Monopoles and instantons on partially compactified D-branes

Kimyeong Lee* and Piljin Yi†
Physics Department, Columbia University, New York, New York 10027
(Received 26 February 1997)

Motivated by the recent D-brane constructions of world-volume monopoles and instantons, we study the supersymmetric SU(N) Yang-Mills theory on $S^1\times R^{3+1}$, spontaneously broken by a Wilson loop. In addition to the usual $N-1$ fundamental monopoles, the Nth Bogomol'nyi-Prasad-Sommerfield monopole appears from the Kaluza-Klein sector. When all N monopoles are present, net magnetic charge vanishes and the solution can be reinterpreted as a Wilson-loop instanton of unit Pontryagin number. The instanton-multimonopole moduli space is explicitly constructed, and seen to be identical to a Coulomb phase moduli space of a $\mathrm{U}(1)^N$ gauge theory in $2+1$ dimensions related to Kronheimer's gauge theory of SU(N)-type. This extends the results by Intriligator and Seiberg to the finite couplings that, in the infrared limit of Kronheimer's theory, the Coulomb phase parametrizes a centered SU(N) instanton. We also elaborate on the case of restored SU(N) symmetry. [S0556-2821(97)02516-2]

PACS number(s): 11.27.+d, 11.25.Mj

I. SOLITONS ON D-BRANES

Recently there has been a considerable interest in low energy D-brane dynamics [1] and its relation to the supersymmetric Yang-Mills systems. When there are N parallel Dp-branes, their low energy dynamics is described by a reduction of the $N=1$ supersymmetric 10-dimensional Yang-Mills system of the gauge group U(N) to the $(p+1)$-dimensional supersymmetric Yang-Mills-Higgs system [2]. The physics of the supersymmetric Yang-Mills systems can be understood by that of the D-brane dynamics, and vice versa, enriching our understanding of both subjects.

Bogomol'nyi-Prasad-Sommerfield (BPS) magnetic monopoles and instantonlike solitons in supersymmetric Yang-Mills systems have been understood in the D-brane language recently. The key point is that the Ramond-Ramond (RR) charge carried by the D-brane [3] can be also carried by world-volume instantons and monopoles. The coupling (between the RR gauge field C_{p+1} and the world-volume gauge field) that is responsible for this is succinctly written as

$$\int_{B_p}\sum_{\delta\geqslant 0} C_{p-2\delta+1}\wedge \mathrm{tr}\, e^{\mathcal{F}/2\pi i}, \tag{1}$$

where the integral is over the world volume of Dp-branes for each p [1]. $\mathcal{F}$ is the world-volume Yang-Mills field strength.

For instance, when an open $D(p-2)$-brane ends on a Dp brane, the RR charge carried by the former must be somehow cancelled by a Yang-Mills soliton on the Dp-brane, of codimension three [4]. From the above coupling, it is clear a soliton of charge

$$\oint_{S^2}\mathcal{F} \tag{2}$$

is a source for C_{p-1} RR gauge field, so a RR charge conservation requires that the boundaries of the $D(p-2)$-brane carry the magnetic charge with respect to Dp-brane world-volume gauge field [5].

On the other hand, Yang-Mills instantons appear, say, when there are N parallel $D4$-branes overlapping each other so that U(N) symmetry is restored. If a $D0$-brane approaches the $D4$-branes from infinity and touches the $D4$-branes, it could melt away, leaving an SU(N) instanton (in real time) on the $D4$-brane [6]. This process conserves the RR charge as both $D0$-branes and instantons carry the same charge. The instanton energy is identical to the $D0$-brane mass and one can interpret the instanton as the threshold bound state between a $D0$-brane and N overlapped $D4$-branes. This picture has been studied in many variations connected under the T duality. In particular, the D-brane configuration has been shown to be connected to the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction of the instanton configurations.

In this paper we consider cases where at least one direction of the space-time is compactified on a circle. A T-dual transformation along the circle maps a $D4$-brane to a $D3$-brane and a $D0$-brane to a $D1$-brane loop. In turn, a $D1$-brane loop that passes through N $D3$-branes along the circle can break up into N $D1$-brane segments ending on $D3$-branes, each of which must behave as a fundamental monopole in the $D3$-brane world-volume theory. The world-volume dynamics of the system we just described is governed by a compactified five-dimensional SU(N) Yang-Mills system, whose gauge symmetry is broken by a nontrivial Wilson loop. Thus, our investigation will lead to a new understanding of magnetic monopoles and solitonic instantons in a compactified Yang-Mills theory with an arbitrary Wilson loop. In the latter half of the paper, we will concentrate on the moduli space of such solitons.

II. THE LOW ENERGY EFFECTIVE FIELD THEORY

We take the spacetime to be $\tilde{S}^1\times R^{3+1}\times T^5$. Let N parallel Dp-branes $(p\geqslant 3)$ overlap with the noncompact part

*Electronic address: klee@phys.columbia.edu
†Electronic address: piljin@phys.columbia.edu

R^{3+1}. Up to various T-dual transformations, we may as well take $p=3$ so that these D-branes lie entirely along R^{3+1}. When N D-branes are all separated from each other, the theory is in the Coulomb phase where it admits magnetic monopole solutions. Furthermore, we will assume that all $D3$-brane positions are aligned along $\tilde{S}^1$, so we are effectively working in $\tilde{S}^1\times R^{3+1}$. Finally, upon a T-dual transformation, we map $\tilde{S}^1$ of circumference $\tilde{L}$ to its dual S^1 of circumference $L=4\pi^2\alpha'/\tilde{L}$, which is now wrapped around by N $D4$-branes.

The world-volume theory in question is then five-dimensional $\mathcal{N}=4$ U(N)=U(1)$\times$SU(N) Yang-Mills compactified on S^1 [2]. The Abelian part U(1) of the gauge group will be ignored in our discussion here. The bosonic part of the effective Lagrangian on $S^1\times R^{3+1}$ is

$$\mathcal{L}=\mu\int_{S^1\times R^{3+1}}\mathrm{tr}\left\{-\frac{1}{2}\mathcal{F}_{MN}\mathcal{F}^{MN}+\sum_P \mathcal{D}_M\Phi_P\mathcal{D}^M\Phi_P+\frac{1}{2}\sum_{P,Q}[\Phi_P,\Phi_Q]^2\right\}. \tag{3}$$

The dimensionful coupling μ is proportional to the $D4$-brane tension τ_4, and, for later use, we express it in terms of the D-string tension $\tilde{\tau}_1$:

$$\mu=2\pi^2\alpha'^2\ \tau_4=\frac{\tilde{\tau}_1\tilde{L}}{8\pi^2}. \tag{4}$$

The circumference $\tilde{L}$ of $\tilde{S}^1$ enters because the T-duality transformation rescales the string coupling $e^{\phi}\propto 1/\tilde{\tau}_p$. The relevant symmetry breaking is via a Wilson loop along S^1, so we shall subsequently ignore the five scalar fields Φ_P.

The theory admits the BPS bound, as usual, and in the absence of electric excitation, the energy functional is

$$\mathcal{E}=\frac{\mu}{2}\int_{S^1\times R^3}\mathrm{tr}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \tag{5}$$

with the Greek indices ranging from 1 to 4. This is bounded by

$$\mathcal{E}\geqslant\frac{\mu}{2}\int_{S^1\times R^3}\mathrm{tr}\ \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\mathcal{F}_{\mu\nu}\mathcal{F}^{\alpha\beta}=8\pi^2\mu\ p_1(\mathcal{F}), \tag{6}$$

where $p_1(\mathcal{F})$ is the Pontryagin number of the Yang-Mills field,

$$p_1(\mathcal{F})\equiv\frac{1}{8\pi^2}\int_{S^1\times R^3}\mathrm{tr}\mathcal{F}\wedge\mathcal{F}, \tag{7}$$

and the bound is saturated when the Yang-Mills field solves the BPS equation

$$\mathcal{F}_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\rho\sigma} \tag{8}$$

on $S^1\times R^3$.

III. FUNDAMENTAL MONOPOLES AND WILSON-LOOP INSTANTONS

Splitting the gauge field $\mathcal{A}_\mu$ into the noncompact part $A_i, i=1,2,3$ and the compact part $\mathcal{A}_4$, the BPS equation is

$$B_i=D_i\mathcal{A}_4-\partial_4A_i. \tag{9}$$

With this notation the BPS bound can be written as

$$\mathcal{E}\geqslant 8\pi^2\mu\ p_1(\mathcal{F})=2\mu\int_{S^1}\int_{R^3}\mathrm{tr}\left[B_iD_i\mathcal{A}_4-\frac{1}{2}\epsilon_{ijk}A_j\partial_4A_k\right]. \tag{10}$$

We first observe that all three-dimensional BPS monopole solutions [7] are also solutions of this equation just by setting A_i independent of the periodic coordinates x^4 and regarding $\mathcal{A}_4$ as an adjoint scalar field. Thus, this theory admits the $N-1$ spherically symmetric, fundamental monopole solutions [8], each carrying a distinct topological winding number in $\pi_2(\mathrm{SU}(N)/\mathrm{U}(1)^{N-1})=Z^{N-1}$. Each of these fundamental solutions is characterized for having only four zero modes.

However, there are other solitonic solutions of the same monopole charges, an infinite number of them as a matter of fact. To see this, it suffices to recall that the BPS solution exists for generic asymptotic value $\mathcal{A}_4(\infty)$. Because $\mathcal{A}_4$ is in fact a component of the gauge connection along the compact direction, it can be shifted under a large gauge transformation as

$$\mathcal{A}_4\rightarrow\mathcal{A}_4+\Delta \tag{11}$$

for some constant Hermitian matrix Δ. Monopole solutions of larger masses are obtained by first solving the three-dimensional BPS equation with the boundary condition $\mathcal{A}_4(\infty)=\langle\mathcal{A}_4\rangle+n\Delta$ and then performing a large gauge transformation back to $\mathcal{A}_4(\infty)=\langle\mathcal{A}_4\rangle$. This works for $n\geqslant 0$ whenever tr $(\langle\mathcal{A}_4\rangle\ \Delta)>0$ and $e^{iL\Delta}$ belongs to the center of the gauge.

Consider the simplest case of SU(2) for $N=2$. We write the generic Wilson loop as

$$\langle\mathcal{A}_4\rangle=\frac{2\pi\eta}{L}\hat{Q} \tag{12}$$

for $0\leqslant\eta<1$ and where $\hat{Q}$ is the unbroken U(1) charge operator normalized to be unit for the massive vector meson. In the unitary gauge, $\hat{Q}$ would be a diagonal matrix diag $(1/2,-1/2)$. The usual BPS monopole is then a solitonic solution where $\mathcal{A}_4$ interpolates between 0 at origin and $2\pi\eta\ \hat{Q}/L$ at asymptotic infinity. The relevant Δ can be chosen as

$$\Delta=\frac{2\pi}{L}\hat{Q}, \tag{13}$$

so the infinite tower of monopole solutions simply correspond to rescaled spherically symmetric BPS solutions that allow $\mathcal{A}_4$ to interpolate between 0 and $2\pi(\eta+k)\ \hat{Q}/L$ for any non-negative integer $k\geqslant 0$. After a large gauge transformation, we may keep $\mathcal{A}_4(\infty)$ at $2\pi\eta\,\hat{Q}/L$ but let $\mathcal{A}_4(0)=-2\pi k\,\hat{Q}/L$. (As long as the Wilson loop at the origin is invariant under the global gauge rotations, such a solution is perfectly regular and acceptable.)

The masses of these monopoles are easy to evaluate. The monopole solution is independent of x^4 when we choose $\mathcal{A}_4(0)=0$, and then its BPS mass formula is similar to that of three-dimensional Yang-Mills theory [7]:

$$\mathcal{E}\geqslant 8\pi^2\mu\; p_1(\mathcal{F})=2\mu\int_{S^1}\int_{R^3}\mathrm{tr}[B_iD_i\mathcal{A}_4]=8\pi^2\mu\;(k+\eta). \tag{14}$$

Note that the monopoles also carry a Pontryagin number $k+\eta$.

This is a perfectly sensible result from the D-brane perspective. There, a fundamental monopole is simply a D-string segment that stretches between an adjacent pair of $D3$-branes along $\tilde{S}^1$ [5]. The $D3$-brane position on $\tilde{S}^1$ is dictated by the Wilson loop. For SU(2), their coordinate $\tilde{x}^4$ $(\in[-\tilde{L}/2,\tilde{L}/2])$ is given by $\pm\eta\tilde{L}/2$, as can be deduced from the fact that both $\eta=0$ and $\eta=1$ correspond to a symmetric phase where the two $D3$-branes must coincide. A D-string of length $\eta\tilde{L}$ stretching between the two $D3$-branes carries a fundamental magnetic charge, and is naturally identified with the fundamental monopole above $(k=0)$. Note that the mass formula obtained from the field theory does agree with the D-brane picture. The mass is

$$\mathcal{E}\geqslant 8\pi^2\mu\eta=8\pi^2\,\frac{\tilde{\tau}_1\tilde{L}}{8\pi^2}\,\eta=\tilde{\tau}_1\,(\eta\tilde{L}), \tag{15}$$

which is simply the D-string tension multiplied by the length.

In addition, we may have a closed D-string wrapping around $\tilde{S}^1$. Such a loop carries no magnetic charge and is of mass $\tilde{\tau}_1\tilde{L}=8\pi^2\mu$. The configurations of a single segment of length $\eta\tilde{L}$ combined with k loops of such closed D-strings has exactly the right mass and charge to form the infinite tower of monopoles obtained above. What is the field theory soliton that corresponds to the closed D-string loop? The answer is obvious once we made the above identification. Removing the fundamental monopole to asymptotic infinity, one obtains k closed loops of D-strings of zero magnetic charge. Its Pontryagin number $p_1(\mathcal{F})$ is k, so each loop of D-string must be realized as an SU(2) instanton on $S^1\times R^3$.

The same reasoning goes through for SU(N). Writing the expectation value $\langle\mathcal{A}_4\rangle$ in a unitary gauge,

$$\langle\mathcal{A}_4\rangle=\mathrm{diag}(\eta_1,\eta_2,\eta_3,\ldots,\eta_N)\,\frac{2\pi}{L},$$

$$\frac{1}{2}\geqslant\eta_1>\eta_2>\cdots>\eta_N\geqslant-\frac{1}{2} \tag{16}$$

the $N-1$ fundamental monopoles are of masses $8\pi^2\mu\,(\eta_a-\eta_{a+1})$, as can be seen from a SU(2) embedding. The infinite tower of monopoles for each fundamental charge can be treated by a simple SU(2) embedding, and this again results in the mass formula

$$\mathcal{E}\geqslant 8\pi^2\mu p_1(\mathcal{F})=8\pi^2\mu\;(k+\eta_a-\eta_{a+1}). \tag{17}$$

Thus the higher mass monopole is again interpretable as a combination of a fundamental monopole and k SU(N) instantons. On the D-brane side, the N $D3$-branes are at $\tilde{x}^4=\eta_i\tilde{L}$, so the string segments between adjacent pairs are of masses $\tilde{\tau}_1\,(\eta_a-\eta_{a+1})\tilde{L}=8\pi^2\mu\,(\eta_a-\eta_{a+1})$ with the fundamental charge, while the chargeless, closed loop of the D-brane has the mass $\tilde{\tau}_1\tilde{L}=8\pi^2\mu$. Comparing this to Eq. (17), it is pretty clear that the closed loop of the D-string corresponds to an instanton of $p_1(\mathcal{F})=1$.

This is in accordance with the fact that, upon the T-duality transformation from $\tilde{S}^1$ to S^1, a single D-string loop crossing N $D3$-branes turn into a $D0$-brane on N $D4$-branes, a natural candidate for an SU(N) instanton. Because the instantons exist even when the gauge symmetry is broken by a Wilson loop, we shall call them the Wilson-loop instantons.

IV. THE WILSON-LOOP INSTANTON AS *N* FUNDAMENTAL MONOPOLES

Such a closed loop of a D-string can break up into segments between adjacent pairs of $D3$-branes. With $D3$-branes separated along $\tilde{S}^1$, there are exactly N such monopolelike segments. However, so far we have isolated only $N-1$ species of fundamental monopoles, each represented by a three-dimensional spherically symmetric solution. We need one more monopole solution in order to match the D-brane picture. In the D-brane picture, this corresponds to a segment stretching between the first and the Nth $D3$-branes directly, and this is clearly possible thanks to the compact nature of $\tilde{S}^1$. On the other hand, since a closed loop of D-string carries no magnetic charge with respect to the unbroken $\mathrm{U}(1)^{N-1}$, the Nth monopole must have a charge opposite to the sum of the other $N-1$ fundamental monopoles.

To see how a BPS solution of a ''wrong'' magnetic charge arises in the Yang-Mills theory, let us again consider the simplest case of SU(2). Here, it is also useful to recall why a pair of BPS monopoles has no static force between them: it is because the vector force is canceled by that of the scalar. In order to flip the magnetic charge and still solve the BPS equation, we need to make the absolute value of $\mathcal{A}_4$ to increase (rather than decrease) toward origin. With $\mathcal{A}_4(\infty)=2\pi\eta\hat{Q}/L$ as above, possible choices for $\mathcal{A}_4(0)$ are $(k+1)2\pi\hat{Q}/L$ for any non-negative integer k.

How do we know a BPS solution with such boundary conditions exists? Because such a configuration can be gauge transformed to the usual BPS solution through a large gauge transformation that shifts $\mathcal{A}_4\to\mathcal{A}_4-(k+1)2\pi\hat{Q}/L$ everywhere, followed by a global gauge rotation $\hat{Q}\to-\hat{Q}$. The resulting configuration is that of the ordinary BPS solution whose $\mathcal{A}_4$ interpolate between 0 at origin to $(k+1-\eta)2\pi\hat{Q}/L$ at asymptotic infinity. The mass is again easily estimated in the latter gauge, and

$$\mathcal{E} \geq 8\pi^2\mu^2\ (k+1-\eta). \tag{18}$$

The infinite tower of monopoles given by this is a precise analog of the infinite towers we encountered above. The minimal case of $k=0$ is again a fundamental monopole in that it carries precisely four zero modes. Generalization to SU(N) proceeds similarly.

This second (or the Nth) fundamental monopole is possible because the x^4 direction is compact [9]. In the gauge where the ordinary fundamental monopoles are x^4 invariant, this solution must have a Kaluza-Klein momentum along x^4. In reality, however, both [or all N for SU(N)] monopoles are on equal footing, since one can always perform a large gauge transformation to get rid of the x^4 dependence of the second (or the Nth) fundamental monopole, as we saw above.

We need one more field theory computation to complete the picture. If the higher mass monopole is indeed a mere sum of a fundamental monopole and k instantons, and if the instanton itself is a sum of N monopoles, the zero-mode counting must reflect this. In particular, the higher mass monopoles of fundamental charge must carry a large number of zero modes which should be also consistent with that of an instanton.

Following Brown *et al.* [10], we write the zero mode equation as

$$\sigma_\mu \mathcal{D}_\mu \Psi = 0, \tag{19}$$

where $\mathcal{D}_\mu$ is the background covariant derivative and $\Psi \equiv \overline{\sigma}_\mu \delta\mathcal{A}_\mu$ satisfies a reality constraint coming from the fact that the zero modes δA_μ are Hermitian matrices. The 2×2 matrices σ_μ are given by (σ_j, i) while $\overline{\sigma}_\mu = (\sigma_j, -i)$.

Now consider the infinite tower of monopoles of the fundamental charge in the SU(2) theory. Performing a Fourier expansion of Ψ with respect to the internal periodic coordinate x^4,

$$\Psi = \sum_{m\in Z} e^{i2\pi m x^4/L}\Psi_{(m)}, \tag{20}$$

the zero-mode equation reduces to a three-dimensional one due to Weinberg [8], now with a bare mass:

$$\sigma_\mu D_\mu \Psi_{(m)} = \frac{2\pi m}{L}\Psi_{(m)}, \tag{21}$$

where $D_j \equiv \mathcal{D}_j$ and $D_4 \equiv \mathcal{D}_4 - \partial_4$. From Ref. [8], one can easily see that this admits four normalizable solutions whenever the bare mass term is smaller than the scale set by the monopole mass, i.e., $|m| < \eta + k$. Since $\eta > 0$ in the broken phase, $2k+1$ Fourier modes contribute four each, and

$$8k+4 \tag{22}$$

is the total number of zero modes. When we consider the SU(N) monopoles, the same reasoning goes through since we may obtain the necessary BPS solutions by embedding the SU(2) solutions. One difference is that there are also some [SU(N)] zero modes that transform as doublets with respect to this embedded SU(2). After this is properly taken into account, we find

$$4Nk+4 \tag{23}$$

as the total number of zero modes.

This zero-mode counting is clearly consistent with the interpretation that the higher mass monopole is a fundamental monopole of the same charge combined with a chargeless collection of Nk fundamental monopoles of four zero modes each. In turn, the index theorem applied to the Wilson-loop instantons of $p_1(\mathcal{F}) = k$ gives the bulk contribution

$$2\times\frac{1}{8\pi^2}\int_{S^1\times R^3}\mathrm{Tr}\mathcal{F}\wedge\mathcal{F} = 4N\times\frac{1}{8\pi^2}\int_{S^1\times R^3}\mathrm{tr}\mathcal{F}\wedge\mathcal{F} = 4Nk \tag{24}$$

while the boundary contribution is expected to be null for integral $p_1(\mathcal{F})$ as in the R^4 case. The first trace Tr is over the SU(N) adjoint representation, and we used the identity $\mathrm{Tr}(\cdots) = 2N\times\mathrm{tr}(\cdots)$ for SU(N). Again the zero-mode counting is consistent with the above picture that the Nk monopoles are in fact k SU(N) instantons. The interpretation of a Wilson-loop instanton as N fundamental monopoles is thus complete in the purely Yang-Mills theory context.

V. THE EXACT MODULI SPACE AND A 3D GAUGE THEORY

A simple consequence is that the one-instanton moduli space is identical to that of N fundamental monopoles. When the N monopoles are well separated, we can infer the approximate form of the metric from their long-range interaction. Following Gibbons and Manton [11], Lee, Weinberg, and Yi constructed the general form of such an approximate metric [12]. Applied to the present case, it gives

$$\mathcal{G} = M_{ab} d\mathbf{x}_a \cdot d\mathbf{x}_b + (M^{-1})_{ab}(d\xi_a + \mathbf{W}_{ac}\cdot d\mathbf{x}_c) \times (d\xi_b + \mathbf{W}_{bd}\cdot d\mathbf{x}_d), \tag{25}$$

where the diagonal components of the $N\times N$ matrix M are

$$M_{aa} = m_1 + \frac{1}{|\mathbf{x}_1 - \mathbf{x}_N|} + \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|}, \quad a=1,$$

$$M_{aa} = m_a + \frac{1}{|\mathbf{x}_a - \mathbf{x}_{a-1}|} + \frac{1}{|\mathbf{x}_a - \mathbf{x}_{a+1}|}, \quad a=2,\ldots,N-1,$$

$$M_{aa} = m_N + \frac{1}{|\mathbf{x}_N - \mathbf{x}_{N-1}|} + \frac{1}{|\mathbf{x}_N - \mathbf{x}_1|}, \quad a=N, \tag{26}$$

with m_a being the (rescaled) mass of the ath monopole, which is located at $\mathbf{x}_a$ in R^3, and the only nonvanishing off-diagonal components are

$$M_{1N} = M_{N1} = -\frac{1}{|\mathbf{x}_1 - \mathbf{x}_N|},$$

$$M_{ab} = -\frac{1}{|\mathbf{x}_a - \mathbf{x}_b|} \quad (|a-b|=1). \tag{27}$$

The vector potential **W** is related to the the scalar potential M by

$$\nabla_c M_{ab} = \nabla_c \times \mathbf{W}_{ab}, \tag{28}$$

which ensures that the metric is hyper-Kähler. The U(1) coordinate $\xi_a/2$ for each a is periodic in 2π and gives rise to integer-quantized dyonic excitation of the ith monopole.

If the Nth monopole is absent (or infinitely far away from the other $N-1$ monopoles), such an approximate metric is known to be exact [12,13]. One compelling physical reason behind this is the fact that the $N-1$ unbroken U(1) gauge symmetry prohibits certain short-distance corrections that allow an electric charge transfer among the $N-1$ fundamental monopoles. With the addition of the Nth monopole, there are still only $N-1$ U(1) symmetries from the original gauge group, so it may appear that a short-distance correction is inevitable.

However, there is another U(1) symmetry, namely the translation along S^1, which acts to preserve an electric charge. One easy way of seeing this is again from the D-brane picture. An electric charge is carried by open fundamental string segments stretching between $D3$-branes. When all N monopoles carry the same (absolute) quantized amount of electric charges, the situation is that of a closed string winding around $\tilde{S}^1$. Upon a T-dual to S^1, this winding number is translated to the conserved momentum along S^1. From the low energy perspective, we can also compute the momentum P_4,

$$P_4 \propto \int_{S^1 \times R^3} \mathrm{tr}\, \mathcal{F}_{0\mu} \mathcal{F}_{\mu 4}, \tag{29}$$

which, for $N=2$ and $k=1$, evaluates to

$$P_4 \propto q_1 \eta + q_2(1-\eta) = q_2 + \eta(q_1 - q_2), \tag{30}$$

when q_1 and q_2 are the electric charges (our convention is such that the total electric gauge charges are zero when $q_1 = q_2$) on the first and the second fundamental SU(2) monopoles, respectively. The translation invariance along x^4 thus preserves a linear combination of the two electric charges.[1]

The N independent conserved electric charges along with the hyper-Kähler property of the moduli space implies that the asymptotic form above must be in fact the exact expression. One should be able to set up an argument similar to those in Ref. [13] and show this explicitly. We have obtained the instanton moduli space of a SU(N) instanton on $S^1 \times R^3$ for an arbitrary Wilson loop from the equivalent multimonopole configuration.

The same form of metric has recently appeared in a work by Intriligator and Seiberg [14], as that of the Coulomb branch moduli space of a $\mathrm{U}(1)^N$ gauge theory with four extended supersymmetry in 2+1-dimensions.[2] The theory has N species of electrons of charge $(1,-1)$ with respect to each adjacent pairs of U(1) gauge groups, and the two moduli space coincides if the bare masses of the electrons vanish.

This identity can be understood by adapting the method of Hanany and Witten [15]. We consider N parallel NS five-branes separated along a circle $\tilde{S}^1$. We will take the rest of the space-time to be noncompact R^{8+1}. Also put one three-brane segment between each adjacent pairs of the five-branes. The corresponding solitons of codimension three in the five-brane world-volume theory are precisely the monopoles we discussed above, or collectively a SU(N) instanton, through a series of S and T dualities as well as some decompactifications.

On the other hand, Hanany and Witten also identified the effective $(2+1)$-dimensional theory on such three-brane segments, and the rule is that each segment produces a U(1) vector multiplet and each adjacent pair of segments gives a hypermultiplet of charge $(1,-1)$ with respects to the two U(1)'s. With the N three-brane segments parallel to $\tilde{S}^1$, then, the gauge theory is $\mathrm{U}(1)^N$ with N species of electrons linking pairs of U(1)'s successively. The Coulomb phase of this theory is parametrized by the three-brane configurations, which are nothing but the instanton-multimonopole configurations from the five-brane perspective. The two moduli spaces are identical, and the three-dimensional $\mathrm{U}(1)^N$ couplings g_a are determined by the monopole masses:

$$\frac{1}{g_a^2} \sim m_a. \tag{31}$$

Furthermore, our assertion that the metric written above is exact is reflected in the fact that the Coulomb phase moduli space metric of the Abelian $\mathrm{U}(1)^N$ theory receives no nonperturbative correction.

One particular linear combination of the U(1) gauge fields is free, and if it is removed, we recover Kronheimer's theory of SU(N) type [14,16]. On the other side of the correspondence, this has the effect of factoring out the center-of-mass motion on $S^1 \times R^3$, so the Coulomb phase of the Kronheimer theory coincides with the relative part of the instanton-multimonopole moduli space.

A special case of this result was anticipated by Intriligator and Seiberg [14]. They noted that the infrared limit of the Kronheimer theory of type SU(N) is ''mirror'' to a U(1) theory with N electrons. The Higgs phase of the latter had been interpreted as the moduli space of an instanton located at a fixed point in R^4, and, under the proposed mirror symmetry, should be mapped to the Coulomb phase of the infinite coupling Kronheimer theory. Such a mirror mapping was subsequently justified by various authors [15,17]. In view of the relationship between the monopole masses m_a and the $\mathrm{U}(1)^N$ couplings g_a, this can be seen to be a special case of our result in the limit of decompactified S^1 (dual to vanishing $\tilde{S}^1$).

[1] P_4 is not properly quantized because it is not the Noether momentum. The latter is found by dropping an η-dependent surface term that is independently conserved.

[2] Intriligator and Seiberg actually considered a $\mathrm{U}(1)^N/\mathrm{U}(1)$ gauge theory due to Kronheimer. But the difference is simply that of whether or not one factors out a trivial part $S^1 \times R^3$ of the moduli space.

VI. SYMMETRIC PHASE AND THE CALABI METRIC

When the Wilson loop becomes trivial, the SU(N) gauge symmetry is restored and the moduli space must reduce to that of a symmetric-phase instanton on $S^1\times R^3$, also known as the periodic instanton. In this limit, the first $N-1$ monopoles are massless ($m_1=\cdots=m_{N-1}=0$). Redefining the position coordinates by $\mathbf{r}_A=\mathbf{x}_A-\mathbf{x}_{A+1}$ for $A=1,\ldots,N-1$ and $\mathbf{R}=\mathbf{x}_N$, and the U(1) phases by $\psi_A=\Sigma_{a=1}^{A}\xi_a$ and $\chi=\Sigma_{a=1}^{N}\xi_a$, the moduli space metric can be rewritten as

$$\mathcal{G}=m_N\, d\mathbf{R}^2+\frac{1}{m_N}\,d\chi^2+\mathcal{G}_{\rm rel},$$

$$\mathcal{G}_{\rm rel}=C_{AB}d\mathbf{r}_A\cdot d\mathbf{r}_B+(C^{-1})_{AB}(d\psi_A+\mathbf{w}_{AC}\cdot d\mathbf{r}_C)\times(d\psi_B+\mathbf{w}_{BD}\cdot d\mathbf{r}_D), \tag{32}$$

where the $(N-1)\times(N-1)$ scalar potentials C_{AB} are

$$C_{AA}=\frac{1}{|\mathbf{r}_A|}+\frac{1}{|\Sigma_A\mathbf{r}_A|},\quad C_{AB}=\frac{1}{|\Sigma_A\mathbf{r}_A|}\quad (A\neq B). \tag{33}$$

The vector potentials satisfy $\nabla_C\times\mathbf{w}_{AB}=\nabla_C C_{AB}$, and the U(1) coordinates ψ_A are all of period 4π. Note that the metric $\mathcal{G}_{\rm rel}$ is devoid of any mass scale. The relative moduli space described by $\mathcal{G}_{\rm rel}$ is thus valid for any size of S^1, and can be considered the moduli space of a symmetric-phase SU(N) instanton located at a point in either $S^1\times R^3$ or R^4. It is precisely the Coulomb phase moduli space of the infinite coupling Kronheimer theory of SU(N) type.[3]

The metric $\mathcal{G}_{\rm rel}$ itself is a degenerate limit of the so-called Calabi metric [18] which is an SU(N)-invariant hyper-Kähler metric. For $\mathcal{G}_{\rm rel}$, the SU(N) isometry is clearly related to the restored SU(N) gauge symmetry. The relative moduli space in the symmetric phase parametrizes the gauge orientation of the instanton beside its size: the principal SU(N) orbit of the Calabi manifold is SU(N)/$U(N-2)$, and the remaining single coordinate must correspond to the instanton size.

The metric $\mathcal{G}_{\rm rel}$ possesses an isolated singularity at origin. For $N=2$, this is particularly easy to see because the Calabi metric is simply that of the Eguchi-Hanson gravitational instanton whose degenerate limit is $R^4/Z_2=R^+\times{\rm SU}(2)/Z_2$. The isolated singularity at origin persists as we break the SU(N) gauge symmetry by a Wilson loop, because the singularity occurs at vanishing instanton size where the scale of the Wilson loop is negligible. Again, this can be seen explicitly for $N=2$: the relative moduli space of a pair of distinct monopoles is always given by Taub-Newman-Unti-Tamborino (NUT) space locally [19], and thus by continuity it has to be a Z_2 orbifold of the Taub-NUT space. The massless limit of the Taub-NUT is R^4.

This massless limit of the instanton-multimonopole moduli space provides us with an interesting explanation of a phenomenon found by Rossi in the late 1970s [20]. Start with 't Hooft's SU(2) multi-instanton solution and line them up along a fixed axis, say x^4, at equal distances. The resulting configuration is periodic along x^4, and effectively a single symmetric-phase instanton on $S^1\times R^3$. As usual, there is a single moduli that paramterizes the instanton size, say ρ, in addition to the moduli that arise from broken global symmetries. Then, it was observed that, as ρ is sent to ∞, the periodic instanton solution of ever-increasing size approached the usual BPS monopole solution in R^3 up to a large gauge transformation.

In our picture, the periodic SU(2) instanton is composed of a pair of distinct fundamental monopoles. In the limit of restored SU(2) gauge symmetry, the Wilson loop is trivial ($\eta=0$) so that the first fundamental monopole is massless. But the second is still massive. The situation is reminiscent of those in Ref. [21]; as the non-Abelian gauge symmetry is restored, some monopoles become massless and dissolve into a charge cloud that shields the (non-Abelian) magnetic charge of the remaining massive monopole. At such a massless limit, some of the collective coordinates acquire new physical significance, and in particular, what used to be the intermonopole distance translates into the size of the cloud.

More generally, when we have the SU(N) gauge group restored, $N-1$ of the monopoles are now massless, and only one, say the Nth, remains massive. There is again a single collective coordinate ρ that parametrizes the cloud size or equivalently the instanton size. In terms of the three-dimensional coordinates above, ρ can be redefined to satisfy a simple relation

$$\rho^2/L=|\mathbf{x}_1-\mathbf{x}_2|+|\mathbf{x}_2-\mathbf{x}_3|+\cdots+|\mathbf{x}_{N-1}-\mathbf{x}_N|+|\mathbf{x}_N-\mathbf{x}_1|, \tag{34}$$

which can be deduced from the study of the moduli space of two distinct massive monopoles in Ref. [21] that arise upon SU($N+2$)$\rightarrow$SU(N)$\times$U(1)^2. The present moduli space results from the latter by putting the two massive monopoles at the same point and identifying their electric charges.

Thus the large instanton limit ($\rho\rightarrow\infty$) is realized if at least one of the to-be-massless monopoles is removed to the asymptotic infinity. In fact, by a SU(N) gauge rotation, this is equivalent to letting $\mathbf{x}_a=\infty$ for $a=1,\ldots,N-1$ simultaneously. Left behind is a single massive monopole at $\mathbf{x}_N$, which certainly can be gauge transformed to a canonical BPS monopole solution.

The smooth interpolation between the monopole picture and the instanton picture also tells us something about the multimonopole configurations in the broken phase. Far away from each other, the individual monopole has a clear identity as magnetic solitons on R^3. As their separations grow smaller, however, the size of the internal S^1 becomes appreciable, and they cannot retain the character of solitons on R^3. Rather, as the length scale progressively decreases, the compact nature of S^1 will no longer be important and they must clump together at a point in $S^1\times R^3$, and look a lot like a very small R^4 instanton. This is a radical departure from what one would expect from ordinary three-dimensional magnetic solitons, and must be responsible for the isolated singularity of the relative moduli space at origin.

[3]The coordinates Intriligator and Seiberg used in Ref. [14] are more like the $\mathbf{x}_a$'s and the ξ_a's above than the proper relative coordinate $\mathbf{r}_A$'s and ψ_A's, so one must be careful to express one of them, say for $a=N$, as functions of those for $a=1,\ldots,N-1$.

VII. CONCLUSION

We have studied field theory aspects of monopoles and instantons on D-branes in a compactified spacetime, and found a consistent picture emerging from a purely field theoretical perspective. For the gauge group SU(N), we also found the exact moduli space of a single Wilson-loop instanton by interpreting it as a collection of N distinct fundamental monopoles. The relative part of this moduli space is subsequently identified with the Coulomb phase moduli space of the three-dimensional Kronheimer theory of type SU(N). In the limit where SU(N) is restored, or in the infrared limit of the Kronheimer theory, the relative moduli space turns out to be the degenerate limit of the Calabi manifold.

There are many directions to explore further. First of all, one may consider the S duality of the type-IIB theory, and look for threshold bound states. Since our instanton moduli space has the maximal triholomorphic Abelian symmetry, we suspect that the generalization of Gibbons construction [22] generates threshold bound states.

Second, there is a generalized Nahm formalism [23] in constructing self-dual solutions on $S^1 \times R^3$ such as our monopoles and instantons [24]. The basic aspect has been explored before and it should be interesting to understand it further in the context of D-brane physics.

Third, we may consider the Yang-Mills theory on compact $D4$-branes, such as on a four-torus. An instanton should persist but the concept of a magnetic monopole is no longer available. One outstanding question is how to construct the moduli space in such cases. It would be most interesting to see if a simple derivation such as ours is also possible.

Note added. After the appearance of this paper, M. Murray drew our attention to Ref. [25] which also discusses relationships between monopoles and instantons.

ACKNOWLEDGMENTS

P.Y. thanks J. Maldacena for a stimulating conversation and also for drawing his attention to compactified Yang-Mills systems. The authors are also grateful to A. Dancer and A. Swann for making their manuscript available prior to its publication. K.L. was supported by the Presidential Young Investigator program. This work was supported in part by the U.S. Department of Energy.

[1] J. Polchinski, ''TASI lectures on D-branes,'' hep-th/9611050, and references therein.
[2] E. Witten, Nucl. Phys. **B460**, 335 (1996).
[3] J. Polchinski, Phys. Rev. Lett. **75**, 4724 (1995).
[4] A. Strominger, Phys. Lett. B **383**, 44 (1996); M. B. Green and M. Gutperle, *ibid.* **377**, 28 (1996); A. A. Tseytlin, Nucl. Phys. **B469**, 51 (1996).
[5] D.-E. Diaconescu, ''D-branes, monopoles and Nahm equations,'' hep-th/9608163; M. R. Douglas and M. Li, ''D-brane realization of $N=2$ super Yang-Mills theory in four dimensions,'' hep-th/9604041.
[6] M. R. Douglas, ''Gauge fields and D-branes,'' hep-th/9604198; E. Witten, Nucl. Phys. **B460**, 541 (1996); M. R. Douglas and G. Moore, ''D-branes, quivers, and ALE instantons,'' hep-th/9603167.
[7] E. B. Bogomol'nyi, Sov. J. Nucl. Phys. **24**, 449 (1976); M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. **35**, 760 (1975); S. Coleman, S. Parke, A. Neveu and C. M. Sommerfield, Phys. Rev. D **15**, 544 (1977).
[8] E. J. Weinberg, Nucl. Phys. **B203**, 445 (1982).
[9] S. Katz and C. Vafa, ''Geometric engineering of $N=1$ quantum field theories,'' hep-th/9611090.
[10] L. S. Brown, R. D. Carlitz, and C. Lee, Phys. Rev. D **16**, 417 (1977).
[11] G. W. Gibbons and N. S. Manton, Phys. Lett. B **356**, 32 (1995).
[12] K. Lee, E. J. Weinberg, and P. Yi, Phys. Rev. D **54**, 1633 (1996).
[13] M. K. Murray, ''A note on the $(1,1,\ldots,1)$ monopole metric,'' hep-th/9605054; G. Chalmers, ''Multimonopole moduli spaces for SU(N) gauge group,'' hep-th/9605182.
[14] K. Intriligator and N. Seiberg, Phys. Lett. B **387**, 513 (1996).
[15] A. Hanany and E. Witten, Nucl. Phys. **B492**, 152 (1997).
[16] P. B. Kronheimer, J. Diff. Geom. **29**, 665 (1989).
[17] J. de Boer, H. Kori, H. Ooguri, and Y. Oz, Nucl. Phys. **B493**, 148 (1997); M. Porrati and A. Zaffaroni, *ibid.* **B490**, 107 (1997).
[18] G. W. Gibbons and P. Rychenkova, ''Hyper-Kähler quotient construction of BPS monopole moduli spaces,'' hep-th/9608085; A. Dancer and A. Swann, J. Geom. Phys. (to be published); E. Calabi, in *The Chern Symposium*, 1979, edited by W.-Y. Hsiang *et al.* (Springer-Verlag, New York, 1980).
[19] K. Lee, E. J. Weinberg, and P. Yi, Phys. Lett. B **376**, 97 (1996); J. P. Gauntlett and D. A. Lowe, Nucl. Phys. **B472**, 194 (1996); S. A. Connell (unpublished).
[20] P. Rossi, Nucl. Phys. **B149**, 170 (1979); B. J. Harrington and H. K. Shepard, Phys. Rev. D **17**, 2122 (1978); D. Gross, R. Pisarksi and L. G. Yaffe, Rev. Mod. Phys. **53**, 43 (1981).
[21] K. Lee, E. J. Weinberg and P. Yi, Phys. Rev. D **54**, 6351 (1996).
[22] G. W. Gibbons, Phys. Lett. B **382**, 53 (1996).
[23] W. Nahm, Phys. Lett. **90B**, 413 (1980).
[24] W. Nahm, in *Group Theoretical Methods in Physics*, edited by G. Denardo *et al.* (Springer-Verlag, Berlin, 1984).
[25] H. Garland and M. Murray, Commun. Math. Phys. **120**, 335 (1988).