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Multiple order-to-chaos transition in an Abelian-Higgs vortex solution

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The chaotic property in the Abelian-Higgs theory is numerically studied for the Nielsen-Olesen vortex solution at a critical coupling constant. Based on the analyses on the induction period necessary to the onset of chaos and the maximal Lyapunov exponents of fields, it is shown that the vortex solution exhibits a multiple order-to-chaos transition. This phenomenon is a very interesting one that can shed new light on the structure of chaos in the Abelian-Higgs theory, compared with the property of chaos in the Yang-Mills-Higgs theory. [S0556-2821(97)01118-1]

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From the viewpoint of chaos, recently, the Nielsen-Olesen vortex solution with cylindrical symmetry has been studied first in the case of space-time dependence of fields to investigate whether there exists a significant difference between the Abelian fields and the non-Abelian ones [1]. Especially, attention was focused on a phenomenon called the order-to-chaos transition and its universality because it has been commonly observed for such topological solutions as the monopole solution, the sphaleron solution, and the chiral soliton of the Yang-Mills-Higgs (YMH) theories [2–5].

The order-to-chaos transition is a phenomenon in which the system suddenly changes from order to chaos in the course of its evolution under perturbations with strength f [3]. Chaos is determined by the maximal Lyapunov exponents σ_L of the fields. This exponent approaches a positive definite value for the chaotic system. Since σ_L is a measure of how chaotic the system is, its value is expected to depend on f. For several topological solutions in the YMH systems, it has been found that σ_L stays zero for small f and then rapidly starts to increase at $f \approx 1$ after the elapse of the induction period [6]. This stepwise behavior of σ_L suggests the order-to-chaos transition to be the inherent phenomenon for these systems.

For the Nielsen-Olesen vortex solution in the Abelian-Higgs theory, on the other hand, it has been reported [1] that the onset of chaos not only needs much stronger perturbation, $f>2$, but also a much longer induction period than in the YMH case. Furthermore, there is an irregular distribution of the Lyapunov exponents that some of them fall down to zero even in the fully chaotic state and they are very sensitive to f, which prevents them from forming the smooth curve expected for the order-to-chaos transition. These results strongly suggest the existence of a characteristic difference on chaos between the Abelian-Higgs system and the YMH one. In order to investigate this issue, in this paper we try a much more extensive study of the Nielsen-Olesen vortex solution.

The Lagrangian density of the Abelian-Higgs model is given by [7]

$$L = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - V(\phi), \qquad (1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu\phi = \partial_\mu\phi + iA_\mu\phi$. The Higgs potential is $V(\phi) = (\kappa/4)(|\phi|^2 - 1)^2$. The coupling constant κ comes from rescaling both the electric charge e and the symmetry-breaking scale v to unity. The Euler-Lagrange equations are

$$\partial_\nu F^{\nu\mu} = \frac{i}{2}[\phi^* D^\mu\phi - (D^\mu\phi)^*\phi], \qquad (2)$$

$$D_\mu D^\mu\phi = \kappa(1 - |\phi|^2)\phi. \qquad (3)$$

Assuming the time-dependent vortex solution with cylindrical symmetry as

$$A_\theta(\xi,\tau) = \frac{n a(\xi,\tau)}{\xi}, \qquad (4)$$

$$\phi(\xi,\tau) = b(\xi,\tau)\exp(in\theta), \qquad (5)$$

we get from Eqs. (2) and (3) the equations of motion as

$$(\partial_\tau^2 - \partial_\xi^2)a = -\frac{1}{\xi}\partial_\xi a + (1-a)b^2, \qquad (6)$$

$$(\partial_\tau^2 - \partial_\xi^2)b = \frac{1}{\xi}\partial_\xi b - \frac{n^2}{\xi^2}(1-a)^2 b + \kappa(1-b^2)b. \qquad (7)$$

Here n is the winding number of the vortex, and ξ and τ denote the rescaled variables of space and time, respectively. The boundary conditions for fields are as follows: $a(\xi,\tau) = 0$, $b(\xi,\tau) = 0$ as $\xi \to 0$ and $a(\xi,\tau) = 1$, $b(\xi,\tau) = 1$ as $\xi \to \infty$. The energy per unit length of the vortex is given by

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$$E(n,\kappa)=2\pi\int_0^\infty d\xi\xi\left(\frac{n^2}{2\xi^2}(\partial_\tau a)^2+\frac{n^2}{2\xi^2}(\partial_\xi a)^2+\frac{n^2}{2\xi^2}(1-a)^2b^2+\frac{1}{2}(\partial_\tau b)^2+\frac{1}{2}(\partial_\xi b)^2+\frac{\kappa}{4}(1-b^2)^2\right). \tag{8}$$

In the present analyses we have to carry out a numerical simulation of the system with keeping the energy conservation. For this purpose, instead of directly using Eqs. (6) and (7) but by replacing $a\to\sqrt{\xi}a$ and $b\to b/\sqrt{\xi}$, we transform them into more desirable form without the first derivative terms as

$$(\partial_\tau^2-\partial_\xi^2)a=-\frac{3a}{4\xi^2}+(1-\sqrt{\xi}a)\frac{b^2}{\xi\sqrt{\xi}}, \tag{9}$$

$$(\partial_\tau^2-\partial_\xi^2)b=\frac{b}{4\xi^2}-\frac{n^2}{\xi^2}(1-\sqrt{\xi}a)^2b+\kappa\left(1-\frac{b^2}{\xi}\right)b. \tag{10}$$

The maximal Lyapunov exponent σ_L is a reliable indicator of chaos in the system. This quantity gives the average rate of the exponential divergence of two nearby fields as [8]

$$\sigma_L=\lim_{\tau\to\infty}\frac{1}{\tau}\ln\frac{d(\tau)}{d(0)}, \tag{11}$$

where $d(\tau)$ is a distance between the two neighboring fields at time τ in the phase space. Therefore a constant positive σ_L means that the system is chaotic, while the ordered system is characterized by $\sigma_L=0$.

To determine σ_L of the system, we need to calculate its long-term evolution under perturbations. For this purpose we write the fields in the form of the sum of the static solution $c_s(\xi)$ and the fluctuation $\delta c(\xi,\tau)$ where c stands for the gauge field a and the Higgs field b. The space coordinate is discretized through $\xi=i\times\Delta\xi$ with a lattice spacing $\Delta\xi$ and a number of lattice points $i=1,2,\ldots,N$. For the fluctuation $\delta c(\xi,\tau)$ we consider two methods: One (A) is based on the Fermi-Pasta-Ulam approach [9] and the other (B) is on the energy dumping approach.

For method A,

$$\delta c(i,\tau)=\sqrt{2/N}\sum_{j=1}^{N-1}\psi_c(j,\tau)\sin\left(\frac{\pi ij}{N}\right), \tag{12}$$

where $\delta c(i,\tau)$ is a combination of various harmonic modes given by N coupled nonlinear oscillators. In order to calculate σ_L we study the long-term evolution after exciting a single mode in Eq. (12). We choose $j_0=N/2$ as this initial mode and excite initially the gauge field alone. The strength of the initial perturbation is therefore given by the amplitude $f\equiv\psi_a(N/2,0)$ in Eq. (12). In this method the chaotic state is expected to appear via the induction phenomenon [6], i.e., the initial energy given to one mode, e.g., $N/2$th mode in our case, begins to be transferred among other normal modes suddenly after a long time τ_I called the induction period.

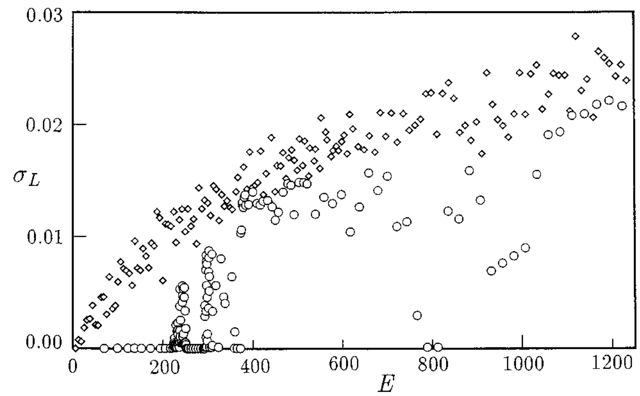

FIG. 1. Maximal Lyapunov exponents σ_L vs energy of the system E of the $n=1$ Nielsen-Olesen vortex solution at κ_c. The circle marks stand for data obtained by method A and the diamond marks by method B.

After the long-term evolution longer than τ_I we can determine σ_L vs f or E, the total energy of the system.

For method B, in contrast to A, we start with a fully chaotic state. Such an initial state can be prepared as follows: we first excite the mode j_0 for large f and then carry out the evolution long enough until all modes are excited. Once we obtain the initial state, we lower the energy of the system by a little amount by adding a dumping term to Eqs. (9) and (10). After we reach the energy, we delete the dumping term and then proceed with the evolution for a given period to determine σ_L.

Figure 1 shows the result on the dependence of σ_L on the total energy E of Eq. (8). In the present analyses we focus on the $n=1$ vortex solution at the critical coupling constant $\kappa=\kappa_c\equiv0.5$. The circle marks are the data obtained by method A after evolution up to $\tau=3\times10^5$. They show some complicated structure: at certain values of E some behavior similar to the order-to-chaos transition is observed while in large portion of E the data points scatter irregularly. Method B marked with a diamond, on the other hand, shows that all points seem to converge on a monotonically increasing curve. Since method A is based on the observation of how the system changes from order to chaos through a long-term evolution after giving an initial perturbation, it is reasonable to expect that the order-to-chaos transition will occur soon after elapsing the induction period τ_I. Thus it also seems reasonable to suppose that method A should produce the same result as method B if the evolution is carried out for sufficient time beyond τ_I. From the result that some points at $E\approx400$ and $E\approx1200$ in method A reach the curve of method B while the data at $E<200$, $E\approx240$, and $E\approx800$ stay almost zero so that the system still remains the order state before the induction period, we find that the induction period is very sensitive to the initial values of E. Indeed, Fig. 1 indicates the possibility that there exists anomalous dependence of τ_I on E, which has not been observed in the case of the YMH systems with spatial dependence of fields.

To search such a possibility we try to determine τ_I as follows: In method A we empirically define τ_I as the time when $\sigma_L>10^{-3}$ because this condition seems to assure the onset of chaos. Figure 2 shows the result on the plot of τ_I vs E, where we have calculated σ_L up to $\tau=1\times10^7$ for the

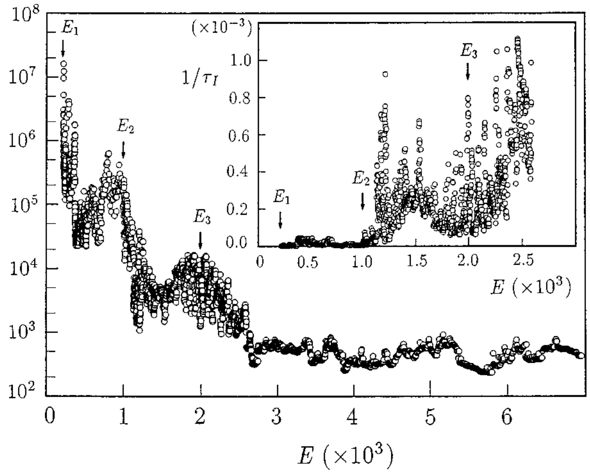

FIG. 2. Induction period τ_I vs energy E of the $n=1$ Nielsen-Olesen vortex solution at κ_c. Three arrows stand for the place of the energy E_i of Eq. (13). The inner box shows the inverse of the induction period vs the energy.

initial states with $E \geqslant 200$. For these regions of E we have observed that almost all initial states give finite τ_I and thus become chaotic. However, the most interesting fact we found here is that the induction period seems to diverge at several values of E, which is observed as bumps in Fig. 2. Apart from seemingly small bumps due to fluctuation in the data, three bumps assigned by E_i $(i=1,2,3)$, lead us to a truly interesting observation through the detailed analysis as follows. Parametrizing the dependence of τ_I on E by

$$\tau_I \approx |E-E_i|^{-\eta}, \tag{13}$$

we try to search the best values of η and E_i to fit the gross structure of the data. For all cases we found that global trend can be well described by straight lines with slope $=-1$ (i.e., $\eta=1$) as shown in Fig. 3, where $E_1 \simeq 212$ $(f_1 \simeq 1.78)$, $E_2 \simeq 995$ $(f_2 \simeq 3.88)$, and $E_3 \simeq 1990$ $(f_3 \simeq 5.49)$. The form of Eq. (13) is typical of the phase transition so that it seems reasonable to regard three values of E_i as the transition points. This strongly suggests the existence of a new phenomena if the feature is compared with the case of the order-to-chaos transition in the YMH theories. In the monopole solution of the SU(2) YMH system, for example, the order-to-chaos transition sharply occurs at a single transition point E_c. This value can be determined by the extrapolation of τ_I^{-1} for smaller E, at which τ_I becomes infinity and then the system becomes order for $E<E_c$. Thus the present result clarifies that in the vortex solution of the Abelian-Higgs theory there exist at least three transition points at which the order-to-chaos transition occurs. We call this phenomenon the multiple order-to-chaos transition. For the region $E>2000$, we also did the similar analyses to find the transition points but we could not find such points there.

All calculations were done by the eighth-order Runge-Kutta method with time-step size $\Delta\tau=0.03$ and lattice size $\Delta\xi=0.1$ to maintain the accuracy of the integration. We put $N=64$ for the lattice and $d(0)=1.0\times 10^{-5}$ for the initial separation of two fields (11). We also have studied the system for different values of N, $\Delta\xi$, and $\Delta\tau$, and checked that the qualitative results remain almost the same.

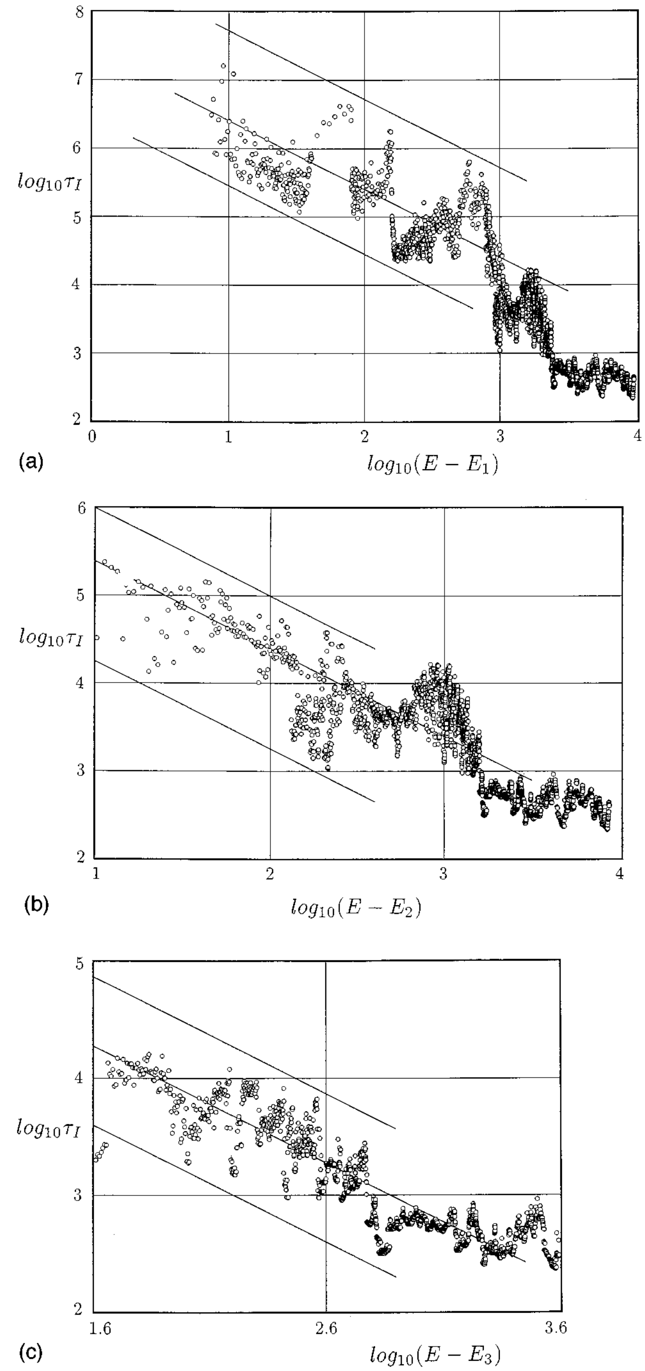

FIG. 3. Doubly logarithmic plot of the induction period τ_I vs energy E of the $n=1$ Nielsen-Olesen vortex solution at κ_c. (a) $E_1=212$, (b) $E_2=995$, and (c) $E_3=1990$. The critical exponent η of Eq. (13) is the slope of the fitted straight lines, whose values are all close to one.

To conclude, in the sense of that the multiple order-to-chaos transition can be observed, the Abelian-Higgs system is quite different from the YMH system. The basic mechanism of the order-to-chaos transition connects with the Arnold diffusion [8] for the system with higher degrees of freedom as pointed out in the case of the YMH system [3]. Thus our result reveals that in the Abelian-Higgs system there exist several processes of mode diffusion depending on the initial energy E or equivalently on the initial amplitude f.

Here we stress that this result comes from the property of the system with many degrees of freedom and is hard to obtain from the so-called homogeneous models with small degrees of freedom of the gauge field theory. In the limit of the spatially homogeneous fields, the gauge-field equations reduce to a simple nonlinear mechanical system described by the Hamiltonian for the motion of a particle in the potential well whose form depends on the Abelian or non-Abelian nature of the theory. If the homogeneous version of the gauge-field theory has the property of a typical Hamiltonian, its feature of chaos is expected to be almost the same one as derived from the dynamical theory in the Hamiltonian system on the KAM tori and its breaking pattern, i.e., the infinite hierarchy between regular and irregular motion in the phase space. Indeed, it has been shown that the uniform Chern-Simon system exhibits such an interesting structure of order-chaos hierarchy in the transition region [10], which will be related to the sensitivity of initial conditions in the intermixing region of stochastic sea and tori. In order to get deeper information on chaos inherent in the gauge-field system, however, it seems essential to study the system with the spatial dependence of fields.

It is open whether this system has other transition points besides E_i $(i=1,2,3)$, for the region $E<200$. It is especially interesting to search the point corresponding to $f\approx 1$ because this point seems a characteristic common to the topological solutions in the YMH systems, which could be clear through substantially expanded computing time. However, it is likely that the onset of chaos in the Nielsen-Olesen vortex solution is harder than in the YMH case due to higher symmetry imposed on the solution, which suppresses both the effect of the nonlinearity and the mode diffusion. In other words, it would be described that the vortex is a more tight object than topological solutions of the YMH systems from the viewpoint of chaos. Our finding on the induction period and the multiple order-to-chaos transition is that it has quite interesting features.

The present study is limited to the vortex at the critical coupling constant κ_c which corresponds to the boundary between the type-I and the type-II superconductivity in Ginzburg-Landau theory [11]. For the static solution, it is well known that the vortices at κ_c do not interact irrespective of the winding number n [12]. However, it is reported that the interaction between vortices with $n\geqslant 2$ could appear even at this Bogomol'nyi limit in the course of evolution [1]. Since the chaos and the stability in the Nielsen-Olesen vortices must be relevant to the applications in cosmology [13], in particle physics [14], and in condensed matter [11], it is a very important issue to extend the present analyses to the case of $n\geqslant 2$ and $\kappa\neq\kappa_c$.

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