Axiomatic approach to electromagnetic and gravitational radiation reaction of particles in curved spacetime

Theodore C. Quinn and Robert M. Wald
Enrico Fermi Institute and Department of Physics, University of Chicago, 5640 S. Ellis Avenue, Chicago, Illinois 60637-1433
(Received 24 October 1996)

The problem of determining the electromagnetic and gravitational ''self-force'' on a particle in a curved spacetime is investigated using an axiomatic approach. In the electromagnetic case, our key postulate is a ''comparison axiom,'' which states that whenever two particles of the same charge e have the same magnitude of acceleration, the difference in their self-force is given by the ordinary Lorentz force of the difference in their (suitably compared) electromagnetic fields. We thereby derive an expression for the electromagnetic self-force which agrees with that of DeWitt and Brehme as corrected by Hobbs. Despite several important differences, our analysis of the gravitational self-force proceeds in close parallel with the electromagnetic case. In the gravitational case, our final expression for the (reduced order) equations of motion shows that the deviation from geodesic motion arises entirely from a ''tail term,'' in agreement with recent results of Mino *et al.* Throughout the paper, we take the view that ''point particles'' do not make sense as fundamental objects, but that ''point particle equations of motion'' do make sense as means of encoding information about the motion of an extended body in the limit where not only the size but also the charge and mass of the body go to zero at a suitable rate. Plausibility arguments for the validity of our comparison axiom are given by considering the limiting behavior of the self-force on extended bodies. [S0556-2821(97)05518-5]

PACS number(s): 04.25.−g, 04.30.−w

I. INTRODUCTION

In this paper, we shall investigate the motion of an isolated body coupled to classical fields in the limit where the spatial extent of the body is small enough that the detailed structure of the body is unimportant, but where the lowest order effects of the ''self-field'' of the body (which are responsible for ''radiation reaction'') are taken into account. Specifically, we shall consider (i) the motion of a charged body coupled to a Maxwell field on an arbitrary, fixed curved background and (ii) the motion of a massive body in an otherwise vacuum spacetime in general relativity.

No difficulty of principle is encountered in the calculation of the motion of any extended body once one has specified what matter fields compose the body and the equations of motion of these matter fields. If one then gives the initial data for these matter fields (as well as for the classical fields to which they are coupled), the complete motion of the body is determined unambiguously by the full set of continuum field equations. However, in practice, the details of the motion of an extended body will be very complicated and will depend on the detailed ''internal structure'' of the body. Thus, if one deals with general, extended bodies, it is highly unlikely that any simple results can be obtained which apply to large classes of systems.

One would expect the details of the internal structure of the body to become less and less important in the limit as one makes the body smaller. Thus, one obvious way to seek a class of simple, general results is to take the limit of the continuum equations as the spatial extent of the body goes to zero, thereby obtaining equations of motion for a ''point particle'' idealization of an extended body. However, as is well known, serious difficulties arise in attempting to take the point particle limit in a straightforward manner, keeping the total charge, e, and mass, m, of the body fixed. In the electromagnetic case, the linearity of Maxwell's equations allows one to make sense of the electromagnetic field of a body of finite charge in the point particle limit. However, the stress-energy of the electromagnetic field becomes singular in this limit, and the stress-energy of the matter fields comprising the body must become correspondingly singular in order for it to ''hold itself together.'' Hence, there is no reason to expect that a well defined point particle limit will exist. In general relativity, the situation is even worse, since the nonlinearity of Einstein's equation does not allow one even to make sense of the ''gravitational field'' of a point mass [1]; physically, an extended body would presumably collapse to a black hole before a point particle limit could be achieved. Thus, it does not appear to be mathematically or physically sensible to attempt to take a point particle limit—holding the charge or mass fixed—in either the electromagnetic or gravitational cases.

Nevertheless, we believe that there should exist some simple and general results regarding the motion of bodies in the limit where the size of the body is sufficiently small (as compared, in particular, with the scale of variation of the background electromagnetic field and/or the radius of curvature of the background spacetime) *and* the charge and mass of the body also are sufficiently small that ''self-field'' effects do not become dominant—but are not negligible. Mathematically rigorous results of this sort presumably would take the form of statements about limits of smooth, one-parameter families of solutions to the full continuum equations in which both the spatial extent of the body and the total charge, e, and/or mass, m, of the body are simultaneously taken to zero in a suitable manner. Geodesic motion

0556-2821/97/56(6)/3381(14)/$10.00

should result in such a limit.[1] For a charged body, the Lorentz force law in the background electromagnetic field should then arise as a correction to geodesic motion, with the acceleration of the body being of order e/m. For such a charged body, we seek in this paper to find the further corrections to this result to order e^2/m. Similarly, for an uncharged body, we seek to obtain the corrections to geodesic motion to order m in the acceleration of the body. These corrections arise from the ''self-field'' of the body and usually are referred to as ''radiation reaction'' or ''self-force'' effects. In both cases, we shall not seek to obtain any additional corrections to the motion due to the finite size and asphericity of the bodies. There is an extensive literature on the equations of motion of extended bodies (see, e.g., Dixon [4] and references cited therein) which takes into account the lowest order deviations from Lorentz force or geodesic motion due to such finite size effects, but neglects the $O(e^2/m)$ and $O(m)$ corrections which concern us here. Presumably, to lowest order, the combined effects of both types of corrections would be obtained by simply adding together the radiation reaction force due to self-field effects given here and ''multipole'' forces due to the background field which can be found elsewhere in the literature.

Thus, the goal of this paper is to obtain effective equations of motion for a point particle which are accurate to the orders specified in the previous paragraph. It should be emphasized that (in contrast to some other analyses of radiation reaction phenomena) our philosophy is *not* to view a point particle of finite charge or mass as a fundamental object, but rather to view the point particle equations of motion we obtain as a formal device to express approximate results for the continuum theory, in the limit where not only the size but also the charge and/or mass of the body are sufficiently small.

As already indicated above, it should be possible to derive the results we seek rigorously, without any additional hypotheses, by considering suitable one-parameter families of solutions in which the size, charge, and mass of the extended body go to zero in an appropriate manner, and suitable assumptions are made about the composition and initial state of the body so that its deviations from sphericity are kept under good control in this limit. However, although we shall take a few first steps in this direction in Sec. II A, we shall not proceed in this manner here. Rather, we shall instead attempt to ''guess'' the correct equations by defining a set of axioms which we believe the total force on the particle should obey. These axioms uniquely determine the force, and we then shall give an explicit prescription which satisfies our axioms.

Our approach bears some similarity to an approach of Penrose [3] for obtaining the electromagnetic self-force in the sense that, in both approaches, the self-force is given by the ordinary Lorentz force associated with a regularized electromagnetic field. However, the regularization is accomplished in very different ways in the two approaches. In the approach of [3], the electromagnetic field of a point particle is regularized by means of the Kirchhoff-d'Adhemar formula, whereas in our approach, the electromagnetic field is effectively regularized by considering differences in the fields associated with different particle trajectories in (possibly different) spacetimes. We have not investigated whether these regularization schemes are equivalent.

In Sec. II, we apply our approach to the case of charged point particles coupled to Maxwell fields on an arbitrary curved background. Our results agree with those of DeWitt and Brehme [5] (as corrected by Hobbs [6]), but the calculations required to obtain the radiation reaction force are considerably simpler and they generalize much more naturally to the gravitational case. In Sec. III, we carry out this generalization to obtain the $O(m)$ correction to geodesic motion for a point particle propagating on an otherwise vacuum background. Our final results agree with those recently obtained by Mino *et al.* [7], but, again, the calculations required in our approach are considerably simpler.

II. ELECTROMAGNETIC RADIATION REACTION

A. Introduction and motivation for the ''comparison axiom''

In this section, we shall consider an arbitrary spacetime (M,g_{ab}) containing an arbitrary timelike world line $z(\tau)$ with tangent u^b representing a point charge of charge e and mass m. We consider an arbitrary solution, F^{ab}, of Maxwell's equations with the point particle as its source

$$\nabla_a F^{ab} = -4\pi e \int \delta(x,z(\tau))u^b(\tau)d\tau \qquad (1)$$

such that F^{ab} is smooth except on the world line of the particle. Our aim is to give a prescription for assigning to each point of the world line a vector f^a—which we shall refer to as the *total electromagnetic force*—such that the point particle equation of motion $a^b = f^b/m$, will be valid to order e^2/m as an equation of motion for a sufficiently small, nearly spherical charged body, as discussed in the previous section.

In order to motivate the axiomatic approach that we shall ultimately adopt, let us see what happens if we attempt to derive a formula for f^a by considering a small, nearly spherical extended body and taking a ''point particle limit.'' To avoid unnecessary complications, we shall initially restrict attention in the discussion below to the case of a body moving in Minkowski spacetime.

The first difficult issue we encounter in this program is that of obtaining a ''representative world line'' in the extended body which can be viewed as describing the motion of the body, so that we may contemplate the limiting behavior of this world line as the body shrinks toward zero spatial extent. We shall not attempt to analyze this issue here, but will merely assume that by methods similar to those of Beiglböck [8], it is possible to define a ''center of mass'' or ''center of motion'' world line such that—to an excellent approximation when the body is sufficiently small and nearly spherical—we have, at each point of the world line,

$$p^a = m u^a, \qquad (2)$$

[1] This should follow directly from the theorem of Geroch and Jang [2] that any world line in a fixed background spacetime having the property that every neighborhood of it admits a conserved stress energy tensor satisfying the dominant energy condition must be a timelike geodesic.

where u^a is the tangent to the representative world line and p^a is given by

$$p^a = \int_{\Sigma} (T^{ab}_{\mathrm{body}}) \epsilon_{bcde}, \tag{3}$$

where T^{ab}_{body} is the stress-energy of the body (*not* including the electromagnetic field) and the integral is taken over the hyperplane Σ perpendicular to u^a. Here ϵ_{abcd} denotes the volume four-form determined by the metric, and the integrand is to be viewed as a vector-valued three-form; the global parallelism of flat spacetime is used here to define the integrals of tensor fields. It should be noted that the difficulties in controlling errors in Eq. (2) and the corresponding equation in curved spacetime probably constitute the most serious obstacle to converting the arguments given in this paper into rigorous theorems about radiation reaction forces which do not rely on any additional axioms.

We define the force on the extended body by

$$f^a = u^b \nabla_b (p^a). \tag{4}$$

Given Eq. (2), we see that $a^a \equiv u^b \nabla_b u^a$ will be given by the projection of f^a/m orthogonal to u^a. [This projection will actually be unnecessary here in the electromagnetic case when we obtain the point particle equations of motion, since our final answer for f^a as defined by Eq. (4) will turn out to be orthogonal to u^a, corresponding to m being independent of τ in Eq. (2).] We have

$$f^a = \frac{d}{d\tau}\left(\int_{\Sigma(\tau)} (T^{ab}_{\mathrm{body}}) \epsilon_{bcde} \right) = \int_{\Sigma(\tau)} \pounds_w ((T^{ab}_{\mathrm{body}}) \epsilon_{bcde}), \tag{5}$$

where w^a is the vector field which generates the map between the successive spatial slices $\Sigma(\tau)$. Note that the value of the integral does not depend upon how we choose to identify successive spatial slices, i.e., it depends only on the normal component of w^a (the lapse function) and not on its spatial projection into $\Sigma(\tau)$ (the shift vector). Using the standard identity

$$\pounds_w \mu = w \cdot d\mu + d(w \cdot \mu) \tag{6}$$

on a differential form μ (where the centered dot denotes contraction into the first index of the form) together with Stokes' theorem, we can rewrite the integrand to obtain

$$f^a = \int_{\Sigma(\tau)} \nabla_b T^{ab}_{\mathrm{body}} w^c d\Sigma_c, \tag{7}$$

where we now have rewritten our volume integral in more standard notation by writing $w^c d\Sigma_c$ in place of $w^c \epsilon_{cdef}$.

We now assume that the body is coupled to a classical electromagnetic field, F_{ab}, which satisfies Maxwell's equations with source given by the charge-current density, J^a, of the body. Thus, F_{ab} includes both the "self-field" of the body and any "background" electromagnetic field which may be present. We denote the stress-tensor of the Maxwell field by T^{ab}_{EM}. For further generality, we also shall allow the body to be coupled to additional classical matter (which does not couple to the electromagnetic field) with stress-energy tensor T^{ab}_{ext}. (This additional matter can be thought of as a "hand" or "string" which pulls on the charged body. We will set $T^{ab}_{\mathrm{ext}} = 0$ when we wish to obtain the equations of motion for a charged body acted upon only by an electromagnetic field, but permitting this extra coupling will allow us to also obtain an expression for the radiation reaction force for arbitrary world lines, when the charge is being "pulled.") By conservation of total stress-energy, we have

$$\nabla_b [T^{ab}_{\mathrm{body}} + T^{ab}_{\mathrm{EM}} + T^{ab}_{\mathrm{ext}}] = 0. \tag{8}$$

By Maxwell's equations, we have

$$\nabla_b T^{ab}_{\mathrm{EM}} = -F^{ab} J_b, \tag{9}$$

and therefore we can rewrite Eq. (7) as

$$f^a = \int_{\Sigma(\tau)} F^{ac} J_c w^b d\Sigma_b + f^a_{\mathrm{ext}}, \tag{10}$$

where

$$f^a_{\mathrm{ext}} \equiv - \int_{\Sigma(\tau)} \nabla_b T^{ab}_{\mathrm{ext}} w^c d\Sigma_c. \tag{11}$$

For a body which accelerates with four-acceleration a^b, the lapse function is given by

$$N \equiv w^a u_a = 1 + r^\gamma a_\gamma \tag{12}$$

where here we have extended the definition of u^a over the hypersurface by parallel transport [i.e., u^a is the unit normal to $\Sigma(\tau)$] and r^γ are the Cartesian coordinate components of the displacement vector on $\Sigma(\tau)$ with origin at the representative world line. Thus, we obtain

$$f^\alpha_{\mathrm{EM}} \equiv f^\alpha - f^\alpha{}_{\mathrm{ext}} = \int_{\Sigma(\tau)} F^{\alpha\beta} J_\beta N dV, \tag{13}$$

where dV denotes the ordinary volume element in the Euclidean three-space $\Sigma(\tau)$ and the Greek indices denote components in a global inertial coordinate system.

Let us now simplify the situation considerably further by assuming that the body is exactly spherically symmetric and "at rest" with respect to u^a at "time" $\Sigma(\tau)$, so that $J^a = \rho(r) u^a$ on $\Sigma(\tau)$. (However, we make no symmetry or other assumptions concerning the electromagnetic field.) Then we have

$$f^\alpha_{\mathrm{EM}} = \int_{\Sigma(\tau)} \rho(r) E^\alpha N dV, \tag{14}$$

where $E^\alpha \equiv F^{\alpha\beta} u_\beta$. We define $q(r)$ to be the total charge within radius r so that

$$\hat{r}^\beta D_\beta q(r) = 4\pi r^2 \rho(r). \tag{15}$$

Then, we have

$$f^{\alpha}_{\rm EM}=\int_{\Sigma(\tau)}\frac{\hat{r}^{\beta}D_{\beta}q(r)}{4\pi r^2}E^{\alpha}NdV=\int_{\Sigma(\tau)}D_{\beta}\left(\frac{\hat{r}^{\beta}q(r)}{4\pi r^2}E^{\alpha}N\right)dV-\int_{\Sigma(\tau)}\frac{q(r)}{4\pi r^2}\hat{r}^{\beta}D_{\beta}(E^{\alpha}N)dV-\int_{\Sigma(\tau)}D_{\beta}\left(\frac{\hat{r}^{\beta}}{4\pi r^2}\right)q(r)E^{\alpha}NdV$$

$$=\int_{r=R}\frac{q(R)}{4\pi R^2}E^{\alpha}NdS-\int_{\Sigma(\tau)}\frac{q(r)}{4\pi r^2}\left[\frac{\partial E^{\alpha}}{\partial r}N+\hat{r}^{\beta}a_{\beta}E^{\alpha}\right]dV-\int_{\Sigma(\tau)}\delta^3(r)q(r)E^{\alpha}NdV$$

$$=e\langle E^{\alpha}N\rangle_R-\int_{\Sigma(\tau)}\frac{q(r)}{4\pi r^2}\left[\frac{\partial E^{\alpha}}{\partial r}N+\hat{r}^{\beta}a_{\beta}E^{\alpha}\right]dV, \tag{16}$$

where R represents the outer radius of the charged body, $e\equiv q(R)$ is the total charge, and $\langle\rangle_R$ denotes the average over the sphere $r=R$.

The difficulties encountered in trying to obtain a prescription for $f^a_{\rm EM}$ in a simple, straightforward manner by taking the point particle limit of the equations of motion for an extended body can be seen directly from Eq. (16). Let us split the total electromagnetic field, F_{ab}, into a smooth ''background piece,'' $F^{(0)}_{ab}$, which has a smooth limit as the size of the body is shrunk to zero and a piece $F^{(1)}_{ab}$ which we may view as being ''due to the charge'' itself. (Most commonly $F^{(1)}_{ab}$ would be taken to be the retarded solution with source J^a.) When the size of the body becomes sufficiently small (in particular, much smaller than the scale of variation of $F^{(0)}_{ab}$), the contribution of $F^{(0)}_{ab}$ to the volume integral in Eq. (16) becomes negligible, whereas the surface term straightforwardly yields the Lorentz force law

$$f^{(0)a}_{\rm EM}=eF^{(0)ab}u_b, \tag{17}$$

since the lapse function, N, may be approximated as 1 when the body shrinks to zero size with E^a remaining bounded. Indeed, even if the body were nonspherical and/or not perfectly ''at rest,'' there would be no significant difficulty in obtaining the lowest order finite size corrections to $f^{(0)a}_{\rm EM}$ (see, e.g., Dixon [9]).

However, the situation is completely different with regard to $F^{(1)}_{ab}$, which contributes a ''self-force'' $f^{(1)a}_{\rm EM}$ of order e^2. If the total charge is fixed, $F^{(1)}_{ab}$ and its spatial derivatives become unboundedly large as the size of the body is made small, so that the integrals appearing in Eq. (16) cannot easily be controlled. These integrals could still have a well defined limit as a result of cancellations over the different portions of the body, but the situation clearly is extremely delicate. In particular, small deviations from exact sphericity or being exactly ''at rest'' (as well as small corrections due to curvature when we consider the motion of charged bodies in curved spacetime) could easily contribute finite corrections to $f^{(1)a}_{\rm EM}$. Indeed, corrections to Eq. (2) itself could be large in the point particle limit.

The singular behavior of the self-field in the point particle limit makes it very difficult to directly extract from Eq. (16) any guidance as to what the electromagnetic force should be, even in the regime where one would expect this force to be essentially composition independent. However, inspection of Eq. (16) provides a possible means of dealing with this difficulty, and this comprises the key new idea of this paper: Suppose that, rather than calculating the electromagnetic force (16) on a particular body, we instead attempt to calculate the *difference* in the electromagnetic force on two bodies of the same (or very similar) composition, which move on different world lines in (possibly different) spacetimes. Then, under appropriate circumstances (see below), it seems plausible that we may identify neighborhoods of the bodies in such a way that the *difference* in the electromagnetic field and its relevant first spatial derivatives for the two bodies can remain bounded as both of the bodies shrink to zero size at the same rate. If so, the volume integral contribution to the *difference* in $f^a_{\rm EM}$ for the two bodies [the last term on the right side of Eq. (16)] will go to zero. The difference in $f^a_{\rm EM}$ for the two bodies will then be given by a version of the Lorentz force law, wherein we take the difference in the electromagnetic fields, average this difference over the surface of the (identified) bodies as in the first term on the right side of Eq. (16) (with $N=1$), and then let the bodies shrink to zero size. Thus, if we consider the difference in electromagnetic forces between two bodies rather than the force on a single body, the ''point particle limit'' should be much less delicate, and, in particular, much less sensitive to small deviations from sphericity, etc. (provided, of course, that these deviations are essentially the same for both bodies).

Indeed, even if the bodies are in different curved spacetimes, it should be possible to keep the difference in the electromagnetic forces on these (suitably identified) bodies under good control. In a curved spacetime, the above calculations would be modified in the following ways. First, we must use parallel transport to define the integral expression for p^a. To do this explicitly, it is convenient to introduce an arbitrary unit vector, k^a, at a point on the representative world line and then parallel transport k^a along the worldline. For each τ, let $\Sigma(\tau)$ be the hypersurface generated by geodesics orthogonal to u^a at point $z(\tau)$ of the representative world line. For each τ, we define k^a on $\Sigma(\tau)$ by parallel transport along these geodesics, thereby defining k^a in a neighborhood of the entire worldline. We define the four-momentum p^a at point $z(\tau)$ by

$$k_a p^a=\int_{\Sigma}k_a(T^{ab}_{\rm body})\epsilon_{bcde}, \tag{18}$$

Again, we assume that a representative world line can be found for the extended body such that Eq. (2) holds to an excellent approximation when the body is sufficiently small and nearly spherical.

If we introduce Riemann normal coordinates at point $z(\tau)$ in place of the Cartesian coordinates of the flat spacetime derivation, the calculation which produced Eq. (16) from Eq. (5) is essentially unchanged, with the curvature of the background introducing only a few corrections to Eq. (16). First, a new term involving $\nabla_b k_a$ explicitly appears in the integrand in Eq. (7). Further corrections also result from the fact that the Riemann normal coordinate components of k^a are not constant. In addition, $w^b d\Sigma_b$ deviates slightly from NdV, where dV is the Riemann normal coordinate volume element for the hypersurface. However, these corrections all decrease with the size of the particle and should become negligible in the point particle limit $R\to 0$.

When will two bodies (possibly in different curved spacetimes) be such that the difference in their electromagnetic fields will be suitably bounded as their size shrinks to zero? To answer this question properly, we would need to carefully examine the behavior of the ''self-field'' of extended bodies as the ''point particle limit'' is approached. We shall not attempt to analyze this here. However, a good guess as to the answer to this question can be obtained by examining the exterior field of a point charge in a curved spacetime, and finding the conditions under which the difference between the fields of two such charges is—with a suitable identification of neighborhoods of the world lines of the particles—suitably bounded as one approaches a point on the world line of the particle.

To do so, we need to study the singular behavior of the electromagnetic field of a point charge in curved spacetime as one approaches the world line of the point charge. As stated above, we are concerned only with solutions to Eq. (1) which are singular precisely on the world line of the particle itself. It follows from the general theory of propagation of singularities (see theorem 26.1.1 of Hormander [10]), that all such solutions have the same singular behavior, i.e., the difference between any two solutions must be smooth on the world line of the particle itself.[2] Thus, to examine the singular behavior, it suffices to focus attention on any particular solution. When (M,g_{ab}) is globally hyperbolic, it is convenient to examine the advanced and retarded solutions. The behavior of these solutions near the world line of the particle can be calculated by the Hadamard expansion techniques detailed by DeWitt and Brehme [5]. The result is

$$
\begin{aligned}
F^{\pm}_{a'b'}(x) = {} & 2e\,\overline{g}_{a'[a}\overline{g}_{|b'|b]}\Big[r^{-2}\kappa^{-1}u^a\Omega^b + \frac{1}{2}r^{-1}\kappa^{-3}a^a u^b \\
& + \frac{1}{8}\kappa^{-5}u^a\Omega^b a^2 - \frac{1}{2}\kappa^{-3}\dot{a}^a\Omega^b \pm \frac{2}{3}\kappa^{-4}\dot{a}^a u^b \\
& + \frac{1}{12}\kappa^{-1}u^a\Omega^b R - \frac{1}{6}\kappa^{-1}u^a R^b{}_c\Omega^c \\
& + \frac{1}{2}\kappa^{-1}\Omega^a R^b{}_c u^c + \frac{1}{12}\kappa^{-1}u^a\Omega^b R_{cd}\Omega^c\Omega^d \\
& + \frac{1}{2}\kappa^{-1}R^a{}_c{}^b{}_d u^c\Omega^d - \frac{1}{12}\kappa^{-3}u^a\Omega^b R_{cd}u^c u^d \\
& + \frac{1}{6}\kappa^{-3}u^a R^b{}_{cde}u^c u^d\Omega^e \mp \frac{1}{3}\kappa^{-2}u^a R^b{}_c u^c\Big] \\
& \pm e\int_{\tau^{\pm}}^{\pm\infty} 2\nabla_{[b'}G^{\pm}_{a']a''}u^{a''}(\tau'')d\tau'' + O(r). \qquad (19)
\end{aligned}
$$

Here, u^a denotes the four-velocity of the point charge at point $z(\tau)$ on its world line, and x denotes a point sufficiently near $z(\tau)$ lying on the hypersurface generated by geodesics from $z(\tau)$ which are orthogonal to u^a. The outward-directed unit tangent at $z(\tau)$ to the geodesic passing through x is denoted by Ω^a, and the affine parameter of x (i.e., the distance of x from the world line) is denoted by r. Primed indices refer to tensors at x, while unprimed indices refer to tensors at $z(\tau)$, and $\overline{g}^{a'}{}_a(x,z(\tau))$ denotes the bitensor of geodesic parallel transport. The quantity κ is defined by $\kappa\equiv\sqrt{-u^a u^b\nabla_a\nabla_b\sigma - a^a\nabla_a\sigma}$, where σ denotes the *biscalar of squared geodesic distance*, which plays a fundamental role in all of these expansions. [The normalization of σ is such that $\sigma(x,z(\tau))=r^2/2$.] We have set $a^a\equiv u^b\nabla_b u^a$ and $\dot{a}^a\equiv u^b\nabla_b a^a$. The last term in the above equation is usually referred to as the ''tail term,'' and it results from the failure of Huygen's principle in curved spacetime. In that term, $G^{\pm}_{a'a''}(x,z(\tau''))$ denotes the advanced/retarded Green's function for the vector potential in the Lorentz gauge, so that

$$
\nabla^b\nabla_b G^{\pm}_{aa'} - R_a{}^b G^{\pm}_{ba'} = -4\pi\overline{g}_{aa'}\delta(x,z), \qquad (20)
$$

and $\tau^{\pm}$ denotes the proper time of the point on the world line of the charged particle which intersects the future/past light cone of x. [The alternating sign in front of the tail term integral in Eq. (19) merely puts the limits in the appropriate time order.] The integral in the tail term runs from $(\tau^{\pm}\pm\epsilon)$ to $\pm\infty$ with the limit $\epsilon\to 0$ then being taken.[3] No distributional component of $G^{\pm}_{a'a''}(x,z(\tau''))$ is encountered in the integral, and we shall assume that the tail term remains smooth as x approaches $z(\tau)$ (as should be the case if suitable asymptotic conditions are placed on the world line and the spacetime).

[2]Note that this implies, in particular, that the advanced minus retarded solution is always smooth throughout the spacetime—including on the world line of the particle—provided only that the advanced and retarded solutions themselves are nonsingular off of the world line of the particle. (However, as illustrated by the example of a uniformly accelerating charge in Minkowski spacetime, the advanced and retarded solutions need not always be nonsingular away from the world line of the particle.)

[3]If the world line is not complete (i.e., if it does not extend to infinite proper time in the future/past), then the upper limit of the tail term integral should be the maximum/minimum proper time values of the curve.

In addition to obvious notational differences, our expression differs from Eq. (5.12) of DeWitt and Brehme [5] in three ways. First, we have added the terms that DeWitt and Brehme omitted due to a trivial calculational error (see Hobbs [6]). Second, we have written the tail term in terms of the Green's function $G^{\pm}_{a'a''}(x,z(\tau''))$ rather than the Hadamard expansion term $v_{a'a''}(x,z(\tau''))$. The latter expression gives the correct form of the tail term only when x is sufficiently close to $z(\tau'')$, and, in general, is not even defined for large separations [when, in particular, there need not be a unique geodesic joining x and $z(\tau'')$]. Finally, our sign convention for the Riemann tensor is that of [11], which is opposite to that of DeWitt and Brehme.

In order to find the singular behavior of $F^{\pm}_{a'b'}$, we need to expand the coordinate components of $\overline{g}_{a'a}$ and κ. Both of these quantities take particularly simple forms in terms of Riemann normal coordinates based at $z(\tau)$. We have

$$\overline{g}^{\alpha'}{}_{\alpha} = \delta^{\alpha'}{}_{\alpha} + \frac{1}{6} r^2 \Omega^{\gamma}\Omega^{\delta} R^{\alpha'}{}_{\gamma\alpha\delta} + O(r^3) \tag{21}$$

and

$$\kappa = \sqrt{1 + r a^{\alpha}\Omega_{\alpha} + \tfrac{1}{3} r^2 u^{\alpha}u^{\beta}\Omega^{\gamma}\Omega^{\delta}R_{\alpha\gamma\beta\delta} + O(r^3)}. \tag{22}$$

Substituting these expansions into Eq. (19), we have

$$\begin{aligned}
F^{\pm}_{\alpha'\beta'}(x) = 2e\Bigg[& r^{-2}u_{[\alpha'}\Omega_{\beta']} - \frac{1}{2} r^{-1}(a^{\alpha}\Omega_{\alpha})u_{[\alpha'}\Omega_{\beta']} + \frac{3}{8}(a^{\alpha}\Omega_{\alpha})^2 u_{[\alpha'}\Omega_{\beta']} + \frac{1}{2} r^{-1} a_{[\alpha'}u_{\beta']} \\
& - \frac{1}{6}u^{\alpha}u^{\beta}\Omega^{\gamma}\Omega^{\delta}R_{\alpha\gamma\beta\delta}u_{[\alpha'}\Omega_{\beta']} - \frac{3}{4}(a^{\alpha}\Omega_{\alpha})a_{[\alpha'}u_{\beta']} + \frac{1}{6}\Omega_{[\beta'}R_{\alpha']\sigma\alpha\tau}u^{\alpha}\Omega^{\sigma}\Omega^{\tau} + \frac{1}{8}u_{[\alpha'}\Omega_{\beta']}a^2 - \frac{1}{2}\dot{a}_{[\alpha'}\Omega_{\beta']} \\
& \pm \frac{2}{3}\dot{a}_{[\alpha'}u_{\beta']} + \frac{1}{12}u_{[\alpha'}\Omega_{\beta']}R - \frac{1}{6}u_{[\alpha'}R_{\beta']\gamma}\Omega^{\gamma} + \frac{1}{2}\Omega_{[\alpha'}R_{\beta']\gamma}u^{\gamma} + \frac{1}{12}u_{[\alpha'}\Omega_{\beta']}R_{\gamma\delta}\Omega^{\gamma}\Omega^{\delta} + \frac{1}{2}R_{[\alpha'|\gamma|\beta']\delta}u^{\gamma}\Omega^{\delta} \\
& - \frac{1}{12}u_{[\alpha'}\Omega_{\beta']}R_{\gamma\delta}u^{\gamma}u^{\delta} + \frac{1}{6}u_{[\alpha'}R_{\beta']\gamma\delta\epsilon}u^{\gamma}u^{\delta}\Omega^{\epsilon} \mp \frac{1}{3}u_{[\alpha'}R_{\beta']\gamma}u^{\gamma}\Bigg] \pm e\int_{\tau\pm}^{\pm\infty}\nabla_{[\beta'}G^{\pm}_{\alpha']\alpha''}u^{\alpha''}(\tau'')d\tau'' + O(r).
\end{aligned} \tag{23}$$

Although this formula is explicitly for the advanced/retarded solution in a globally hyperbolic spacetime, as noted above, the singular behavior of F_{ab} will be the same as in Eq. (23) for any solution of Maxwell's equations with source (1) in a (possibly non-globally-hyperbolic) spacetime, provided only that F_{ab} is smooth away from the world line of the particle.

From Eq. (23), it can be seen that the divergent terms in $F^{\pm}_{a'b'}$ depend only upon the four-velocity and the four-acceleration of the world line at $z(\tau)$. In particular, they do not depend upon the spacetime curvature or derivatives of the acceleration. Furthermore, although many of the finite terms (which do depend upon the curvature and $\dot{a}^a$) are direction-dependent and thus have singular angular derivatives on the world line, the radial derivatives of these terms [which is all that enters the volume term in Eq. (16)] are bounded. Therefore, it seems plausible that if we have two bodies with the same magnitude of acceleration at corresponding points P and $\widetilde{P}$ on their representative world lines and if we identify neighborhoods of P and $\widetilde{P}$ using Riemann normal coordinates, with u^a aligned with $\widetilde{u}^a$ and a^a aligned with $\widetilde{a}^a$, then the singular contributions of the "self-fields" to $f^a_{\rm EM}$ in the point particle limit should cancel. Thus, the difference in $f^a_{\rm EM}$ for the two bodies in the point particle limit should be given by a version of the Lorentz force law wherein we average the difference in the electromagnetic fields over a surface of radius r as in the first term on the right side of Eq. (16) (with $N=1$), and then let $r \to 0$. This provides the motivation for axiom 1 below.

B. The axioms

We now are ready to state our main axiom, the motivation for which was given in the previous subsection.

Electromagnetic axiom 1 (comparison axiom). Consider two points, P and $\widetilde{P}$, each lying on timelike world lines in possibly different spacetimes which contain Maxwell fields F_{ab} and $\widetilde{F}_{ab}$ sourced by particles of charge e on the world lines. If the four-accelerations of the world lines at P and $\widetilde{P}$ have the same magnitude, and if we identify the neighborhoods of P and $\widetilde{P}$ via Riemann normal coordinates such that the four-velocities and four-accelerations are identifed, then the difference in the electromagnetic forces $f^a_{\rm EM}$ and $\widetilde{f}^a_{\rm EM}$ is given by the limit as $r \to 0$ of the Lorentz force associated with the difference of the two fields averaged over a sphere at geodesic distance r from the world line at P.

$$f^a_{\rm EM} - \widetilde{f}^a_{\rm EM} = \lim_{r\to 0}(\langle F^{ab} - \widetilde{F}^{ab}\rangle)u_b. \tag{24}$$

Axiom 1 is a very powerful one, since it enables us to compute the difference in electromagnetic force between any two particles which have the same instantaneous acceleration. Thus, to obtain $f^a_{\rm EM}$ for an arbitrary trajectory in an arbitrary curved spacetime, it suffices to know $f^a_{\rm EM}$ for a uniformly accelerating particle—with arbitrary acceleration a^a—in Minkowski spacetime, with the electromagnetic field chosen to be, say, the half-advanced, half-retarded solution.

Let us, then, consider this special case. By symmetry, $f^a_{\rm EM}$ must be proportional to a^a. If the proportionality factor were constant, such a force would correspond merely to a ''mass renormalization,'' and could be redefined away. On the other hand, such a redefinition would not be possible if the proportionality factor varied with acceleration. We see no argument from symmetry considerations alone which would forbid the presence of such a term. However, this spacetime, world line, and Maxwell field possess a time reversal symmetry about each point on the world line, which suggests that the particle always should be absorbing as much electromagnetic energy as it radiates, so the electromagnetic field should be doing ''no net work'' on the particle. This, in turn, strongly suggests that $f^a_{\rm EM}=0$ in this case. Indeed, if we did not have $f^a_{\rm EM}=0$, the type of calculation given in section 17.2 of Jackson [12] would show that our resulting prescription for $f^a_{\rm EM}$ would fail to conserve energy for a point particle trajectory which begins and ends in inertial motion (where, in this calculation, the infinite self-energy of the Coulomb field of the particle is discarded at the initial and final times). This motivates the following additional axiom, which agrees with standard claims made in textbooks (see, e.g., Jackson [12]):

Electromagnetic axiom 2 (flat spacetime axiom). If (M,g_{ab}) is Minkowski spacetime, the world line is uniformly accelerating, and F_{ab} is the half-advanced, half-retarded solution, $F_{ab}=\frac{1}{2}[F^-_{ab}+F^+_{ab}]$, then $f^a=0$ at every point on the world line.

Note that, since the advanced and retarded solutions for F_{ab} for a uniformly accelerating charge in Minkowski spacetime coincide in a neighborhood of the world line (indeed, within the entire ''Rindler wedge'' containing the world line [13]), it follows immediately from axiom 1 that we also have $f^a=0$ when F_{ab} is given by the advanced solution, F^+_{ab}, or by the retarded solution, F^-_{ab}. Thus, we would obtain an equivalent axiom if we replaced the half-advanced, half-retarded solution by the advanced solution or the retarded solution.

In the next subsection, we shall use axioms 1 and 2 together with Eq. (23) to compute $f^a_{\rm EM}$ for an arbitrary charged particle trajectory in an arbitrary curved spacetime.

C. The prescription

Let P be a point on the world line of a charged particle in a curved spacetime (M,g_{ab}) containing a Maxwell field F_{ab} satisfying Eq. (1), where F_{ab} is singular only on the world line of the particle. For simplicity we assume that (M,g_{ab}) is globally hyperbolic so a unique retarded Green's function exists; as explained at the end of this subsection, our formulas can easily be generalized to the non-globally-hyperbolic case. Let u^a denote the four-velocity of the world line at P and let a^a denote its acceleration at P.

We may view the tangent space at P as a copy of Minkowski spacetime. We shall denote the origin of this tangent space by $\tilde{P}$. In this Minkowski spacetime, consider a uniformly accelerating trajectory passing through $\tilde{P}$ with four-velocity $\tilde{u}^a=u^a$ and acceleration $\tilde{a}^a=a^a$. By axiom 2, the electromagnetic force on this uniformly accelerating Minkowski trajectory vanishes when the electromagnetic field is given by the (Minkowski) half-advanced, half-retarded solution.

In (M,g_{ab}), we write $F^{\rm in}_{ab}=F_{ab}-F^-_{ab}$, where F^-_{ab} denotes the retarded solution, and it is assumed that $F^{\rm in}_{ab}$ is smooth on the world line of the particle. Near the actual trajectory of the particle [in (M,g_{ab})], F^-_{ab} is given by Eq. (23). On the other hand, near the uniformly accelerating trajectory in the tangent space at $\tilde{P}$, the (Minkowski) half-advanced, half-retarded solution $\tilde{F}_{ab}\equiv\frac{1}{2}(\tilde{F}^-_{ab}+\tilde{F}^+_{ab})$ also is given by Eq. (23) except that the ''tail term'' and all of the terms involving the curvature are absent, and there is cancellation of terms involving $\dot{a}$ and a^2. Axiom 1 instructs us to subtract this Minkowski retarded solution from F_{ab} (using the exponential map—or, equivalently, Riemann normal coordinates—to compare them), average this difference over a sphere of radius r, and then let $r\to 0$. The electromagnetic force on the particle at P is then just the Lorentz force associated with the resulting field. We obtain

$$f^a_{\rm EM}=e(F^{\rm in})^{ab}u_b+\frac{2}{3}e^2(\dot{a}^a-a^2u^a)+\frac{1}{3}e^2(R^a{}_bu^b+u^aR_{bc}u^bu^c)+e^2u_b\int_{-\infty}^{\tau^-}\nabla^{[b}(G^-)^{a]c'}u_{c'}(\tau')d\tau'. \tag{25}$$

The corresponding equation of motion of a charged particle subject to no additional (i.e., nonelectromagnetic) external forces is then simply $f^a_{\rm EM}=ma^a$.

Our result (25) agrees with that of DeWitt and Brehme [5] as corrected by Hobbs [6]. Although we, of course, made crucial use of the Hadamard expansion for the retarded Green's function (23), no other lengthy computations were needed in our approach, since we did not need to compute the behavior of the electromagnetic stress-energy tensor near the world line of the particle.

Note that the first term in Eq. (25) is the ordinary Lorentz force due to the incoming field. The second term corresponds to the familar flat spacetime Abraham-Lorentz damping term. The third term is a local curvature term, whose presence is necessary to maintain conformal invariance of $f^a_{\rm EM}$. Finally, the fourth term is the so-called ''tail term'' resulting from the failure of Huygen's principle in curved spacetime.

Due to the presence of the Abraham-Lorentz term, the equation of motion $f^a_{\rm EM}=ma^a$ shares the unphysical ''runaway solutions'' of the ordinary flat spacetime equation of motion. As in the flat spacetime case, this difficulty can be resolved through the reduction of order technique. An exposition of the rational for this technique as well as an explanation of how to implement it in a general context can be found in Sec. IV D of [16]. To implement it here, we view $\epsilon\equiv e^2/m$ as a ''small parameter.'' We differentiate Eq. (25) (with $f^a_{\rm EM}$ set equal to ma^a) to obtain an expression for $\dot{a}^a$, and then substitute this expression back in Eq. (25), neglecting terms which are higher than first order in ϵ. We then similarly eliminate the terms involving a^a from the right side of Eq. (25). The result is

$$\begin{aligned} a^a = & \frac{e}{m}(F^{\rm in})^{ab}u_b + \frac{2}{3}\frac{e^2}{m}\left(\frac{e}{m}u^c\nabla_c(F^{\rm in})^{ab}u_b \right. \\ & \left. + \frac{e^2}{m^2}(F^{\rm in})^{ab}F^{\rm in}_{bc}u^c - \frac{e^2}{m^2}u^a(F^{\rm in})^{bc}u_c F^{\rm in}_{bd}u^d\right) \\ & + \frac{1}{3}\frac{e^2}{m}(R^a{}_b u^b + u^a R_{bc}u^b u^c) \\ & + \frac{e^2}{m}u_b\int_{-\infty}^{\tau^-}\nabla^{[b}(G^-)^{a]c}u_c(\tau')d\tau'. \end{aligned} \tag{26}$$

We believe that this equation properly describes the motion of a small, nearly spherical charged body in a curved spacetime, taking into account the leading order effects of the body's ''self-field.''

Inasmuch as they require the retarded solution to be singled out, expressions (25) and (26) are applicable as they stand only for a particle in a globally hyperbolic spacetime. However, since axiom 1 did not require global hyperbolicity, it is clear that our axioms also determine the electromagnetic force and equations of motion in the non-globally-hyperbolic case as well. Perhaps the simplest way of generalizing our formulas to the non-globally-hyperbolic case is as follows: If we wish to obtain $f^a_{\rm EM}$ at a point $z(\tau)$ on the world line of a charged particle in a non-globally-hyperbolic spacetime, simply choose a (sufficiently small) globally hyperbolic neighborhood of $z(\tau)$. Equations (25) and (26) then hold at $z(\tau)$, where $F^{\rm in}_{ab}$ and the tail term are defined in the appropriate manner, relative to that neighborhood.

III. GRAVITATIONAL RADIATION REACTION

In this section, we seek to obtain the gravitational analog of our formula (25) above for the total electromagnetic force (including radiation reaction) on a charged particle, as well as the analog of our equation of motion (26) above. The latter will provide us with the lowest order correction to geodesic motion of a particle resulting from radiation reaction effects. In our approach, we shall not make any of the slow motion or post-Newtonian approximations common to most other treatments of gravitational radiation reaction. On the other hand, the applicability of our results will be limited to the motion of a small, nearly spherical body.

There are many physical and mathematical similarities in the analyses of the electromagnetic and gravitational radiation reaction forces, and our analysis of gravitational radiation reaction will ultimately closely parallel that of the electromagnetic case. However, there also are a number of very significant differences between these two cases. We begin our analysis of the gravitational case by explaining in detail the nature of these differences.

Probably the most significant difference between the electromagnetic and gravitational cases concerns the formulation of the question which we would like to pose. As discussed in detail in the Introduction, we do not view a ''point particle'' as a fundamental object, but, instead, view the ''point particle limit'' as a convenient mathematical means of summarizing results concerning the behavior of one-parameter families of extended body solutions in the limit where not only the size but also the charge and mass of the body go to zero in a suitable manner. Nevertheless, in the electromagnetic case, there is no difficulty in making sense of solutions to Maxwell's equations if we let the size of the body shrink to zero keeping its charge fixed. This enabled us to pose (and propose an answer to) the following idealized question in Sec. II: Given a solution to Eq. (1) for a Maxwell field with point particle source, what is the total electromagnetic force on the charged particle? On the other hand, the corresponding question in the gravitational case would be the following: Given a solution to Einstein's equation with a point particle source, what is the total ''gravitational force'' on the particle? However, as already noted in the Introduction, this question makes no sense, since there is no notion of a solution to Einstein's equation with a ''point mass'' source [1].

A resolution of this difficulty is suggested by the fact that we are really interested in the case of (small) extended bodies whose self-gravity is ''weak.''[4] Thus, it should be adequate to treat the gravitational effects of the body via linearized perturbation theory off of a background vacuum spacetime. For linear equations, there is no difficulty in making sense of solutions with distributional sources, so, when working with the linearized equations, it becomes mathematically legal to let the size of the body shrink to zero, keeping its mass fixed. This suggests that we pose the following question, which is directly analogous to the question posed in Sec. II: Let $(M,g^{(0)}_{ab})$ be a spacetime satisfying the vacuum Einstein equation, let $z(\tau)$ be an arbitrary timelike world line in $(M,g^{(0)}_{ab})$, and let γ_{ab} be a solution of the linearized Einstein equation sourced by a particle following this worldline. What is the total ''gravitational force'' on the particle?

Unfortunately, the above question also suffers from serious mathematical inconsistencies: By the linearized Bianchi identity, the linearized Einstein equation implies exact conservation of the stress-energy of the (linearized) source with respect to the background metric. In the limit where the source is a point particle, this conservation requires the world line of the particle to be a geodesic of the background metric. Thus, if $z(\tau)$ is not chosen to be a geodesic of $g^{(0)}_{ab}$, the above question makes no sense since there does not exist any solution γ_{ab} whatsoever to the linearized Einstein equation with this source. But, a knowledge of the total ''gravitational force'' only for geodesics of $g^{(0)}_{ab}$ would not be adequate for obtaining the self-consistent motion of the particle under the influence of its own gravitational ''self-force,'' since such a particle will deviate from geodesic motion.

The origin of this difficulty can be understood as follows. Even for an extended body with very weak self-gravity, the linearized Einstein equation does not hold exactly; rather there are nonlinear corrections to this equation. Although these nonlinear terms make only a very small correction to

[4]However, we do not wish to preclude the possibility of eventually extending our analysis to small bodies with strong self-gravity; see [14].

γ_{ab}, it is precisely the presence of these terms which are responsible for the deviations from geodesic motion. By throwing away the nonlinear terms in γ_{ab}, we exclude from the outset the possibility that the particle fails to move on a geodesic of the background metric, thereby making it mathematically inconsistent to study departures from geodesic motion.

To see this more explicitly, consider the exact Einstein equation for the metric $g^{(0)}_{ab}+\gamma_{ab}$, written in the form of the linearized Einstein equation for γ_{ab} in the Lorentz gauge, with the nonlinear terms in γ_{ab} moved to the right side of the equation (in a schematic manner) to aid us in viewing them as an additional ''source term:''

$$\nabla^{(0)c}\nabla^{(0)}_{c}\overline{\gamma_{ab}}-2R^{(0)c}{}_{ab}{}^{d}\overline{\gamma_{cd}} = -16\pi T_{ab}+[\text{nonlinear terms in}\,\gamma_{ab}], \tag{27}$$

$$\nabla^{(0)a}\overline{\gamma_{ab}}=0, \tag{28}$$

where $\overline{\gamma_{ab}}\equiv\gamma_{ab}-\frac{1}{2}\gamma g^{(0)}_{ab}$.

As already noted above, for a body with weak self-gravity, the matter stress-energy T_{ab} should dominate the ''nonlinear terms in γ_{ab}.'' More precisely, T_{ab} is of order m, whereas if there is no incoming gravitational radiation, the nonlinear terms should have magnitude of order m^2 and higher, where m denotes the mass of the body. As we shall see in more detail below, a knowledge of the resulting dominant $O(m)$ contribution to γ_{ab} from T_{ab} will suffice for determining the leading order contribution to the self-force, so we should make little error by dropping the nonlinear terms. However, if we do so, there are no solutions to Eqs. (27) and (28) unless $\nabla^{(0)}_{a}T^{ab}=0$.

However, a means of dealing with this difficulty is suggested by the form in which we have written the equations. Even when $\nabla^{(0)}_{a}T^{ab}\neq 0$, no mathematical inconsistencies occur in Eq. (27) alone when the nonlinear terms are dropped. It is only when the Lorentz gauge condition (28) is adjoined to this equation that inconsistencies arise. Thus, we propose to simply relax the Lorentz gauge condtion so that it holds only to the required accuracy, i.e., to $O(m)$. (This can be ensured by simply requiring that any ''incoming radiation'' contributions to γ_{ab} satisfy the Lorentz gauge condition; i.e., $\nabla^{b}\overline{\gamma^{\rm in}_{ab}}=0$, where $\overline{\gamma^{\rm in}_{ab}}\equiv\overline{\gamma_{ab}}-\overline{\gamma^{-}_{ab}}$.) The resulting system of equations should then have the accuracy needed to obtain the leading order contribution to the gravitational self-force, but should not suffer from the mathematical inconsistencies which would occur if the linearized Einstein equation were used to relate γ_{ab} to T_{ab}. We note that our viewpoint appears to correspond to that taken in [7], and similar procedures for relaxing field equations or gauge conditions at appropriate orders also occur in many other approaches to obtaining self-consistent equations of motion (see, e.g., [15]).

Having reformulated the equations for γ_{ab} in this manner, we now may consider the point particle limit and pose the following question: Let $(M,g^{(0)}_{ab})$ be a spacetime satisfying the vacuum Einstein equation, let $z(\tau)$ be an arbitrary timelike world line in $(M,g^{(0)}_{ab})$, and let γ_{ab} be a solution of

$$\nabla^{c}\nabla_{c}\overline{\gamma_{ab}}-2R^{c}{}_{ab}{}^{d}\overline{\gamma_{cd}} = -16\pi m\int\delta(x,z(\tau))u_a(\tau)u_b(\tau)d\tau, \tag{29}$$

where $\overline{\gamma^{\rm in}_{ab}}\equiv\overline{\gamma_{ab}}-\overline{\gamma^{-}_{ab}}$ satisfies the Lorentz gauge condition (28). What is the total ''gravitational force'' on the particle?

Although the above question is closely analogous to the question posed at the beginning of Sec. II, there still remain a several notable differences between the electromagnetic and gravitational cases. First, since we have made a linearized approximation, it is necessary here that $\gamma^{\rm in}_{ab}$ be ''small'' compared with the background metric $g^{(0)}_{ab}$. No corresponding restriction on $F^{\rm in}_{ab}$ was necessary in the electromagnetic case. This restriction on $\gamma^{\rm in}_{ab}$ will have an important bearing on the final form of the reduced order equations of motion which we shall obtain at the end of this section. However, it should be noted that this restriction on $\gamma^{\rm in}_{ab}$ does not actually impose any physical restriction on the applicability of our results, since if we wished to consider a situation where the incoming, free gravitational radiation is ''large,'' we could simply incorporate this radiation into the background metric $g^{(0)}_{ab}$. Indeed, there would be no (physical) loss of generality in demanding that $\gamma^{\rm in}_{ab}=0$, but we choose not to do so, since there are a wide variety of circumstances where it is both appropriate and convenient to treat the incoming radiation as a linearized perturbation.

The second difference concerns the status of ''external forces.'' In the electromagnetic case, we were free to assume that $T^{ab}_{\rm ext}$ had no coupling to the electromagnetic field. However, in the gravitational case, it is not consistent to assume that $T^{ab}_{\rm ext}$ has no gravitational coupling; we must include $T^{ab}_{\rm ext}$ on the right side of Eq. (29), and take into account its contributions to γ_{ab}. Since, ultimately, we will set $T^{ab}_{\rm ext}=0$ to get the equations of motion of a freely falling particle, this will not be relevant for our final formula for the equations of motion. However, in our expression for f^{a}_{G}, the presence of $T^{ab}_{\rm ext}$ will make a contribution to γ_{ab}, which must be included.

A third important difference concerns the gauge invariance of our results. In the electromagnetic case, both the Maxwell field, F_{ab}, and the world line, $z(\tau)$, of the particle are gauge invariant. Most importantly, all of the information concerning the motion of the particle is contained in the specification of $z(\tau)$. However, in the gravitational case, neither γ_{ab} nor $z(\tau)$ are gauge invariant, since both can be changed by diffeomorphisms. Indeed, $z(\tau)$ can be changed arbitrarily by diffeomorphisms. Thus, the specification of $z(\tau)$ alone provides no information about the motion of the particle. Rather, this information is encoded in the joint specification of both $z(\tau)$ *and* γ_{ab}.

Despite the above differences, our analysis of the gravitational self-force will now proceed in close parallel with the electromagnetic case. In order to motivate the axioms which we ultimately will adopt, we consider a small, nearly spherical extended body with weak self-gravity, so that the spacetime metric, g_{ab}, deviates only slightly from a vacuum solution, $g^{(0)}_{ab}$. We seek to obtain an equation of motion for a suitable representative world line in the body, expressed in terms of the structures associated with the ‘‘background spacetime’’ $(M,g^{(0)}_{ab})$. To do so, we view the exact four-momentum density, $(T^{ab}_{\rm body})\epsilon_{bcde}$ in the spacetime (M,g_{ab}) from the perspective of the background spacetime $(M,g^{(0)}_{ab})$. In parallel with Eq. (18), at a point $z(\tau)$ on a representative world line in the body with tangent u^a, we define the four-momentum p^a by

$$k_a p^a = \int_{\Sigma^{(0)}} k^{(0)}_a (T^{ab}_{\rm body})\epsilon_{bcde}. \tag{30}$$

Here $\Sigma^{(0)}$ is the hypersurface generated by geodesics of $g^{(0)}_{ab}$ which are orthogonal (with respect to $g^{(0)}_{ab}$) to u^a, and the vector field $k^{(0)}_a$ is given the superscript ‘‘0’’ in order to emphasize that we are extending k^a off of the world line by parallel transporting it with respect to the background metric $g^{(0)}_{ab}$ (as opposed to g_{ab}). As in the electromagnetic case, we assume that a representative world line can be chosen so that—to an excellent approximation when the body is sufficiently small and spherical—we have at each point of the world line

$$p^a = m u^a. \tag{31}$$

Again, the difficulties in justifying this assumption would provide one of the more formidable obstacles to converting the motivational arguments given here into theorems about radiation reaction forces.

From the perspective of the spacetime $(M,g^{(0)}_{ab})$, the force on the body is given by

$$f^a \equiv u^b \nabla^{(0)}_b p^a. \tag{32}$$

Although we would expect $g_{ab}p^a p^b$ to be constant along the world line to the order to which we shall work, there is no reason why $g^{(0)}_{ab}p^a p^b$ need be constant to this order. Equivalently, if we normalize u^a so that $g^{(0)}_{ab}u^a u^b = -1$, there is no reason why the parameter m in Eq. (31) need be constant along the curve. If $g^{(0)}_{ab}p^a p^b$ fails to be constant, f^a will fail to be perpendicular to u^a (in the metric $g^{(0)}_{ab}$); we shall retain the component of f^a parallel to u^a in our formula for the gravitational force below. However, the deviation from geodesic motion, $u^b\nabla^{(0)}_b u^a$, is determined entirely by the projection of f^a perpendicular to u^a in the metric $g^{(0)}_{ab}$, i.e., we have

$$m u^b \nabla^{(0)}_b u^a = h^{(0)a}{}_b f^b, \tag{33}$$

where $h^{(0)}_{ab} = g^{(0)}_{ab} + u_a u_b$ and all indices here are raised and lowered using $g^{(0)}_{ab}$. Thus, we ultimately will project f^a perpendicular to u^a when we wish to obtain the equation of motion of the particle.

The calculation of f^a proceeds in parallel with the calculation in the electromagnetic case. Taking k^a to be parallel transported (with respect to $g^{(0)}_{ab}$) along the world line, we have

$$k_a f^a = u^b \nabla^{(0)}_b (k_a p^a) = \int_{\Sigma^{(0)}} \pounds_w (k^{(0)}_a T^{ab}_{\rm body}\epsilon_{bcde}). \tag{34}$$

Applying the identity (6) and using Stokes’ theorem, we obtain

$$k_a f^a = \int_{\Sigma^{(0)}} [\nabla_b k^{(0)}_a T^{ab}_{\rm body} + k^{(0)}_a \nabla_b T^{ab}_{\rm body}] w^c \epsilon_{cdef}. \tag{35}$$

In the first term, we rewrite $\nabla_b k^{(0)}_a$ as

$$\nabla_b k^{(0)}_a = \nabla^{(0)}_b k^{(0)}_a - C^c{}_{ba} k_c, \tag{36}$$

where

$$C^c{}_{ba} \equiv \frac{1}{2} g^{(0)cd} (\nabla^{(0)}_b \gamma_{ad} + \nabla^{(0)}_a \gamma_{bd} - \nabla^{(0)}_d \gamma_{ba}). \tag{37}$$

Although $\nabla^{(0)}_b k^{(0)}_a$ will make a nonvanishing contribution to the integrand due to the background curvature, this contribution is easily seen to vanish in our final expression for f^a when we take the point particle limit,[5] so we will drop this contribution as well as the other background curvature corrections mentioned in the electromagnetic derivation as they arise in the calculations below.

As in the electromagnetic case, for generality, we allow the body to be coupled to additional classical matter with stress-energy tensor $T^{ab}_{\rm ext}$. By conservation of total stress-energy, we have

$$\nabla_b [T^{ab}_{\rm body} + T^{ab}_{\rm ext}] = 0, \tag{38}$$

so that

$$\nabla_b T^{ab}_{\rm body} = -\nabla_b T^{ab}_{\rm ext}. \tag{39}$$

Substituting these results in Eq. (35), we have

$$k_a f^a = \int_{\Sigma^{(0)}} -k^{(0)}_a C^a{}_{bc} T^{bc}_{\rm body} w^d d\Sigma_d + k_a f^a_{\rm ext}, \tag{40}$$

where

$$k_a f^a_{\rm ext} \equiv \int_{\Sigma^{(0)}} -\nabla_b T^{ab}_{\rm ext} w^d d\Sigma_d. \tag{41}$$

We now approximate the body to be ‘‘at rest’’ at time $\Sigma^{(0)}$, so that $T^{bc}_{\rm body} = \rho u^b u^c$, where u^b is the unit normal (in the metric $g^{(0)}_{ab}$) to $\Sigma^{(0)}$. We obtain

$$\begin{aligned} k_a f^a_{\rm G} &\equiv k_a (f^a - f^a_{\rm ext}) \\ &= \int_{\Sigma^{(0)}} k^{(0)}_a \rho \left(\frac{1}{2} u^b u^c \nabla^{(0)a} \gamma_{bc} \right. \\ &\qquad \left. - u^b u^c \nabla^{(0)}_b \gamma_c{}^a \right) w^d d\Sigma_d. \end{aligned} \tag{42}$$

This formula corresponds to Eq. (14) in the electromagnetic case, with the expression in parentheses playing the role of the electric field E^a which appeared there. Therefore, we can immediately write down the gravitational analog of Eq. (16). We obtain

[5]Note that neither $\nabla_b k_a$ nor $\nabla^{(0)}_b k_a$ would be negligible in the point particle limit if k^a were defined by parallel transport with respect to g_{ab} rather than $g^{(0)}_{ab}$.

$$f_{\mathrm{G}}^{\alpha}=m\langle E_{\mathrm{G}}^{\alpha}N\rangle_R-\int_{\Sigma(\tau)}\frac{m(r)}{4\pi r^2}\left[\frac{\partial E_{\mathrm{G}}^{\alpha}}{\partial r}N+\hat{r}^{\beta}a_{\beta}E_{\mathrm{G}}^{\alpha}\right]dV$$
$$+[\text{terms which vanish as } R\rightarrow 0], \tag{43}$$

where

$$E_{\mathrm{G}}^{\alpha}\equiv\frac{1}{2}u^{\beta}u^{\gamma}\nabla^{\alpha}\gamma_{\beta\gamma}-u^{\beta}u^{\gamma}\nabla_{\beta}\gamma_{\gamma}^{\alpha}. \tag{44}$$

In this equation and in all equations henceforth, it is to be understood that all quantities except γ_{ab} refer to the background structure, and the superscript ''0'' will be omitted on the background metric and its derivative operator. As in the electromagnetic case, a ''point particle'' limit of the right side of Eq. (43) cannot be taken in a straightforward manner. However, we can again consider the *difference* in f_{G}^{α} on two bodies of similar composition that move on different world lines in (possibly different) background spacetimes. In order to find the conditions under which such a difference will remain bounded as $R\rightarrow 0$, we once again study the singular behavior of the exterior field of a point source in curved spacetime using the Hadamard expansion techniques of DeWitt and Brehme [5]. Since the trace-reversed metric perturbation $\overline{\gamma_{ab}}$ satisfies a wave equation (29) very similar to the equation for the electromagnetic vector potential, the Hadamard expansion goes through in close parallel with the electromagnetic case. The covariant expansions for $\overline{\gamma_{a'b'}^{\pm}}$ and $\nabla_{c'}\overline{\gamma_{a'b'}^{\pm}}$ [the gravitational analogs of Eq. (19)] are given by

$$\overline{\gamma_{a'b'}^{\pm}}(x)=2m\overline{g}_{a'(a}\overline{g}_{|b'|b)}[2r^{-1}u^au^b\pm 4\kappa^{-2}a^au^b]$$
$$\pm m\int_{\tau^{\pm}}^{\pm\infty}G_{a'b'a''b''}^{\pm}u^{a''}(\tau'')u^{b''}(\tau'')d\tau''+O(r) \tag{45}$$

and

$$\nabla_{c'}\overline{\gamma_{a'b'}^{\pm}}(x)=2m\overline{g}_{c'c}\overline{g}_{a'(a}\overline{g}_{|b'|b)}\Big[-2r^{-2}\kappa^{-1}u^au^b\Omega^c-4r^{-1}\kappa^{-3}a^au^bu^c-r^{-1}\kappa^{-3}u^au^ba^c-\frac{1}{4}\kappa^{-5}a^2u^au^b\Omega^c$$
$$+\kappa^{-5}u^au^bu^c\dot{a}^d\Omega_d+2\kappa^{-3}\dot{a}^au^b\Omega^c+2\kappa^{-3}a^aa^b\Omega^c\pm\frac{2}{3}\kappa^{-6}a^2u^au^bu^c\mp 4\kappa^{-4}\dot{a}^au^bu^c\mp 4\kappa^{-4}a^aa^bu^c$$
$$\mp 4\kappa^{-4}a^au^ba^c\mp\frac{2}{3}\kappa^{-4}u^au^b\dot{a}^c-\frac{2}{3}\kappa^{-3}R_{def}{}^{c}u^au^bu^d\Omega^eu^f-2\kappa^{-3}R_{def}{}^{a}u^bu^cu^d\Omega^eu^f+2\kappa^{-1}R^{c}{}_{ed}{}^{a}u^bu^d\Omega^e$$
$$-2\kappa^{-1}R_d{}^a{}_e{}^b\Omega^cu^du^e\mp 2\kappa^{-2}R^c{}_{de}{}^au^bu^du^e\pm 2\kappa^{-2}R_d{}^a{}_e{}^bu^cu^du^e\Big]$$
$$\pm m\int_{\tau^{\pm}}^{\pm\infty}\nabla_{c'}G_{a'b'a''b''}^{\pm}u^{a''}(\tau'')u^{b''}(\tau'')d\tau''+O(r)\,. \tag{46}$$

[In these formulas, we have normalized the advanced and retarded Green's functions $G_{aba'b'}^{\pm}$ so that they satisfy Eq. (29) with source $-16\pi\overline{g}_{aa'}\overline{g}_{bb'}\delta(x,z)$.]

Expanding $\overline{g}_{\alpha'\alpha}$ and κ, we find that the Riemann normal coordinate components of $\nabla_c\overline{\gamma_{a'b'}^{\pm}}$, in the same notation as Eq. (23), are

$$\nabla_{\gamma'}\overline{\gamma_{\alpha'\beta'}^{\pm}}(x)=2m\Big[-2r^{-2}u_{\alpha'}u_{\beta'}\Omega_{\gamma'}-4r^{-1}a_{(\alpha'}u_{\beta')}u_{\gamma'}-r^{-1}u_{\alpha'}u_{\beta'}a_{\gamma'}+r^{-1}u_{\alpha'}u_{\beta'}\Omega_{\gamma'}a^{\delta}\Omega_{\delta}+6a_{(\alpha'}u_{\beta')}u_{\gamma'}a^{\delta}\Omega_{\delta}$$
$$+\frac{3}{2}u_{\alpha'}u_{\beta'}a_{\gamma'}(a^{\delta}\Omega_{\delta})-\frac{3}{4}u_{\alpha'}u_{\beta'}\Omega_{\gamma'}(a^{\delta}\Omega_{\delta})^2+u_{\alpha'}u_{\beta'}u_{\gamma'}\dot{a}^{\delta}\Omega_{\delta}+\frac{1}{3}u_{\alpha'}u_{\beta'}\Omega_{\gamma'}R_{\delta\lambda\epsilon\kappa}u_{\delta}u^{\epsilon}\Omega^{\lambda}\Omega^{\kappa}$$
$$-\frac{2}{3}R_{\alpha'\delta\lambda\epsilon}u^{\lambda}\Omega^{\delta}\Omega^{\epsilon}u_{\beta'}\Omega_{\gamma'}-\frac{1}{4}a^2u_{\alpha'}u_{\beta'}\Omega_{\gamma'}+2\dot{a}_{(\alpha'}u_{\beta')}\Omega_{\gamma'}+2a_{\alpha'}a_{\beta'}\Omega_{\gamma'}\pm\frac{2}{3}a^2u_{\alpha'}u_{\beta'}u_{\gamma'}\mp 4\dot{a}_{(\alpha'}u_{\beta')}u_{\gamma'}$$
$$\mp 4a_{\alpha'}a_{\beta'}u_{\gamma'}\mp 4a_{(\alpha'}u_{\beta')}a_{\gamma'}\mp\frac{2}{3}u_{\alpha'}u_{\beta'}\dot{a}_{\gamma'}-\frac{2}{3}R_{\delta\epsilon\lambda\gamma'}u_{\alpha'}u_{\beta'}u^{\delta}\Omega^{\epsilon}u^{\lambda}-2R_{\delta\epsilon\lambda(\alpha'}u_{\beta')}u_{\gamma'}u^{\delta}\Omega^{\epsilon}u^{\lambda}$$
$$+2R_{\gamma'\epsilon\delta(\alpha'}u_{\beta')}u^{\delta}\Omega^{\epsilon}-2R_{\delta\alpha'\epsilon\beta'}\Omega_{\gamma'}u^{\delta}u^{\epsilon}\mp 2R_{\gamma'\delta\epsilon(\alpha'}u_{\beta')}u^{\delta}u^{\epsilon}\pm 2R_{\delta\alpha'\epsilon\beta'}u_{\gamma'}u^{\delta}u^{\epsilon}\Big]$$
$$\pm m\int_{\tau^{\pm}}^{\pm\infty}\nabla_{\gamma'}G_{\alpha'\beta'\alpha''\beta''}^{\pm}u^{\alpha''}(\tau'')u^{\beta''}(\tau'')d\tau''+O(r). \tag{47}$$

We have verified Eqs. (45), (46), and (47) using the software package MathTensor. Apart from differences in notation and sign conventions, there are two differences between Eqs. (45) and (46) and the corresponding Eqs. (2.77) and (2.33) of [7]. The first is that we write the ''tail term'' integrands in terms of the full retarded and advanced Green's functions, while the authors of [7] write the ''tail term'' integrands in terms of the Hadamard expansion term $v_{aba'b'}$, in parallel with DeWitt and Brehme [5]. Second, the authors of [7] have dropped all $O(r^0)$ terms in the expression for $\nabla_{c'}\overline{\gamma^{\pm}_{a'b'}}$ which contain time derivatives of the particle's four-velocity in accordance with an approximation scheme adapted to their specific calculation.

Aside from the obvious complexity introduced by the additional index structure, there are two important differences between the above formulas and the corresponding formulas in the electromagnetic case. First, as stated above, we have assumed here that the background spacetime is a solution of the vacuum Einstein equation, so in the gravitational case, no terms are present which involve the Ricci curvature. Although it would be possible to repeat the above analysis by perturbing off of a nonvacuum solution, the perturbations of the metric and background matter would become coupled at linear order, so Eq. (29) no longer would hold, and the entire analysis would have to be redone. Second, due to the presence of several terms of alternating sign in the above expressions which do not depend upon curvature, we see that the advanced and retarded expressions for γ_{ab} and its first spatial derivative for a uniformly accelerating trajectory in flat spacetime do not agree in a neighborhood of the world line of the particle. In parallel with the electromagnetic case, the advanced and retarded solutions can be shown to be gauge equivalent within the entire ''Rindler wedge'' containing the worldline. However, unlike the electromagnetic case, the analog of the Lorentz force, $-C^a{}_{bc}u^bu^c$, is not gauge invariant in this case, and it differs for the advanced and retarded solutions even in the limit as $r\to 0$. Nevertheless, it can be verified that this difference between the forces for the advanced and retarded solutions is parallel to the four-velocity of the particle. Therefore, when we project the force perpendicular to the particle's four-velocity to produce an equation of motion, the difference will vanish and the situation is effectively the same as in the electromagnetic case.

Despite the above differences, Eq. (47) shares the most important property of the analogous electromagnetic expression (23), namely the divergent terms as $r\to 0$ depend only upon the four-velocity and four-acceleration of the particle at $z(\tau)$. Therefore, in direct analogy with axiom 1 in the electromagnetic case. We postulate the following.

Gravitational axiom 1 (comparison axiom). Consider two points, P and $\widetilde{P}$, each lying on timelike world lines in (possibly different) spacetimes which contain linearized metric perturbations sourced by particles of mass m on the world lines [see Eq. (29)]. If the four-accelerations of the world lines at P and $\widetilde{P}$ have the same magnitude, and if we identify the neighborhoods of P and $\widetilde{P}$ via Riemann normal coordinates such that the four-velocities and four-accelerations are identifed, then the difference in the gravitational forces $f^a_{\rm G}$ and $\widetilde{f}^a_{\rm G}$, is given by the limit as $r\to 0$ of the difference of the effective gravitational forces averaged over a sphere at geodesic distance r from the world line at P,

$$f^a_{\rm G}-\widetilde{f}^a_{\rm G}=\lim_{r\to 0}\left(\left\langle\left(\frac{1}{2}\nabla^a\gamma_{bc}-\nabla_b\gamma^a{}_c\right)-\left(\frac{1}{2}\nabla^a\widetilde{\gamma}_{bc}-\nabla_b\widetilde{\gamma}^a{}_c\right)\right\rangle\right)u^bu^c. \tag{48}$$

In analogy with axiom 2, we also postulate the following.

Gravitational axiom 2 (flat spacetime axiom). If (M,g_{ab}) is Minkowski spacetime, the world line is uniformly accelerating, and γ_{ab} is the half-advanced, half-retarded solution, $\gamma_{ab}=\frac{1}{2}[\gamma^-_{ab}+\gamma^+_{ab}]$, then $f^a_{\rm G}=0$ at every point on the world line.

As noted above, since $-C^a{}_{bc}u^bu^c$ differs for the advanced and retarded solutions, it does matter in this case that we use the half-advanced, half-retarded solution in this axiom, rather than, say, the advanced or retarded solution, although this difference does not affect the projection of the force perpendicular to the world line of the particle.

In parallel with the electromagnetic case, the above axioms yield the following prescription for the gravitational force. Let (M,g_{ab}) be a solution of the vacuum Einstein equation and let γ_{ab} be a solution of Eq. (29). At a point P on the particle's world line, we compare γ_{ab} with the half-advanced, half-retarded solution for a uniformly accelerating trajectory in the tangent space (using the exponential map to make the comparison). The gravitational force is then given by calculating the difference in $-C^a{}_{bc}u^bu^c=(\frac{1}{2}\nabla^a\gamma_{bc}-\nabla_b\gamma^a{}_c)u^bu^c$ for these two fields, averaging over a sphere of radius r, and letting $r\to 0$. If we write γ_{ab} as $\gamma_{ab}=\gamma^{\rm in}_{ab}+\gamma^-_{ab}$, the resulting expression is

$$f^a_{\rm G}=m\left(\frac{1}{2}\nabla^a\gamma^{\rm in}_{bc}-\nabla_b(\gamma^{\rm in})^a{}_c\right)u^bu^c-m^2\left(\frac{11}{3}\dot{a}^a+\frac{1}{3}a^2u^a\right)+m^2u^bu^c\int_{-\infty}^{\tau^-}\left(\frac{1}{2}\nabla^aG^-_{bca'b'}-\nabla_b(G^-)_c{}^a{}_{a'b'}\right)\times u^{a'}u^{b'}d\tau'. \tag{49}$$

As anticipated, this expression contains contributions to $f^a_{\rm G}$ parallel to the four-velocity u^a, which merely describe the effect of the metric perturbation on the normalization of u^a. To obtain the equations of motion, we project $f^a_{\rm G}$ perpendicular to u^a as in Eq. (33) above. This yields

$$a^a=\left(\frac{1}{2}\nabla^a\gamma^{\rm in}_{bc}-\nabla_b(\gamma^{\rm in})^a{}_c-\frac{1}{2}u^au^d\nabla_d\gamma^{\rm in}_{bc}\right)u^bu^c-\frac{11}{3}m(\dot{a}^a-a^2u^a)+mu^bu^c\int_{-\infty}^{\tau^-}\left(\frac{1}{2}\nabla^aG^-_{bca'b'}-\nabla_b(G^-)_c{}^a{}_{a'b'}-\frac{1}{2}u^au^d\nabla_dG^-_{bca'b'}\right)u^{a'}u^{b'}d\tau'. \tag{50}$$

It should be noted that an Abraham-Lorentz term of the form $(\dot{a}^a - a^2 u^a)$ appears in Eq. (50), but with a sign opposite to that of the electromagnetic case, corresponding to the ''antidamping'' phenomenon found by Havas [17]. (See also Carmeli [18].)

Finally, we apply to this equation of motion the same reduction of order techniques that we applied to the electromagnetic equation of motion. When we do so, the terms on the right side involving $m\dot{a}^a$ and ma^a get eliminated in favor of terms involving $m\gamma^{\rm in}_{ab}$. However, our equation (50) is valid only to linear order in both m and $\gamma^{\rm in}_{ab}$, so it is not consistent to keep terms involving products of these two quantities. [This contrasts strongly with the electromagnetic case, we we worked only to linear order in e^2/m, but $F^{\rm in}_{ab}$ was allowed to be as large as we liked, so that $(e/m)F^{\rm in}_{ab}$ could be treated as being of order unity.] Consequently, the reduction of order procedure in this case effectively drops the Abraham-Lorentz terms, leaving the ''tail term'' as the only contribution to the ''self-force:''[6]

$$a^a = \left(\frac{1}{2}\nabla^a \gamma^{\rm in}_{bc} - \nabla_b(\gamma^{\rm in})^a{}_c - \frac{1}{2}u^a u^d \nabla_d \gamma^{\rm in}_{bc}\right)u^b u^c + m u^b u^c \int_{-\infty}^{\tau^-}\left(\frac{1}{2}\nabla^a G^-_{bca'b'} - \nabla_b (G^-)_c{}^a{}_{a'\,b'} - \frac{1}{2}u^a u^d \nabla_d G^-_{bca'b'}\right) u^{a'} u^{b'} d\tau'. \quad (51)$$

This formula agrees with the results of Mino *et al.* [7], although they did not include $\gamma^{\rm in}_{ab}$ in their expression and, as noted previously, they wrote their ''tail term'' integrand in terms of the Hadamard expansion term $v_{aba'b'}$ rather than the full advanced/retarded Green's functions. Note that Eq. (51) has a very simple interpretation: To lowest nontrivial order, the particle moves on a geodesic of $g^{(0)}_{ab} + \gamma_{ab}$, where $\gamma_{ab} = \gamma^{\rm in}_{ab} + \gamma^{\rm tail}_{ab}$ and $\overline{\gamma^{\rm tail}_{ab}} = \gamma^{\rm tail}_{ab} - \frac{1}{2} g^{(0)}_{ab}\gamma^{\rm tail}$ is the last term in Eq. (45).

IV. CONCLUSIONS

In this paper, we have taken an axiomatic approach to obtain the lowest order electromagnetic and gravitational ''self-forces'' on a small, nearly spherical body of sufficiently small charge and/or mass. Our final result for the total electromagnetic force on a body (possibly acted upon by an external electromagnetic field as well as additional, nonelectromagnetic ''external forces'') is given by Eq. (25). If such a body is subject only to electromagnetic forces, our final result for the (reduced order) equation of motion of the body is given by Eq. (26). The corresponding results for the gravitational case are given by Eqs. (49) and (51).

The above electromagnetic results were derived from the two axioms given in Sec. II B, and the corresponding gravitational results were obtained from the axioms of Sec. III. Although plausibility arguments in support of these axioms were given, we did not attempt to prove that the electromagnetic axioms follow as a consequence of Maxwell's equations in curved spacetime together with conservation of total stress-energy, nor did we attempt to prove that the gravitational axioms follow from Einstein's equation. Nevertheless, we believe that our plausibility arguments have provided the first steps in that direction. In any case, the problem of providing a rigorous justification for the electromagnetic and gravitational self-forces and the corresponding equations of motion has been reduced to the problem of providing a rigorous justification of our axioms.

Finally, we note that on account of the ''tail term,'' our equations of motion in both the electromagnetic and gravitational cases are integro-differential equations which, in principle, require us to know the entire past history of the particle. However, if the curvature of spacetime is sufficiently small and the motion of the body is sufficiently ''slow,'' one would expect the ''tail term'' to become effectively local, since the contributions to the ''tail term'' arising from portions of the orbit distant from the present position of the particle should become negligible. Indeed, if the tail term becomes effectively local, arguments using the standard dipole formula for radiated energy together with conservation of energy (see [12] and the analogous gravitational calculation given below) suggest that if $F^{\rm in}_{ab}=0$ and $R_{ab}=0$, the ''tail term'' of Eq. (26) should reduce to the familiar Abraham-Lorentz damping force

$$f^i_{\rm EM} = \frac{2}{3}e^2 \frac{d^3x^i}{dt^3}. \quad (52)$$

That this is indeed the case was established by DeWitt and DeWitt [19] for a charged particle in a slow, circular orbit in linearized Schwarzschild spacetime. It should be emphasized that the ''true'' Abraham-Lorentz force actually vanishes here—since $a^a = \dot{a}^a = 0$ for geodesic motion—but, remarkably, the tail term mocks up an effective Abraham-Lorentz term.

In the gravitational case, the standard ''quadrupole formula'' for radiated power in the slow motion, weak field limit is

$$P = \frac{1}{45}\frac{d^3Q^{ij}}{dt^3}\frac{d^3Q_{ij}}{dt^3}, \quad (53)$$

where i and j are spatial indices $(i,j=1,2,3)$ for a set of global inertial coordinates and Q^{ij} is the traceless quadrupole moment

$$Q^{ij} \equiv q^{ij} - \frac{1}{3}q\,\delta^{ij}, \quad (54)$$

$$q^{ij} \equiv 3\int T^{00}x^i x^j d^3x. \quad (55)$$

So, for a point particle,

$$Q^{ij} = 3m\left(x^i x^j - \frac{1}{3}x^k x_k \delta^{ij}\right). \quad (56)$$

[6] This contrasts sharply with the analysis of [17] which effectively neglected the dominant ''tail term.''

By conservation of energy, we should have

$$\int f^i_{\rm G} v_i dt = -\frac{1}{45}\int \frac{d^3Q^{ij}}{dt^3}\frac{d^3Q_{ij}}{dt^3}dt = -\frac{1}{45}\int \frac{d^5Q^{ij}}{dt^5}\frac{dQ_{ij}}{dt}dt$$

$$= -\frac{6}{45}m\int \frac{d^5Q^{ij}}{dt^5}\left(v_i x_j - \frac{1}{3}v^k v_k \delta_{ij}\right)dt$$

$$= -\frac{2}{15}m\int \left(\frac{d^5Q^{ij}}{dt^5}x_j\right)v_i dt, \tag{57}$$

which suggests that the radiation reaction force should be given by (see section 36.8 of Misner, Thorne, and Wheeler [11])

$$f^i_{\rm G} = -\frac{2}{15}m\frac{d^5Q^{ij}}{dt^5}x_j. \tag{58}$$

It would be interesting to perform the gravitational analog of the analysis of DeWitt and DeWitt [19] to see if this formula does, indeed, arise from the tail term of Eq. (51) in the slow motion, weak field limit.

ACKNOWLEDGMENTS

We wish to thank Eanna Flanagan for reading the manuscript. This research was supported in part by NSF Grant No. PHY 95-14726.

[1] R. Geroch and J. Traschen, Phys. Rev. D **36**, 1017 (1987).
[2] R. Geroch and P. S. Jang, J. Math. Phys. (N.Y.) **16**, 65 (1975).
[3] R. Penrose and W. Rindler, *Spinors and Space-time* (Cambridge University Press, Cambridge, England, 1984), Vol. I; W. G. Unruh, Proc. R. Soc. London **A348**, 447 (1976).
[4] W. G. Dixon, Trans. R. Soc. London **A277**, 59 (1974).
[5] B. S. DeWitt and R. W. Brehme, Ann. Phys. (N.Y.) **9**, 220 (1960).
[6] J. M. Hobbs, Ann. Phys. (N.Y.) **47**, 141 (1968).
[7] Y. Mino, M. Sasaki, and T. Tanaka, Phys. Rev. D **55**, 3457 (1997).
[8] W. Beiglböck, Commun. Math. Phys. **5**, 106 (1967).
[9] W. G. Dixon, Nuovo Cimento **34**, 319 (1964).
[10] L. Hörmander, *The Analysis of Linear Partial Differential Operators IV* (Springer, Berlin, 1985).
[11] C.W. Misner, K.S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1970); R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
[12] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975).
[13] D. Boulware, Ann. Phys. (N.Y.) **124**, 169 (1980).
[14] J. B. Hartle and K. S. Thorne, Phys. Rev. D **31**, 1815 (1985).
[15] A. Einstein, L. Infeld, and B. Hoffmann, Ann. Math. **39**, 65 (1938).
[16] E. E. Flanagan and R. M. Wald, Phys. Rev. D **54**, 6233 (1996).
[17] P. Havas, Phys. Rev. **108**, 1351 (1957) [a crucial sign error in Eq. (10) of the reference was corrected in P. Havas and J. N. Goldberg, *ibid.* **128**, 398 (1962)].
[18] M. Carmeli, Ann. Phys. (N.Y.) **34**, 465 (1965).
[19] B. S. DeWitt and C. M. DeWitt, Physics (Long Island City, NY) **1**, 3 (1964).