Perturbations in a spherically symmetric inhomogeneous cosmological model with the self-similar region

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To clarify the evolution of inhomogeneous substructures in large-scale structures such as voids, we study the perturbations in a spherically symmetric inhomogeneous model with the self-similar region outside the inner low-density homogeneous region, first based on the Gerlach-Sengupta general-relativistic formulation for perturbations in spherically symmetric inhomogeneous models. Owing to self-similarity, the analysis for all kinds of perturbations at the early stage is simplified in a similar way to homogeneous models. Next we take the approximate treatment due to the local homogeneous model with the average density parameter which is considered at each point, in order to study the behavior of perturbations on small scales. It is found that the growth rate of density perturbations in the outside self-similar region can be larger by about 30−20% (for $z=2-3$) than that in the corresponding ordinary low-density homogeneous model. [S0556-2821(97)05218-1]

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I. INTRODUCTION

There are various large-scale dynamical structures such as superclusters and voids which are evolving with cosmic expansion. In these structures, substructures such as galaxies appear as a result of the growth of perturbations. The behavior of perturbations in general inhomogeneous structures is very complicated, but it may be rather simplified in spherically symmetric structures.

The general-relativistic theory of perturbations in a spherically symmetric inhomogeneous model was derived by Gerlach and Sengupta [1]. They gave the elegant gauge-invariant formulation, but the solutions in cosmological problems have not been derived so far. This may be due to the situation that their perturbation equations are partial differential equations inconvenient for analytical treatments and intuitive insight, and that three kinds of perturbations are not decoupled and their coupling is complicated, as in anisotropic cosmological models [2–4]. So far the perturbations of spherically symmetric inhomogeneous cosmological models have been studied in a few treatments [5,6], but they were limited to spherically symmetric perturbations.

In this paper we study the perturbations in an inhomogeneous background model with the self-similar region outside the inner low-density homogeneous region, which includes pressureless matter. In this self-similar region, dynamics is described using a single coordinate $\xi \equiv ct/r$ [7–10], where t and r are the cosmic time and a radial coordinate, and the perturbation equations are also reduced to ordinary differential equations (see also a treatment in the partially symmetric case [11] for comparison). Moreover, the background in this region is isotropic and homogeneous in the limit $\xi \to 0$, so that the behavior of perturbations is similar to that in the Friedmann model in this limit. Because of these situations, the perturbations can be most easily analyzed in a self-similar model among general inhomogeneous models. The perturbations are studied first due to Gerlach and Sengupta's formalism, and next in the approximate treatment due to local homogeneous models.

In Sec. II the background model is shown in connection with the Gerlach-Sengupta formalism. In Sec. III the perturbation equations in the self-similar model are derived from the equations by Gerlach and Sengupta [1], and in Sec. IV the analytic solutions in a series expansion around $\xi=0$ and qualitative behavior of numerical solutions in the region of $\xi \gtrsim 1$ are shown. In Sec. V, it is shown that the perturbations in the local homogeneous models are useful for the approximate treatment of the perturbations on small spatial scales and, assuming that the inside and outside of a spherical void are expressed by homogeneous and self-similar models, respectively, the growth of density perturbations in the outside self-similar region of the void is shown in comparison with that in the corresponding ordinary homogeneous model in Appendixes A, B, C, and D, the tensor calculus in the submanifolds, basic equations in the even parity, various formulas in the series expansion, and the perturbations in a homogeneous model in Gerlach and Sengupta's formalism, respectively, which are necessary for calculations in the text.

II. BACKGROUND MODEL

The Lemaitre-Tolman-Bondi solution for spherically symmetric inhomogeneous cosmological models with pressureless matter (Tolman [12], Landau and Lifshitz [13]) is represented by the line element

$$ds^2 = \tilde{g}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu = -c^2 dt^2 + S^2(t,r)\left\{\frac{(1+rS'/S)^2}{1-k\alpha(r)r^2}dr^2 + r^2 d\Omega^2\right\}, \tag{2.1}$$

where $(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = (ct, r, \theta, \varphi)$, $d\Omega^2 = d\theta^2 + \sin^2 d\varphi^2$, $\alpha(r)$ is an arbitrary function of coordinate radius r, and $S' = \partial S/\partial r$. The equation for $\dot{S}(=\partial S/\partial t)$ is given by

$$(\dot{S})^2 = c^2(\beta S^{-1} - k\alpha), \tag{2.2}$$

with the sign $k(=\pm 1)$ of the spatial curvature, and its solution is expressed as

$$S(t,r)=\frac{\beta(r)}{2k\alpha(r)}\left[1-\frac{d\sigma(\eta)}{d\eta}\right], \tag{2.3}$$

$$c[t-\tau(r)]=\frac{\beta(r)}{2k[\alpha(r)]^{3/2}}[\eta-\sigma(\eta)], \tag{2.4}$$

where $\beta(r)$ and $\tau(r)$ are arbitrary functions, and

$$\sigma(\eta)=\sin\eta,\ \ \sinh\eta \ \text{ for } \ k=1,-1, \tag{2.5}$$

respectively. The matter density is expressed as

$$\rho=\frac{3c^2\overline{\beta(r)}}{8\pi G}S^{-3}(1+rS'/S)^{-1}, \tag{2.6}$$

where $\overline{\beta(r)}=\beta(r)+\frac{1}{3}r\beta'(r)$. In this paper we adopt $k=-1$, assume $\tau(r)=0$, and use the unit $c=1$ for simplicity.

For convenience we assume that the spacetime consists of the inner homogeneous region ($r<r_1$) and the outer self-similar inhomogeneous region ($r>r_1$), as in Ref. [8]. In the limit $r_1\to 0$, the model is covered by the latter region. In the inner homogeneous region we have $\alpha(r)/\alpha_0=\beta(r)/\beta_0=1/(r_1)^2$, where $\beta_0=H_0{}^{-1}(r_1)^2$ and $H_0\equiv(\dot{S}/S)_{t=t_0}$. The Hubble constant H_0 corresponds to the present epoch t_0. Then Eqs. (2.3) and (2.6) at epoch t_0 lead to

$$\Omega_0\equiv\rho_{\rm in}(t_0)\Big/\left(\frac{3H_0{}^2}{8\pi G}\right)=1/(H_0S_0)^3 \tag{2.7}$$

and

$$\alpha_0/(r_1)^2=(1-\Omega_0)/(\Omega_0)^{2/3}, \tag{2.8}$$

for $S_0\equiv S(t_0,0)$. The solutions are expressed as

$$S/S_0=\frac{\Omega_0}{2(1-\Omega_0)}(\cosh\eta-1), \tag{2.9}$$

$$H_0t=\frac{\Omega_0}{2(1-\Omega_0)^{3/2}}(\sinh\eta-\eta), \tag{2.10}$$

$$\rho_{\rm in}=\left(\frac{3H_0{}^2\Omega_0}{8\pi G}\right)\left(\frac{S}{S_0}\right)^{-3}. \tag{2.11}$$

Here the parameter α_0 or r_1 is arbitrary, and the positions of observers are also arbitrary. If an observer is at the center and we consider a light ray which is emitted at the redshift z_1 from the boundary and reaches the observer at the center at present epoch [8], the light-ray equation leads to

$$\sqrt{\alpha_0}=\frac{4\sqrt{1-\Omega_0}}{\Omega_0{}^2(1+z_1)}\left[1-\Omega_0+\frac{\Omega_0}{2}(1+z_1)\right.$$
$$\left.+\left(\frac{\Omega_0}{2}-1\right)\sqrt{1+\Omega_0z_1}\right], \tag{2.12}$$

by which r_1 is related to z_1 using Eq. (2.8). For $z_1\ll 1$, we have $\alpha_0\simeq(1-\Omega_0)z^2$, $r_1\simeq\Omega_0^{1/3}z_1$, and, for $z_1=1$ and $2(\Omega_0=0.2)$, we have $\alpha_0=0.398,1.095$, $r_1=0.412,0.684$, respectively.

In the self-similar region, $S(t,r)$ is a function of only $\xi(\equiv t/r)$, and $\alpha(r)$ and $\beta(r)$ are taken to be

$$\alpha(r)/\alpha_0=\beta(r)/\beta_0=1/r^2, \tag{2.13}$$

so that $\alpha(r)r^2$ and $\beta(r)r^2$ may be constant and continuous at the boundary $r=r_1$. Equations (2.3) and (2.6) lead to Eq. (2.9) and

$$\xi\equiv\frac{t}{r}=\frac{H_0{}^{-1}(r_1)^{-1}\Omega_0}{2(1-\Omega_0)^{3/2}}(\sinh\eta-\eta), \tag{2.14}$$

$$\rho=\frac{3H_0{}^2\Omega_0}{8\pi G}\left(\frac{r_1}{r}\right)^2\left(\frac{S}{S_0}\right)^{-3}(1-\xi S_{,\xi}/S)^{-1}, \tag{2.15}$$

where $,\xi$ denotes the partial derivative $\partial/\partial\xi$.

Line element (2.1) in the self-similar region is expressed as

$$ds^2=-dt^2+S^2(\xi)\left\{\frac{(1-\xi S_{,\xi}/S)^2}{1-k\alpha_0}dr^2+r^2d\Omega^2\right\}. \tag{2.16}$$

If we use for r a radial coordinate x defined by $r/r_1\equiv\exp(x)$, the line element is

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu=r^2d\overline{s^2}, \tag{2.17}$$

$$d\overline{s^2}=\overline{g_{\mu\nu}}dx^\mu dx^\nu=-d\xi^2-2\xi d\xi dx+M^2(\xi)dx^2$$
$$+S^2(\xi)d\Omega^2, \tag{2.18}$$

where $(x^0,x^1,x^2,x^3)=(\xi,x,\theta,\varphi)$,

$$M^2\equiv N^2(\xi)-\xi^2, \tag{2.19}$$

and

$$N^2(\xi)\equiv S^2(1-\xi S_{,\xi}/S)^2/(1+\alpha_0). \tag{2.20}$$

The metric components are

$$g_{\mu\nu}=r^2\overline{g_{\mu\nu}},\quad g^{\mu\nu}=r^{-2}\overline{g^{\mu\nu}}, \tag{2.21}$$

with

$$(\overline{g_{00}},\overline{g_{01}},\overline{g_{11}})=(-1,-\xi,M^2),$$
$$(\overline{g^{00}},\overline{g^{01}},\overline{g^{11}})=(-M^2,-\xi,1)/N^2. \tag{2.22}$$

The conformal spacetime with $d\overline{s^2}$ is homogeneous in the x direction in the sense that it is invariant for the spatial transformation $x\to x+\text{const}$. For the mathematical analysis, the coordinates (ξ,x) are convenient, while the coordinates (t,r) are better for the physical interpretation. Accordingly, these two sets of coordinates will be used in the following.

Gerlach and Sengupta's gauge-invariant theory for perturbations on general spherically symmetric spacetimes is used in the next section. In their notation the background metrics in the self-similar region are written as

$$ds^2 = g_{AB} dx^A dx^B + R^2(x^C) d\Omega^2, \tag{2.23}$$

$$d\overline{s^2} = ds^2/r^2 = \overline{g}_{AB} dx^A dx^B + S^2(x^C) d\Omega^2, \tag{2.24}$$

where capital latin indices A,B,C refer to time and radial coordinates, latin indices a,b,c refer to θ and φ, and $R = S(\xi) r_1 \exp(x)$. On the two-dimensional submanifold spanned by $x^C (C=0,1)$, $R(x^C), v_A (\equiv R_{,A})$, and $g_{AB}(x^C)$ are regarded as scalar, vector, and tensor fields, and the barred quantities corresponding to them are defined as the quantities obtained after extracting all factors of r from them. Then the components of v_A and v^A are

$$(v_0, v_1) = (\overline{v}_0, \overline{v}_1) = (S_{,0}/S, 1),$$

$$r^2 (v^0, v^1) = (\overline{v}^0, \overline{v}^1) = (M^2 S_{,0}/S + 1, \xi S_{,0}/S - 1). \tag{2.25}$$

The energy-momentum tensor for pressureless matter is expressed as

$$\tilde{t}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu = \tilde{t}_{00} (d\tilde{x}^0)^2, \tag{2.26}$$

$$t_{\mu\nu} dx^\mu dx^\nu = t_{00} (dx^0)^2 + 2 t_{01} dx^0 dx^1, \tag{2.27}$$

in the (t,r) and (ξ,x) coordinates, respectively. Here $\tilde{t}^{00} = t_{00} = \overline{t}_{00} = r^2 \rho [= \overline{\rho} \propto 1/(S^2 N)]$, $t_{01} = \overline{t}_{01} = \xi t_{00}$, and $t_{11} = 0$, because $t_{00} = (\partial t/\partial \xi)^2 \tilde{t}_{00}$, $t_{01} = \partial t/\partial \xi \partial r/\partial x \, \tilde{t}_{01}$. From Eq. (2.15) it is found that $\overline{t}_{00}$ and $\overline{t}_{01}$ depend only on ξ, as well as $\overline{v}_0$ and $\overline{v}_1$.

The quantities t_{AB} and v_A satisfy the background equations

$$\tfrac{1}{2} \kappa t_{AB} = -2(v_{A|B} + v_{B|A}) + (2 v_C{}^{|C} + 3 v_C v^C - R^{-2}) g_{AB} \equiv \mathcal{G}_{AB}, \tag{2.28}$$

$$v_C{}^{|C} + v_C v^C - \mathcal{R} = 0, \tag{2.29}$$

where $\kappa = 16\pi G/c^4$, $|A$ denotes the covariant derivative with respect to x^A in the two-dimensional submanifold with metric g_{AB}, and $\mathcal{R}$ is the Gaussian curvature of the submanifold. In the barred quantities they are

$$\tfrac{1}{2} \kappa \overline{t}_{AB} = -2(\overline{v}_{A|B} + \overline{v}_{B|A}) + 4(\delta^1_A \overline{v}_B + \delta^1_B \overline{v}_A) + (2 \overline{v}_C{}^{\|C} + 3 \overline{v}_C \overline{v}^C - S^{-2} - 4 \overline{v}^1) \overline{g}_{AB} \equiv \overline{\mathcal{G}}_{AB}, \tag{2.30}$$

$$\overline{v}_C{}^{\|C} + \overline{v}_C \overline{v}^C - \overline{\mathcal{R}} = 0, \tag{2.31}$$

where $\|A$ denotes the covariant derivative with respect to x^A in the submanifold with metric $\overline{g}_{AB}$ and $\overline{g}^{AB}$, and $\overline{\mathcal{R}} = r^2 \mathcal{R}$.

III. PERTURBED QUANTITIES IN THE SELF-SIMILAR SPACETIME

The angular dependence of all perturbations can be expressed using spherical harmonics such as $Y_{lm}(\theta,\varphi)$ with angular integers l and m, and they have odd parity $[(-1)^{l+1}]$ or even parity $[(-1)^l]$, corresponding to the rotation on the two-dimensional submanifold with $x^0 = \text{const}$ and $x^1 = \text{const}$ (Regge and Wheeler [14], Gerlach and Sengupta [15]). In the following these two kinds of perturbations are separately considered.

A. Odd-parity perturbations

The perturbations of metric and energy-momentum tensors are expressed as

$$h_{\mu\nu} dx^\mu dx^\nu = 2 h_A(x^C) S_a(\theta,\varphi) dx^A dx^a + h(x^C)(S_{a;b} + S_{b;a}) dx^a dx^b, \tag{3.1}$$

$$\Delta t_{\mu\nu} dx^\mu dx^\nu = 2 \Delta t_A(x^C) S_a(\theta,\varphi) dx^A dx^a, \tag{3.2}$$

where semicolons indicate the covariant derivatives on the unit sphere and the vector harmonics S_a is transverse $(S_a{}^{;a} = S^a{}_{;a} = 0)$. Since we treat pressureless matter, we have no components with $\mu = 2,3$, i.e., L in Eq. GS (7b) of Ref. [1] vanishes. In the following, GS denotes the equation number in Ref. [1].

The gauge-invariant quantities corresponding to $h_{\mu\nu}$ and $\Delta t_{\mu\nu}$ are

$$k_A = h_A - R^2 (h/R^2)_{,A}, \tag{3.3}$$

$$L_A = \Delta t_A, \tag{3.4}$$

where $,A$ denotes $\partial/\partial x^A$. The gauge transformations are shown in their paper.

Because of self-similarity, $\overline{g}_{\mu\nu}, \overline{g}^{\mu\nu}, \overline{t}_{\mu\nu}$, and $\overline{t}^{\mu\nu}$ depend only on ξ, and they can be regarded as quantities in the conformal spacetime with metric $d\overline{s^2}$, as in the previous section. In the same way it is imposed that $(e^{-2x} h_{\mu\nu}, \Delta t_{\mu\nu})$ or $(e^{-2x} k_A, L_A)$ are regarded as quantities in the conformal spacetime, which is homogeneous in the x direction. Then we can assume that they are Fourier expanded in the form of $\exp(ipx)$, where p is a wave number. Each component is expressed as

$$h_A = e^{2x} \overline{h}_A, \quad h = e^{2x} \overline{h}, \quad \Delta t_A = \Delta \overline{t}_A, \tag{3.5}$$

and the gauge-invariant quantities are

$$k_A = e^{2x} \overline{k}_A, \quad L_A = \overline{L}_A, \tag{3.6}$$

where $\overline{h}_A, \overline{h}, \overline{k}_A, \Delta \overline{t}_A, \overline{L}_A$ have the common factor $\exp(ipx)$. The relations between them are

$$\overline{k}_0 = \overline{h}_0 - S^2 (\overline{h}/S^2)_{,0}, \tag{3.7}$$

$$\overline{k}_1 = \overline{h}_1 - ip\overline{h}, \tag{3.8}$$

where $,0$ means $\partial/\partial\xi$.

The quantities with upper indices are defined as

$$\overline{k^A}=\overline{g^{AB}}\,\overline{k_B}, \quad \overline{L^A}=\overline{g^{AB}}\,\overline{L_B}, \tag{3.9}$$

so that

$$k^A=\overline{k^A}, \quad L^A=e^{-2x}\overline{L^A}. \tag{3.10}$$

The Einstein equations for gauge-invariant quantities [GS (9a), (9b), and (14)] reduce to the following ordinary differential equations. First, we obtain from GS (9a)

$$\overline{k^A}_{\|A}+2\,\overline{k^1}=0 \quad \text{or} \quad (N\,\overline{k^0})_{,0}/N+(ip+2)\,\overline{k^1}=0 \quad (l\geqslant 2) \tag{3.11}$$

and from GS (14)

$$S^2\overline{L^A}_{\|A}+2S^2\overline{L^1}=0$$

or

$$(S^2N\overline{L^0})_{,0}/N+S^2(ip+2)\overline{L^1}=0 \quad (l\geqslant 1), \tag{3.12}$$

where $\|A$ denotes the covariant derivative in the submanifold with metric $\overline{g_{AB}}$, introduced in the previous section. If we define B^{AC} by

$$B^{AC}=R^4[(R^{-2}k^A)^{|C}-(R^{-2}k^C)^{|A}], \tag{3.13}$$

GS (16b) is expressed as

$$-[(\overline{B^{AC}})_{\|C}+2\overline{B^{A1}}]+(l-1)(l+2)\,\overline{k^A}=\kappa S^2\overline{L^A}, \tag{3.14}$$

where $B^{AB}=\overline{B^{AB}}$, the nonvanishing component of $\overline{B^{AB}}$ is

$$\overline{B^{01}}=-\overline{B^{10}}=(S/N)^2[-ip\,\overline{k_0}+S^2(\overline{k_1}/S^2)_{,0}], \tag{3.15}$$

and

$$\overline{B^{AC}_{\|C}}=\overline{B^{AC}_{,C}}+(N_{,0}/N)\overline{B^{AC}}. \tag{3.16}$$

For $A=0$, therefore, we obtain

$$-(ip+2)\overline{B^{01}}+(l-1)(l+2)\,\overline{k^0}=\kappa S^2\overline{L^0} \tag{3.17}$$

and for $A=1$,

$$\overline{B^{01}_{,0}}+(N_{,0}/N)\overline{B^{01}}+(l-1)(l+2)\,\overline{k^1}=\kappa S^2\overline{L^1}. \tag{3.18}$$

If we eliminate $\overline{B^{01}}$ from Eqs. (3.17) and (3.18) and use Eq. (3.11), we obtain Eq. (3.12) again. If we use the relations $\overline{k_0}=-\overline{k^0}-\xi\overline{k^1}$ and $\overline{k_1}=-\xi\overline{k^0}+M^2\overline{k^1}$ with Eq. (3.11), Eq. (3.17) reduces to

$$\left(\frac{\overline{k^0}}{S^2}\right)_{,00}+\left[2\left(\frac{S_{,0}}{S}+\frac{M_{,0}}{M}\right)+\frac{N_{,0}}{N}+\frac{2(1+ip)\xi}{M^2}\right]\left(\frac{\overline{k^0}}{S^2}\right)_{,0}$$
$$+\left[\left(2\frac{S_{,0}}{S}+\frac{N_{,0}}{N}\right)_{,0}+\left(2\frac{S_{,0}}{S}+\frac{N_{,0}}{N}\right)\left(2\frac{M_{,0}}{M}+\frac{ip\xi}{M^2}\right)\right.$$
$$\left.+\frac{2+p^2-ip}{M^2}+(l^2+l-2)\left(\frac{N}{MS}\right)^2\right]\frac{\overline{k^0}}{S^2}=\left(\frac{N}{M^2S^4}\right)\kappa\overline{L^0}. \tag{3.19}$$

As a result we derive $\overline{k^0}$ and $\overline{L^0}$ solving Eqs. (3.12) and (3.19) and get $\overline{k^1}$ from Eq. (3.11).

In this parity, perturbations include rotational motions and gravitational waves. If $\overline{L^0}=0$, the perturbations consist of free gravitational waves. The rotational motions with $\overline{L^0}\neq 0$ have conserved angular momentum satisfying Eq. (3.12).

B. Even-parity perturbations

The perturbations of metric and matter tensors are

$$h_{\mu\nu}dx^\mu dx^\nu=h_{AB}(x^C)Y(\theta,\varphi)dx^Adx^B+2h_AY_{,a}dx^Adx^a$$
$$+R^2[KY(\theta,\varphi)\gamma_{ab}+GY_{,a;b}]dx^adx^b, \tag{3.20}$$

$$\Delta t_{\mu\nu}dx^\mu dx^\nu=\Delta t_{AB}(x^C)Y(\theta,\varphi)dx^Adx^B$$
$$+2\Delta t_A(x^C)Y_{,a}dx^Adx^a, \tag{3.21}$$

where Δ^1 and Δ^2 in GS (4b) do not appear because of pressureless matter. The corresponding gauge-invariant quantities are

$$k_{AB}=h_{AB}-(p_{A|B}+p_{B|A}), \tag{3.22}$$

$$k=K-2v^Ap_A, \tag{3.23}$$

$$T_{AB}=\Delta t_{AB}-t^{|C}_{AB}p_C-(t_{CA}p^C_{|B}+t_{CB}p^C_{|A}), \tag{3.24}$$

$$T_A=\Delta t_AK-2t^C_Ap_C, \tag{3.25}$$

where

$$p_A=h_A-\tfrac{1}{2}R^2G_{,A} \tag{3.26}$$

and T^1 and T^2 in GS (8b) vanish.

In the same way as in the odd-parity case, it is imposed that $\overline{h}_{\mu\nu}(=r^{-2}h_{\mu\nu})$ and $\Delta\,\overline{t}_{\mu\nu}(=\Delta t_{\mu\nu})$ depend only on ξ, i.e.,

$$h_{AB}=r^2\,\overline{h}_{AB}(\xi), \quad h_A=r^2\,\overline{h}_A(\xi), \quad p_A=r^2\,\overline{p}_A(\xi),$$
$$(K,G)=(\overline{K}(\xi),\overline{G}(\xi)), \quad \Delta t_{AB}=\Delta\,\overline{t}_{AB}(\xi),$$
$$\Delta t_A=\Delta\,\overline{t}_A(\xi), \tag{3.27}$$

and the gauge-invariant counterparts are

$$k_{AB}=r^2\,\overline{k}_{AB}, \quad k=\overline{k}, \quad T_{AB}=\overline{T}_{AB}, \quad T_A=\overline{T}_A. \tag{3.28}$$

Then Einstein equations for barred quantities are reduced to ordinary differential equations with respect to ξ. Equations for the gauge-invariant quantities are given in GS (10a)–(10d) and GS (15a) and (15b). In the following we show them using the above barred quantities.

First we obtain, from GS (15a),

$$S^{-2}(S^2\overline{T^A})_{\|A}+2\,\overline{T^1}=\tfrac{1}{2}\,\overline{t^{AB}}\,\overline{k_{AB}}, \tag{3.29}$$

where $\overline{T^1}$ vanishes because of pressureless matter. From GS (15b) we obtain similarly

$$S^{-2}(S^2\overline{T^B_A})_{\|B}+2\overline{T^1_A}-\delta^1_A\overline{T^C_C}-l(l+1)S^{-2}\overline{T_A}$$
$$=\tfrac{1}{2}\overline{k_{AC\|A}}\,\overline{t^{BC}}+\overline{k^{CB}_{\|B}}\,\overline{t_{CA}}-\tfrac{1}{2}\overline{k^C_{C\|B}}\,\overline{t^B_A}-\overline{k_{,C}}\,\overline{t^C_A}$$
$$+\overline{k^{BC}}(2\overline{v_B}\,\overline{t_{CA}}+\overline{t_{CA\|B}}-\delta^1_A\overline{t_{BC}}). \qquad (3.30)$$

In the derivation of these and following equations we used the relations shown in Appendix A.

Moreover we obtain, from GS (10d),

$$\overline{k^C_C}=0, \qquad (3.31)$$

since $k^C_C=\overline{k^C_C}$. From GS (10b),

$$\overline{k_A}-\overline{k^C_{A\|C}}-2\overline{k^1_A}+\delta^1_A\overline{k^C_C}+(\overline{k^C_C})_{,A}-\overline{v_A}\,\overline{k^C_C}=-\kappa\overline{T_A} \qquad (3.32)$$

and, from GS (10c),

$$\overline{k^{\|C}_{\|C}}+2\overline{v^C}-\overline{k_{,C}}-\overline{k^{CD}_{\|DC}}-[N(2\overline{k^{1C}}-\overline{g^{C1}}\,\overline{k^B_B})]_{,C}-2\overline{v_C}\,\overline{k^{CD}_{\|D}}$$
$$-2(\overline{v_{C\|D}}+\overline{v_C}\,\overline{v_D})\overline{k^{CD}}+(\overline{k^D_D})^{\|C}{}_{\|C}+\overline{v^C}(\overline{k^D_D})_{\|C}$$
$$+[\overline{\mathcal{R}}-l(l+1)S^{-2}]\overline{k^C_C}=0. \qquad (3.33)$$

The above equations can be reduced to forms more convenient for solving them with respect to $\overline{k},\overline{k^{00}}$, $\overline{k^{01}}$, $\overline{T^0}/\overline{\rho}$, $\overline{T^{00}}/\overline{\rho}$, and $\overline{T^{01}}/\overline{\rho}$, which are shown in Appendix B.

Here we examine the physical meaning of the gauge-invariant quantities in comparison with those in a homogeneous and isotropic model. In the scalar perturbations the metric components k and k^{00} correspond to Φ_H and Φ_A in Bardeen's gauge-invariant formalism [16,17], and $\overline{T^{00}}/\overline{\rho}$ is a contrast of energy density, while $\overline{T^{01}}/\overline{\rho}$ and $\overline{T^0}/\overline{\rho}$ are velocities in the radial and angular directions. In order to define the density contrast corresponding to Bardeen's ϵ_m, we first consider T^{AB} in the submanifold with metric $\tilde{g}_{AB}$ and $\tilde{\rho}$. The present definition of ϵ_m is given as the density contrast in the synchronous and comoving reference system. Then we obtain in this system

$$T^{tt}=t^{tt}\epsilon_m-t^{tt}_{,t}p^t-2t^{tt}p^{|t}_t, \qquad (3.34)$$

$$T^t=t^{tt}p_t, \qquad (3.35)$$

and

$$k^{tt}=-k^t_t=2p^{|t}_t \qquad (3.36)$$

from the definition of $\tilde{T}^{AB}$, $\tilde{T}^A$, $\tilde{p}_A$, and $\tilde{k}^{AB}$, where we put $\tilde{T}^{00}=T^{tt}$, $\tilde{T}^0=T^t$, and so on. Accordingly, it is found that

$$\epsilon_m=\frac{T^{tt}}{\rho}+\frac{\rho_{,t}}{\rho}\frac{T^t}{\rho}+k^{tt}. \qquad (3.37)$$

By transformation from (t,r) to (ξ,x), we obtain

$$\epsilon_m=\frac{\overline{T^{00}}}{\overline{\rho}}+2\xi\frac{\overline{T^{01}}}{\overline{\rho}}+\frac{\overline{\rho_{,0}}}{\overline{\rho}}\frac{\overline{T^0}}{\rho}+\left(\frac{N}{M}\right)^2(\overline{k^{00}}+2\xi\overline{k^{01}}), \qquad (3.38)$$

where $\overline{T^{00}}=\overline{T^{\xi\xi}}$, $\overline{T^{01}}=\overline{T^{\xi x}}$, $\overline{t^{00}}=\overline{t^{\xi\xi}}$, and $\overline{t^{00}_{,0}}=\overline{t^{\xi\xi}_{,\xi}}$.

IV. BEHAVIOR OF PERTURBATIONS IN THE SELF-SIMILAR SPACETIME

In anisotropic cosmological models, three types of perturbations (scalar, vector, and tensor) are not generally decoupled and their behavior is complicated, as can be seen in perturbations of the Bianchi type I anisotropic model [2–4]). In the present case, however, it is possible to classify the perturbations into the above three types around $\xi=0$. This is because in the limit $\xi\rightarrow 0$ the background model tends to an isotropic spacetime, as follows. In this limit background metric (2.1) leads to

$$ds^2=-c^2dt^2+(S_1)^2\left(\frac{t}{r}\right)^{4/3}\left\{\frac{dr^2}{9(1+\alpha_0)}+r^2d\Omega^2\right\}, \qquad (4.1)$$

because $S(\xi)=S_1\xi^2$, where S_1 is constant. By the transformation $r\rightarrow\zeta\equiv r^{1/3}$ the metric reduces to

$$ds^2=-c^2dt^2+(S_1)^2(ct)^{4/3}\left\{\frac{d\zeta^2}{1+\alpha_0}+\zeta^2d\Omega^2\right\}. \qquad (4.2)$$

The perturbations are accordingly supposed to have a similar behavior to that in the homogeneous and isotropic model in the limit $\xi\rightarrow 0$.

In the following, the solutions in the series expansion are first derived in connection with the perturbations in a homogeneous, isotropic model, which are shown in Appendix D in the Gerlach and Sengupta formalism. Here we call the three classified perturbations as *primordial* scalar, vector, and tensor perturbations, in the sense that the classification is done in the neighborhood of $\xi=0$. Next the solutions in the whole region of ξ are derived numerically, using the solutions in the series expansion as their initial conditions.

A. Odd parity

First, we expand $\overline{k^0}/S^2$ as

$$\overline{k^0}/S^2=\xi^\alpha(1+a_1X+a_2X^2+\cdots), \qquad (4.3)$$

where

$$X\equiv(6\xi/\xi_*)^{2/3}, \qquad (4.4)$$

with

$$\xi_*\equiv\frac{H_0{}^{-1}(r_1)^{-1}\Omega_0}{2(1-\Omega_0)^{3/2}}. \qquad (4.5)$$

Substituting Eq. (4.3) and expansions of the background quantities (shown in Appendix C) into Eq. (3.19), we obtain the following solutions.

1. Primordial tensor perturbation

We assume $\overline{L^0}=0$. Then α satisfies $3\alpha^2+7\alpha+2=0$ or $\alpha=-2,-1/3$, which is consistent with Eq. (D19) in the limit $t\to 0$. The coefficients a_1 and a_2 are

for $\alpha=-2$

$$a_1=\tfrac{1}{120}[-123-180\sigma_0+10(l^2+l-2)(\sigma_0-1)+90\sigma_0(p^2-3ip)], \quad (4.6)$$

$$a_2=\tfrac{1}{5600}\{1933+2520\sigma_0+8400\sigma_0^2+\tfrac{280}{3}(l^2+l-2)\times(31-26\sigma_0-5\sigma_0^2)-840(28+95\sigma_0)\sigma_0p^2+140ip\sigma_0[33+20(l^2+l-2)(1-\sigma_0)+450\sigma_0-180p^2\sigma_0]\}, \quad (4.7)$$

for $\alpha=-1/3$

$$a_1=\tfrac{1}{40}[14-10\sigma_0+5(l^2+l-2)(\sigma_0-1)-15\sigma_0(3p^2+ip)], \quad (4.8)$$

$$a_2=\tfrac{1}{5600}\{1933+2520\sigma_0-8400\sigma_0^2+35(l^2+l-2)\times(-101+221\sigma_0-120\sigma_0^2)+315(53-180\sigma_0)\sigma_0p^2+315ip\sigma_0[7+180p^2\sigma_0+20(l^2+l-2)(\sigma_0-1)]\}, \quad (4.9)$$

where $\sigma_0\equiv(\alpha_0+1)/\alpha_0$ and α_0 is given in Eq. (2.13). These perturbations give free gravitational waves.

2. Primordial vector perturbations

Next let us consider the case $\overline{L^0}\neq 0$ and examine the orders of magnitude of $\overline{L_A}$. For pressureless matter we have $t^{\mu\nu}=\rho c^2u^\mu u^\nu$, and in the odd-parity case $(\Delta t^{0a},\Delta^{1a})=\rho c^2(u^0,u^1)u_a$. Since u^1 and u^a are of first order, Δt^{1a} is of second order, so that $L^1(=\Delta t_{1a})$ vanishes in our linearized treatment. On the other hand, $L_0=\Delta t_0=\rho c^2u_0u_a$, and $L^0=\Delta t^0=-\rho c^2u^0u_a$. The integration of Eq. (3.12) leads to

$$\overline{L^0}=(\overline{L^0})_0/(S^2N). \quad (4.10)$$

This relation means the conservation of angular momentum, because $\rho\propto 1/(S^2N),u^0\sim 1$, and $u_a=R^2u^a=\text{const}$.

Here we show an inhomogeneous solution in which the lowest term with respect to ξ corresponds to the inhomogeneous term in the right-hand side of Eq. (3.19). In this solution we have $\alpha=-4/3$ and the coefficients

$$a_1=\tfrac{1}{20}[-18.8-8\sigma_0+2(l^2+l-2)(\sigma_0-1)+6\sigma_0(3p^2-5ip)], \quad (4.11)$$

$$a_2=-\tfrac{1}{8400}\{1648.8-3444\sigma_0-1120\sigma_0{}^2+\tfrac{56}{3}(l^2+l-2)\times(106-121\sigma_0+15\sigma_0^2)-168(91+135\sigma_0)\sigma_0p^2+84ip\sigma_0[-37+30\sigma_0-180p^2\sigma_0+20(l^2+l-2)(1-\sigma_0)]\}. \quad (4.12)$$

These perturbations express rotational motions of pressureless matter.

According to the numerical solution, we find as general behavior that the solutions change smoothly following the power-series solutions, if $\xi\ll S/p$ and, after the epoch $\xi\sim S/p$, the solutions oscillate generally.

As ξ increases, M^2 decreases and becomes negative for ξ larger than a critical value $\xi_{\rm cr}$ at which $X\sim 1/\sigma_0(<1)$. For negative M^2, the hypersurface of $\xi=$const is partially timelike, and so the expansion in terms of exp(ipx) is insignificant, except for the case with small values of l and p. In the latter case it is found by numerical analysis that the behavior of perturbations is consistent with those in homogeneous models at the nearly isotropic stage of $\eta<2.0$ (cf Sec. V).

B. Even parity

1. Primordial scalar perturbations

The metric components of scalar perturbations in a homogeneous model satisfy the relations $k=k_r^r$ and $k^{tr}=0$, as can be seen in (D1) of Appendix D. Using the condition $k_t^t+k_r^r=0$ and transforming the coordinates from (t,r) to (ξ,x), we obtain the relations $\overline{k^{00}}=(1+\xi^2/N^2)\overline{k}$ and $\overline{k^{01}}=-[\xi/(N^2+\xi^2)]\overline{k^{00}}$. If we expand the metric perturbations as

$$\overline{k}=\xi^\alpha(1+a_1X+\cdots), \quad (4.13)$$

$$\overline{k^{00}}=\xi^\alpha(1+b_1X+\cdots), \quad (4.14)$$

$$\overline{k}=-\sigma_0\xi^{\alpha-1}(1+c_1X+\cdots), \quad (4.15)$$

the constants α,a_1,b_1,c_1 satisfy the relations

$$b_1=a_1+\sigma_0+\delta b_1 \quad (4.16)$$

and

$$c_1=a_1+\tfrac{1}{10}+\delta c_1, \quad (4.17)$$

where δb_1 and δc_1 vanish in the homogeneous model. In the present self-similar model, δb_1 and δc_1 do not vanish and must be determined together with α and a_1.

For this purpose, we use first Eq. (B10) or (C14) to get

$$(ip+5/3)\delta b_1=(\alpha+7/3)\delta c_1. \quad (4.18)$$

For the metric perturbations in Eqs. (4.13)–(4.15), we have $\overline{T^{00}},\overline{T^{01}}$, and $\overline{T^0}$ shown in Eqs. (C15)–(C17), and the consistency conditions that they should obey Eqs. (B2)–(B4) are given by Eqs. (C18)–(C20). In the present case with $b_0=c_0=1$ we obtain $\alpha(\alpha-5/3)=0$ from the condition in the lowest order, and from the conditions in the next order and Eq. (4.18), we obtain the following two sets of consistent constants:

$$\alpha=0{:}\,a_1=-\tfrac{9}{35},\quad \delta b_1=\tfrac{2}{5},\quad \delta c_1=\tfrac{6}{35}(ip+\tfrac{5}{3}), \quad (4.19)$$

$$\alpha=-\tfrac{5}{3}{:}\,\delta b_1=-a_1,\quad \delta c_1=-\tfrac{3}{2}(ip+\tfrac{5}{3})a_1, \quad (4.20)$$

where a_1 has not yet been determined for $\alpha=-5/3$. This is because the vector perturbations can be also included for

$\alpha=-5/3$, as well as scalar perturbations. To determine the value of a_1, it is necessary to impose the condition of no rotation.

The vorticity tensor $\omega_{\mu\nu}$ is

$$\omega_{\mu\nu}\equiv\frac{1}{2}P^{\lambda}_{\mu}P^{\sigma}_{\nu}(u_{\lambda;\sigma}-u_{\sigma;\lambda}), \tag{4.21}$$

where $P_{\mu\nu}=g_{\mu\nu}+u_\mu u_\nu$ and ; denotes four-dimensional covariant differentiation. Its components reduce to $\omega_{23}=0$, and

$$(\omega_{12},\omega_{13})=[T_{tr}/\rho-(T_t/\rho)_{,r}](Y_{,2},Y_{,3}) \tag{4.22}$$

in the (t,r) coordinates, where $\rho(=t^{tt})$ is the background matter density. The condition is, therefore, expressed as

$$T_{tr}/\rho=(T_t/\rho)_{,r}. \tag{4.23}$$

Here since $T_{AB}=\overline{T_{AB}}$, $T_A=\overline{T_A}$, and $\rho=r^{-2}\overline{\rho}$, we have

$$T_{tr}/\rho=r^{-2}(T_{01}-\xi T_{00})/\rho=(\overline{T_{01}}-\xi\overline{T_{00}})/\overline{\rho} \tag{4.24}$$

and

$$(T_t/\rho)_{,r}=(r^{-1}T_0/\rho)_{,r}r^{-1}-\xi(T_0/\rho)_{,0}r^{-2}$$
$$=(r\overline{T_0}/\overline{\rho})_{,1}r^{-1}-\xi(\overline{T_0}/\overline{\rho})_{,0} \tag{4.25}$$

in terms of the $(\xi,x)[=(x^0,x^1)]$ coordinates. The above condition is rewritten as

$$\overline{T_{01}}-\xi\overline{T_{00}}=\overline{T_0}(ip+1)-\xi\overline{\rho}(\overline{T_0}/\overline{\rho})_{,0} \tag{4.26}$$

or

$$N^2\overline{T^{01}}=\overline{T^0}(ip+1)-\xi\overline{\rho}(\overline{T^0}/\overline{\rho})_0, \tag{4.27}$$

where $\overline{T_0}=-\overline{T^0}, \overline{T_{00}}=\overline{T^{00}}+2\xi\overline{T^{01}}$, and $\overline{T_{01}}=(\xi^2-M^2)\overline{T^{01}}+\xi\overline{T^{00}}$.

Substituting Eqs. (C15)–(C17) into Eq. (4.27), we obtain another condition for constants. For $\alpha=-5/3$ with $b_0=c_0=1$, the lowest-order condition is automatically satisfied and in the next order it gives

$$a_1=\frac{-5+14\sigma_0+4(3+\sigma_0)ip}{5(10+9ip)}. \tag{4.28}$$

The physical contrast of matter density is defined in Eq. (3.38) and shown in Eq. (C21) in Appendix C. In the present case we have

$$\epsilon_m=\frac{1}{12}[2\sigma_0-1+(l^2+l-2)(\sigma_0-1)-3ip\sigma_0+9p^2\sigma_0]X$$
$$+0(X^2) \quad \text{for} \quad \alpha=0 \tag{4.29}$$

and

$$\epsilon_m=\frac{1}{12}[1-28\sigma_0-1+(l^2+l-2)(\sigma_0-1)$$
$$-33ip\sigma_0+9p^2\sigma_0]X/\xi^{5/3}+0(X^2/\xi^{5/3}) \quad \text{for}$$
$$\alpha=-5/3. \tag{4.30}$$

The result shows that the cases of $\alpha=0$ and $\alpha=-5/3$ are the growing and decaying modes of density perturbations, respectively, whose behavior is quite same as those in a homogeneous model, with respect to the powers in the lowest order.

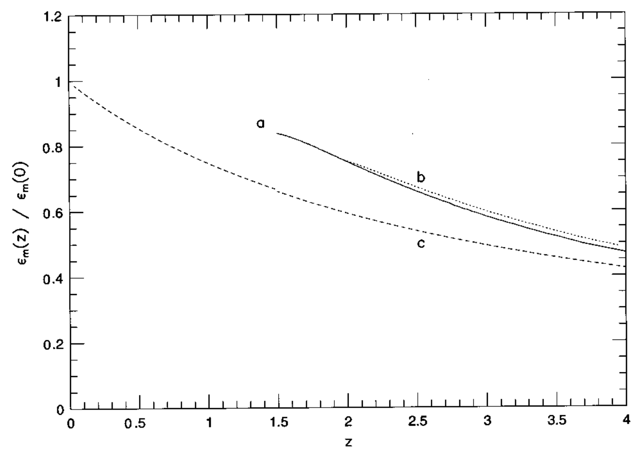

FIG. 1. The ratio $\epsilon_m(z)/\epsilon_m(0)$ in the self-similar region is shown for the inhomogeneous models of (a) $\Omega_0=0.2$, $z_1=1.5$, and (b) $\Omega_0=0.2$, $z_1=2.0$. For comparison the ratio in (c) the ordinary homogeneous model with $\Omega_0=0.2$ is also shown.

Next we can analyze the perturbations in larger values of ξ by numerically solving ordinary differential equations (B2), (B3), (B4), (B9), (B10), and (B13) for $\overline{k}_{,0},(\overline{k^{00}})_{,0},(\overline{k^{01}})_{,0},(\overline{T^{00}}/\overline{\rho})_{,0},(\overline{T^{01}}/\overline{\rho})_{,0},(\overline{T^0}/\overline{\rho})_{,0}$, where the above solutions in the series expansion are used as the initial conditions at $\xi=\xi_i\ll 1$. As the result we find that at the interval $\xi_i<\xi\lesssim 1$ the numerical solutions can be well reproduced by those in the series expansions (4.13)–(4.15) and (C9)–(C17).

At a point $\xi=\xi_{\rm cr}$ with $X\sim 1/\sigma_0(<1)$, the factor M^2 in denominators vanishes and, as was stated in the odd-parity case, the present treatment cannot be used in $\xi>\xi_{\rm cr}$, except for small values of l and p. It should be noted, however, that at this point there is no physical singularity, because the vanishing numerators exist always corresponding to them, and so the numerical calculations can be continued after the point in which $X>1/\sigma_0$. It is found as the result that it is larger than ϵ_m in the homogeneous model with the inner density parameter Ω_0 by the factor which is 1.23 at the boundary $r=r_1$ at epoch z_1, and that ϵ_m at the emission epochs with $z>z_1$ is always larger than that in the ordinary homogeneous model. Here the radial coordinate r is connected with z and η (at the emission epoch) by Eqs. (3.6) and (3.9) in Ref. [6], and the initial value $(\epsilon_m)_i$ is given for arbitrary r at the initial hypersurface $t=t_i$, so that the initial value η_i in the self-similar region is related to $(\eta_{in})_i$ in the inner region by

$$\frac{2(1-\Omega_0)^{3/2}}{\Omega_0}H_0t_i=\sinh(\eta_{in})_i-(\eta_{in})_i=\frac{r}{r_i}(\sinh\eta_i-\eta_i), \tag{4.31}$$

where Eqs. (2.10) and (2.14) were used. The behavior of ϵ_m at the emission epochs is shown in Fig. 1 in an example in which $\Omega_0=0.2$ and $z_1=1.5$ and 2.0.

2. Primordial vector perturbations

The metric perturbations at $\xi \simeq 0$ can be given in the same form as **1** and the constant coefficients reduce to the set in Eq. (4.20), while the set in Eq. (4.19) gives only a density perturbation. In the present case, a_1 should be determined under the condition that the divergence of the velocity vanishes. Here it should be noted that, as was shown in **1**, the velocity vector $(T^{tr}Y, T^{t}Y_2, T^{t}Y_3)/\rho$ is gradient in the lowest order for $\alpha = -5/3$ and $b_0 = 1$, but at the same time the physical density contrast $\epsilon_m = 0$. Accordingly, the vector is meaningless in the lowest order. Here we will consider only the terms in the next order $(\tilde{T}^{tr}Y, \tilde{T}^{t}Y_2, \tilde{T}^{t}Y_3)/\rho$. Then the condition is given by

$$(\tilde{T}^{tr}/\rho \cdot Y)_{|r} + g^{22}(\tilde{T}^{t}/\rho \cdot Y_{,2})_{|2} + g^{33}(\tilde{T}^{t}/\rho \cdot Y_{,3})_{|3} = 0 \tag{4.32}$$

in the (t,r) coordinates and in the (ξ,x) coordinates it reduces to

$$\xi(\widetilde{\overline{T^{01}}}/\overline{\rho})_{,0} + \left[\frac{\xi^2 S_{,00}/S}{\xi S_{,0}/S - 1} + 2\xi\frac{S_{,0}}{S} - (2+ip)\right]\widetilde{\overline{T^{01}}}/\overline{\rho} + \frac{l(l+1)}{S^2}\frac{\widetilde{\overline{T^{0}}}}{\overline{\rho}} = 0, \tag{4.33}$$

where $\tilde{T}_{AB} = \widetilde{\overline{T_{AB}}}$ and $\rho = \overline{\rho}/r^2$. Imposing Eq. (4.33) for $\alpha = -5/3$ and $b_0 = 1$, we obtain

$$a_1 = \frac{1}{30}\,\frac{1 - (62 - 45p^2 + 111ip)\sigma_0}{20 + 30\sigma_0 + 6p(8i - 3p)\sigma_0 - (l^2 + l - 2)(\sigma_0 - 1)}. \tag{4.34}$$

In the next order, ϵ_m does not vanish but it has no meaning as the density contrast, because the velocity is not gradient.

3. Primordial tensor perturbations

For tensor perturbations a more complicated form of metric perturbations is necessary, in the same way as in a homogeneous model. Here we assume the form with constants b_0 and c_0 shown in Eqs. (C10)–(C12) in Appendix C. Since there are neither density perturbations nor rotational velocity perturbations in the present case, we impose that $\epsilon_m = 0$ and the velocity vector is gradient. From the condition $\epsilon_m = 0$ and Eq. (C21) we obtain first in the lowest order

$$\alpha(1 - b_0) = 0. \tag{4.35}$$

For $\alpha = 0$, the other constants are determined as follows by the consistency conditions (C13), (C18)–(C20), and $\epsilon_m = 0$ in the next order in Eq. (C21):

$$b_0 = \frac{3[18p^2\sigma_0 - (l^2 + l - 2)(1 - \sigma_0) - 12ip\sigma_0]}{10 - 20\sigma_0 + 7(l^2 + l - 2)(1 - \sigma_0) - 6(ip + 21p^2)\sigma_0}, \tag{4.36}$$

$$c_0 = b_0 + \tfrac{3}{5}(b_0 - 1)ip, \tag{4.37}$$

$$a_1 = -\tfrac{9}{140}[(10\sigma_0 - 1)b_0 + (14b_0 + 31)p^2\sigma_0 - \tfrac{5}{2}(l^2 + l - 2)\times(b_0 + 1)(1 - \sigma_0) + ip\sigma_0\tfrac{28}{45}(52\sigma_0 - 101)], \tag{4.38}$$

$$b_1 = \frac{1}{56\sigma_0}[(76\sigma_0 - 2)b_0 + 90p^2\sigma_0 - 5(l^2 + l - 2)(b_0 + 1)\times(1 - \sigma_0) + \tfrac{6}{5}ip\sigma_0(41b_0 - 66)], \tag{4.39}$$

$$c_1 = \frac{1}{70c_0}\{20a_1 + b_0(4 + 50b_1 + 50\sigma_0) - 100c_0\sigma_0 + 3ip[-1 - 10a_1 + b_0(1 + 10b_1 + 10\sigma_0) - 20c_0\sigma_0]\}. \tag{4.40}$$

For $b_0 = 1$, the consistency conditions (C18)–(C20) in the lowest order lead to

$$\alpha = -5/3 \tag{4.41}$$

and

$$c_0 = \tfrac{1}{2}(1 + b_1/\sigma_0). \tag{4.42}$$

The conditions (C14), (C18)–(C20), and (C21) in the next order give two relations between three parameters a_1, b_1, and c_1. In the same way as in the scalar perturbations for $\alpha = -5/3$, we use the condition of vanishing vorticity vector given by Eq. (4.27). Then the three parameters are determined as

$$a_1 = \tfrac{1}{5}[14\sigma_0 - 5 + 4(3 + \sigma_0)ip]/(10 + 9ip), \tag{4.43}$$

$$b_1 = -\frac{1}{15(6 + ip)}[3a_1 + 2 + (220 - 90p^2)\sigma_0 - 10(l^2 + l - 2)\times(\sigma_0 - 1) + 315ip], \tag{4.44}$$

$$c_1 = \tfrac{1}{20}[1 + 3b_1/\sigma_0 - 60a_1]/(1 + b_1/\sigma_0). \tag{4.45}$$

Thus we have two modes of tensor perturbations with powers $\alpha = 0$ and $-5/3$. They correspond to two modes of tensor perturbations in a homogeneous model, though the dependence of constants on p and l is very complicated.

The behavior of the perturbations in the region $\xi \sim 1$ is shown by solving Eqs. (B2), (B3), (B4), (B9), (B13), and (B14) numerically and using the above solutions as the initial conditions. It is found as the result that for $\xi \ll 1$ they are smooth and follow the solutions in series expansion, and for $p\xi \gtrsim 1$, they become oscillatory as expected from the comparison with the perturbations in homogeneous models.

V. APPROXIMATE TREATMENT DUE TO PERTURBATIONS IN LOCAL HOMOGENEOUS MODELS

In this section we consider an approximate treatment in which the background inhomogeneous model is replaced by a stepwise series of virtual local homogeneous models and the perturbations are assumed to be caused in these homogeneous models. Because the homogeneous models with $S = S(t)$ must be isotropic at the same time, the perturbations are those in the Friedmann model, which is shown in Appendix D.

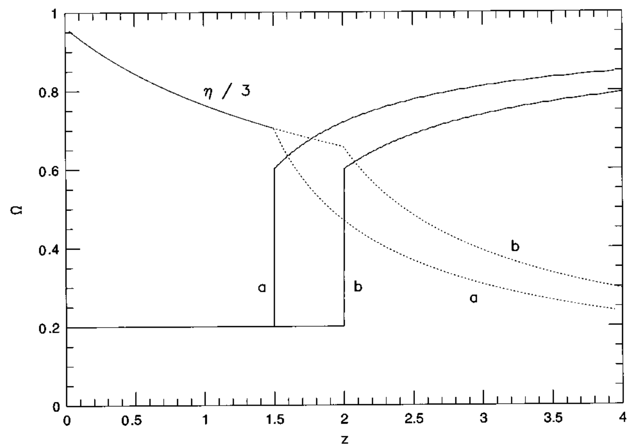

FIG. 2. The local value of the density parameter Ω at present epoch is shown in solid lines. The density parameter Ω_0 in the inner homogeneous region is assumed to be 0.2. The values of $\eta/3$ are also shown in dotted lines. The models with $z_1 = 1.5$ and 2.0 are denoted by a and b, respectively.

To determine the model parameter of the local homogeneous models, we define the expansion rates H_t, H_r, and Θ in the transverse and radial directions, and in the average as

$$H_t \equiv \xi(\ln S)_{,\xi}, \quad H_r \equiv \xi[\ln(S - \xi S_{,\xi})]_{,\xi}, \tag{5.1}$$

and

$$\Theta \equiv (2H_t + H_t)/3, \tag{5.2}$$

respectively. Their values at the present epoch (i.e., in the $t = t_0$ hypersurface) are the corresponding Hubble constants $(H_t)_0, (H_r)_0$, and Θ_0. The Hubble constant in the inner region is H_0, the present cosmic time is t_0, and $(H_t)_0$ is equal to Eq. (2.18) in Ref. [8]. The local density parameters can be defined in the two directions, and the average parameter is

$$\overline{\Omega} \equiv \rho(t_0) \Bigg/ \left[\frac{3(\Theta_0)^2}{8\pi G}\right]$$
$$= \frac{2}{3}\,\frac{(\cosh\overline{\eta} - 1)}{\sinh^2\overline{\eta}} \Bigg/ \left[1 - \frac{\sinh\overline{\eta}(\sinh\overline{\eta} - \overline{\eta})}{(\cosh\overline{\eta} - 1)^2}\right], \tag{5.3}$$

where a mistake in the original expression was corrected (cf. [9]) and $\overline{\eta}$ is the present value of η defined by Eqs. (2.19) in Ref. [8]. The radial coordinate r is connected with z and η (at the emission epoch) by Eqs. (3.6) and (3.9) in Ref. [8]. The behavior of $\overline{\Omega}$ and η (at the emission epoch) is also shown in Fig. 2 as a function of z.

Here we assume that the local homogeneous models have the average density parameter $\overline{\Omega}$, and draw attention to the scalar perturbation. The density contrast ϵ_m is derived by solving Eq. (D55) from an initial epoch t_i to the emission epoch t_e, where the initial value of ϵ_m is assumed to be equal everywhere. The relation between η and the corresponding $\tilde{\eta}$ in the local homogeneous model is given so that the cosmic time t may be equal. Accordingly, we have the following two relations at the initial and emission epochs:

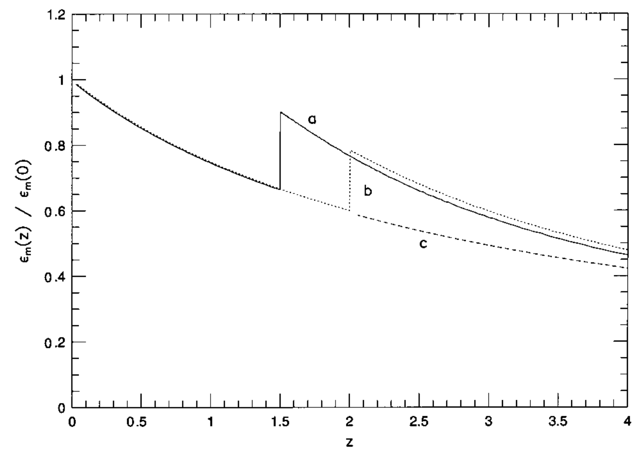

FIG. 3. The ratio $\epsilon_m(z)/\epsilon_m(0)$ is shown for the local homogeneous models corresponding to the cases of (a) $\Omega_0 = 0.2$, $z_1 = 1.5$, and (b) $\Omega_0 = 0.2$, $z_1 = 2.0$. For comparison the ratio in (c) the ordinary homogeneous model with $\Omega_0 = 0.2$ is also shown.

$$t_i = \frac{\Omega_0/H_0}{(1-\Omega_0)^{3/2}}[\sinh(\eta_{in})_i - (\eta_{in})]$$
$$= \frac{\Omega/\Theta_0}{(1-\Omega)^{3/2}}(\sinh\tilde{\eta}_i - \tilde{\eta}_i) \tag{5.4}$$

and

$$t_e = \frac{r_i}{r}\,\frac{\Omega_0/H_0}{(1-\Omega_0)^{3/2}}(\sinh\eta_e - \eta_e) = \frac{\Omega/\Theta_0}{(1-\Omega)^{3/2}}(\sinh\tilde{\eta}_e - \tilde{\eta}_e), \tag{5.5}$$

where $(\eta_{in})_i$ is the initial value of η in the inner homogeneous region and η_e is the value of η in the self-similar region at an emission epoch with redshift z.

As a result of numerical calculations, the behavior of $\epsilon_m(z)/\epsilon_m(0)$ at each emission epoch is shown in Fig. 3 as a function of z at the emission epochs, in an example in which the boundary between the inner homogeneous region and the self-similar region corresponds to the epoch $z_1 = 1.5$ and 2.0. In these two cases, the ratio $\epsilon_m(z)/\epsilon_m(0)$ is about 30–20% (for $z = 2$–3) larger than that in the ordinary homogeneous model. If we compare this figure with the corresponding one in the previous section, this approximate treatment is found to be good enough and useful.

This consistency may be due to the situation in which the anisotropy is comparatively small and so the coupling between different modes is negligible. In fact, the anisotropy is defined by

$$A \equiv H_t/\Theta - 1, \tag{5.6}$$

and it is shown that, for $\eta \ll 1$, $A \simeq 0.033\eta^2$, and for $\eta = 1.0$, 1.5, 2.0, and 3.0, $A = 0.031$, 0.065, 0.104, and 0.177, respectively. On the other hand, the values of η at the emission and present epochs in the self-similar region are nearly equal to or smaller than 2.0 and 3.0, respectively.

VI. CONCLUDING REMARKS

In this paper the gravitational instability in the self-similar region in an inhomogeneous cosmological model was first studied by paying attention to the self-similar hypersurface with $\xi=$const. The perturbation equations reduced to the ordinary differential equations with respect to ξ by expanding perturbations in terms of $\exp[ip\ln(r/r_i)]$. However, the formalism was found to be useless for $\xi>\xi_{\rm cr}$, except for those on large scales such that $p=0\sim1$ and $l\approx1$, because the above hypersurface becomes timelike in the radial direction.

Next, perturbations were treated approximately as those in local homogeneous (isotropic) models with the same density parameter $\Omega(r)$ as in the inhomogeneous model. This approximation is good for perturbations on small scales, but it was found that it gives the result consistent with that in the above treatment on large scales (for $p=0$ and $l\approx1$). In the super-horizon-scale inhomogeneous (void) model consisting of the inner low-density homogeneous ($z\leqslant z_1$) and the outer self-similar region ($z\geqslant z_1$), the density contrast in the latter region was shown to be larger by about 20% than that in the corresponding one in the ordinary homogeneous model in the case of $\Omega_0=0.2$ and $z_1=1.5$–2.0. If we take a steeper void model with a larger density gap in the boundary between the inner and outer regions, the density contrast in the outer region will be larger, compared with that of the ordinary homogeneous model.

The latter treatment will be improved by taking into account the deviation of the background from its local inhomogeneity as an expansion with respect to the ratio of the spatial scales of perturbations to the scale of the background inhomogeneity. The present analysis is in the zeroth order of this expansion. In the next order the coupling will appear between two modes such as scalar and tensor perturbations.

APPENDIX A: TENSOR CALCULUS IN THE SUBMANIFOLDS

In the two submanifolds M_2 and $\overline{M_2}$ with metrics g_{AB} and $\overline{g}_{AB}(=g_{AB}/r^2)$, the covariant derivatives to vectors k^A and $\overline{k^A}$ are defined as

$$k^A{}_{|B}=k^A{}_{,B}+\Gamma^A_{BC}k^C, \tag{A1}$$

$$\overline{k^A}_{\|B}=\overline{k^A}_{,B}+\overline{\Gamma^A_{BC}}\,\overline{k^C}. \tag{A2}$$

The upper and lower indices of tensors in M_2 and $\overline{M_2}$ are changed using the metric g_{AB} and $\overline{g}_{AB}$, respectively. The Christoffel symbols $\Gamma^C_{AB}[=1/2g^{CD}(g_{DA,B}+g_{DB,A}-g_{AB,D})]$ and $\overline{\Gamma^C_{AB}}[=1/2\,\overline{g}^{CD}(\overline{g}_{DA,B}+\overline{g}_{DB,A}-\overline{g}_{AB,D})]$ are related by

$$\overline{\Gamma^C_{AB}}=\Gamma^C_{AB}+\delta^1_B\delta^C_A+\delta^1_A\delta^C_B-\overline{g}^{C1}\overline{g}_{AB}. \tag{A3}$$

Accordingly, we have, for $k^A=r^n\overline{k^A}, k_A=r^{n+2}\overline{k_A}$,

$$k^A{}_{|B}=r^n[\overline{k^A}_{\|B}+(n+1)\delta^1_B\overline{k^A}-\overline{g}^{A1}\overline{k_B}+\delta^A_B\overline{k^1}], \tag{A4}$$

$$k_{A|B}=r^{n+2}[(\overline{k_A})_{\|B}+(n+1)\delta^1_B\overline{k_A}-\delta^1_A\overline{k_B}+\overline{g}_{AB}\overline{k^1}], \tag{A5}$$

where $n=0$ for k^A in Sec. III. In the same way we have, for $k^{AB}=r^n\overline{k^{AB}},\ k_{AB}=r^{n+4}\overline{k_{AB}}$,

$$k^{AB}{}_{|C}=r^n[k^{AB}{}_{\|C}+(n+2)\delta^1_C\overline{k^{AB}}+\delta^B_C\overline{k^{1A}}+\delta^A_C\overline{k^{1B}} -\overline{g}^{B1}\overline{k^A_C}-\overline{g}^{A1}\overline{k^B_C}], \tag{A6}$$

$$k_{AB|C}=r^{n+4}[k_{AB\|C}+(n+2)\delta^1_C\overline{k_{AB}}+\overline{g}_{BC}\overline{k^1_A}+\overline{g}_{AC}\overline{k^1_B} -\delta^1_B\overline{k_{AC}}-\delta^1_A\overline{k_{BC}}], \tag{A7}$$

where $n=-2$ for k^{AB} in Sec. III.

The components of barred symbols in coordinates $(x^0,x^1)=(\xi,x)$ are

$$(\overline{\Gamma^0_{00}},\overline{\Gamma^1_{00}})=(\xi,-1)/N^2, \tag{A8}$$

$$(\overline{\Gamma^0_{01}},\overline{\Gamma^1_{01}},\overline{\Gamma^0_{11}},\overline{\Gamma^1_{11}})=(-\xi,1,M^2,\xi)MM_{,0}/N^2. \tag{A9}$$

In an arbitrary tensor $\overline{l}_{ABC}$ in $\overline{M}_2$, we have a formula

$$(r^2\overline{l}_{ABC})^{|C}=(\overline{l}_{ABC})^{\|C}-\delta^1_A\overline{l}_{DB}{}^D-\delta^1_B\overline{l}_{AD}{}^D+\overline{l}^1{}_{BA} +\overline{g}^{1C}\overline{l}_{ACB}. \tag{A10}$$

Here it should be noticed that $\delta^1_{A|B}$ vanishes in $\overline{M}_2$, but $\delta^1_{A\|B}$ does not vanish as

$$\delta^1_{A\|B}=-\overline{\Gamma^1_{AB}}. \tag{A11}$$

Using Eq. (A10), we obtain for k_{AB} with $n=-2$ the following relation, for example,

$$\begin{aligned}R^{-2}(R^2k_{AB|C})^{|C}=&\overline{k}_{AB\|C}-2\delta^1_A(\overline{k}_{BC})^{\|C}-2\delta^1_B(\overline{k}_{AC})^{\|C}\\&+2\overline{k^1_{A\|B}}+2\overline{k^1_{B\|A}}-2\delta^1_A\overline{k^1_B}-2\delta^1_B\overline{k^1_A}\\&+2\delta^1_A\overline{k^1_B}\,\overline{k^C_C}-2\overline{g}^{11}\overline{k}_{AB}+2\overline{g}_{AB}\overline{k}^{11}\\&+\overline{\Gamma^1_{AC}}\,\overline{k^C_B}+\overline{\Gamma^1_{BC}}\,\overline{k^C_A},\end{aligned} \tag{A12}$$

$$\begin{aligned}&R^{-2}[R^2(k_{AB|C}-k_{AC|B}-k_{BC|A})]^{|C}\\&=2[(S_{,0}/S)\overline{g}^{C0}+\overline{g}^{C1}][\overline{k}_{AB\|C}-\overline{k}_{AC\|B}-\overline{k}_{BC\|A}\\&\quad-2\overline{g}_{AB}\overline{k^1_C}+2\delta^1_C\overline{k}_{AB}]+k_{AB\|C}{}^{\|C}+\delta^1_B\overline{g}^{CD}\overline{k}_{DC\|A}\\&\quad-2\overline{g}_{AB}\overline{k^1_C}{}^{\|C}-2\overline{g}^{CD}\overline{\Gamma^1_{CD}}\,\overline{k}_{AB},\end{aligned} \tag{A13}$$

and

$$\begin{aligned}R^{-2}(R^2k_{AC})^{|AC}=&r^{-2}[(\overline{k^{AC}})_{\|AC}+(N\overline{k^{1C}})_{,C}/N-(\overline{k^C_C})^{\|1}\\&+2\overline{v}^C(\overline{k^A_{C\|A}}+2\overline{k^1_C}-\delta^1_C\overline{k^D_D})\\&+2S^{-2}(S^2\overline{v}_A\overline{k^A_C})_{\|C}+4\overline{v}_A\overline{k}^{B1}].\end{aligned} \tag{A14}$$

APPENDIX B: BASIC EQUATIONS IN THE EVEN PARITY

From Eq. (3.29) we obtain

$$\overline{k}^{11}=-(\overline{g}_{00}\overline{k}^{00}+2\overline{g}_{01}\overline{k}^{01})/\overline{g}_{11} \tag{B1}$$

and from Eq. (3.27)

$$(\overline{T}^{0}/\overline{\rho})_{,0}=\tfrac{1}{2}\overline{k}_{00}=\tfrac{1}{2}(N/M)^2(\overline{k}^{00}+2\xi\overline{k}^{01}). \tag{B2}$$

From Eq. (3.28) we get for $A=0$

$$\begin{aligned}(\overline{T}^{00}/\overline{\rho})_{,0}=&-(ip+2-2\xi N_{,0}/N)(\overline{T}^{01}/\overline{\rho})\\&+l(l+1)S^{-2}(\overline{T}^{0}/\overline{\rho})-\tfrac{3}{2}(\overline{k}^{00})_{,0}-2\xi(\overline{k}^{01})_{,0}\\&-\overline{k}^{0}+f_{00}\overline{k}^{00}+f_{01}\overline{k}^{01}\end{aligned} \tag{B3}$$

and for $A=1$

$$\begin{aligned}(\overline{T}^{01}/\overline{\rho})_{,0}=&-2(N_{,0}/N)(\overline{T}^{01}/\overline{\rho})-\tfrac{3}{2}M^{-2}(\overline{k}^{00}+2\xi\overline{k}^{01})\\&+h_{00}\overline{k}^{00}+h_{01}\overline{k}^{01},\end{aligned} \tag{B4}$$

where

$$f_{00}=-\tfrac{3}{2}\frac{ip\xi}{M^2}-\xi\left(\frac{1}{N^2}+\frac{1}{M^2}\right)+\frac{\xi^2}{N^2}\frac{M_{,0}}{M}, \tag{B5}$$

$$\begin{aligned}f_{01}=&-\left(1+\frac{3\xi^2}{M^2}\right)ip+\left(3+\frac{4\xi^2}{M^2}\right)\left(\frac{\xi MM_{,0}}{N^2}-1\right)\\&+\xi^2\left(\frac{2}{M^2}-\frac{1}{N^2}\right)-\xi\left(\frac{2M_{,0}}{M}+\frac{N_{,0}}{N}\right),\end{aligned} \tag{B6}$$

$$h_{00}=\frac{ip}{2M^2}+\left(\xi\frac{M_{,0}}{M}-1\right)\left(\frac{1}{N^2}-\frac{1}{M^2}\right), \tag{B7}$$

$$h_{01}=\xi\left(\frac{1}{N^2}-\frac{2}{M^2}\right)+\left(1+\frac{2\xi^2}{M^2}\right)\frac{\xi^2}{N^2}\frac{M_{,0}}{M}-\frac{\xi^2}{M^2}\frac{N_{,0}}{N}+\frac{ip\xi}{M^2}. \tag{B8}$$

From Eq. (3.30) we get, for $A=0$,

$$\begin{aligned}(M/N)^2\overline{k}_{,0}+(\overline{k})^{00}_{,0}=&\kappa\overline{T}^{0}-ip\xi N^{-2}\overline{k}-2(N_{,0}/N)\overline{k}^{00}\\&-(ip+2)\overline{k}^{01}\end{aligned} \tag{B9}$$

and, for $A=1$,

$$(\overline{k}^{01})_{,0}=N^{-2}(-\xi\overline{k}_{,0}+ip\overline{k})+e_{00}\overline{k}^{00}+e_{01}\overline{k}^{01}, \tag{B10}$$

where

$$e_{00}=\frac{1}{N^2}-\frac{1}{M^2}\left(ip+2+\frac{\xi MM_{,0}}{N^2}\right), \tag{B11}$$

$$e_{01}=-\frac{N_{,0}}{N}-\frac{2}{M^2}(ip\xi+2\xi+MM_{,0}). \tag{B12}$$

Moreover, from Eq. (3.32) we obtain, for $A=B=0$,

$$\begin{aligned}&2\left(\frac{M}{N}\right)^2\left\{\left[\left(\frac{M}{N}\right)^2\frac{S_{,0}}{S}+\frac{N_{,0}}{N}+\frac{\xi}{N^2}\right]\overline{k}_{,0}+\left(\frac{S_{,0}}{S}+\frac{\xi}{M^2}\right)\overline{k}^{00}_{,0}\right\}\\&=\kappa\overline{T}^{00}+\left[\frac{2ipM^2}{N^4}\left(-\xi\frac{S_{,0}}{S}+2-\xi\frac{M_{,0}}{M}\right)-\frac{2p^2}{N^2}\right.\\&\left.\quad-\left(\frac{M}{N}\right)^2\frac{l^2+l-2}{S^2}\right]\overline{k}+c_{00}\overline{k}^{00}+c_{01}\overline{k}^{01},\end{aligned} \tag{B13}$$

for $A=0,\ B=1$,

$$\begin{aligned}&\frac{2}{N^2}\left[\left(\frac{M}{N}\right)^2\left(\xi\frac{S_{,0}}{S}+\xi\frac{M_{,0}}{M}+2\frac{\xi^2}{N^2}\right)+ip\right]\overline{k}_{,0}-\frac{2}{N^2}\left(1-\xi\frac{S_{,0}}{S}\right)\overline{k}^{00}\\&=\kappa\overline{T}^{01}+\frac{1}{N^2}\left\{2ip\left(\frac{M}{N}\right)^2\left[-\left(1+\frac{2\xi^2}{M^2}\right)\frac{S_{,0}}{S}+\frac{M_{,0}}{M}+\frac{2\xi}{M^2}\right]\right.\\&\left.\quad-\xi\frac{l^2+l-2}{S^2}\right\}\overline{k}+d_{00}\overline{k}^{00}+d_{01}\overline{k}^{01},\end{aligned} \tag{B14}$$

where

$$\begin{aligned}c_{00}=&\kappa\overline{t}^{00}-\frac{2}{N^2}\left(NN_{,00}+\xi\frac{N_{,0}}{N}+1\right)+\frac{2ip}{N^2}\left(-1+\xi\frac{S_{,0}}{S}\right)\\&-\frac{2MM_{,0}}{N^2}\left(\frac{M^2}{N^2}+2\right)\frac{S_{,0}}{S}-\frac{2M^2}{N^2}\left[\frac{S_{,00}}{S}+\left(\frac{S_{,0}}{S}\right)^2\right.\\&\left.+\left(\frac{2}{M^2}+\frac{1}{N^2}\right)\xi\frac{S_{,0}}{S}\right]-\frac{l(l+1)}{S^2},\end{aligned} \tag{B15}$$

$$c_{01}=\xi\kappa\overline{t}^{00}-\frac{4M^2}{N^2}\left(\frac{S_{,0}}{S}+\frac{\xi}{M^2}\right)(2+ip), \tag{B16}$$

$$\begin{aligned}d_{00}=&\frac{\xi}{2M^2}\kappa\overline{t}^{00}-\frac{2ip}{N^2}\left(\frac{S_{,0}}{S}\right)+\frac{\xi}{M^2}-\frac{2\xi}{N^2}\left[\frac{S_{,00}}{S}+\left(\frac{S_{,0}}{S}\right)^2\right]\\&+\frac{2\xi}{N^4}\left(1-\xi\frac{S_{,0}}{S}\right)-\frac{2\xi M^2}{N^4}\frac{M_{,0}}{M}\left(\frac{S_{,0}}{S}+\frac{\xi}{M^2}\right),\end{aligned} \tag{B17}$$

$$\begin{aligned}d_{01}=&\left(1+\frac{2\xi^2}{M^2}\right)\frac{1}{2}\kappa\overline{t}^{00}-\frac{l(l+1)}{S^2}\\&+2\left(\frac{-N_{,00}}{N}+\frac{1}{N^2}-\frac{\xi}{N^2}\frac{N_{,0}}{N}\right)-\frac{4}{N^2}\left(\frac{S_{,0}}{S}+\frac{\xi^2}{M^2}\right)\\&\times\left[ip\xi+(M^2+2\xi^2)\frac{M_{,0}M}{N^2}\right]-\frac{4\xi}{N^2}\left(2\frac{S_{,0}}{S}-\frac{M_{,0}}{M}+\frac{2\xi^2}{M^2}\right)\\&+\frac{4}{N^2}\left(1-\frac{\xi MM_{,0}}{N^2}\right)\left(1-\xi\frac{S_{,0}}{S}\right)-\frac{2M^2}{N^4}\left(1-\frac{\xi M_{,0}}{M}\right).\end{aligned} \tag{B18}$$

APPENDIX C: SERIES EXPANSION

At the early stage ($\xi\simeq 0$), S and ξ in Eqs. (2.9) and (2.14) are expanded in terms of η and, by expanding them inversely, the following relations are obtained in terms of ξ and X, where X is defined in Eqs. (4.4) and (4.5). First we have

$$\eta = X^{1/2}[1-\tfrac{1}{60}X+\tfrac{1}{1400}X^2+\cdots], \tag{C1}$$

$$S/S_* = \tfrac{1}{2}X[1+\tfrac{1}{20}X-\tfrac{3}{2800}X^2+\cdots], \tag{C2}$$

where

$$S_* \equiv \frac{\Omega_0}{2(1-\Omega_0)}S_0 = \sqrt{\alpha_0}\xi_* . \tag{C3}$$

Moreover,

$$S_{,0}/S = \frac{2}{3\xi}[1+\tfrac{1}{20}X-\tfrac{13}{2800}X^2+\cdots], \tag{C4}$$

$$N = \frac{\xi_*}{6\sqrt{\sigma_0}}X[1-\tfrac{1}{20}X+\tfrac{9}{2800}X^2+\cdots], \tag{C5}$$

$$M^2 = N^2-\xi^2 = \frac{(\xi_*)^2}{36\sigma_0}X^2[1-(\tfrac{1}{10}+\sigma_0)X+\tfrac{1}{112}X^2+\cdots], \tag{C6}$$

$$N_{,0}/N = \frac{2}{3\xi}[1-\tfrac{1}{20}X+\tfrac{11}{2800}X^2+\cdots], \tag{C7}$$

where

$$\sigma_0 = (1+\alpha_0)/\alpha_0 . \tag{C8}$$

For the energy density,

$$\kappa \overline{t}^{00} = \kappa\rho = \tfrac{8}{3}\xi^{-2}[1-\tfrac{1}{20}X+\tfrac{23}{560}X^2+\cdots]. \tag{C9}$$

For the metric perturbations,

$$\overline{k} = \xi^{\alpha}(1+a_1X+\cdots), \tag{C10}$$

$$\overline{k}^{00} = b_0\xi^{\alpha}(1+b_1X+\cdots), \tag{C11}$$

$$\overline{k}^{01} = -\sigma_0 c_0 \xi^{\alpha-1}(1+c_1X+\cdots). \tag{C12}$$

From Eq. (B10) we first obtain

$$(\alpha+\tfrac{5}{3})(c_0-1) = (\tfrac{5}{3}+ip)(b_0-1) \tag{C13}$$

in the lowest order and

$$c_1 = -[30(\alpha+7/3)c_0]^{-1}\{-3\alpha-4b_0-3c_0-a_1(20+30\alpha) -50b_0b_1+50(2c_0-b_0)\sigma_0+ip[3-3b_0+30a_1 -30b_0b_1+30(2c_0-b_0)\sigma_0]\} \tag{C14}$$

in the next order. From Eqs. (B9), (B13), and (B14), we obtain

$$(\overline{T}^{00}/\overline{\rho})\xi^{-\alpha} = \tfrac{1}{2}\alpha(2+b_0)+\frac{X}{120}\{6\alpha+40(2+3\alpha)a_1+6\alpha b_0 +20(2+3\alpha)b_0b_1+5(l^2+l-2)(1+b_0) \times(\sigma_0-1)+10[-9\alpha+(10+3\alpha)b_0 -12c_0+9p^2]\sigma_0+30(-2+b_0-4c_0)ip\sigma_0\}, \tag{C15}$$

$$(\overline{T}^{01}/\overline{\rho})\xi^{1-\alpha}/(\sigma_0X) = \tfrac{1}{4}[(4-b_0)\alpha+ip(3\alpha+2b_0)] +\frac{X}{240}\{36\alpha+80(2+3\alpha)a_1 +(20-3\alpha)b_0-20(2+3\alpha)b_0b_1 +20c_0+10(l^2+l-2) \times(\sigma_0-1)(1-c_0) +20(3\alpha+9b_0-7c_0)\sigma_0+[12+27\alpha +60(2+3\alpha)a_1+24(1+5b_1)b_0 +60(-1+3b_0-4c_0)\sigma_0]ip\}, \tag{C16}$$

$$(\overline{T}^{0}/\overline{\rho})\xi^{-1-\alpha} = \frac{1}{8}(3\alpha+4b_0+3\alpha b_0) +\frac{X}{160}[3\alpha+20(2+3\alpha)a_1+3\alpha b_0 +60(2+\alpha)b_0b_1-60(\alpha+2c_0)\sigma_0 +60(1-c_0)ip\sigma_0], \tag{C17}$$

Moreover, we obtain from Eqs. (B2), (B3), and (B4):

$$ch00 \equiv \alpha[2+2\alpha+(3+\alpha)b_0]+\frac{X}{90}\{21\alpha+9\alpha^2+20(10+21\alpha+9\alpha^2)a_1+(20+21\alpha+9\alpha^2)b_0+10(22+39\alpha+90\alpha^2)b_0b_1 +5(l^2+l-2)(1-b_0)(\sigma_0-1)+15\alpha(1-9\alpha)\sigma_0+(320+135\alpha+45\alpha^2)b_0\sigma_0-540(1+\alpha)c_0\sigma_0+90(1-b_0)p^2\sigma_0 +60(-1+3\alpha+6b_0-5c_0-3\alpha c_0)ip\sigma_0\}=0, \tag{C18}$$

$$ch01 \equiv \alpha[4+4\alpha+(1-\alpha)b_0+ip(3+3\alpha+2b_0)]+\frac{X}{180}\{12(11\alpha+9)\alpha+80(10+21\alpha+9\alpha^2)a_1+(100+93\alpha-9\alpha^2)b_0$$
$$+20(2-3\alpha-9\alpha^2)b_0b_1-20(5+3\alpha)c_0+(l^2+l-2)(\sigma_0-1)10(5+3\alpha)(1-c_0)+60[5\alpha+3\alpha^2+19b_0$$
$$+15\alpha b_0-18c_0(1+\alpha)]\sigma_0+3ip[20+45\alpha+27\alpha^2+20(10+21\alpha+9\alpha^2)a_1+4(5+6\alpha)b_0+40(2+3\alpha)b_0b_1$$
$$-20(5+3\alpha)\sigma_0+180(1+\alpha)b_0\sigma_0-80(2+3\alpha)c_0\sigma_0]\}$$
$$=0, \quad \text{(C19)}$$

$$ch0 \equiv \alpha(3+3\alpha+7b_0+3\alpha b_0)+4(2c_0\sigma_0-b_0\sigma_0-b_0b_1)+\frac{X}{20}(\alpha+\tfrac{5}{3})[3\alpha+20(2+3\alpha)a_1+3\alpha b_0+120(1+\alpha)b_0b_1$$
$$-60(\alpha+2c_0)\sigma_0+60(1-c_0)ip\sigma_0]=0. \quad \text{(C20)}$$

From Eq. (3.38) we obtain

$$\epsilon_m/\xi^\alpha = \tfrac{1}{4}\alpha(1-b_0)+\frac{X}{120}[10(2+3\alpha)(a_1-b_0b_1)$$
$$+(10\sigma_0-2)b_0+5(l^2+l-2)(1+b_0)(\sigma_0-1)$$
$$+30(8\alpha+7b_0-\alpha b_0-6c_0+3p^2)\sigma_0$$
$$+30(-5+6\alpha+5b_0-c_0)ip\sigma_0]. \quad \text{(C21)}$$

APPENDIX D: PERTURBATIONS IN A HOMOGENEOUS MODEL

The perturbation behavior in a homogeneous model is shown in the Gerlach and Sengupta formalism and the gauge-invariant quantities in this formalism are compared with those in the Bardeen formalism. Tensor and vector spherical harmonics derived by Gerlach and Sengupta Ref. [15] are used. The following background metric is used here:

$$ds^2=-c^2dt^2+S^2(t)\left[\frac{dr^2}{1+\alpha r^2}+r^2d\Omega^2\right], \quad \text{(D1)}$$

where $x^0=ct$, $x^1=r$, $x^2=\theta$, and $x^3=\varphi$. Scalar spherical harmonics Q of order n satisfies

$$Q\big|_i^i=-(n^2+1)Q \quad \text{(D2)}$$

in the open model ($k=-1$), where $|i$ is a covariant derivative in the $t=$const hypersurface, and Q is expanded using the usual spherical harmonics Y_{lm} as

$$Q=\Pi_l^n(r)Y_{lm}(\theta,\varphi). \quad \text{(D3)}$$

Then we obtain from the above two equations

$$r(1+\alpha r^2)\Pi_{,11}+(2+3\alpha r^2)\Pi_{,1}$$
$$+[(n^2+1)\alpha r-l(l+1)/r]\Pi$$
$$=0. \quad \text{(D4)}$$

1. Odd parity

a. Tensor perturbations

Tensor harmonics G_{ij} with the odd parity satisfies

$$G_{ik}\big|_j^j=-(n^2+3)G_{ik}, \quad G_{i|k}^k=0, G_i^i=0, \quad \text{(D5)}$$

and the components are expressed using Π_l^n and Y_{im} as

$$G_{11}=0, \quad \text{(D6)}$$

$$G_{22}=\sqrt{\alpha}r^2\sqrt{1+\alpha r^2}\left(\Pi_{n,1}^l+\frac{2}{r}\Pi_n^l\right)(-X_{lm}/\sin\theta), \quad \text{(D7)}$$

$$G_{33}=\sqrt{\alpha}r^2\sqrt{1+\alpha r^2}\left(\Pi_{n,1}^l+\frac{2}{r}\Pi_n^l\right)X_{lm}\sin\theta, \quad \text{(D8)}$$

$$G_{23}=\sqrt{\alpha}r^2\sqrt{1+\alpha r^2}\left(\Pi_{n,1}^l+\frac{2}{r}\Pi_n^l\right)W_{lm}\sin\theta, \quad \text{(D9)}$$

$$G_{12}=\sqrt{\frac{\alpha}{1+\alpha r^2}}(l-1)(l+2)\Pi_n^l(-Y_{lm,2}/\sin\theta), \quad \text{(D10)}$$

$$G_{13}=\sqrt{\frac{\alpha}{1+\alpha r^2}}(l-1)(l+2)\Pi_n^lY_{lm,3}\sin\theta, \quad \text{(D11)}$$

where

$$X_{lm}=2(Y_{lm,23}-\cot\theta Y_{lm,3}), \quad \text{(D12)}$$

$$W_{lm}=Y_{lm,22}-\cot\theta Y_{lm,2}-Y_{lm,33}/\sin^2\theta. \quad \text{(D13)}$$

The metric components are

$$h_0=0, \quad h_1=\nu(t)\frac{(l-1)(l+2)}{\sqrt{1+\alpha r^2}}\Pi_n^l, \quad \text{(D14)}$$

$$h=\nu(t)r^2\sqrt{1+\alpha r^2}\left(\Pi_{n,1}^l+\frac{2}{r}\Pi_n^l\right), \quad \text{(D15)}$$

and the gauge-invariant quantities are

$$k_0=-k^0=\left(2\frac{S_{,0}}{S}\nu-\nu_{,0}\right)r^2\sqrt{1+\alpha r^2}\left(\Pi^l_{n,1}+\frac{2}{r}\Pi^l_n\right), \quad \text{(D16)}$$

$$k_1=\frac{S^2}{1+\alpha r^2}k^1=\frac{(n^2+1)\alpha r^2}{\sqrt{1+\alpha r^2}}\nu\Pi^l_n, \quad \text{(D17)}$$

where we used Eq. (D4) for Π^l_n.

From GS (9a) with $L=0$ we obtain an equation for $\nu(t)$:

$$\nu_{,00}-\frac{S_{,0}}{S}\nu_{,0}+\left(\frac{n^2+1}{S^2}\alpha-2\frac{S_{,00}}{S}\right)\nu=0. \quad \text{(D18)}$$

In the limit $t\rightarrow 0$, we have two solutions

$$\nu\propto t^{1/3}[1+0(t^{2/3})], \quad t^{4/3}[1+0(t^{2/3})], \quad \text{(D19)}$$

so that the lowest order of $k^0\propto t^{-2/3}$, t.

b. Vector (rotational) perturbations

Vector harmonics V_i satisfies

$$V_{i|k}{}^{|k}=-(n^2+2)V_i, \quad V_i{}^{|i}=0. \quad \text{(D20)}$$

Their components are expressed as

$$V_1=0, \quad \text{(D21)}$$

$$V_2=-\sqrt{\alpha}r\Pi^n_l Y_{,3}/\sin\theta, \quad \text{(D22)}$$

$$V_3=\sqrt{\alpha}r\Pi^n_l Y_{,2}\sin\theta. \quad \text{(D23)}$$

These V_i $(i=1,2,3)$ are different from the two-dimensional harmonics S_a $(a=2,3)$ on the hypersurface $r=$const, where $S_2=-Y_{,3}/\sin\theta$, $S_3=Y_{,2}\sin\theta$. The metric components are

$$h_0=B(t)r\Pi^n_l, \quad \text{(D24)}$$

$$h_1=H(t)r\left(\Pi^n_{l,1}-\frac{1}{r}\Pi^n_l\right), \quad \text{(D25)}$$

$$h=H(t)r\Pi^n_l, \quad \text{(D26)}$$

and the gauge-invariant quantities are

$$k_0=-k^0=\Psi r\Pi^n_l, \quad \Psi\equiv B-S^2(H/S^2)_{,0}, \quad \text{(D27)}$$

$$k_1=\frac{S^2}{1+\alpha r^2}k^1=0. \quad \text{(D28)}$$

For the vector perturbations in pressureless matter, we obtain from GS (14) with $L^1=L=0$

$$(S^3L^0)_{,0}=0 \ \text{ or } \ L^0\propto S^{-3} \quad \text{(D29)}$$

and GS (9b) with $A=0$ and Eq. (D4) give us

$$k^0\propto\Psi=-\frac{\kappa(S^3L^0)}{(n^2+4)\alpha Sr}, \quad \text{(D30)}$$

where $\kappa=16\pi G/c^4$. The other equations are also consistent with Eqs. (D29) and (D30). This shows that k^0/S corresponds to Ψ in the Bardeen formalism. Since L^0S_a is $\rho u^0 u_a(\propto\rho u^a R^2, a=2,3)$, $V_c\equiv(L^0/\rho)/R$ is the velocity in the angular direction, corresponding to v_c in the Bardeen formalism.

2. Even parity

a. Scalar perturbations.

Scalar spherical harmonics Π satisfies Eq. (D2) and are expressed as Eq. (D3). The metric components are expressed as

$$h_{00}=-2A\Pi, \quad \text{(D31)}$$

$$h_{01}=SB\Pi_{,1}, \quad h_0=SB\Pi, \quad \text{(D32)}$$

$$h_1=2S^2H_T(\Pi_{,1}-\Pi/r), \quad \text{(D33)}$$

$$h_{11}=S^2\left[2H_1\frac{\Pi}{1+\alpha r^2}+2H_T\left(\Pi_{,11}+\frac{\alpha r}{1+\alpha r^2}\Pi_{,1}\right)\right], \quad \text{(D34)}$$

$$K=2H_1\Pi+2H_T(1+\alpha r^2)\Pi_{,1}/r, \quad \text{(D35)}$$

$$G=2H_T\Pi/r^2. \quad \text{(D36)}$$

The gauge-invariant quantities are

$$k_{00}=-2\left[A+S\left(B_{,0}+\frac{S_{,0}}{S}B\right)-S^2\left(H_{T,00}+2\frac{S_{,0}}{S}H_{T,0}\right)\right]\Pi, \quad \text{(D37)}$$

$$k_{01}=0, \quad \text{(D38)}$$

$$k_{11}=\frac{2S^2}{1+\alpha r^2}\left[H_1+\frac{S_{,0}}{S}(SB-S^2H_{T,0})\right]\Pi, \quad \text{(D39)}$$

$$k=k^1_1=\frac{1+\alpha r^2}{S^2}k_{11}, \quad \text{(D40)}$$

where k^0_0 and k^1_1 correspond to $2\Phi_A$ and $2\Phi_H$, respectively. GS (10d) gives $k^A_A=k^0_0+k^1_1=0$ or $\Phi_A+\Phi_H=0$.

For pressureless matter we have $\delta\rho=\delta\cdot\rho\Pi$, $\delta u^0=-A\Pi$, and $\delta u^i=v[(1+\alpha r^2)\Pi_{,1}Y, r^{-2}\Pi Y_{,a}]$, and, for the energy-momentum tensor $t_{\mu\nu}=\rho u_\mu u_\nu$, we have its perturbations

$$\Delta t_{00}=(\delta\cdot\rho+2\rho A)\Pi, \quad \text{(D41)}$$

$$(\Delta t_0,\Delta t_{01})=-\rho(S^2v+SB)(\Pi,\Pi_{,1}), \quad \text{(D42)}$$

and

$$\Delta t_1=\Delta t_{11}=0. \quad \text{(D43)}$$

Corresponding gauge-invariant quantities are

$$T_0\equiv-\rho S^2V_s=-\rho S^2(v+H_{T,0}), \quad \text{(D44)}$$

$$T_1=T_{11}=0, \quad \text{(D45)}$$

$$T_{00}=\rho(\epsilon_g\Pi+k^0_0), \quad \text{(D46)}$$

$$T_{01}=-\rho S^2V_{s,1}, \quad \text{(D47)}$$

where

$$\epsilon_g \equiv \delta - 3\frac{S_{,0}}{S}(SB - S^2 H_{T,0}). \tag{D48}$$

The energy density contrast ϵ_m defined in a comoving synchronous system is

$$\epsilon_m \Pi = \epsilon_g \Pi - 3\frac{S_{,0}}{S}S^2 V_s = \frac{1}{\rho}\left(T_{00} + 3\frac{S_{,0}}{S}T_0\right) - k^0_0. \tag{D49}$$

GS (10b) and GS (15a) give

$$(Sk)_{,0} = \tfrac{1}{2}\kappa(\rho S^3)V_s \tag{D50}$$

and

$$(S^2 V_s)_{,0} = \tfrac{1}{2}k_{00} = \tfrac{1}{2}k, \tag{D51}$$

respectively, from which we obtain

$$V_{s,00} + 5\frac{S_{,0}}{S}V_{s,0} + \left[3\frac{S_{,00}}{S} + 3\left(\frac{S_{,0}}{S}\right)^2 + \frac{\alpha}{S^2}\right]V_s = 0. \tag{D52}$$

GS (10a) with $A = B = 0$ leads to

$$\kappa\rho\epsilon_m \Pi = \kappa\left(T_{00} + \rho k + 3\frac{S_{,0}}{S}T_0\right) = \frac{2}{S^2}(n^2+4)\alpha k. \tag{D53}$$

From Eqs. (D50)–(D53) we obtain

$$\epsilon_{m,00} + 2\frac{S_{,0}}{S}\epsilon_{m,0} + 3\frac{S_{,00}}{S}\epsilon_m = 0 \tag{D54}$$

or

$$\epsilon_m'' + \frac{S'}{S}\epsilon_m' + 3\left(\frac{S'}{S}\right)'\epsilon_m = 0, \tag{D55}$$

where $' = \partial/\partial\eta$ and $c\,dt = \alpha^{-1/2}S\,d\eta$. In the limit $t \to 0 (S \propto t^{2/3})$, we obtain

$$V_s \propto t^{\beta}, \quad \epsilon_m \propto t^{\beta+1} (\beta = -1/3, -2). \tag{D56}$$

b. Vector (rotational) perturbations.

The vector harmonics s_a in the even parity satisfying Eq. (D20) is

$$s_1 = \frac{l(l+1)}{r\sqrt{1+\alpha r^2}}\Pi Y_{lm}, \tag{D57}$$

$$(s_2, s_3) = r\sqrt{1+\alpha r^2}(\Pi_{,1} + \Pi/r)(Y_{lm,2}, Y_{lm,3}/\sin^2\theta), \tag{D58}$$

and the metric components are

$$h_{00} = 0, h_0 = B(t) r\sqrt{1+\alpha r^2}(\Pi_{,1} + \Pi/r), \tag{D59}$$

$$h_{01} = B(t) l(l+1)\Pi/(r\sqrt{1+\alpha r^2}), \tag{D60}$$

$$h_1 = H(t) r\sqrt{1+\alpha r^2} \times \left[-\frac{2}{r}\Pi_{,1} + \frac{-(n^2+2)\alpha r^2 + 2(l^2+l-2)}{r^2(1+\alpha r^2)}\Pi\right], \tag{D61}$$

$$h_{11} = H(t)\frac{2l(l+1)}{r\sqrt{1+\alpha r^2}}(\Pi_{,1} - \Pi/r), \tag{D62}$$

$$K = 2H(t) l(l+1)\Pi\sqrt{1+\alpha r^2}/(S^2 r^2), \tag{D63}$$

$$G = 2H(t) r\sqrt{1+\alpha r^2}(\Pi_{,1} + \Pi/r)/(S^2 r^2). \tag{D64}$$

The gauge-invariant quantities are

$$k_{00} = -2\Psi_{,0} r\sqrt{1+\alpha r^2}(\Pi_{,1} + \Pi/r), \tag{D65}$$

$$k_{01} = \Psi\frac{n^2\alpha r}{\sqrt{1+\alpha r^2}}\Pi, \tag{D66}$$

$$k_{11} = 2SS_{,0}\Psi\frac{r}{\sqrt{1+\alpha r^2}}(\Pi_{,1} + \Pi/r), \tag{D67}$$

$$k = k^1_1, \tag{D68}$$

where

$$\Psi = B - S^2(H/S^2)_{,0}. \tag{D69}$$

The sum $k^A_A = 0$ gives

$$\Psi_{,0} + \frac{S_{,0}}{S}\Psi = 0 \quad \text{or} \quad \Psi \propto 1/S. \tag{D70}$$

From GS (10b) we obtain

$$\kappa T_0 = [\kappa\rho\Psi + (n^2+4)\alpha\Psi/S^2] r\sqrt{1+\alpha r^2}(\Pi_{,1} + \Pi/r), \tag{D71}$$

where $T_0 s_a = \rho u_0 u_a = -\rho u^a R^2$, and so

$$V_s \equiv \frac{T_0}{\rho R} = \frac{\Psi}{S}\sqrt{1+\alpha r^2}(\Pi_{,1} + \Pi/r) + V_c, \tag{D72}$$

$$V_c \equiv \frac{(n^2+4)\alpha}{2\rho S^3}\Psi\sqrt{1+\alpha r^2}(\Pi_{,1} + \Pi/r) \propto 1/S. \tag{D73}$$

In these equations, V_s, V_c, and Ψ/S correspond to Bardeen's v_s, v_c, and Ψ. Moreover, we obtain

$$T_{00}/\rho = \left(2\Psi_{,0} - 3\frac{S_{,0}}{S}\Psi\right) r\sqrt{1+\alpha r^2}(\Pi_{,1} + \Pi/r), \tag{D74}$$

and $T_{01} = T_{0,1}$, $T_1 = T_{11} = 0$.

c. Tensor perturbations

The tensor harmonics in the even parity satisfying Eq. (D5) is

$$G_{11}=\frac{L}{r^2(1+\alpha r^2)}\Pi Y_{lm}, \tag{D75}$$

$$(G_{22},G_{33})=\left[\frac{L}{2}\Pi Y_{lm}+\mathcal{G}_l^n W_{lm}\right](1,-\sin^2\theta), \tag{D76}$$

$$G_{23}=\mathcal{G}_l^n X_{lm}, \tag{D77}$$

$$G_{12}=(l-1)(l+2)(\Pi_{,1}+\Pi/r)Y_{lm,2}, \tag{D78}$$

where $L\equiv l(l+1)(l-1)(l+2)$,

$$\mathcal{G}_l^n=r(1+\alpha r^2)\Pi_{,1}+[\tfrac{1}{2}(l^2+l+2)-(n^2-1)\alpha r^2]\Pi, \tag{D79}$$

and X_{lm},W_{lm} are defined in Eqs. (D12) and (D13).

The metric components are

$$h_0=h_{00}=h_{01}=0, \tag{D80}$$

$$h_1=\frac{\nu(t)}{l(l+1)}(\Pi_{,1}+\Pi/r), \tag{D81}$$

$$h_{11}=\frac{\nu(t)\Pi}{r^2(1+\alpha r^2)}, \tag{D82}$$

$$K=\frac{\nu(t)}{S^2r^2}\left[-\frac{1}{2}\Pi+\frac{\mathcal{G}_l^n}{(l-1)(l+2)}\right], \tag{D83}$$

$$G=\frac{2\nu(t)}{L}\frac{\mathcal{G}_l^n}{S^2r^2}, \tag{D84}$$

and the gauge-invariant quantities are

$$k_{00}=[S^2(\nu/S^2)_{,0}]_{,0}\frac{2}{L}\mathcal{G}_l^n, \tag{D85}$$

$$k_{01}=-\frac{2n^2\alpha}{L}\left(\nu_{,0}-2\frac{S_{,0}}{S}\nu\right)(r^2\Pi)_{,1}, \tag{D86}$$

$$k_{11}=-\frac{2}{L}\frac{\mathcal{G}_l^n}{1+\alpha r^2}\left[SS_{,0}\left(\nu_{,0}-2\frac{S_{,0}}{S}\nu\right)+(n^2+1)\alpha\nu\right], \tag{D87}$$

$$T_0/\rho=-\tfrac{1}{2}S^2(\nu/S^2)_{,0}\frac{2}{L}\mathcal{G}_l^n, \tag{D88}$$

and $T_{01}=T_{0,1}$, $T_{00}/\rho=-k_{00}-3(\dot{S}/S)T_0/\rho$, $T_1=T_{11}=0$, where we used Eq. (D4). The sum $k_A^A=k_0^0+k_1^1=-k_{00}+(1+\alpha r^2)k_{11}/S^2$ vanishes owing to GS (10d), so that we have, from Eqs. (D85) and (D87),

$$\nu_{,00}-\frac{S_{,0}}{S}\nu_{,0}+\left(\frac{n^2+1}{S^2}\alpha-2\frac{S_{,00}}{S}\right)\nu=0 \tag{D89}$$

or, for $\tilde{\nu}=\nu/S^2$,

$$\tilde{\nu}_{,00}+3\frac{S_{,0}}{S}\tilde{\nu}+\left(\frac{n^2+1}{S^2}\right)\alpha\tilde{\nu}=0. \tag{D90}$$

In the limit $t\to 0$, $\nu=t^\beta(1+\nu_0t^{2/3}+\cdots)$, where ν_0 is a constant coefficient. Solving Eq. (D88), we obtain $\beta=1/3$ or 4/3. For $\beta=1/3$, we have $k_{00}\simeq k\propto t^{-5/3}$, and for $\beta=4/3$, $k_{01}/k_{00}\propto t$, $k_{00}\simeq 1/2k\propto$const, and $k_{01}/k_{00}\propto t$.

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