 + \frac{2i\theta}{\eta}\partial_\mu\phi^3\tilde{F}_\mu\right]$$
$$= \int d^3x\left[\frac{1}{4g^2}F_{\mu\nu}^2 - \frac{1}{2}\phi^3\Box\phi^3 - \frac{2i\theta}{\eta}\phi^3\partial_\mu\tilde{F}_\mu\right] + 2i\theta q, \quad (6.4)$$

where $\tilde{F}_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda}F_{\nu\lambda}$ and $q = \int d^2S_\mu\tilde{F}_\mu$. The ϕ^3 field is eliminated by Gaussian integration with the transformation $\phi^3(x)\to\phi'^3(x) = \phi^3(x) + (2i\theta/\eta)\Box^{-1}(x)\partial_\mu\tilde{F}_\mu(x)$. Care is needed with the definition of the inverse D'Alembertian, the action of which upon an arbitrary function $f(x)$ is,

$$\Box^{-1}(x)f(x) = \int d^3x'\Box^{-1}(x-x')f(x'). \quad (6.5)$$

The action of the D'Alembertian gives the correct result allowing the interpretation of $\Box^{-1}$ as a Green's function:

$$\Box(x)\Box^{-1}(x-x') = \delta(x-x'). \quad (6.6)$$

Therefore, integrating out the ϕ^3 field we obtain

$$S(A,\phi^3)=\int d^3x\left[\frac{1}{4g^2}F_{\mu\nu}^2-\frac{2\theta^2}{\eta^2}\int d^3x'\Box^{-1}\right.$$
$$\left.\times(x-x')\partial_\mu\tilde{F}_\mu(x')\partial_\nu\tilde{F}_\nu(x)\right]+2i\theta q. \tag{6.7}$$

We are working in the limit of zero scalar potential here. The contribution of one monopole at point x is, therefore,

$$W_M^1(x,C)\propto\exp\left(-\frac{a}{g^2}+b+i[\eta(x,C)+2\theta q]\right), \tag{6.8}$$

where b comes from the evaluation of the θ^2 term in Eq. (6.7), which can be written as

$$\frac{2\theta^2g^2}{\Lambda^2}\int d^3xd^3x'\Box^{-1}(x-x')\partial_\mu\tilde{F}_\mu(x')\partial_\nu\tilde{F}_\nu(x). \tag{6.9}$$

Immediately, we see that Eq. (6.9) has the mass dimension of -1 in agreement with the θ^2 modification of z, Eq. (5.8). From [2], we know the configuration of the A field due to monopoles gives rise to

$$\tilde{F}_\mu=\frac{1}{2}\left(\frac{x_\mu}{|x|^3}-4\pi\delta_{\mu3}\theta(x_3)\delta(x_1)\delta(x_2)\right). \tag{6.10}$$

As in [2], each monopole is surrounded with a sphere of radius R such that $M_W^{-1}\ll R\ll|x_{ab}|$ where x_{ab} is the distance between two monopoles, located at x_a and x_b. Inside the sphere, Eq. (6.10) is not valid and the influence of other monopoles may be neglected. This is the region that gives rise to the so-called self-pseudoenergy of the monopoles, a. Monopoles of charge >1 are neglected as they can be considered as the limit of two or more monopoles in close proximity and these configurations have been shown to be inessential [2]. Only far separated monoples are important in the infrared region. For a large separation,

$$\partial_\mu\tilde{F}_\mu\simeq-2\pi\delta^3(x). \tag{6.11}$$

Writing Eq. (6.9) in momentum space we obtain

$$\frac{2\theta^2g^2}{\Lambda^2}4\pi^2\int d^3xd^3x'\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}\int\frac{d^3k''}{(2\pi)^3}k^{-2}\exp\{i[k(x-x')+k'x'+k''x]\}\propto\frac{\theta^2g^2}{\Lambda}. \tag{6.12}$$

This is in direct agreement with Eq. (5.8). Our proposed Lagrangian for low-energy QED$_3$ hence gives an equivalent form of Eq. (6.3) with the modifications of m^2 replaced by m'^2 and $\eta(x,C)$ replaced by $\eta'(x,C)$ where $m'^2=m^2\exp[-\text{const}\times\theta^2g^2/\Lambda]$ and $\eta'(x,C)=\eta(x,C)+2\theta q$. The modified form of the photon mass will not change the rest of the formalism of [6]. The constant shift in η will have an effect upon the monopole configuration, or shape of the surface Σ_C that minimizes the action. Because of the integration over all x, which is equivalent to a sum over all angles, such a constant shift should have no effect within the formalism of [6].

It is interesting to contrast this result to that of Diamantini, Quevedo, and Trugengerger [11], who introduced a θ term in a four-dimensional compact U(1) theory and proceeded to compute the low-energy effective action for the confining string in a derivative expansion. In [11] the θ term is written in the Kalb-Ramond action,

$$S(B_{\mu\nu})=\int d^4x\left[\frac{1}{12\Lambda^2}H_{\mu\nu\alpha}H_{\mu\nu\alpha}+\frac{1}{4e^2}B_{\mu\nu}B_{\mu\nu}\right.$$
$$\left.+\frac{i\theta}{64\pi^2}B_{\mu\nu}\epsilon_{\mu\nu\alpha\beta}B_{\alpha\beta}\right], \tag{6.13}$$

which produces a shift in the mass of the field:

$$m_\theta=\frac{e\Lambda}{4\pi}\sqrt{\left(\frac{4\pi}{e^2}\right)^2+t^2},$$

$$t\equiv\frac{\theta}{2\pi}, \tag{6.14}$$

$$\Lambda\propto\frac{\Lambda_0}{4}\exp\left[-\frac{a'}{g^2}\right],$$

where Λ_0 is the UV cutoff and a' is a constant. To compare this with our calculation above we shall write the results of [11] in a more convenient form:

$$m_\theta^2=m_0^2\left(1+\frac{4e^4\theta^2}{(4\pi)^4}\right)\simeq m_0^2\exp\left[\frac{4e^4\theta^2}{(4\pi)^4}\right],$$

$$m_0^2=\frac{\Lambda^2}{e^2}\propto\frac{\Lambda_0^2}{16e^2}\exp\left[-\frac{2a'}{g^2}\right]. \tag{6.15}$$

We see that the results of the calculation above and those of [11] are in qualitative agreement—the inclusion of a θ term in a compact U(1) theory in $2+1$ and $3+1$ dimensions leads to a θ^2 shift in the mass of the corresponding confining string theory. Here we should also note another recent paper about the confining string corresponding to compact U(1) theory in four Euclidean dimensions [12].

VII. CONCLUSION

We have found that it is much more natural to include a scalar Higgs field to consider a θ term in QED$_3$. The theory without a scalar field gives the same result of being independent of θ but excludes the limit of spatial infinity. From consideration of the θ term in the non-Abelian Lagrangian of the SU(2) 't Hooft—Polyakov monopole we propose such a term in QED_3 in which the gauge and scalar fields are coupled. The term we propose is exactly of the form of the topological term proposed by Affleck, Harvey, and Witten [8]. We find that this term is expected if the $(2+1)$-dimensional theory is considered as a result of dimensional reduction of a purely gauge U(1) theory with a θ term in $3+1$ dimensions.

The gauge sector of QED_3 is found to have a mass and a vacuum energy that are independent of θ for weak coupling in the limit of large UV cutoff. The independence from θ of the vacuum energy of the gauge sector is in agreement with [3]. The nonperturbative dynamical mass generation for the photon, the vacuum energy density, and the expectation value of the Wilson loop are all in agreement with [7]. In both [3] and [7], QED_3 was considered without a scalar field.

Further, we find that the vacuum energy of the scalar field is dependent upon θ, but that this dependence is suppressed. It is in the BPS limit of zero scalar potential and in the limit of a very massive scalar field (large scalar potential) that the θ dependence is most greatly suppressed. The θ dependence is in terms of $O(z^{3/2})$ in the BPS limit but, in the limit of a very massive scalar field the scalar sector, and hence the total vacuum energy, becomes independent of θ in direct agreement with [3].

A nonperturbative dynamical mass proportional to θ is generated for the scalar field which does not disappear in the limit of zero scalar potential.

It is clear from the calculation of the string tension that the expectation value of the Wilson loop obeys the area law and leads to confinement. Its dependence upon θ is greatly suppressed for weak coupling in the limit of a large UV cutoff.

An extension of Polyakov's work on confining strings [6] has shown that our proposed Lagrangian for low-energy QED_3, with a scalar field and a θ term, is equivalent to a nonstandard string theory. This string theory is of the same form as that found by Polyakov to be equivalent to purely gauge field compact QED_3 with a θ^2, but subleading in UV momentum cutoff, modification of the photon mass, and a θ-dependent shift of the shape of the minimal surface. The modification of the photon mass is in direct agreement with our variational calculation.

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APPENDIX A

In this appendix we shall give details of the evaluation of integrals over the singular function χ_ν by considering the example

$$Z_\nu^{-1}\int D\chi_\nu \partial_i\chi_\nu G^{-2}\partial_i\chi_\nu \exp\bigg[-\frac{1}{4g^2}\partial_i\chi_\nu G^{-1}\partial_i\chi_\nu - \frac{4\theta^2}{\eta^2}\pi^2\rho K\rho\bigg]. \tag{A1}$$

We use the transformation

$$\partial_i\chi_\nu = \epsilon_{ij}\partial_j\psi. \tag{A2}$$

The singularities in χ_ν are angular functions in two dimensions and so we can use the standard definition

$$\chi_\nu = -\frac{i}{2}\sum_{\alpha,\beta}\left[\ln\left(\frac{z-z_\alpha}{\overline{z}-\overline{z}_\alpha}\right)-\ln\left(\frac{z-z_\beta}{\overline{z}-\overline{z}_\beta}\right)\right]. \tag{A3}$$

The form of ψ is, therefore,

$$\psi = \sum_{\alpha,\beta}\left[\ln\left(\frac{1}{|z-z_\alpha|}\right)-\ln\left(\frac{1}{|z-z_\beta|}\right)\right], \tag{A4}$$

so that $\partial_j^2\psi = -2\pi\rho$ or, in momentum space, $\psi = -2\pi\rho k^{-2}$ where ρ is the distribution function of the vortices or singularities of χ_ν and is defined as

$$\rho(x) = \sum_{\alpha,\beta}\delta(x-x_\alpha)-\delta(x-x_\beta). \tag{A5}$$

So Eq. (A1) is now transformed to

$$4\pi^2 k^{-2}G^{-2}\langle\rho(x)\rho(y)\rangle = 8\pi^2\int\frac{d^2k}{(2\pi)^2}k^{-2}G^{-2}z. \tag{A6}$$

$\langle\rho\rho\rangle$ is calculated to $O(z)$ in Sec. V A.

APPENDIX B

We shall functionally minimize the vacuum energy density with respect to the scalar and vector propagators to obtain the forms of the masses and propagators of the fields. From Eq. (5.8) we note that

$$\frac{\delta z}{\delta G(k)} = \frac{1}{4g^2}k^{-2}G^{-2}(k)z,$$

$$\frac{\delta z}{\delta K(k)} = -\frac{\theta^2}{\eta^2}z. \tag{B1}$$

First, we consider the minimization of $(1/V)\langle H^\theta\rangle$ with respect to $G(k)$:

$$\frac{\delta\langle H_A^\theta\rangle}{\delta G(k)} = \frac{1}{4}\bigg\{k^2 - G^{-2}(k) + \frac{4\pi^2}{g^2}\bigg[2zk^{-2}G^{-3}(k) - \frac{\delta z}{\delta G(k)}4\pi^2\int\frac{d^2p}{(2\pi)^2}p^{-2}G^{-2}(p)\bigg]\bigg\}. \tag{B2}$$

Assuming that at large momenta $G(k)\rightarrow k^{-1}$, the ratio of the fourth term to the third term in Eq. (B2) is

$$\frac{\delta z}{\delta G(k)}\,\frac{4\pi^2\int\frac{d^2p}{(2\pi)^2}p^{-2}G^{-2}(p)}{2zk^{-2}G^{-3}(k)}\propto\frac{\Lambda^2}{g^2k}. \tag{B3}$$

This is much greater than 1, at weak coupling, for any value of k and so we omit the third term from Eq. (B2). Also, using

$$\frac{\delta\langle H^{\theta}_{\phi^3}\rangle}{\delta G(k)}=\frac{16\pi^4\theta^2}{\eta^2}\,\frac{\delta z}{\delta G(k)}\int\frac{d^2p}{(2\pi)^2}[1-K^2(p)(p^2+\lambda\eta^2)], \tag{B4}$$

we obtain the minimization equation

$$0=\frac{1}{4}[k^2-G^{-2}(k)]-\frac{\pi^4}{g^4}k^{-2}G^{-2}(k)z\times\int\frac{d^2p}{(2\pi)^2}p^{-2}G^{-2}(p)+\frac{4\theta^2\pi^4}{\eta^2g^2}k^{-2}G^{-2}(k)z\times\int\frac{d^2p}{(2\pi)^2}[1-K^2(p)(p^2+\lambda\eta^2)]. \tag{B5}$$

Now, consider the minimization with respect to $K(k)$:

$$\frac{\delta\langle H^{\theta}_{A}\rangle}{\delta K(k)}=-\frac{4\pi^4}{g^2}\,\frac{\delta z}{\delta K(k)}\int\frac{d^2p}{(2\pi)^2}p^{-2}G^{-2}(p), \tag{B6}$$

$$\frac{\delta\langle H^{\theta}_{\phi^3}\rangle}{\delta K(k)}=\frac{1}{4}\Bigg[(k^2+\lambda\eta^2)-K^{-2}(k)-\frac{32\pi^2\theta^2}{\eta^2}zK(k)\times(k^2+\lambda\eta^2)+\frac{64\pi^4\theta^2}{\eta^2}\,\frac{\delta z}{\delta K(k)}\int\frac{d^2p}{(2\pi)^2}\times[1-K^2(p)(p^2+\lambda\eta^2)]\Bigg]. \tag{B7}$$

Assuming that at large momenta $K^2(k)\rightarrow k^{-2}+k^{-4}a^2$, where a is the constant coefficient of the second term in the expansion, the ratio of the penultimate to the last term in $\delta\langle H^{\theta}_{\phi}\rangle/\delta K(k)$ is

$$\frac{K(k)(k^2+\lambda\eta^2)}{2\frac{\theta^2}{\eta^2}\int\frac{d^2p}{(2\pi)^2}[1-K^2(p)(p^2+\lambda\eta^2)]}\propto\frac{k(1+\lambda\eta^2k^{-2})}{\frac{\theta^2}{\eta^2}(a^2+\lambda\eta^2)\ln\Lambda}. \tag{B8}$$

This is much less than 1 for nonzero θ in the UV limit for any value of λ and so the penultimate term is ignored. So, we obtain another simple minimization equation

$$0=\frac{1}{4}\Bigg[\frac{16\pi^4\theta^2}{g^2\eta^2}z\int\frac{d^2p}{(2\pi)^2}\Bigg(p^{-2}G^{-2}(p)-\frac{4\theta^2g^2}{\eta^2}[1-K^2(p)\times(p^2+\lambda\eta^2)]\Bigg)+(k^2+\lambda\eta^2)-K^{-2}(k)\Bigg]. \tag{B9}$$

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