Perturbations of solutions of the Einstein equations which represent colliding plane waves

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(Received 13 November 1996)

Using expressions for the complete perturbations of solutions of the Einstein equations with sources in terms of potentials, the perturbations of the space-time representing the collision of plane polarized plane gravitational waves possibly coupled with electromagnetic waves or with neutrinos are obtained in the regions prior to the collision; in the case where there is a background electromagnetic field, the problem is reduced to solve a system of two second-order linear ordinary differential equations for two complex scalar functions, called the master equations. When the source is a neutrino field, the master equations are a system of four first-order linear ordinary differential equations for four complex scalar functions. [S0556-2821(97)06716-7]

PACS number(s): 04.20.Jb, 04.40.Nr

I. INTRODUCTION

The study of collisions of gravitational waves coupled with electromagnetic waves or with neutrino waves has shown, among other things, the appearance of singularities as a result of mutual focusing of the colliding waves, such as curvature singularities, Killing-Cauchy horizons, etc. [1–4]. The following natural step in these lines is to investigate the stability of these properties against external perturbations, for example, infinitesimal perturbations of different sorts. To this end, Yurtsever has studied the persistence of some singularities that appear in the collisions of plane gravitational waves and of almost plane gravitational waves under perturbations that preserve the planar symmetry [5], although these plane symmetric perturbations have the unsatisfactory property that they carry infinite energy. With the purpose of considering perturbations that do not respect the planar symmetry and motivated by the negative results obtained by Chandrasekhar and Xanthopoulos in the study of the gravitational and electromagnetic perturbations of the Bell-Szekeres solution [6], Xanthopoulos has considered the coupled perturbations for the most general metric representing plane waves bound for a collision, and his conclusion was that among the perturbations of this general metric there are no purely incoming perturbations [7]. In all these works, the method employed to obtain the coupled perturbations consists in solving the linearized Newman-Penrose equations for the perturbations, which amounts to considering a high number of differential equations to solve. Moreover, using this approach, in Ref. [8] Xanthopoulos does not succeed in expressing the perturbations of the Newman-Penrose quantities completely algebraically in terms of a single function; this difficulty is solved in the present paper with another approach described below. In addition, in a recent paper [9] it has been demonstrated that in the regions prior to the collision in the Bell-Szekeres solution there exist nontrivial purely incoming perturbations. These new results have been obtained using a pair of scalar potentials which determine completely the gravitational and electromagnetic perturbations of a solution of the Einstein-Maxwell equations with a null background electromagnetic field. The results of Ref. [9] were obtained by means of Wald's method of adjoint operators which applies when we can obtain a decoupled set of equations from the original equations for the perturbations [10–14]. The use of such scalar potentials requires the solution of few differential equations and automatically gives the correct relative normalization between all the components of the perturbations [15,16].

In this paper we make use of the expressions found in Ref. [9] to study the perturbations of the general metric considered in Ref. [8]; we find that in all cases there exist nontrivial purely incoming perturbations, contrary to the results of Ref. [7], and furthermore, in contrast with Ref. [8], we also show that when the perturbations also depend on the advanced time, all of them can be written in terms of a single function that obeys a second-order ''master equation.'' Thus, our description and results differ in radical aspects from those reached in Refs. [7,8]. The main reason of this significant discrepancy is that in Refs. [7,8] it is considered that all the perturbed quantities may have a common dependence of the form $e^{i(k_1x^1+k_2x^2+k_3u)}$ in the ignorable coordinates u,x^1,x^2 of the metric for waves bound to the collision; however, this is not necessarily the case, because some perturbed quantities must contain the factors $e^{i(k_1x^1+k_2x^2+k_3u)}$ and $e^{-i(k_1x^1+k_2x^2+k_3u)}$ simultaneously (see, e.g., Refs. [14,17]). This issue will be extended later on. Specifically, in Sec. II we obtain the components of the electromagnetic field perturbations and the components of the Weyl spinor perturbations; these expressions are written down for future reference and finally will be used in Sec. III to find the perturbations of the solution that represents plane waves in the preinteraction regions. Similarly, in Sec. IV we find the perturbations of solutions to the Einstein-Weyl equations in the precollision regions such that the flux vector of the neutrino field is normal to the wave front.

II. PERTURBATIONS OF SOLUTIONS OF THE EINSTEIN-MAXWELL EQUATIONS

When we consider a solution of the Einstein-Maxwell equations with a null background electromagnetic field (and a possibly nonzero cosmological constant), we can take $\varphi_0=0=\varphi_1$ and, therefore, $\kappa=0=\sigma$ and $\Psi_0=0=\Psi_1$. Under these conditions, the *real* complete metric and vector potential perturbations are given by [9]

$$h_{\mu\nu}=-2\{l_\mu l_\nu[(\delta+3\beta+\bar\alpha-\tau)(\delta+4\beta+3\tau)-\bar\lambda(D+4\varepsilon+3\rho)]+m_\mu m_\nu(D+3\varepsilon-\bar\varepsilon-\rho)(D+4\varepsilon+3\rho)-l_{(\mu}m_{\nu)}[(D+3\varepsilon+\bar\varepsilon-\rho+\bar\rho)(\delta+4\beta+3\tau)+(\delta+3\beta-\bar\alpha-\tau-\bar\pi)(D+4\varepsilon+3\rho)]]\}\psi_G+\text{c.c.}, \tag{1}$$

$$b_\mu=\tfrac{1}{2}[l_\mu(\delta+2\beta+\tau)-m_\mu(D+2\varepsilon+\rho)]\psi_E+\text{c.c.} \tag{2}$$

(modulo gauge transformation), where the two complex scalar potentials ψ_E and ψ_G satisfy the equations [9]

$$[(\Delta+3\gamma-\bar\gamma+\bar\mu)(D+4\varepsilon+3\rho)-(\bar\delta+3\alpha+\bar\beta-\bar\tau)(\delta+4\beta+3\tau)-3\Psi_2]\psi_G-\varphi_2\psi_E=0,$$

$$2\bar\varphi_2(D+3\varepsilon-\bar\varepsilon-\rho)(D+4\varepsilon+3\rho)\psi_G+[(\Delta+\gamma-\bar\gamma+\bar\mu)(D+2\varepsilon+\rho)-(\bar\delta+\alpha+\bar\beta-\bar\tau)(\delta+2\beta+\tau)]\psi_E=0. \tag{3}$$

Then, the components of the electromagnetic field perturbations with respect to the original tetrad are given by [9]

$$\bar\varphi_0^B\equiv l^\mu\bar m^\nu F^B_{\mu\nu}=\tfrac{1}{2}(D+\varepsilon-\bar\varepsilon-\rho)(D+2\varepsilon+\rho)\psi_E,$$

$$\bar\varphi_1^B\equiv\tfrac{1}{2}(l^\mu n^\nu+m^\mu\bar m^\nu)F^B_{\mu\nu}=\tfrac{1}{4}[(D+\varepsilon+\bar\varepsilon-\rho+\bar\rho)(\delta+2\beta+\tau)+(\delta+\beta-\bar\alpha-\tau-\bar\pi)(D+2\varepsilon+\rho)]\psi_E,$$

$$\bar\varphi_2^B\equiv m^\mu n^\nu F^B_{\mu\nu}=\tfrac{1}{2}[(\delta+\beta+\bar\alpha-\tau)(\delta+2\beta+\tau)-\bar\lambda(D+2\varepsilon+\rho)]\psi_E+\varphi_2(D-\varepsilon+3\bar\varepsilon-\bar\rho)(D+4\bar\varepsilon+3\bar\rho)\bar\psi_G, \tag{4}$$

where $F^B_{\mu\nu}=\partial_\nu b_\nu-\partial_\nu b_\mu$ represents the electromagnetic field perturbation. The components of the Weyl spinor perturbations with respect to the unperturbed tetrad can be obtained from Eq. (1) making use of the formula

$$\Psi^B_{ACDE}=\tfrac{1}{2}\nabla^{R'}{}_{(A}\nabla^{S'}{}_{C}h_{DE)R'S'}+\tfrac{1}{2}h_{(AC}{}^{R'S'}\Phi_{DE)R'S'}.$$

In this manner, we find that

$$2\bar\Psi_0^B=(D+\varepsilon-3\bar\varepsilon-\rho)(D+2\varepsilon-2\bar\varepsilon-\rho)(\bar m^\mu\bar m^\nu h_{\mu\nu}),$$

$$8\bar\Psi_1^B=[(\delta+\beta-3\bar\alpha-3\bar\pi-\tau)(D+2\varepsilon-2\bar\varepsilon-\rho)+(D+\varepsilon-\bar\varepsilon-\rho+\bar\rho)(\delta+2\beta-2\bar\alpha-2\bar\pi-\tau)](\bar m^\mu\bar m^\nu h_{\mu\nu})$$
$$+2(D+\varepsilon-\bar\varepsilon-\rho+\bar\rho)(D+2\varepsilon-\rho+\bar\rho)(n^\mu\bar m^\nu h_{\mu\nu}),$$

$$24\bar\Psi_2^B=-(D+\bar\varepsilon+\varepsilon+2\bar\rho-\rho)(D+2\bar\varepsilon+2\varepsilon+2\bar\rho-\rho)(n^\mu n^\nu h_{\mu\nu})-(D+\bar\varepsilon+\varepsilon+2\bar\rho-\rho)[\bar\lambda(m^\mu m^\nu h_{\mu\nu}-2\bar m^\mu\bar m^\nu h_{\mu\nu})]-(\bar\delta+\bar\beta$$
$$-\alpha+\pi+2\bar\tau)(\bar\delta+2\bar\beta-2\alpha+\pi+2\bar\tau)(m^\mu m^\nu h_{\mu\nu})-[(\delta+\beta-\bar\alpha-\tau-2\bar\pi)(\delta+2\beta-2\bar\alpha-2\bar\pi-\tau)-3\bar\lambda(D+2\varepsilon-2\bar\varepsilon$$
$$-\rho)](\bar m^\mu\bar m^\nu h_{\mu\nu})+[(D+\bar\varepsilon+\varepsilon+2\bar\rho-\rho)(\bar\delta+2\bar\beta+2\pi+2\bar\tau)+(\bar\delta+\bar\beta-\alpha+\pi+2\bar\tau)(D+2\bar\varepsilon-2\rho+2\bar\rho)](n^\mu m^\nu h_{\mu\nu})$$
$$-2[(D+\bar\varepsilon+\varepsilon+2\bar\rho-\rho)(\delta+2\beta-\tau-\bar\pi)+(\delta+\beta-\bar\alpha-\tau-2\bar\pi)(D+2\varepsilon-\rho+\bar\rho)](n^\mu\bar m^\nu h_{\mu\nu}), \tag{5}$$

$$16\bar\Psi_3^B=-[(\delta+\bar\alpha+\beta-\bar\pi-\tau)(D+2\bar\varepsilon+2\varepsilon-\rho+2\bar\rho)+(D+3\bar\varepsilon+\varepsilon-\rho+3\bar\rho)(\delta+2\bar\alpha+2\beta-\tau)](n^\mu n^\nu h_{\mu\nu})-(\delta+\bar\alpha+\beta-\bar\pi$$
$$-\tau)[\lambda(m^\mu m^\nu h_{\mu\nu})]-(D+3\bar\varepsilon+\varepsilon-\rho+3\bar\rho)[\nu(m^\mu m^\nu h_{\mu\nu})]-[(\bar\delta+3\bar\beta-\alpha+\pi+3\bar\tau)(\Delta+2\bar\gamma-2\gamma+\mu)+(\Delta+\bar\gamma-\gamma$$
$$-\bar\mu+\mu)(\bar\delta+\pi+2\bar\beta-2\alpha+2\bar\tau)](m^\mu m^\nu h_{\mu\nu})+2(\delta+\bar\alpha+\beta-\bar\pi-\tau)[\bar\lambda(\bar m^\mu\bar m^\nu h_{\mu\nu})]+2\bar\lambda(\delta+2\bar\beta-2\bar\alpha-\tau-2\bar\pi)$$
$$\times(\bar m^\mu\bar m^\nu h_{\mu\nu})+[(\delta+\bar\alpha+\beta-\bar\pi-\tau)(\bar\delta+2\pi+2\bar\beta+2\bar\tau)+(D+3\bar\varepsilon+\varepsilon-\rho+3\bar\rho)(\Delta+2\bar\gamma+2\mu)+(\bar\delta+3\bar\beta-\alpha+\pi$$
$$+3\bar\tau)(\delta+2\bar\alpha-2\tau)+(\Delta+\bar\gamma-\gamma-\bar\mu+\mu)(D+2\bar\varepsilon-2\rho+2\bar\rho)](n^\mu m^\nu h_{\mu\nu})+2(D+3\bar\varepsilon+\varepsilon-\rho+3\bar\rho)[\bar\lambda(n^\mu\bar m^\nu h_{\mu\nu})]$$
$$-2[(\delta+\bar\alpha+\beta-\bar\pi-\tau)(\delta+2\beta-\tau-\bar\pi)-2\bar\lambda(D+2\varepsilon-\rho+\bar\rho)](n^\mu\bar m^\nu h_{\mu\nu}),$$

$$2\bar\Psi_4^B=[(\delta+\beta+3\bar\alpha-\tau)(\delta+2\beta+2\bar\alpha-\tau)-\bar\lambda(D+2\varepsilon+2\bar\varepsilon-\rho+2\bar\rho)](n^\mu n^\nu h_{\mu\nu})+[(\Delta-\gamma+3\bar\gamma+\mu)(\Delta-2\gamma+2\bar\gamma+\mu)$$
$$+\nu(\delta-2\beta+2\bar\alpha)-\bar\nu(\bar\delta-2\alpha+2\bar\beta+\pi+2\bar\tau)+\{(\delta+3\beta+\bar\alpha-\tau)\nu-\lambda\bar\lambda\}](m^\mu m^\nu h_{\mu\nu})-[(\Delta-\gamma+3\bar\gamma+\mu)(\delta+2\bar\alpha$$
$$-2\tau)+(\delta+\beta+3\bar\alpha-\tau)(\Delta+2\bar\gamma+2\mu)-\bar\nu(D+2\bar\varepsilon-2\rho+2\bar\rho)-\bar\lambda(\bar\delta+2\bar\beta+2\pi+2\bar\tau)](n^\mu m^\nu h_{\mu\nu})-2[(\delta-\beta+3\bar\alpha$$
$$-\bar\pi)\bar\lambda](n^\mu\bar m^\nu h_{\mu\nu})+2\bar\lambda\lambda(\bar m^\mu\bar m^\nu h_{\mu\nu})-\tfrac{1}{2}\Phi_{22}(m^\mu m^\nu h_{\mu\nu}),$$

with

$$\begin{aligned} n^\mu n^\nu h_{\mu\nu} &= -2[(\delta+3\beta+\overline{\alpha}-\tau)(\delta+4\beta+3\tau) \\ &\quad -\overline{\lambda}(D+4\varepsilon+3\rho)]\psi_G+\text{c.c.}, \\ \overline{m^\mu}\,\overline{m^\nu} h_{\mu\nu} &= -2(D+3\varepsilon-\overline{\varepsilon}-\rho)(D+4\varepsilon+3\rho)\psi_G, \\ n^\mu \overline{m^\nu} h_{\mu\nu} &= -[(D+3\varepsilon+\overline{\varepsilon}-\rho+\overline{\rho})(\delta+4\beta+3\tau) \\ &\quad +(\delta+3\beta-\overline{\alpha}-\tau-\overline{\pi})(D+4\varepsilon+3\rho)]\psi_G, \end{aligned} \tag{6}$$

which is a consequence of Eq. (1). In addition, in order to determine the perturbations of the spin coefficients, for example, σ, we make use of the formulas $\sigma=-l^\mu m^\nu \nabla_\nu m_\mu$ and $g^{\mu\nu}=2l^{(\mu}n^{\nu)}-2m^{(\mu}\overline{m^{\nu)}}$ and obtain

$$\sigma^B=(\overline{\alpha}-\beta)m_\mu l^{B\mu}-(\rho m_\mu-\tau l_\mu)m^{B\mu}-l^\mu m^\nu(\nabla_\nu m_\mu)^B, \tag{7}$$

where, as in the preceding equations, the superscript B denotes the corresponding perturbations; furthermore, it can be shown that

$$(\nabla_\nu m_\mu)^B=\nabla_\nu m_\mu^B-\tfrac{1}{2}m^\rho(\nabla_\mu h_{\rho\nu}+\nabla_\nu h_{\rho\mu}-\nabla_\rho h_{\mu\nu}),$$

$$m_\mu m^{B\mu}=\tfrac{1}{2}m^\mu m^\nu h_{\mu\nu},$$

$$l^\mu m_\mu^B=-m_\mu l^{B\mu},$$

thus, using these expressions and Eq. (1), one easily finds that Eq. (7) becomes

$$\begin{aligned} \sigma^B &= (\delta-2\beta-\tau)m_\mu l^{B\mu}-(D+2\overline{\varepsilon}-2\varepsilon-3\overline{\rho}-\rho) \\ &\quad \times(D+3\overline{\varepsilon}-\varepsilon-\overline{\rho})(D+4\overline{\varepsilon}+3\overline{\rho})\overline{\psi_G}, \end{aligned} \tag{8}$$

and similarly for the remaining spin coefficients. The quantity $m_\mu l^{B\mu}$ corresponds to one of six degrees of tetrad gauge freedom that the perturbations of the null tetrad $(n,l,m,\overline{m})$ have [7]; making a comparison, from Eqs. (4)–(6), we can see that the perturbations of the electromagnetic field and of the Weyl spinor do not contain similar terms depending on this gauge freedom since, as we have indicated, these expressions correspond to the perturbations of the fields projected on the unperturbed tetrad (which is assumed fixed), without having to take into account the perturbed tetrad; thus, they are fully specified by Eqs. (1) and (2), unlike the similar quantities in Ref. [8], which involve the perturbed tetrad. The reason to prefer quantities that do not involve the perturbed tetrad, like those of Eqs. (4)–(6), it is that they are more appropriate to compute, for example, fluxes of energy [18], and to match the perturbations between the different regions occurring in the collision of plane waves [19].

III. COLLIDING WAVES

The metric in the regions prior to the collision of the colliding plane waves can be specified by the null tetrad [20]

$$D=\frac{\sqrt{2}}{U}\,\partial_u, \quad \Delta=\frac{\sqrt{2}}{U}\,\partial_v, \tag{9}$$

$$\delta=\frac{1}{\sqrt{2}}\,(1-v^2)^{-1/2}(\chi^{1/2}\partial_{x^1}-i\chi^{-1/2}\partial_{x^2}),$$

$$\overline{\delta}=\frac{1}{\sqrt{2}}\,(1-v^2)^{-1/2}(\chi^{1/2}\partial_{x^1}+i\chi^{-1/2}\partial_{x^2}),$$

where u,v,x^1,x^2 are real coordinates and

$$U=U(v), \quad \chi=\chi(v). \tag{10}$$

The only nonvanishing spin coefficients are given by [20]

$$\lambda=-\tfrac{1}{2}\Delta\,\ln\chi, \quad \mu=\Delta\,\ln(1-v^2)^{1/2}, \quad \gamma=-\tfrac{1}{2}\Delta\,\ln U, \tag{11}$$

the only nonvanishing component of the curvature is

$$\begin{aligned} \Phi_{22} &= \frac{1}{v^2}\,(\Delta\,\ln\sqrt{1-v^2})^2+2(\Delta\,\ln U)(\Delta\,\ln\sqrt{1-v^2}) \\ &\quad -\tfrac{1}{4}(\Delta\,\ln\chi)^2, \end{aligned} \tag{12}$$

and the only nonvanishing component of the electromagnetic field is given by

$$\varphi_2=\sqrt{\Phi_{22}/2}, \tag{13}$$

where we ignore a physically irrelevant phase factor. In this manner, we can apply the results presented in the preceding section with the null tetrad given in Eqs. (9). From this null tetrad, we can see that the coordinates u,x^1,x^2 can be ignored, then we seek for solutions of Eqs. (3) of the form

$$\psi_E=f(v)e^{i(k_1x^1+k_2x^2+k_3u)}, \quad \psi_G=g(v)e^{i(k_1x^1+k_2x^2+k_3u)}, \tag{14}$$

where k_1, k_2, and k_3 are constants. Substituting Eqs. (9)–(11) and (14) into Eqs. (3), one obtains the following set of linear ordinary differential equations, which will be referred to as the master equations:

$$\begin{aligned} &2ik_3\,\frac{d}{dv}\,[U^{-2}g]+\frac{k_1^2\chi+k_2^2\chi^{-1}}{2(1-v^2)}\,g-2ik_3U^{-2}\,\frac{v}{1-v^2}\,g \\ &\quad =\varphi_2 f, \\ &2ik_3\,\frac{d}{dv}\,[U^{-1}f]+\frac{U^3(k_1^2\chi+k_2^2\chi^{-1})}{2(1-v^2)}\,f-2ik_3U\,\frac{v}{1-v^2}\,f \\ &\quad =4k_3^2U\varphi_2 g. \end{aligned} \tag{15}$$

Although we cannot find the general solutions to Eqs. (15) for arbitrary functions $U(v)$ and $\chi(v)$, the perturbations can be expressed, in principle, in terms of the scalar potentials ψ_G and ψ_E; from Eqs. (4)–(6) and (9)–(13), one obtains

$$\overline{\varphi_0^B}=-\frac{k_3^2}{U^2}\,f(v)e^{i(k_1x^1+k_2x^2+k_3u)}, \tag{16}$$

$$\overline{\varphi_1^B}=-i\,\frac{U(k_2\chi^{-1/2}+ik_1\chi^{1/2})}{2k_3\sqrt{1-v^2}}\,\overline{\varphi_0^B},$$

$$\overline{\varphi_2^B}=-\frac{U^2}{2k_3^2}\left[\frac{(k_2\chi^{-1/2}+ik_1\chi^{1/2})^2}{2(1-v^2)}+\frac{ik_3^2}{U^2}\frac{d\ \ln\chi}{dv}\right]\overline{\varphi_0^B}-\frac{2\varphi_2}{U^2}\overline{k_3^2 g(v)e^{i(k_1x^1+k_2x^2+k_3u)}},$$

$$\overline{\Psi_0^B}=-\frac{4k_3^4}{U^4}\overline{g(v)e^{i(k_1x^1+k_2x^2+k_3u)}},$$

$$\overline{\Psi_1^B}=-i\frac{U(k_2\chi^{-1/2}+ik_1\chi^{1/2})}{2k_3\sqrt{1-v^2}}\overline{\Psi_0^B},$$

$$\overline{\Psi_2^B}=\frac{U^2}{8k_3^2}\left[\frac{(k_2\chi^{-1/2}+ik_1\chi^{1/2})^2}{(1-v^2)}+\frac{2ik_3}{U^2}\frac{d\ \ln\chi}{dv}\right]\overline{\Psi_0^B},$$

$$\overline{\Psi_3^B}=-i\frac{U^3(k_2\chi^{-1/2}+ik_1\chi^{1/2})}{8k_3^3\sqrt{1-v^2}}\left[\frac{(k_2\chi^{-1/2}+ik_1\chi^{1/2})^2}{2(1-v^2)}+\frac{3ik_3}{U^2}\frac{d\ \ln\ \chi}{dv}\right]\overline{\Psi_0^B},$$

$$\overline{\Psi_4^B}=\frac{U^4}{4k_3^4}\left\{\frac{(k_2\chi^{-1/2}+ik_1\chi^{1/2})^2}{2(1-v^2)}\left[\frac{6ik_3}{U^2}\frac{d\ \ln\ \chi}{dv}+\frac{(k_2\chi^{-1/2}+ik_1\chi^{1/2})^2}{2(1-v^2)}\right]-\frac{3k_3^2}{U^4}\left(\frac{d\ \ln\chi}{dv}\right)^2\right\}\overline{\Psi_0^B}-\frac{U^2}{4k_3^4(1-v^2)}$$
$$\times\left\{ik_3(k_1^2\chi-k_2^2\chi^{-1})\frac{d\ \ln\chi}{dv}-\frac{U^2(k_1^2\chi+k_2^2\chi^{-1})^2}{4(1-v^2)}-4ik_3(k_1^2\chi+k_2^2\chi^{-1})\frac{d\ \ln U}{dv}\right\}\overline{\Psi_0^B}+\frac{1}{k_3^2}\left[\frac{2}{U}\frac{d^2U}{dv^2}-10\left(\frac{d\ \ln U}{dv}\right)^2\right.$$
$$\left.-\frac{4v}{1-v^2}\frac{d\ \ln U}{dv}+\frac{1}{4}\left(\frac{d\ \ln\chi}{dv}\right)^2\right]\overline{\Psi_0^B}-\frac{2k_3}{U^2}\left[\frac{4k_3}{U^2}\left(3\frac{d\ \ln U}{dv}+\frac{v}{1-v^2}\right)-i\frac{(k_1^2\chi+k_2^2\chi^{-1})}{1-v^2}\right]\overline{\frac{dg}{dv}e^{i(k_1x^1+k_2x^2+k_3u)}}$$
$$+\frac{4k_3^2}{U^4}\overline{\frac{d^2g}{dv^2}e^{i(k_1x^1+k_2x^2+k_3u)}}.$$

These perturbations may diverge at $v=1$, depending on the explicit expressions for U and χ. These expressions differ from those given in Ref. [8], since, as can be seen in Eqs. (16), some perturbed quantities must contain simultaneously terms proportional to $e^{i(k_1x^1+k_2x^2+k_3u)}$ and $e^{-i(k_1x^1+k_2x^2+k_3u)}$, in disagreement with the assumption made in Ref. [8], that all the perturbed quantities have the same dependence on x^1, x^2, and u of the form $e^{i(k_1x^1+k_2x^2+k_3u)}$, suppressing incorrectly this factor and preserving the same symbols as describing the amplitudes of the corresponding perturbations in the equations, although some of these equations considered in that paper clearly contain the perturbations of the electromagnetic field, of the conformal curvature, and of the spin coefficients and their complex conjugates (see, e.g., Eqs. (R.d), (R.f), (R.o), (R.r), (B.d), (B.g), (B.h) of Ref. [7]). This error has been committed systematically in most works on the subject, whose clarification has been part of the aim of Refs. [9,18].

The u-independent perturbations are especially important, since they correspond to purely ingoing perturbations, which correspond to $k_3=0$; these perturbations were studied by Xanthopoulos [7], who found that there exist no nontrivial u-independent perturbations, contrary to the results presented below; in this case it is convenient to define the complex variable

$$z\equiv\frac{\chi^{-1/2}x^1+i\chi^{1/2}x^2}{\sqrt{2}} \tag{17}$$

and its complex conjugate to replace the real coordinates x^1 and x^2. Then, the only relevant members of null tetrad (9) are $\delta=(1-v^2)^{-1/2}\partial_z$ and $\overline{\delta}=(1-v^2)^{-1/2}\partial_{\overline{z}}$. Assuming that the potentials ψ_G and ψ_E do not depend on u, Eqs. (3) become

$$(1-v^2)^{-1}\partial_z\partial_{\overline{z}}\psi_G+\varphi_2\psi_E=0,\quad \partial_z\partial_{\overline{z}}\psi_E=0, \tag{18}$$

whose solutions are given by

$$\psi_G=(1-v^2)^2[\overline{zF}(v,z)+G(v,z)],\qquad \psi_E=-\frac{(1-v^2)}{\varphi_2}\partial_zF(v,z), \tag{19}$$

where $F(v,z)$ and $G(v,z)$ are arbitrary functions and the factors $(1-v^2)$ are introduced for convenience. From Eqs. (4)–(6), the only nonvanishing components of the electromagnetic field perturbations and of the Weyl spinor perturbations are

$$\overline{\varphi_2^B}=-\frac{1}{2\varphi_2}\partial_z^3F(v,z),\qquad \overline{\Psi_4^B}=-\overline{z}\overline{\partial_z^4}F(v,z)-\partial_z^4(v,z). \tag{20}$$

Thus, there exist nontrivial u-independent perturbations which need not diverge at $v=1$. Furthermore, when the electromagnetic field perturbations vanish (i.e., $\partial_z^3F=0$), the nonvanishing purely gravitational perturbations (20) not only correspond to a solution of the linearized Einstein-Maxwell equations, but they also correspond to an *exact* solution of the Einstein-Maxwell equations [21].

IV. THE EINSTEIN-WEYL CASE

The metric (9) is also a solution of the Einstein-Weyl equations, assuming that the flux vector of the neutrino field is parallel to D, which defines the direction of propagation of the colliding wave; then, denoting by η_A the components of the neutrino field, $\eta_0=0$ and η_1 depends on v only. The Einstein field equations reduce to

$$\Phi_{22}=2ik(\eta_1\Delta\overline{\eta_1}-\overline{\eta_1}\Delta\eta_1),$$

with Φ_{22} given by Eq. (12), and k being a real constant. The complete perturbations of solutions of the Einstein-Weyl equations such that the flux vector of the neutrino field is tangent to a shear-free congruence of geodesics can also be expressed in terms of complex scalar potentials [22] which, in the present case, satisfy the following set of equations:

$$\overline{\delta}M_{1'}-(\Delta+2\gamma+\mu)M_{0'}=\eta_1\psi_G,$$

$$DM_{1'}-\delta M_{0'}=\eta_1\psi_N, \tag{21}$$

$$(\Delta+\gamma+\mu)\psi_N-\delta\psi_G=-ik\overline{\eta_1}(DM_{1'}+\delta M_{0'}),$$

$$\overline{\delta}\psi_N-D\psi_G=-2ik\overline{\eta_1}DM_{0'},$$

where $M_{0'}$ and $M_{1'}$ are two auxiliary potentials.

The components of the perturbations of the neutrino field are given by [22]

$$\overline{\eta_0^B}=\frac{1}{4ik}D\psi_N,\quad \overline{\eta_1^B}=\frac{1}{4ik}\delta\psi_N, \tag{22}$$

while $h_{\mu\nu}$ is given in Eq. (1) [thus, the perturbations of the Weyl spinor are those given in Eqs. (5) and (6)].

Now, we seek for solutions of Eqs. (21) of the form

$$\psi_G=g(v)e^{i(k_1x^1+k_2x^2+k_3u)},$$

$$\psi_N=f(v)e^{i(k_1x^1+k_2x^2+k_3u)}, \tag{23}$$

$$M_{0'}=h_1(v)e^{i(k_1x^1+k_2x^2+k_3u)},$$

$$M_{1'}=h_2(v)e^{i(k_1x^1+k_2x^2+k_3u)}.$$

Substituting Eqs. (9), (11), and (23) into Eqs. (21), we obtain the following set of linear ordinary differential equations, which correspond to the master equations:

$$\frac{dh_1}{dv}-\left[\frac{d}{dv}\ln U+\frac{v}{1-v^2}\right]h_1=i\,\frac{U(k_1\chi^{1/2}+ik_2\chi^{-1/2})}{2\sqrt{1-v^2}}\,h_2-\frac{U\eta_1}{\sqrt{2}}\,g,$$

$$\frac{df}{dv}-\left[\frac{1}{2}\frac{d}{dv}\ln U+\frac{v}{1-v^2}\right]f=kk_3\overline{\eta_1}h_2+\frac{U}{2\sqrt{1-v^2}}(k_1\chi^{1/2}-ik_2\chi^{-1/2})(k\overline{\eta_1}h_1+ig),$$

$$\eta_1 f=\frac{\sqrt{2}ik_3}{U}\,h_2-\frac{i}{\sqrt{2}\sqrt{1-v^2}}(k_1\chi^{1/2}-ik_2\chi^{-1/2})h_1,$$

$$k_3g=\frac{U(k_1\chi^{1/2}+ik_2\chi^{-1/2})}{2\sqrt{1-v^2}}\,f+2ikk_3\overline{\eta_1}h_1. \tag{24}$$

Assuming that $k_3\neq 0$, the most general solution of Eqs. (24) for f and g is given by

$$f=C_1\frac{U^{1/2}}{\sqrt{1-v^2}}\,e^{iF_1(v)},$$

$$g=2ik\overline{\eta_1}C_2\frac{(1-v^2)^{1/2}}{U}\,e^{iF_2(v)}-i\,\frac{U^{3/2}(k_1\chi^{1/2}+ik_2\chi^{-1/2})}{2k_3(1-v^2)}\,C_1e^{iF_1(v)}, \tag{25}$$

where C_1 and C_2 are arbitrary constants and F_1 and F_2 are real functions given by

$$F_1(v)=-\frac{1}{\sqrt{2}}\int U\left[\frac{U(k_1^2\chi+k_2^2\chi^{-1})}{2\sqrt{2}k_3(1-v^2)}+k\eta_1\overline{\eta_1}\right]dv,$$

$$F_2(v)=\frac{1}{\sqrt{2}}\int U\left[\frac{U(k_1^2\chi+k_2^2\chi^{-1})}{2\sqrt{2}k_3(1-v^2)}+2k\eta_1\overline{\eta_1}\right]dv. \tag{26}$$

Thus, from Eqs. (22), one easily finds that

$$\overline{\eta_0^B}=\frac{\sqrt{2}k_3}{4kU}\,f(v)e^{i(k_1x^1+k_2x^2+k_3u)},$$

$$\overline{\eta_1^B}=\frac{U(k_1\chi^{1/2}-ik_2\chi^{-1/2})}{2k_3\sqrt{1-v^2}}\,\overline{\eta_0^B}, \tag{27}$$

and the perturbations of the Weyl spinor are given in Eqs. (16) with f and g given in this case by expressions (25).

When $k_3=0$, and assuming that the potentials ψ_N and ψ_G do not depend on u, Eqs. (21) for these potentials become

$$\left(\lambda\,\delta+\mu\,\overline{\delta}+\gamma\,\overline{\delta}-ik\,\eta_1\,\overline{\eta_1}\,\overline{\delta}+\frac{\Delta\,\overline{\eta_1}}{\overline{\eta_1}}\,\overline{\delta}\right)\psi_N=\overline{\delta}\delta\psi_G,$$
$$\delta\overline{\delta}\psi_N=0. \tag{28}$$

Making use of the definition (17) for the variables z and $\overline{z}$, the solution can be expressed in the form

$$\psi_N=(1-v^2)^{-1}\partial_z F(v,z),$$

$$\psi_G=(1-v^2)^{-1/2}\left[\lambda\,\partial_z+\left(\mu+\gamma-ik\,\eta_1\,\overline{\eta_1}+\frac{\Delta\,\overline{\eta_1}}{\overline{\eta_1}}\right)\partial_{\overline{z}}\right]\overline{z}\overline{F}(v,z)+G(v,z), \tag{29}$$

where $F(v,z)$ and $G(v,z)$ are arbitrary functions [in these expressions for ψ_N and ψ_G we have ignored terms of the form $H(v,\overline{z})$ because they yield trivial perturbations]. From Eqs. (5), (6), and (22), we find that the only nonvanishing components of the neutrino field perturbations and of the Weyl spinor perturbations are

$$\overline{\eta_1^B}=\frac{1}{4ik(1-v^2)^{3/2}}\,\partial_z^2 F(v,z),$$

$$\overline{\Psi_4^B}=-\frac{1}{(1-v^2)^{3/2}}\left[\mu+\gamma-ik\,\eta_1\,\overline{\eta_1}+\frac{\Delta\,\overline{\eta_1}}{\overline{\eta_1}}+\lambda\,\overline{z}\,\overline{\partial_z}\right]\partial_z^4 F(v,z)-(1-v^2)^{-2}\partial_z^4 G(v,z). \tag{30}$$

When the neutrino field perturbations vanish (i.e., $\partial_z^2 F=0$), the nonvanishing purely gravitational perturbations (30) also correspond to an exact solution of the Einstein-Weyl equations [21].

V. CONCLUDING REMARKS

In the approach followed in Ref. [8], in addition to the computations that one needs to make in order to find the perturbations of the Newman-Penrose quantities, these quantities cannot be expressed completely algebraically in terms of the solutions of what is called the master equation of the theory in that reference; in the present work we have obtained by a clear and direct way the expressions for the perturbations of several Newman-Penrose quantities, being remarkable the fact that all these quantities can be expressed completely algebraically in terms of some few scalar potentials or their derivatives, contrary to the findings of Ref. [8]. [Note that, by eliminating the function f from Eqs. (15), one obtains a single master equation for g, which is analogous to the master equation found in Ref. [8].] As we have already seen, these potentials are solutions of *''true'' master equations*, because the problem is reduced completely to study the mathematical properties of these equations; for example, some particular background solutions that lead to singularities [23], which possibly will be the aim of a future investigation. It also would be interesting to study the matching of these perturbations with those in the interaction region of the colliding waves.

ACKNOWLEDGMENTS

This work was partially supported by CONACYT (Mexico) and the Sistema Nacional de Investigadores (Mexico). The author would like to thank Dr. G. F. Torres del Castillo for useful comments.

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