Thermodynamics of $(3+1)$-dimensional black holes with toroidal or higher genus horizons

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We examine counterparts of the Reissner-Nordström–anti–de Sitter black hole spacetimes in which the two-sphere has been replaced by a surface Σ of constant negative or zero curvature. When horizons exist, the spacetimes are black holes with an asymptotically locally anti–de Sitter infinity, but the infinity topology differs from that in the asymptotically Minkowski case, and the horizon topology is not S^2. Maximal analytic extensions of the solutions are given. The local Hawking temperature is found. When Σ is closed, we derive the first law of thermodynamics using a Brown-York-type quasilocal energy at a finite boundary, and we identify the entropy as one-quarter of the horizon area, independent of the horizon topology. The heat capacities with constant charge and constant electrostatic potential are shown to be positive definite. With the boundary pushed to infinity, we consider thermodynamical ensembles that fix the renormalized temperature and either the charge or the electrostatic potential at infinity. Both ensembles turn out to be thermodynamically stable, and dominated by a unique classical solution. [S0556-2821(97)00818-7]

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I. INTRODUCTION

Isolated black holes created in astrophysical processes are expected to be well described by Einstein spacetimes that are asymptotic to Minkowski space near a spacelike or null infinity. A familiar example is the Kerr-Newman family of Einstein-Maxwell black holes [1]. However, there is mathematical interest in black holes with other kinds of asymptotic infinities. One alternative is to consider black holes that are asymptotically anti–de Sitter in the sense of Refs. [2–4], so that the topology at infinity is the same as that in asymptotically flat spacetimes. An example of this in four spacetime dimensions is the Kerr-Newman–anti–de Sitter black hole family [5–7], which generalizes the Kerr-Newman family to accommodate a negative cosmological constant. Examples in other dimensions include the Bañados-Teitelboim-Zanelli (BTZ) black hole [8,9] and its dimensionally continued relatives [10].

In this paper we examine a class of four-dimensional black holes that are asymptotically anti–de Sitter, but whose topology near infinity differs from that in the asymptotically Minkowski case. These spacetimes solve the Einstein-Maxwell equations with a negative cosmological constant: they generalize the Reissner-Nordström–anti–de Sitter solutions, replacing the round two-sphere by a two-dimensional space Σ of constant negative or vanishing curvature. These spacetimes emerge as the generic solution family from a sufficiently general form of Birkhoff's theorem, and their local geometry is well understood [11,12]. The purpose of the present paper is to examine the global structure of these spacetimes appropriate for a black hole interpretation, and the thermodynamics of the black hole spacetimes. In particular, we shall address the thermodynamical stability of these black holes under suitable boundary conditions, both with a finite boundary and with an asymptotic infinity. Our results generalize those obtained previously in Refs. [13–22]. Preliminary results were briefly mentioned in Refs. [18,23,24].

We begin, in Sec. II, by describing the local and global structure of the spacetimes. All the spacetimes have one or more asymptotically anti–de Sitter infinities, and we can use Killing time translations at infinity to define Arnowitt-Deser-Misner (ADM) mass and charge. These quantities turn out to be finite if Σ is closed. The number and character of the Killing horizons depends on the parameters in the metric. Whenever a nondegenerate Killing horizon exists, the spacetime has an interpretation as a black hole, and the (outer) Killing horizon has an interpretation as a black hole horizon. The (outer) Killing horizon bifurcation two-space has the topology of Σ. If the additive constant in the ADM mass is chosen so that this mass vanishes for the solutions that are locally anti–de Sitter, we find that black holes with flat Σ necessarily have positive ADM mass, but when Σ has negative curvature, there are black hole solutions with either sign of the ADM mass. The spacetimes with a degenerate Killing horizon are not black holes, in contrast to (say) the extreme Reissner-Nordström black hole [25]; the reason for this difference is that the negative cosmological constant makes the future null infinity in our spacetimes connected.

Section III addresses the thermodynamics of the black hole spacetimes. The local Hawking temperature is found from the Unruh effect, or from the periodicity of Euclidean time, in terms of the surface gravity at the horizon. Taking Σ closed, we introduce a boundary with the topology of Σ and fixed size, and we find the Brown-York-type quasilocal en-

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ergy at this boundary. Interpreting this quasilocal energy as the internal thermodynamical energy, and using the local Hawking temperature, we write the first law of black hole thermodynamics. We find that for all the horizon topologies, the entropy is one-quarter of the horizon area. This result extends the Bekenstein-Hawking area law to toroidal and higher genus horizons. In the limit of a large box, we show that the heat capacities with fixed ADM charge and fixed electrostatic potential are always positive.

In Sec. IV we consider the thermodynamics in the limit where the boundary is pushed strictly to infinity. As the local Hawking temperature vanishes at infinity, we focus on the renormalized temperature that is obtained by multiplying the local temperature by the redshift factor. As with the conventional Reissner-Nordström–anti–de Sitter black holes [26–30], this turns out to yield a first law from which the entropy emerges as one-quarter of the horizon area. We consider the canonical ensemble, in which one fixes the ADM charge, and the grand canonical ensemble, in which one fixes the electrostatic potential difference between the horizon and the infinity with respect to the Killing time. The (path) integral expression for the (grand) partition function is obtained by adapting to our symmetries the Hamiltonian reduction techniques of Refs. [30–36]. Both ensembles turn out to be thermodynamically stable, and always dominated by a unique classical black hole solution.

Section V contains a brief summary and discussion. Some of the technical detail on the heat capacities is collected in the Appendix.

We work throughout in Planck units, $\hbar = c = G = 1$.

II. BLACK HOLE SPACETIMES

A. Local curvature properties

We consider spacetimes whose metric can be written locally in the form

$$ds^2 = -F dT^2 + F^{-1} dR^2 + R^2 d\Omega_k^2, \tag{2.1a}$$

where

$$F := k - \frac{2M}{R} + \frac{Q^2}{R^2} - \frac{\Lambda R^2}{3}. \tag{2.1b}$$

The parameters M, Q, and Λ are real and continuous. The discrete parameter k takes the values 1, 0, and -1, and $d\Omega_k^2$ is the metric on a two-dimensional surface Σ_k of constant Gaussian curvature k. In local coordinates (θ,φ) on Σ_k, we can write

$$d\Omega_k^2 = \begin{cases} d\theta^2 + \sin^2(\theta) d\varphi^2, & k=1, \\ d\theta^2 + \theta^2 d\varphi^2, & k=0, \\ d\theta^2 + \sinh^2(\theta) d\varphi^2, & k=-1. \end{cases} \tag{2.2}$$

Σ_k is locally homogeneous [37,38], with the local isometry group SO(3) for $k=1$, E^2 for $k=0$, and $SO_c(2,1)$ [the connected component of SO(2,1)] for $k=-1$. The local isometries of Σ_k are clearly inherited by the four-dimensional metric (2.1). The vector $\partial/\partial T$ is a Killing vector, timelike for $F>0$ and spacelike for $F<0$. We refer to T as the Killing time, and to the coordinates (T,R) as the curvature coordinates. Without loss of generality, we can assume $R>0$.

The metric (2.1) solves the Einstein-Maxwell equations with the cosmological constant Λ and the electromagnetic potential one-form

$$\mathbf{A} = \frac{Q}{R}\,\mathbf{d}T. \tag{2.3}$$

Indeed, the metric (2.1) with the electromagnetic potential (2.3) emerges from a sufficiently general form of Birkhoff's theorem as the generic family of Einstein-Maxwell spacetimes admitting the local isometry group SO(3), E^2, or $SO_c(2,1)$ with two-dimensional spacelike orbits [11]. Our electromagnetic potential (2.3) yields a vanishing magnetic field, but the spacetimes with a nonvanishing magnetic field can be obtained from Eq. (2.3) by the electromagnetic duality rotation.

B. Global properties

We now examine the global properties of the spacetimes (2.1) with $\Lambda<0$. We write $\Lambda = -3\ell^{-2}$ with $\ell>0$.

The first issue is in the global geometry of Σ_k. To exclude spacetime singularities that would result solely from singularities in the two-dimensional geometry of Σ_k, we take Σ_k to be complete. We can then write $\Sigma_k = \tilde{\Sigma}_k/\Gamma$, where $\tilde{\Sigma}_k$ is the universal covering space of Σ_k, and Γ is a freely and properly discontinuously acting subgroup of the full isometry group of $\tilde{\Sigma}_k$. If the action of Γ on $\tilde{\Sigma}_k$ is nontrivial, Σ_k is multiply connected.

For $k=1$, $\tilde{\Sigma}_1$ is S^2 with the round metric. The isometry group is O(3). The only multiply connected choice for Σ_1 is $\mathbb{R}P^2 = S^2/\mathbb{Z}_2$, where the nontrivial element of $\mathbb{Z}_2$ is the antipodal map [37].

For $k=0$, $\tilde{\Sigma}_0$ is $\mathbb{R}^2$ with the flat metric. The isometry group is $E^2 \times_s \mathbb{Z}_2$, where the nontrivial element of $\mathbb{Z}_2$ is the reflection about a prescribed geodesic, and $\times_s$ stands for the semidirect product. The multiply connected choices for Σ_0 are the cylinder, the Möbius band, the torus, and the Klein bottle [37].

For $k=-1$, $\tilde{\Sigma}_{-1}$ is $\mathbb{R}^2$ with the hyperbolic metric. The isometry group is $SO_c(2,1) \times_s \mathbb{Z}_2$, where the nontrivial element of $\mathbb{Z}_2$ is the reflection about a prescribed geodesic. The closed and orientable choices for Σ_{-1} are the closed Riemann surfaces of genus $g>1$ (see, for example, Ref. [39]). The multiply connected but not closed choices for Σ_{-1} include the cylinder [18,21] and the Möbius band, as well as surfaces with an arbitrary finite number of infinities [23,24,40].

When Σ_k is closed, we denote its area by V. For $k=1$, both S^2 and $\mathbb{R}P^2$ are closed, and we have, respectively, $V=4\pi$ and $V=2\pi$. For $k=0$, the closed choices are the torus and the Klein bottle, and V can in either case take arbitrary positive values. For $k=-1$, with Σ_{-1} closed, the Gauss-Bonnet theorem (see, for example, Refs. [41, 42]) implies $V=-2\pi\chi$, where χ is the Euler number of Σ_{-1}, and V is therefore completely determined by the topology. In the orientable case, we have $\chi = 2(1-g)$ and $V = 4\pi(g-1)$.

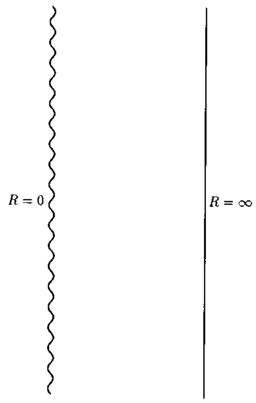

FIG. 1. The Penrose diagram for $M<M_{\rm crit}$. The straight line indicates an infinity and the wavy line a singularity.

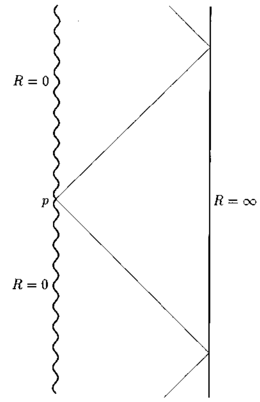

FIG. 2. The Penrose diagram for $M=M_{\rm crit}$, if $Q\neq 0$ or $k=-1$ or both. The point p is an internal spacelike infinity, and the singularity consists of countably many connected components. The infinity, which is both spacelike and (future) null, consists of a single connected component. As the past of the infinity consists of all of the spacetime, the spacetime does not have an interpretation as a black hole.

We next turn to the infinity structure of the metric (2.1). At $R\to\infty$, the dominant behavior of the metric is determined by the cosmological constant for any values of M and Q. In the special case $M=0=Q$, the spacetime is locally isometric to anti–de Sitter space [10,18,21]. We can therefore regard the infinity at $R\to\infty$ as an asymptotically locally anti–de Sitter infinity for any values of M and Q. The precise sense of this asymptotic structure has been examined in Refs. [3,4,10,30] for $k=1$, and the Hamiltonian falloff analyses of Refs. [10,30] can be readily adapted to cover also the cases $k=0$ and $k=-1$. The infinity is both a spacelike and a null infinity. In a Penrose diagram that suppresses Σ_k, the infinity can be represented by a vertical line.

For $k=1$ and $\Sigma_1=S^2$, the asymptotic anti–de Sitter symmetry at $R\to\infty$ allows one to introduce a Hamiltonian formulation with a well-defined Arnowitt-Deser-Misner (ADM) Hamiltonian [3,4,10,30]. This Hamiltonian generates translations of the spacelike hypersurfaces at infinity with respect to the asymptotic Killing time, normalized as the coordinate T in Eq. (2.1). It is straightforward to adapt the techniques of Refs. [10,30] to show that the same conclusion holds for all of our metrics for which Σ_k is closed. If one normalizes the additive constant in the Hamiltonian so that the Hamiltonian vanishes for $M=0=Q$, one finds that the contribution of an infinity to the ADM Hamiltonian is $(V/4\pi)M$, and the contribution to the analogously defined ADM electric charge is $(V/4\pi)Q$. When Σ_k is not closed, however, the infinite area of Σ_k implies infinite values for both the Hamiltonian and the charge.

Consider next the singularity structure of the metric (2.1). The metric has a curvature singularity at $R\to 0$ except when $M=0=Q$. When $M=0=Q$, the spacetime is locally anti–de Sitter, and the behavior at $R\to 0$ depends on the topology of Σ_k. If Σ_k is simply connected, the spacetime (2.1) with $R>0$ is isometric to a certain region of anti–de Sitter space [10,18,21]: $R\to 0$ is then a mere coordinate singularity, and the spacetime can be continued past $R=0$ to all of anti–de Sitter space. If Σ_k is not simply connected, the spacetime (2.1) with $R>0$ is isometric to a quotient space of a certain region of anti–de Sitter space with respect to a discrete subgroup of the isometry group, and the possibilities of continuing the spacetime past $R=0$ depend on how these discrete isometries extend to the rest of anti–de Sitter space. Typically, the extended spacetime is singular in its topological structure [18,21], in analogy with Misner space [25] or the BTZ black hole [8,9]. We shall not attempt to classify these singularities here.

We can now turn to the horizon structure. As usual [43], the positive values of R at which the function $F(R)$ [Eq. (2.1b)] vanishes are coordinate singularities on null hypersurfaces. The vector $\partial/\partial T$ is a globally defined Killing vector, timelike in the regions with $F>0$, spacelike in the regions with $F<0$, and null on the hypersurfaces with $F=0$. The regions with $F>0$ are therefore static, and the hypersurfaces with $F=0$ are Killing horizons.

For examining the (positive) zeroes of $F(R)$, it is useful to define the quantity

$$M_{\rm crit}(Q):=\frac{\ell}{3\sqrt{6}}\left(\sqrt{k^2+12(Q/\ell)^2}+2k\right)\times\left(\sqrt{k^2+12(Q/\ell)^2}-k\right)^{1/2}. \tag{2.4}$$

For $k=1$, a complete analysis can be found in Refs. [7,10,30]. We shall therefore from now on only consider the cases $k=0$ and $k=-1$.

Suppose first that $Q\neq 0$. For $M<M_{\rm crit}$, F has no zeros. For $M=M_{\rm crit}$, F has a degenerate zero, and for $M>M_{\rm crit}$, F has two distinct nondegenerate zeros. The Penrose diagrams of the analytic extensions are shown in Figs. 1–3.[1]

Suppose next that $Q=0$ and $k=-1$. We now have $M_{\rm crit}=-\ell/(3\sqrt{3})$. For $M<M_{\rm crit}$, F has no zeros. For $M=M_{\rm crit}$, F has a degenerate zero, and for $M_{\rm crit}<M<0$, F has two distinct nondegenerate zeros. The Penrose diagrams of the analytic extensions are again as in Figs. 1–3. For $M\geqslant 0$, F has just one nondegenerate zero. When $M>0$,

[1]These statements hold without change also for $k=1$, in which case we obtain the well-known Reissner-Nordström–anti–de Sitter spacetimes [7,10,30].

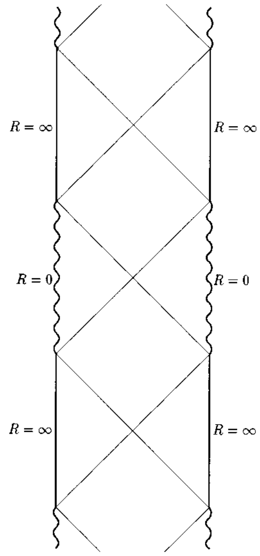

FIG. 3. The Penrose diagram for $M>M_{\mathrm{crit}}$ if $Q\neq 0$, and for $M_{\mathrm{crit}}<M<0$ if $Q=0$ and $k=-1$. There is both an outer horizon and an inner horizon.

$R=0$ is a curvature singularity, and the Penrose diagram is shown in Fig. 4. When $M=0$, $R=0$ is not a curvature singularity, as discussed above; however, provided Σ_{-1} is not simply connected, we regard $R=0$ as a topological singularity, and the Penrose diagram is again as in Fig. 4. When $M=0$ and Σ_{-1} is simply connected, the Penrose diagram in our coordinates is as in Fig. 4, but the singularity at $R=0$ is only a coordinate one.

Suppose finally that $Q=0$ and $k=0$. We now have $M_{\mathrm{crit}}=0$. For $M<0$, F has no zeros, and for $M>0$, F has a single nondegenerate zero. The Penrose diagrams of the analytic extensions are respectively as in Figs. 1 and 4. In the special case $M=0$, F has no zeros, and the space is locally anti–de Sitter. The Penrose diagram is shown in Fig. 5. The status of $R=0$ is then as above: if Σ_0 is multiply connected, we regard $R=0$ as a topological singularity, whereas if Σ_0 is simply connected, $R=0$ is just a coordinate singularity.

We have therefore obtained Penrose diagrams that faithfully depict the causal structure of the spacetimes, with the sole exception of $M=0=Q$ and Σ_k simply connected. With this exception, we see that all the spacetimes in which F has a nondegenerate zero can be interpreted as black holes. The connected components of the infinities displayed in the Penrose diagrams are genuine future null infinities, and the

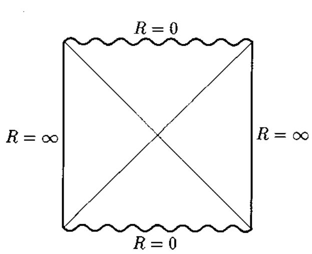

FIG. 4. The Penrose diagram for $M>0$ if $Q=0$, and for $M=0$ if $Q=0$ and $k=-1$. If Σ_{-1} is simply connected, the singularity in the latter case is a coordinate one.

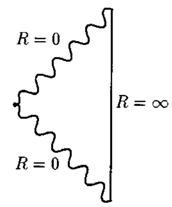

FIG. 5. The Penrose diagram for $M=0$, if $Q=0$ and $k=0$. If Σ_0 is simply connected, the singularity is a coordinate one.

boundaries of their causal pasts are black hole horizons. When a second zero of F exists, it can be interpreted as an inner horizon, as in the Reissner-Nordström–anti–de Sitter spacetime [7,10]. The topology of the horizon bifurcation two-manifold is that of Σ_k, and thus different from S^2. The theorems about spherical horizon topology [44–47] do not apply because the negative cosmological constant can be interpreted as a negative vacuum energy density.[2]

In the spacetimes in which F has a degenerate zero, it is seen from Fig. 2 that the future null infinity consists of a single connected component, and the past of this infinity is all of the spacetime. The Killing horizons in these spacetimes therefore do not have an interpretation as black hole horizons. Note that this differs from the extreme Reissner-Nordström solutions [25], in which the future null infinity is not connected, and the past of each connected component has a boundary along a Killing horizon.

The existence criterion for a nondegenerate horizon is $M>M_{\mathrm{crit}}$. For $k=0$, we have $M_{\mathrm{crit}}\geqslant 0$, and black holes therefore only occur with positive values of M. For $k=-1$, however, M_{crit} is negative for $|Q|<\ell/2$, so that black holes occur even with negative values of M. Note also that when $k=-1$ and $Q=0$, the internal structure of the black hole changes qualitatively at $M=0$: for $-\ell/(3\sqrt{3})<M<0$, we have two horizons and the singularities are timelike (Fig. 3), whereas for $M>0$, we only have one horizon and the singularities are spacelike (Fig. 4). Provided Σ_{-1} is not simply connected, we regard the limiting case $M=0$ as belonging to the latter category.

Instead of the pair (M,Q), it is more convenient to parametrize the black hole spacetimes in terms of the pair (R_h,Q), where R_h is the value of R at the (outer) horizon. For given Q, R_h can take the values $R_h>R_{\mathrm{crit}}(Q)$, where

$$R_{\mathrm{crit}}(Q):=\frac{\ell}{\sqrt{6}}\left[\sqrt{k^2+12(Q/\ell)^2}-k\right]^{1/2}. \tag{2.5}$$

The mass is then given in terms of Q and R_h as

$$M=\frac{R_h}{2}\left(\frac{R_h^2}{\ell^2}+k+\frac{Q^2}{R_h^2}\right). \tag{2.6}$$

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[2]For discussions of $k=1$ with the $\mathbb{R}P^2$ horizon topology but without a cosmological constant, see Refs. [44, 48].

III. THERMODYNAMICS WITH FINITE BOUNDARY

In this section we consider the thermodynamics of a black hole in a finite size box. First we calculate the local Hawking temperature for the black hole both by using the surface gravity formula, and by identifying the periodicity in the time coordinate in the Euclideanized metric. Then, we put the black hole in a box and use the Brown-York quasilocal energy formalism to calculate what we call the thermodynamical internal energy for the system. Upon varying this energy with respect to the extensive variables M and Q, using the expression for the local Hawking temperature, and assuming that the first law of black hole thermodynamics holds, we identify the entropy and the electrostatic potential for the system. Finally, we calculate the signs of the heat capacities C_Q, C_{Φ_B}, and C_{ϕ_h}. We include the three cases $k=1$, $k=0$, and $k=-1$ throughout the section.

A. Local Hawking temperature

The local Hawking temperature for a static eternal black hole can be calculated using the Unruh effect in curved spacetime or finding the periodicity in the time coordinate in the Euclidean version of the black hole metric covering the outer region. (See, for example, Ref. [49].)

In the Unruh effect one considers how an observer outside the black hole,[3] following the timelike Killing flow, would experience a quantum field that is in the Hartle-Hawking vacuum state. The Hartle-Hawking vacuum is a globally nonsingular vacuum invariant under the Killing flow. The result is that the observer will experience a thermal state with local temperature

$$T_H(R)=\frac{\kappa_h}{2\pi\sqrt{-\chi_\alpha\chi^\alpha}}, \tag{3.1a}$$

where κ_h is the surface gravity evaluated at the horizon,

$$\kappa_h:=\sqrt{-\tfrac{1}{2}\nabla^\alpha\chi^\beta\nabla_\alpha\chi_\beta}\Big|_{R=R_h}, \tag{3.1b}$$

and χ^α is the Killing vector field generating the event horizon. The interpretation of this physical process is that we have a black hole in thermal equilibrium with its surroundings.

For the black holes (2.1) considered here, the Killing field is

$$\chi^\alpha\frac{\partial}{\partial x^\alpha}=\frac{\partial}{\partial T}. \tag{3.2}$$

We therefore have

$$\kappa=\tfrac{1}{2}F'(R), \tag{3.3a}$$

where the prime indicates derivative with respect to R, and

$$T_H(R)=\frac{F'(R_h)}{4\pi\sqrt{F(R)}}=\frac{(k-Q^2/R_h^2+3R_h^2/\ell^2)}{4\pi R_h\sqrt{k-2M/R+Q^2/R^2+R^2/\ell^2}}. \tag{3.3b}$$

Note that $T_H(R)$ does not depend on the normalization of the Killing vector field χ^α. In the limits $R\to R_h$ and $R\to\infty$, we have, respectively, $T_H(R)\to\infty$ and $T_H(R)\to 0$. Unlike in the asymptotically flat case, the black hole therefore does not have a finite, nonvanishing physical temperature at infinity.

It is of interest to define the renormalized temperature, denoted by T_∞, as the product of $T_H(R)$ and the redshift factor $\sqrt{-\chi^\alpha\chi_\alpha}$ [26–28]. The result is

$$T_\infty=\frac{F'(R_h)}{4\pi}=\frac{(k-Q^2/R_h^2+3R_h^2/\ell^2)}{4\pi R_h}. \tag{3.4}$$

Although T_∞ does not appear to have a physical interpretation as the temperature experienced by a family of observers,[4] we shall see below that it emerges as the counterpart of temperature in the infinite space limit of the first law of thermodynamics [26–29].

For given k, both $T_H(R)$ and T_∞ are independent of the topology of the two-space Σ_k. Also, both $T_H(R)$ and T_∞ vanish for the extremal solutions, $M=M_{\rm crit}$, as $F(R)$ then has a double root at $R=R_h$.

These results for the Hawking temperature can also be derived by Euclidean methods [26]. When a nondegenerate horizon exists, regularity of the Euclidean version of the metric (2.1) at the horizon requires the Euclidean time, $\tau:=iT$, to be periodic with period $P=4\pi/F'(R_h)$. When a Green's function that is regular on the Euclidean section is analytically continued to the Lorentzian section, it retains periodicity in imaginary time. The local temperature can then be identified as the inverse Euclidean period divided by the redshift factor, with the result $T_H(R)=(Pg_{00})^{-1}=F'(R_h)[4\pi\sqrt{F(R)}]^{-1}$.

B. First law and entropy

We wish to verify that the black holes satisfy the first law of black hole thermodynamics (BHTD) and to identify the black hole entropy. As $T_H(R)\to 0$ in the limit $R\to\infty$, we first formulate the first law with a boundary at a finite value of R. To have a black hole spacetime, we assume throughout $M>M_{\rm crit}$. To make the thermodynamical quantities finite, we take Σ_k closed. As in Sec. II, V denotes the (dimensionless) area of Σ_k.

We introduce a boundary at $R=R_B$, and we regard the boundary scale factor R_B as a prescribed, finite parameter. A spacelike snapshot of the boundary history then has the topology of Σ_k. The Brown-York quasilocal energy formalism [50] can be readily used to define the thermodynamical internal energy of this system on a constant T hypersurface. Denoting the internal energy by $U(R_B)$, we find

[3]The Unruh effect gives rise to a Hawking temperature even without a black hole, as long as the spacetime contains a bifurcate Killing horizon, there exists a Killing field timelike in the outer region, and the Hartle-Hawking vacuum exists [49].

[4]T_∞ coincides with $T_H(R)$ at the locations where the redshift factor equals unity. However, these locations depend on the normalization of the Killing vector χ^α.

$$U(R_B) = -R_B^2 V\left(\frac{\sqrt{F(R_B)}}{4\pi R_B} + \epsilon_0(R_B)\right), \tag{3.5}$$

where $\epsilon_0(R)$ is an arbitrary function that arises from the freedom of adding surface terms to the gravitational action. A specific choice for $\epsilon_0(R)$ will not be needed for what follows.

We note in passing that one natural criterion for choosing $\epsilon_0(R)$ would be to require that $U(R_B)$ vanishes for the locally anti–de Sitter solutions, for which $M=0=Q$. This leads to

$$\epsilon_0(R) = -\frac{\sqrt{k + R^2/\ell^2}}{4\pi R}. \tag{3.6}$$

For $k=1$ and $k=0$, we then have $U(R_B) \geqslant 0$, but for $k=-1$, $U(R_B)$ does not have a definite sign. In particular, for $k=-1$, $U(R_B)<0$ when $Q=0$ and $M<0$.

Variation of $U(R_B)$ with respect to M and Q (or, equivalently, R_h and Q) gives

$$\begin{aligned} dU(R_B) &= \frac{V}{8\pi\sqrt{F(R_B)}}\left(k - \frac{Q^2}{R_h^2} + \frac{3R_h^2}{\ell^2}\right) dR_h \\ &\quad + \frac{QV}{4\pi\sqrt{F(R_B)}}\left(\frac{1}{R_h} - \frac{1}{R_B}\right) dQ \\ &= T_H(R_B)\, d\left(\tfrac{1}{4} V R_h^2\right) + \tilde{\Phi}(R_B)\, de, \end{aligned} \tag{3.7}$$

where we have used the Hawking temperature (3.3b) and defined

$$e := (V/4\pi)Q, \tag{3.8a}$$

$$\tilde{\Phi}(R_B) := \frac{Q}{\sqrt{F(R_B)}}\left(\frac{1}{R_h} - \frac{1}{R_B}\right). \tag{3.8b}$$

As mentioned in Sec. II, e is the ADM charge. Comparing Eq. (3.8b) to Eq. (2.3) shows that $\tilde{\Phi}(R_B)$ is equal to the electrostatic potential difference between the horizon and the boundary, with the electromagnetic gauge chosen as in Eq. (2.3), and with respect to a time coordinate that agrees with the proper time of a static observer at the boundary. We can think of $\tilde{\Phi}(R_B)$ as the electrostatic potential difference between the horizon and the boundary, appropriately redshifted to the boundary.

Comparing Eq. (3.7) with the desired form of the first law of BHTD,

$$dU = T dS + \tilde{\Phi} de, \tag{3.9}$$

we identify the entropy of the black hole as

$$S = \tfrac{1}{4} V R_h^2 = \tfrac{1}{4} A_h, \tag{3.10}$$

where A_h is the area of the event horizon. This area law holds for all the closed horizon topologies that occur with our black hole spacetimes. In the special case $k=1$ and $\Sigma_1 = S^2$, we recover the Bekenstein-Hawking area law.

It would be possible to vary $U(R_B)$ also with respect to R_B. The first law (3.7) would then contain the additional term $-p_B dR_B$, where p_B is the surface pressure, thermodynamically conjugate to R_B [51,52]. The expression for p_B would, however, depend on the choice of the term $\epsilon_0(R)$ in Eq. (3.5).

In the limit $R_B \to \infty$, both sides of Eq. (3.7) vanish. Nevertheless, multiplying first both sides by $\sqrt{F(R_B)}$ and then taking the limit $R_B \to \infty$, we recover the finite equation

$$d\left(\frac{VM}{4\pi}\right) = T_\infty d\left(\tfrac{1}{4} V R_h^2\right) + \phi de, \tag{3.11}$$

where

$$\phi := \frac{Q}{R_h}. \tag{3.12}$$

From Sec. II we recall that $(V/4\pi)M$ is the ADM energy at infinity and ϕ is the electrostatic potential difference between the horizon and infinity, both with respect to the Killing time coordinate of the metric (2.1). We can therefore identify Eq. (3.11) as the first law of BHTD in the absence of a boundary. If T_∞ is postulated to have an interpretation as a temperature, we obtain for the entropy the area law (3.10) [26]. Conversely, if the area law (3.10) for the entropy is postulated to hold, T_∞ emerges as a temperature [29]. We reemphasize, however, that T_∞ is not the physical temperature measured by an observer at infinity.

Note that the first laws (3.7) and (3.11) only determine the entropy up to an additive constant. In the identification (3.10) we have chosen this constant so that the entropy is equal to one-quarter of the area. One could, however, add to Eq. (3.10) an arbitrary function of any quantities that our variations treat as fixed. In particular, one could add an arbitrary function of the cosmological constant and the topology of Σ_k.

Finally, we note that the above thermodynamical discussion has regarded V as fixed. It does not appear possible to relax this assumption in a way that would promote V into an independent thermodynamical variable. For $k=1$ and $k=-1$, V only takes discrete values, and continuous variations in V are not possible. For $k=0$, the possible values of V form a continuum; however, changes in V can then be absorbed into redefinitions of R, M, and Q.

C. Thermodynamical stability

We now turn to the thermodynamical stability of the black holes. In this section we consider a black hole in a box with a prescribed, finite value of R_B. The limit $R_B \to \infty$ will be addressed in Sec. IV.

The response function whose sign determines the thermodynamical stability is the heat capacity (see, for example, Ref. [53])

$$C_X = T\left(\frac{\partial S}{\partial T}\right)_X, \tag{3.13}$$

where S is the entropy, T is the temperature, and X indicates the quantities that are held fixed. With a finite boundary, the relevant temperature is the local Hawking temperature (3.3b), and the entropy is given by the area law (3.10). For the fixed quantity X, we consider three choices: the ADM charge e (3.8a) (or, equivalently, the parameter Q), the redshifted electrostatic potential difference $\tilde{\Phi}(R_B)$ [Eq. (3.8b)] between the horizon and the boundary [33], and the redshifted electrostatic potential difference between the boundary and infinity, given by

$$\Phi_B := \frac{Q}{R_B\sqrt{F(R_B)}}. \tag{3.14}$$

We write $\tilde{\Phi}(R_B) := \tilde{\Phi}_B$.

The technical details of analyzing the three heat capacities C_Q, $C_{\tilde{\Phi}_B}$, and C_{Φ_B} are given in the Appendix. When R_B is so large that the box-dependent features of the heat capacities become negligible, we find that these heat capacities are positive definite for $k=0$ and $k=-1$, but indefinite for $k=1$. In this sense, the black holes with $k=0$ and $k=-1$ have a wider range of thermodynamical stability than the conventional black holes with $k=1$. However, as discussed in the Appendix, there exist choices for the fixed quantity X that would render also the black holes with $k=0$ and $k=-1$ thermodynamically unstable.

For the conventional black holes with $k=1$, the heat capacities C_Q, $C_{\tilde{\Phi}_B}$, and C_{Φ_B} diverge at the places where they change sign in the (M,Q) parameter space. In the asymptotically flat context, this phenomenon was discussed by Davies [54,55].

IV. INFINITE SPACE THERMODYNAMICAL ENSEMBLES

In this section we consider thermodynamics in the limit where the boundary is pushed to infinity. For $k=1$ and $\Sigma_1 = S^2$, this problem was analyzed in Refs. [29,30]. We take here $k=0$ or $k=-1$, and assume throughout that Σ_k is closed.

As discussed in Sec. III, both the local Hawking temperature $T_H(R_B)$ (3.3b) and the redshifted electrostatic potential difference $\tilde{\Phi}(R_B)$ (3.8b) vanish in the limit $R_B \to \infty$. Relying on the infinite space form (3.11) of the first law, we adopt the viewpoint that the appropriate counterparts of $T_H(R_B)$ and $\tilde{\Phi}(R_B)$ are, respectively, the renormalized temperature T_∞ (3.3b) and the (unredshifted) Killing time electrostatic potential difference ϕ (3.12). We write $\beta_\infty = T_\infty^{-1}$.

It would be straightforward to proceed as in Sec. III and show that the heat capacities at fixed e (3.8a) and ϕ (3.12) are both positive definite. However, we wish to go further and construct full quantum thermodynamical equilibrium ensembles that fix, in addition to β_∞, either e or ϕ. Following the terminology of Refs. [30–36], we refer to the ensemble that fixes β_∞ and e as the canonical ensemble, and to the ensemble that fixes β_∞ and ϕ as the grand canonical ensemble.

One way to approach this problem would be within the Euclidean path-integral formalism, performing a Hamiltonian reduction of the action as in Refs. [31–33]. Another way would be to perform a Hamiltonian reduction in the Lorentzian theory, and then take the trace of an analytically continued evolution operator under suitably chosen boundary conditions as in Refs. [30,34–36]. The boundary conditions in the two approaches are identical by construction, and one may argue that the only difference between the two approaches is in the order of quantization and Euclideanization. For our spacetimes, the appropriate boundary conditions and boundary terms are easily found by adapting to our symmetries the Lorentzian Hamiltonian analysis of the $k=1$ case in Ref. [30]. Adapting to our symmetries the details of the Lorentzian Hamiltonian reduction of Ref. [30] would require more work, and we have not pursued this in detail; instead, we appeal to the Euclidean reduction formalism [31–33] to argue that only the boundary terms survive after the reduction. This yields the reduced actions through steps that follow the cited references so closely that we shall not repeat the details of the analysis here. Instead, we just state the results for the reduced Euclidean actions, and proceed to the thermodynamical analysis.

A. Grand canonical ensemble

The reduced Euclidean action with fixed β_∞ and ϕ is given by

$$I^*_{\rm gc}(\mathbf{R}_h, \mathbf{q}) := \frac{V}{4\pi}[\beta_\infty(\mathbf{m} - \mathbf{q}\phi) - \pi\mathbf{R}_h^2], \tag{4.1}$$

where

$$\mathbf{m} := \tfrac{1}{2}\mathbf{R}_h(\mathbf{R}_h^2\ell^{-2} + k + \mathbf{q}^2\mathbf{R}_h^{-2}). \tag{4.2}$$

The variables in $I^*_{\rm gc}$ are $\mathbf{R}_h$ and $\mathbf{q}$, and their domain is specified by the inequalities

$$\mathbf{R}_h > \sqrt{-k/3}\,\ell, \tag{4.3a}$$

$$\mathbf{q}^2 < \mathbf{R}_h^2(k + 3\mathbf{R}_h^2\ell^{-2}). \tag{4.3b}$$

The reduction has eliminated the constraints, but it has not used the full Einstein equations. For generic values of $\mathbf{R}_h$ and $\mathbf{q}$, $I^*_{\rm gc}(\mathbf{R}_h, \mathbf{q})$ is therefore not equal to the Euclidean action of any of the classical black holes of Sec. II. However, $I^*_{\rm gc}(\mathbf{R}_h, \mathbf{q})$ is the Euclidean action of a spacetime with the same topological and asymptotic properties. In particular, $\mathbf{R}_h$ is the value of the ‘‘scale factor’’ associated with Σ_k at the horizon, and the ADM charge at infinity is $(V/4\pi)\mathbf{q}$.

$I^*_{\rm gc}$ has precisely one stationary point, at

$$\mathbf{R}_h = \mathbf{R}_h^+ := \frac{2\pi\ell^2}{3\beta_\infty}\left[1 + \sqrt{1 + \frac{3\beta_\infty^2(\phi^2 - k)}{4\pi^2\ell^2}}\right], \tag{4.4a}$$

$$\mathbf{q} = \mathbf{q}^+ := \phi\mathbf{R}_h^+, \tag{4.4b}$$

and this stationary point is the global minimum. It is straightforward to verify that this stationary point is the black hole spacetime of Sec. II with the specified values of β_∞ and ϕ. $\mathbf{R}_h^+$ is equal to the value of R_h in this spacetime, $\mathbf{q}^+$ is equal to the value of Q, and $\mathbf{m}^+ := \mathbf{m}(\mathbf{R}_h^+, \mathbf{q}^+)$ is equal the value of M.

The grand partition function of the thermodynamical grand canonical ensemble is obtained as the integral

$$\mathcal{Z}(\beta_\infty, \phi) = \int_{\mathcal{A}} \tilde{\mu}\, d\mathbf{R}_h d\mathbf{q}\, \exp(-I^*_{\rm gc}), \tag{4.5}$$

where the integration domain $\mathcal{A}$ is given by Eq. (4.3). The weight factor $\tilde{\mu}$, which depends on the details of quantization [30–34,56], is assumed to be positive and slowly varying. The qualitative properties of the ensemble are then determined by the exponential factor in Eq. (4.5).

The integral in Eq. (4.5) is convergent, and when the stationary point approximation is good, the dominant contribution comes from the global minimum at the stationary point (4.4). Denoting by $\langle E \rangle$ and $\langle e \rangle$ the thermal expectation values of, respectively, the energy and the charge, we have

$$\langle E \rangle = \left(-\frac{\partial}{\partial \beta_\infty} + \beta_\infty^{-1} \phi \frac{\partial}{\partial \phi} \right) (\ln \mathcal{Z}) \approx (V/4\pi) \mathbf{m}^+, \tag{4.6a}$$

$$\langle e \rangle = \beta_\infty^{-1} \frac{\partial (\ln \mathcal{Z})}{\partial \phi} \approx (V/4\pi) \mathbf{q}^+. \tag{4.6b}$$

It follows from the construction of the grand canonical ensemble that the constant ϕ heat capacity, $C_\phi = \beta_\infty^2 (\partial^2 (\ln \mathcal{Z}) / \partial \beta_\infty^2)$, is positive, and also that $(\partial \langle e \rangle / \partial \phi)$ is positive: when the stationary point approximation is good, these statements can be easily verified observing that $\partial \mathbf{R}_h^+ / \partial \beta_\infty < 0$ and $\partial \mathbf{q}^+ / \partial \phi > 0$. The system is therefore stable under thermal fluctuations in both the energy and the charge.

When the stationary point dominates, we obtain for the entropy

$$S = \left(1 - \beta_\infty \frac{\partial}{\partial \beta_\infty} \right) (\ln \mathcal{Z}) \approx \tfrac{1}{4} V (\mathbf{R}_h^+)^2 = \tfrac{1}{4} A_h. \tag{4.7}$$

This agrees with the area law (3.10).

B. Canonical ensemble

In the canonical ensemble, we wish to fix β_∞ and the ADM charge e. The reduced Euclidean action with these fixed quantities is

$$I_c^*(\mathbf{R}_h) := \frac{V}{4\pi} (\beta_\infty \mathbf{m} - \pi \mathbf{R}_h^2), \tag{4.8}$$

where $\mathbf{m}$ is given by Eq. (4.2) with $\mathbf{q} = (4\pi/V) e$. The only variable in I_c^* is $\mathbf{R}_h$, and its domain is $\mathbf{R}_h > R_{\mathrm{crit}}(\mathbf{q})$, where the function R_{crit} was defined in Eq. (2.5). Again, the reduction has eliminated the constraints but not used the full Einstein equations, and for generic values of $\mathbf{R}_h$, $I_c^*(\mathbf{R}_h)$ is not equal to the Euclidean action of any of the black holes of Sec. II. Instead, $I_c^*(\mathbf{R}_h)$ is the Euclidean action of a spacetime with the same topological and asymptotic properties, and $\mathbf{R}_h$ is the value of the "scale factor" of Σ_k at the horizon of this spacetime.

I_c^* has precisely one stationary point, at the unique root of the equation

$$\frac{3\mathbf{R}_h^4}{\ell^2} - \frac{4\pi \mathbf{R}_h^3}{\beta_\infty} + k \mathbf{R}_h^2 - \mathbf{q}^2 = 0 \tag{4.9}$$

in the domain $\mathbf{R}_h > R_{\mathrm{crit}}(\mathbf{q})$. This stationary point is the global minimum of I_c^*. It is straightforward to verify that this stationary point is the black hole spacetime of Sec. II with the specified values of β_∞ and e. $\mathbf{R}_h^+$ is equal to the value of R_h in this spacetime, $Q = \mathbf{q} = (4\pi/V) e$, and $\mathbf{m}(\mathbf{R}_h^+)$ is equal the value of M.

The partition function of the thermodynamical canonical ensemble reads

$$Z(\beta_\infty, e) = \int_{R_{\mathrm{crit}}(\mathbf{q})}^{\infty} \tilde{\tilde{\mu}} d\mathbf{R}_h \exp(-I_c^*), \tag{4.10}$$

where we again assume the weight factor $\tilde{\tilde{\mu}}$ to be positive and slowly varying compared with the exponential. The integral is convergent, and the positivity of the constant e heat capacity, $C_e = \beta_\infty^2 (\partial^2 (\ln Z) / \partial \beta_\infty^2)$, is guaranteed by construction. When the stationary point approximation is good, the dominant contribution comes from the unique stationary point. For the thermal expectation values of the energy and the electric potential, we find

$$\langle E \rangle = -\frac{\partial (\ln Z)}{\partial \beta_\infty} \approx (V/4\pi) \mathbf{m}, \tag{4.11a}$$

$$\langle \phi \rangle = -\beta_\infty^{-1} \frac{\partial (\ln Z)}{\partial e} \approx \frac{\mathbf{q}}{\mathbf{R}_h}, \tag{4.11b}$$

which are related to the parameters of the dominating classical solution in the expected way. When the approximation in Eq. (4.11a) for $\langle E \rangle$ holds, the positivity of C_e can be verified observing that at the critical point $\partial \mathbf{m} / \partial \beta_\infty < 0$. For the entropy we again recover the area law (3.10),

$$S = \left(1 - \beta_\infty \frac{\partial}{\partial \beta_\infty} \right) (\ln Z) \approx \tfrac{1}{4} A_h. \tag{4.12}$$

V. DISCUSSION

In this paper we have discussed the thermodynamics of asymptotically anti–de Sitter black holes in which the round two-sphere of the Reissner-Nordström–anti–de Sitter spacetimes has been replaced by a two-dimensional space Σ of constant negative or vanishing curvature. The local properties of these spacetimes are well known [11]. The main new feature for black hole interpretation is that the topology of the horizon is not spherical but that of Σ. This allows a toroidal horizon when Σ is flat, and a horizon with the topology of any closed higher genus Riemann surface when Σ has negative curvature. More possibilities arise if Σ is not closed.

All the spacetimes have one or more asymptotically anti–de Sitter infinities, and one can use the asymptotic Killing time translations to define ADM mass and charge. These quantities are finite whenever Σ is closed. If the additive constant in the ADM mass is chosen so that the mass vanishes for the solutions that are locally anti–de Sitter, black holes with flat Σ have positive ADM mass, but when Σ has negative curvature, there are black hole solutions with either sign of the ADM mass.

The thermodynamical analysis was carried out via a straightforward generalization of the techniques previously applied to the Reissner-Nordström–anti–de Sitter spacetimes. The local Hawking temperature was found from the Unruh effect, or from the periodicity of Euclidean time. Taking Σ closed, we introduced a boundary with the topology of Σ, and we interpreted the Brown-York-type quasilocal energy at the boundary as the internal thermodynamical energy. The first law of black hole thermodynamics then led to the conclusion that the entropy is one-quarter of the horizon area. This result extends the Bekenstein-Hawking area law to our toroidal and higher genus horizons.

Examination of heat capacities with fixed ADM charge, or with fixed appropriate electrostatic potentials, showed that our black holes are thermodynamically more stable than the Reissner-Nordström–anti–de Sitter black hole. In particular, in the limit of a large box, our black holes are always thermodynamically stable under these boundary conditions. With the boundary pushed fully to infinity, we constructed quantum equilibrium ensembles that fixed a renormalized temperature and either the ADM charge or the electrostatic potential. We found that these ensembles are well defined, and always dominated by a unique black hole solution. This provides another piece of evidence for thermodynamical stability of our black holes.

All our black hole spacetimes belong to the family (2.1). This family arises as the generic solution family from a Birkhoff's theorem that assumes the spacetime to admit the local isometry group E^2 (leading to flat Σ), $SO_c(2,1)$ (leading to negatively curved Σ), or SO(3) (leading to the Reissner-Nordström–anti–de Sitter solutions), with two-dimensional spacelike orbits [11]. This suggests seeking a black hole interpretation also for spacetimes that have the same local isometries but do not fall within the family (2.1). The most promising candidate would seem to be the Nariai-Bertotti-Robinson family [11]. In this family, the spacetime has the product form $M_L \times M_E$, where M_L (M_E, respectively) is a two-dimensional Riemannian manifold of signature $(-+)$ $[(++)]$ and constant Gaussian curvature K_L (K_E). The curvatures satisfy

$$K_E + K_L = 2\Lambda, \tag{5.1a}$$

$$K_E - K_L \geqslant 0. \tag{5.1b}$$

The electromagnetic two-form with a vanishing magnetic field is

$$\mathbf{F} = \pm \sqrt{\tfrac{1}{2}(K_E - K_L)}\,\boldsymbol{\omega}_L, \tag{5.2}$$

where $\boldsymbol{\omega}_L$ is the volume two-form on M_L, and the case of a nonvanishing magnetic field is obtained via the electromagnetic duality rotation.[5] The local symmetries of the spacetime are clear from the construction. To create a black hole in analogy with the BTZ construction [8,9,18,21], one would now like to take the quotient with respect to a suitable discrete isometry group. The crucial question is whether satisfactory discrete isometries exists.

Note added. After the present work was completed, Ref. [61] was posted. The results therein overlap with ours for $Q = 0$ and in the absence of a finite boundary. The main difference is that the subtraction procedure of Ref. [61] to make the Euclidean action finite generates for $k = -1$ a horizon contribution that is not present in our Hamiltonian subtraction procedure in Sec. IV. As a result, the entropy obtained in Ref. [61] for $k = -1$ differs from Eq. (3.10) by an additive constant, such that the values of the entropy span the whole positive real axis. Also, the additive constant in the ADM energy in Ref. [61] is chosen so that the ADM energy takes all positive values both for $k = 0$ and $k = -1$.

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APPENDIX: HEAT CAPACITIES IN A BOX

In this appendix we calculate the heat capacities C_Q, $C_{\tilde{\Phi}_B}$, and C_{Φ_B}, defined in Sec. III. We consider both $k = 1$, $k = 0$, and $k = -1$.

As explained in Sec. III, we consider a black hole in a box with a prescribed, finite boundary scale factor R_B. The potential $\tilde{\Phi}_B := \tilde{\Phi}(R_B)$ is defined by Eq. (3.8b), and it equals the electrostatic potential difference between the boundary and the horizon, with respect to a time coordinate normalized to a static observer's proper time at the boundary. Similarly, the potential Φ_B was defined by Eq. (3.14), and it equals the electrostatic potential difference between the boundary and the infinity, with respect to a time coordinate normalized to a static observer's proper time at the boundary.

The heat capacity C_X at constant value of the thermodynamical variable X is defined by Eq. (3.13), where S is the entropy and T the temperature. For us, S and $T = T_H(R_B)$ are given, respectively, by Eqs. (3.10) and (3.3b).

It is useful to regard S and T_H as functions of the two independent variables R_h and Q^2. M becomes then a dependent variable, determined by Eq. (2.6). From Eq. (3.13), we obtain

$$C_X = \tfrac{1}{2} V R_h T_H \left[T_{R_h} + T_{Q^2} \left(\frac{dQ^2}{dR_h} \right)_X \right]^{-1}, \tag{A1}$$

where

$$T_{R_h} := \frac{\partial T_H}{\partial R_h} = \frac{1}{4\pi\sqrt{F(R)}} \left[\frac{1}{R_h^2} \left(-k + \frac{3R_h^2}{\ell^2} + \frac{3Q^2}{R_h^2} \right) + \frac{1}{2R_B R_h F(R_B)} \left(k + \frac{3R_h^2}{\ell^2} - \frac{Q^2}{R_h^2} \right)^2 \right], \tag{A2a}$$

$$T_{Q^2} := \frac{\partial T_H}{\partial (Q^2)} = \frac{1}{4\pi\sqrt{F(R_B)}} \left[-\frac{1}{R_h^3} + \frac{1}{2R_B R_h F(R_B)} \left(k + \frac{3R_h^2}{\ell^2} - \frac{Q^2}{R_h^2} \right) \times \left(\frac{1}{R_h} - \frac{1}{R_B} \right) \right]. \tag{A2b}$$

The range of the parameters is

[5] The special case $\mathbf{F} = 0$ yields flat spacetime for $\Lambda = 0$, the Nariai solution [57] for $\Lambda > 0$, and a negative curvature analogue of the Nariai solution for $\Lambda < 0$. The special case $\Lambda = 0$ yields the Bertotti-Robinson solution [58–60] for $\mathbf{F} \neq 0$ and flat spacetime for $\mathbf{F} = 0$.

$$\sqrt{\max(0,-\tfrac{1}{3}k\ell^2)}<R_h<R_B, \tag{A3a}$$

$$0\leqslant Q^2<R_h^2\left(\frac{3R_h^2}{\ell^2}+k\right). \tag{A3b}$$

Here, Eq. (A3b) and the leftmost inequality in Eq. (A3a) are the conditions for the existence of a nondegenerate horizon. The rightmost inequality in (A3a) is the condition that the black hole fit in the box.

We shall mainly discuss the sign of the heat capacities in the limit where R_B is taken to infinity while the parameters R_h and Q^2 remain in some prescribed finite range. This means neglecting the second terms in Eq. (A2). In this limit, T_{Q^2} is always negative, and T_{R_h} is positive except when the following set of conditions holds:

$$k=1,$$

$$Q^2/\ell^2<1/36,$$

$$\frac{1}{6}\left[1-\sqrt{1-36(Q/\ell)^2}\right]<R_h^2/\ell^2<\frac{1}{6}\left[1+\sqrt{1-36(Q/\ell)^2}\right]. \tag{A4}$$

Consider first C_Q. With $X=Q$, we have $(dQ^2/dR_h)_X=0$, and the sign of C_Q agrees with the sign of T_{R_h}. Hence, in the limit $R_B\to\infty$, C_Q is positive for all configurations except those satisfying Eq. (A4). As the second term in Eq. (A2a) is positive definite, taking R_B finite would increase C_Q, preserving the stability for $k=0$ and $k=-1$ and widening the domain of stability for $k=1$.

Consider next $C_{\tilde{\Phi}_B}$. With $X=\tilde{\Phi}_B$, we now have[6]

$$\left(\frac{dQ^2}{dR_h}\right)_{\tilde{\Phi}_B}=\frac{2Q^2}{R_h}[1+O(R_B^{-1})], \tag{A5}$$

and the denominator in (A1) becomes

$$T_{R_h}+T_{Q^2}\left(\frac{dQ^2}{dR_h}\right)_{\tilde{\Phi}_B}=\frac{1}{4\pi\sqrt{F(R_B)}}\left\{\frac{Q^2}{R_h^4}\left[1+\frac{R_h^2}{Q^2}\left(\frac{3R_h^2}{\ell^2}-k\right)\right]+O(R_B^{-1})\right\}. \tag{A6}$$

In the limit $R_B\to\infty$, the expression in Eq. (A6) is positive definite for $k=0$ and $k=-1$. For $k=1$, however, it becomes negative when the following set of conditions holds:

$$k=1,$$

$$R_h<\frac{\ell}{\sqrt{3}},$$

$$Q^2<R_h^2\left(1-\frac{3R_h^2}{\ell^2}\right). \tag{A7}$$

Thus, in the limit $R_B\to\infty$, $C_{\tilde{\Phi}_B}$ is positive definite for $k=0$ and $k=-1$, but indefinite for $k=1$. For finite R_B, it can be verified that the terms omitted from Eq. (A6) are positive definite: $C_{\tilde{\Phi}_B}$ is positive for $k=0$ and $k=-1$ also with a finite boundary, whereas for $k=1$, taking the boundary finite widens the domain of stability.

Consider finally C_{Φ_B}. With $X=\Phi_B$, we have

$$\begin{aligned}\left(\frac{dQ^2}{dR_h}\right)_{\Phi_B}&=Q^2\left(k+\frac{3R_h^2}{\ell^2}-\frac{Q^2}{R_h^2}\right)\\&\quad\times\left[R_B\left(k+\frac{R_B^2}{\ell^2}\right)-R_h\left(k+\frac{R_h^2}{\ell^2}\right)\right]^{-1}\\&=\frac{Q^2\ell^2}{R_B^3}\left(k+\frac{3R_h^2}{\ell^2}-\frac{Q^2}{R_h^2}\right)[1+O(R_B^{-1})],\end{aligned} \tag{A8}$$

and the denominator in Eq. (A1) becomes

$$T_{R_h}+T_{Q^2}\left(\frac{dQ^2}{dR_h}\right)_{\Phi_B}=\frac{1}{R_h^2}\left(-k+\frac{3R_h^2}{\ell^2}+\frac{3Q^2}{R_h^2}\right)\times[1+O(R_B^{-1})]. \tag{A9}$$

In the limit $R_B\to\infty$, the expression in Eq. (A9) is positive definite for $k=0$ and $k=-1$. For $k=1$, it is negative when the conditions (A4) hold. Hence, in the limit $R_B\to\infty$, C_{Φ_B} is positive definite for $k=0$ and $k=-1$ and indefinite for $k=1$.

These results show that for $k=0$ and $k=-1$, the heat capacities C_Q, $C_{\tilde{\Phi}_B}$, and C_{Φ_B} are positive definite in the limit $R_B\to\infty$. There exist, however, choices for X such that C_X can be negative for $k=0$ and $k=-1$: from Eq. (A2), it is seen that this happens whenever $(dQ^2/dR_h)_X>R_h(-k+3R_h^2/\ell^2+3Q^2/R_h^2)$. Saturating this inequality corresponds to $X=F'(R_h)$, which is equivalent to holding the renormalized temperature T_∞ constant.

The signs of our three heat capacities, in the limit of a large box, may be summarized in the following table:

	C_Q	$C_{\tilde{\Phi}_B}$	C_{Φ_B}
$k=-1$	+	+	+
$k=0$	+	+	+
$k=1$	±	±	±

(A10)

[6]Note that in the limit $R_B\to\infty$ we have $(dQ^2/dR_h)_{\tilde{\Phi}_B}=(dQ^2/dR_h)_{\phi_h}$, where $\phi_h:=QR_h^{-1}$ is the quantity held fixed in the grand canonical ensemble in Sec. IV. This provides a check on the positivity of the heat capacity C_ϕ discussed in Sec. IV.

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